The Sum of Two S-Units Being a Square Where $S = \{2, p\}$

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Suppose $S = \{2, p\}$, where $p \geq 3$ is a prime and let $\tilde{S}$ denote the set of positive rational integers which have no prime divisors outside of S. We study the equation

$$x + y = z^2$$

in x, y $\tilde{S}$-units, and $z \in \mathbb{Q}$, where the set of $\tilde{S}$-unit in this case is defined as

$$\{\pm 2^{x_1} p^{x_2} \mid x_i \in \mathbb{Z} \text{ for } i = 1, 2\}.$$

Clearing denominators, without loss of generality, we may study the equation

$$x + y = z^2$$

where

$$\begin{cases} x \in \tilde{S}, & \pm y \in \tilde{S}, \\ x \geq y, & z \in \mathbb{Z}, \\ z > 0, & \gcd(x, y) \text{ is squarefree.} \end{cases}$$

In other words,

$$\begin{cases} x \geq 0, & \operatorname{rad}(x) | 2p, \\ y \in \mathbb{Z}, & \operatorname{rad}(y) | 2p, \\ x \geq y, & z \in \mathbb{Z}, \\ z > 0, & \gcd(x, y) \text{ is squarefree.} \end{cases}$$

Hence let $x = 2^a p^b$ and $y = 2^c p^d$. Since

$$\gcd(x, y) = \gcd(2^a p^b, 2^b p^d) = 2^{\min(a,c)} p^{\min(b,d)}$$

is necessarily squarefree, it follows that $\min(a, c) \leq 1$ and $\min(b, d) \leq 1$. This leaves us several cases to consider.

Case	$x + y = z^2$	Additional Conditions
1.	$2^a + p^d = z^2$ $-2^a + p^d = z^2$ $2^a - p^d = z^2$	$a \geq 0,\ d \geq 0$
2.	$2^a p + p^d = z^2$ $-2^a p + p^d = z^2$ $2^a p - p^d = z^2$	$a \geq 0,\ d \geq 1$
3.	$2^a p^b + 1 = z^2$ $-2^a p^b + 1 = z^2$ $2^a p^b - 1 = z^2$	$a \geq 0,\ b \geq 0$
4.	$2^a p^b + p = z^2$ $-2^a p^b + p = z^2$ $2^a p^b - p = z^2$	$a \geq 0,\ b \geq 1$
5.	$2^a + 2p^d = z^2$ $-2^a + 2p^d = z^2$ $2^a - 2p^d = z^2$	$a \geq 1,\ d \geq 0$
6.	$2^a p + 2p^d = z^2$ $-2^a p + 2p^d = z^2$ $2^a p - 2p^d = z^2$	$a \geq 1,\ d \geq 1$
7.	$2^a p^b + 2 = z^2$ $-2^a p^b + 2 = z^2$ $2^a p^b - 2 = z^2$	$a \geq 1,\ b \geq 0$
8.	$2^a p^b + 2p = z^2$ $-2^a p^b + 2p = z^2$ $2^a p^b - 2p = z^2$	$a \geq 1,\ b \geq 1$