Computing elliptic curves over $\mathbb{Q}$ via Thue-Mahler equations, and related problems

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Thue-Mahler equations

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  - a is a fixed integer
  - $x, y, z_1, \ldots, z_v$ are unknown integers

Our main objective

Given S, F and a, find all solutions $(x, y, z_1, \ldots, z_v)$ satisfying

$$F(x, y) = a p_1^{z_1} \cdots p_v^{z_v}$$

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- **Hambrook (2011)**: Implementation of a Thue-Mahler solver

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Elliptic Curves

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$$E : y^2 = x^3 + ax + b$$

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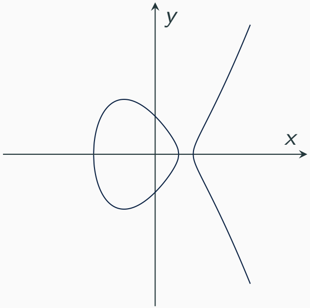
- A curve is nonsingular $\iff \Delta_E = 4a^3 + 27b^2 \neq 0$.

Visualizing subgroups of E

$E : y^2 = x^3 - 2x + 1$ over $\mathbb{R}$



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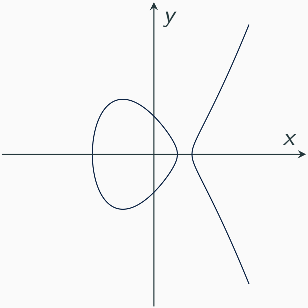
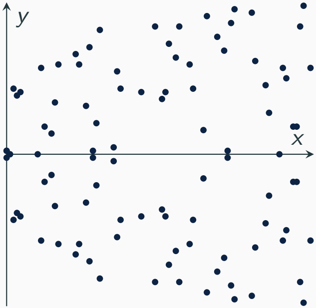


$E : y^2 = x^3 - 2x + 1$ over $\mathbb{F}_{89}$



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- $\Delta_E = p_1^{a_1} \cdots p_n^{a_n} \implies N = p_1^{b_1} \cdots p_n^{b_n}$

Computing Elliptic Curves

Some history

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- **Shafarevich (1963)**: There are at most finitely many elliptic curves having good reduction outside $S = \{p_1, \ldots, p_v\}$
- **Taniyama, Weil (1950s, 1960's)**: A conjecture about elliptic curves of conductor N
    - $S = \{2\}$: Ogg (1966)
    - $S = \{2, 3\}$: Coghlan (1967)
    - $S = \{p\}$ for certain small primes p: Setzer (1975)
    - $S = \{11\}$: Agrawal, Coates, Hunt, and van der Poorten (1980)

The modern approach

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    - Antwerp IV (1972): $N \leq 200$
    - Tingley (1975): $N \leq 320$
    - Cremona (2019): $N \leq 500000$

The Thue-Mahler approach

- Reduce problem to solving a number of Thue-Mahler equations

$$F(x, y) = c_0x^3 + c_1x^2y + c_2xy^2 + c_3y^3 = ap_1^{z_1} \cdots p_v^{z_v}$$

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- Goal: compute all curves having conductor $N \leq 10^6$

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- Ideal Goal: $N \leq 10^8$

Theorem (Bennett, G., Rechnitzer)

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = 2^{\alpha}3^{\beta}N_0$ where N_0 is coprime to 6.

Then there exists an integral binary cubic form F of discriminant

$$D_F = sign(\Delta_E)2^{\alpha_0}3^{\beta_0}N_1,$$

and relatively prime integers u and v with

$$F(u,v) = c_0u^3 + c_1u^2v + c_2uv^2 + c_3v^3 = 2^{\alpha_1}3^{\beta_1}\prod_{p \mid N_0} p^{\kappa_p}$$

such that E is isomorphic over $\mathbb{Q}$ to $E_{\mathcal{D}}$, where

$$E_{\mathcal{D}} : 3^{[\beta_0/3]}y^2 = x^3 - 27\mathcal{D}^2H_F(u,v)x + 27\mathcal{D}^3G_F(u,v).$$

Theorem (Bennett, G., Rechnitzer)

Here, $N_1 \mid N_0$,

$$(\alpha_0, \alpha_1) = \begin{cases} (2,0) \text{ or } (2,3) & \text{if } \alpha = 0 \\ (3, \geq 3) \text{ or } (2, \geq 4) & \text{if } \alpha = 1 \\ (2,1), (4,0) \text{ or } (4,1) & \text{if } \alpha = 2 \\ (2,1), (2,2), (3,2), (4,0) \text{ or } (4,1) & \text{if } \alpha = 3 \\ (2, \geq 0), (3, \geq 2), (4,0) \text{ or } (4,1) & \text{if } \alpha = 4 \\ (2,0) \text{ or } (3,1) & \text{if } \alpha = 5 \\ (2, \geq 0), (3, \geq 1), (4,0) \text{ or } (4,1) & \text{if } \alpha = 6 \\ (3,0) \text{ or } (4,0) & \text{if } \alpha = 7 \\ (3,1) & \text{if } \alpha = 8, \end{cases}$$

Theorem (Bennett, G., Rechnitzer)

$$(\beta_0, \beta_1) = \begin{cases} (0,0) & \text{if } \beta = 0 \\ (0, \geq 1) \text{ or } (1, \geq 0) & \text{if } \beta = 1 \\ (3,0), (0, \geq 0), \text{ or } (1, \geq 0) & \text{if } \beta = 2 \\ (\beta, 0) \text{ or } (\beta, 1) & \text{if } \beta \geq 3, \end{cases}$$

$$\mathcal{D} = \prod_{p \mid \gcd(c_4(E), c_6(E))} p^{\min\{[\nu_p(c_4(E))/2], [\nu_p(c_6(E))/3]\}},$$

and $\kappa_p \in \mathbb{Z}_{>0}$ *with* $\kappa_p \in \{0,1\}$ *whenever* $p^2 | N_1$.

Further,

$$\text{if } \beta_0 \geq 3, \text{ then } 3|c_1 \text{ and } 3|c_2$$

and

$$\text{if } \nu_p(N) = 1, \text{ for } p \geq 3, \text{ then } p \mid D_F F(u,v)$$

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3. Check "local" conditions and output the elliptic curves that arise

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2. Solve the corresponding Thue-Mahler equations

- Let $S = \{2, 3, 5, 7, 11, 13\}$

Representative forms: an example

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- Let $S = \{2, 3, 5, 7, 11, 13\}$
- There are 7893 corresponding forms which need to be solved
    - $F(x, y) = 19x^3 + 69x^2y + 76xy^2 + 126y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
    - $F(x, y) = 22x^3 + 22x^2y + 55xy^2 + 100y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
    - $F(x, y) = 46x^3 + 24x^2y + 85xy^2 + 7y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
    - $F(x, y) = 13x^3 + 3x^2y + 18xy^2 + 14y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
    - $F(x, y) = 17x^3 + 36x^2y + 39xy^2 + 26y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
    - $F(x, y) = 2x^3 + 18x^2y + 21xy^2 + 65y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
    - $F(x, y) = x^3 + 12x^2y + 18xy^2 + 149y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$

$$\vdots$$

Applying the Thue-Mahler solver

Some timing examples

- A nice example

$$x^3 + 3xy^2 + 44xy^2 + 66y^3 = 3^{z_1} \cdot 11^{z_2} \cdot 17^{z_3} \cdot 23^{z_4} \cdot 31^{z_5}$$

- A less nice example

$$3x^3 + 3xy^2 + 44xy^2 + 66y^3 = 3^{z_1} \cdot 11^{z_2} \cdot 17^{z_3} \cdot 23^{z_4} \cdot 31^{z_5}$$

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Total time: 247.580

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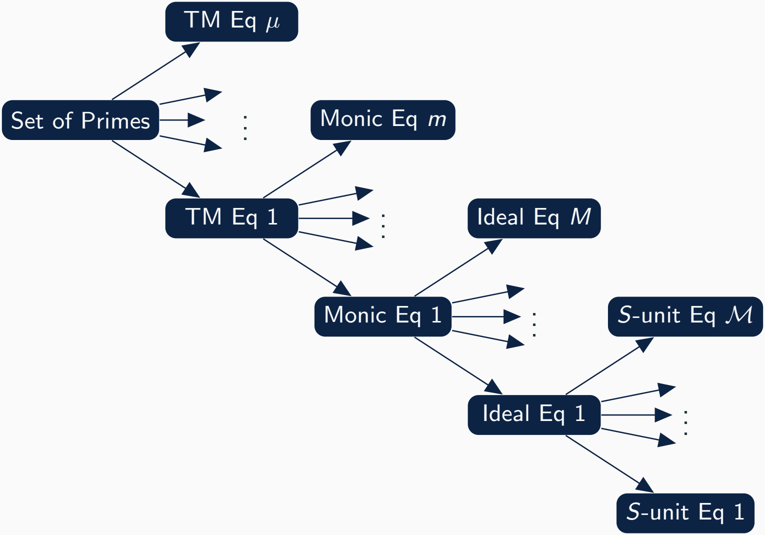
Algorithms for solving Thue-Mahler equations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Overview

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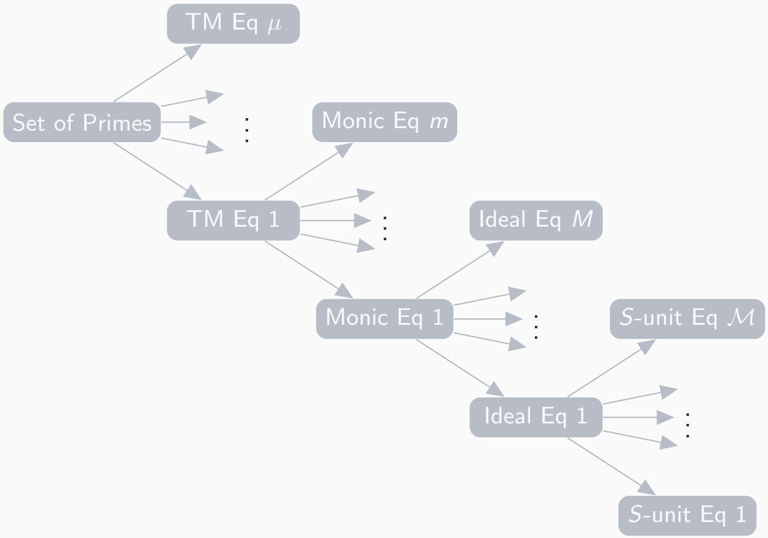

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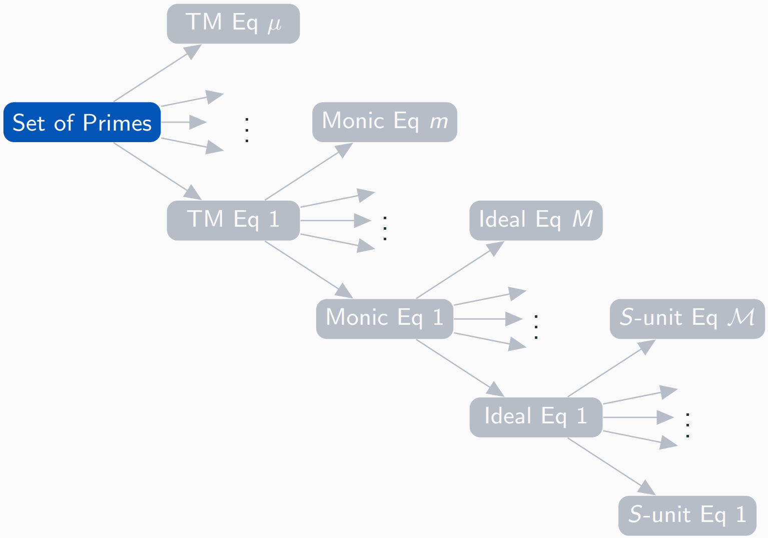
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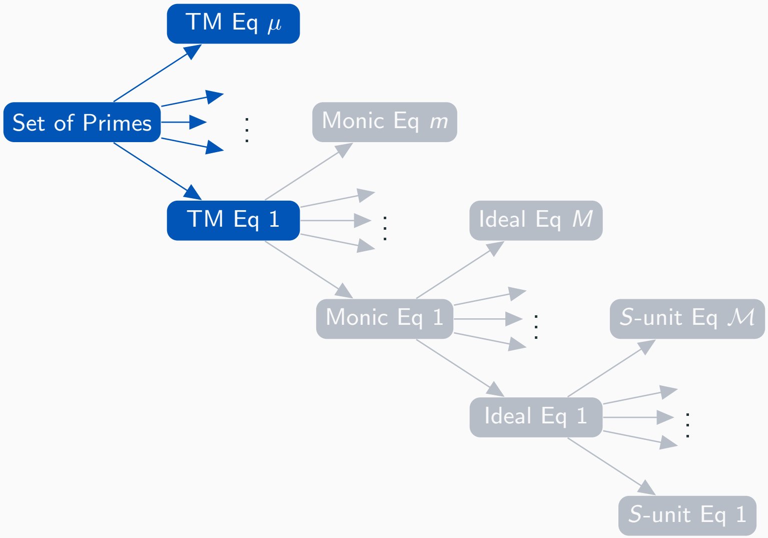
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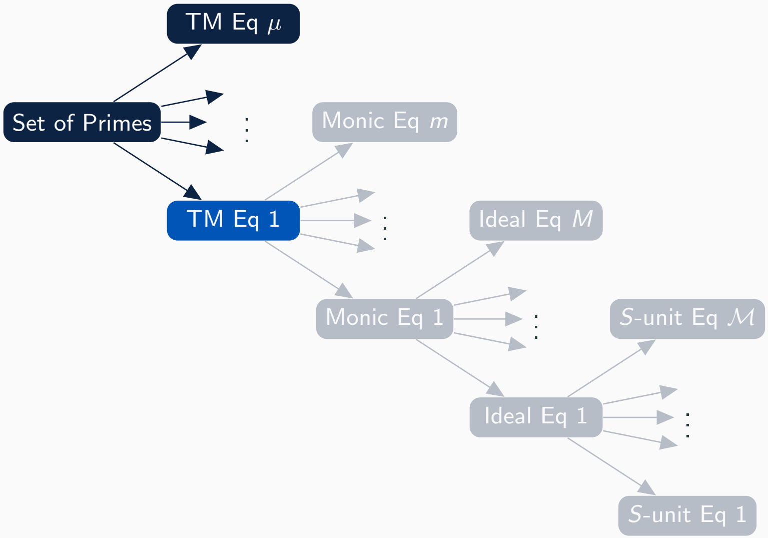
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- Without loss of generality, we may assume

$$(x, y) = (y, c_0) = 1 \quad \text{and} \quad (a, p_1, \ldots, p_v) = 1$$

First steps

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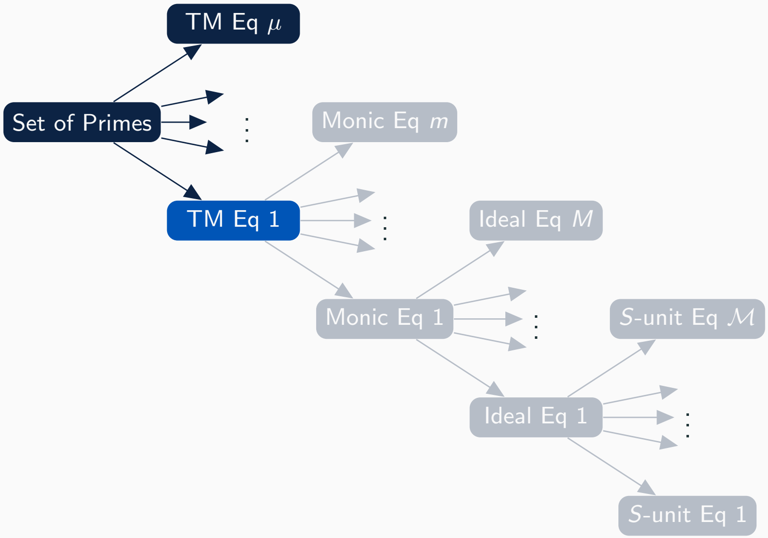
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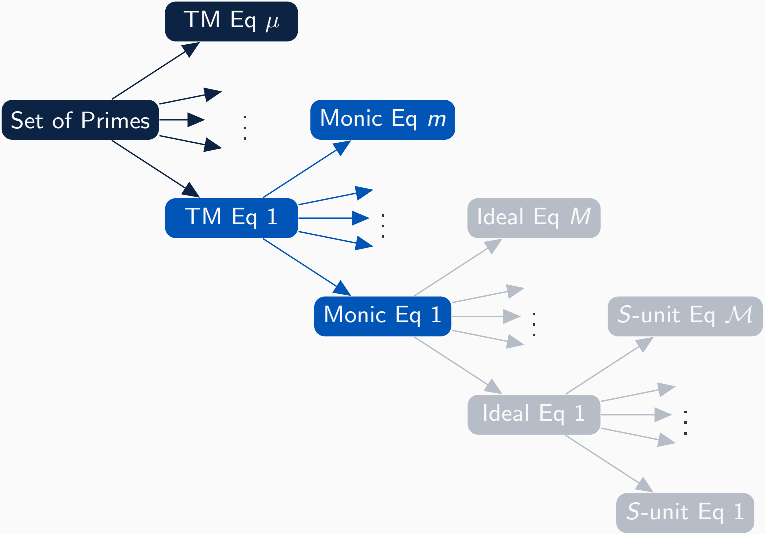
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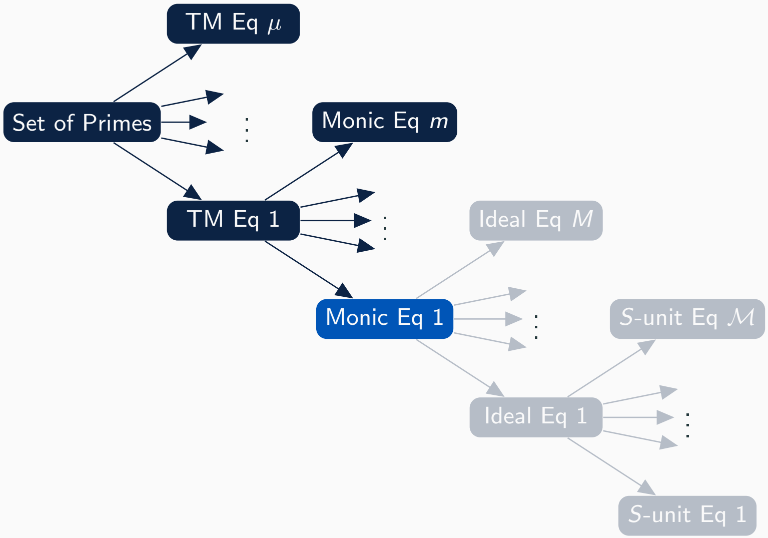
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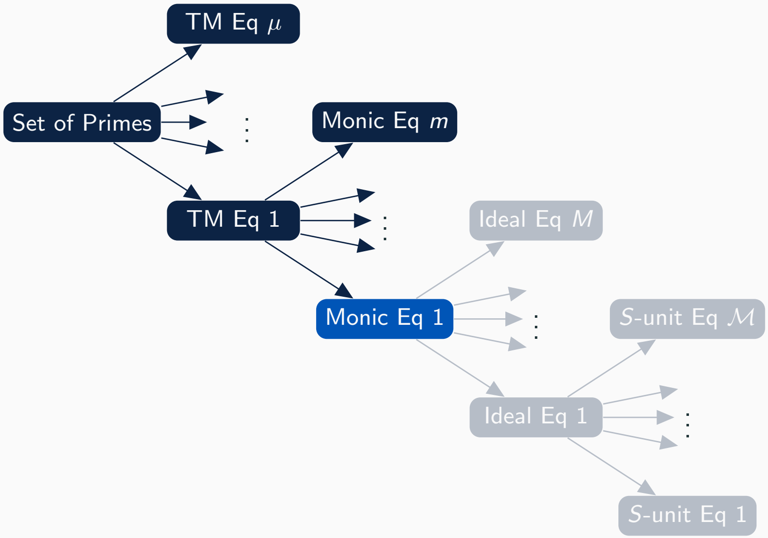
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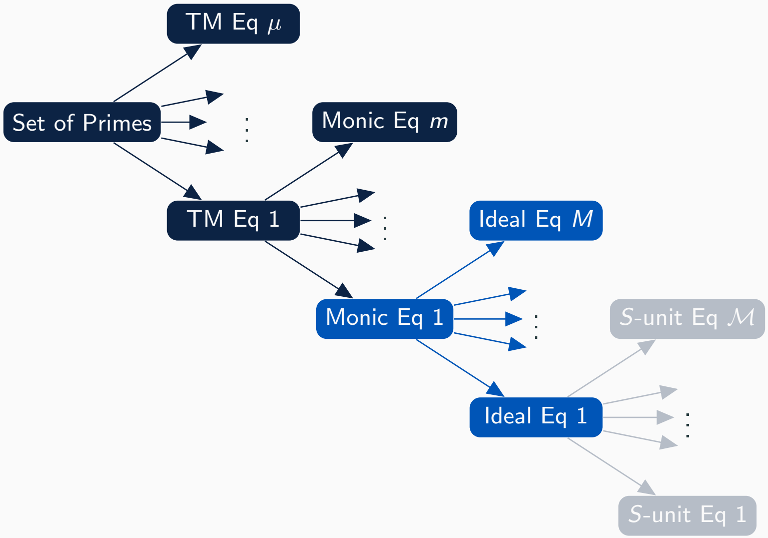

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Example: the number of ideal equations

- A nice example

$$x^3 + 20xy^2 + 24xy^2 + 15y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3} \cdot 37^{z_4} \cdot 53^{z_5}$$

- A less nice example

$$7x^3 + xy^2 + 29xy^2 - 25y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3}$$

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- Number of Ideal Equations: 32
- A less nice example

$$7x^3 + xy^2 + 29xy^2 - 25y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3}$$

- Number of Ideal Equations: 26,136

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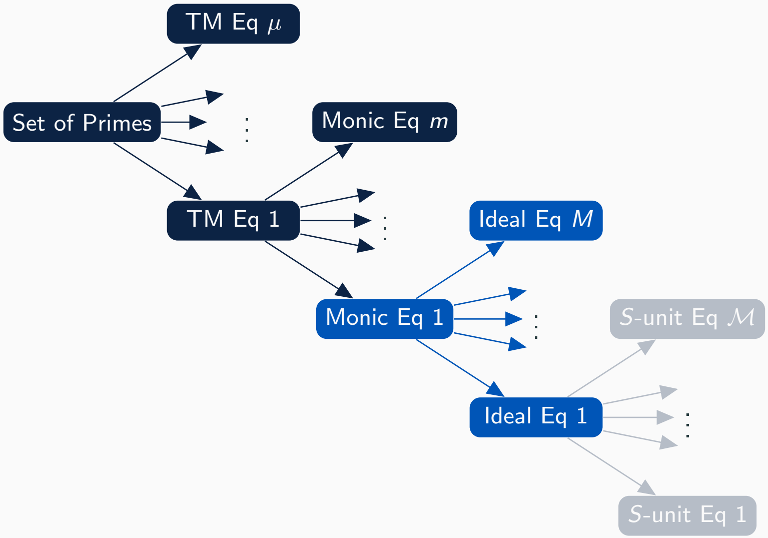

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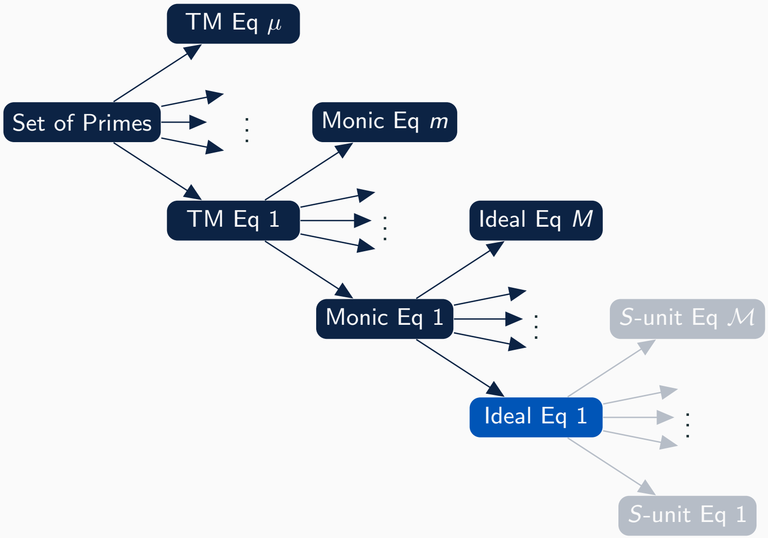
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Principalization tests

- Applying a number of principalization tests, we are left with a set of *S-unit equations*

$$x - y\theta = \alpha\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r}\gamma_1^{n_1}\cdots\gamma_\nu^{n_\nu}$$

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  - $\alpha, \varepsilon_i, \gamma_i$ are computed directly from the ideal equation

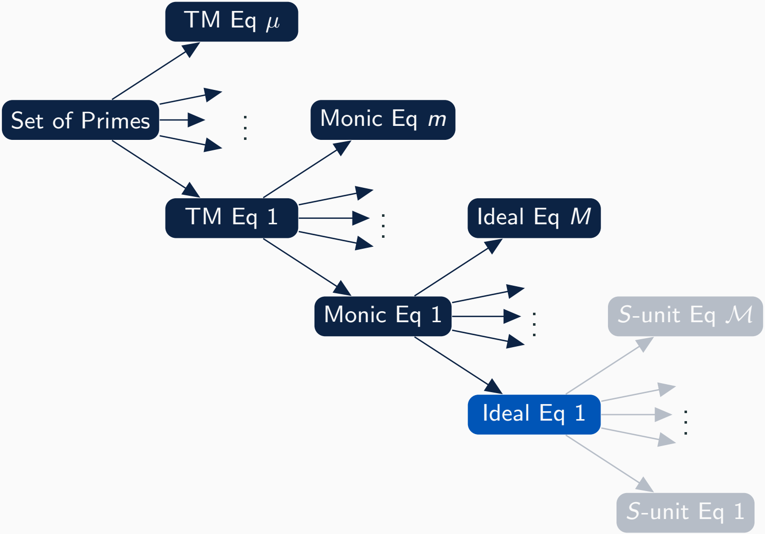
Principalization tests

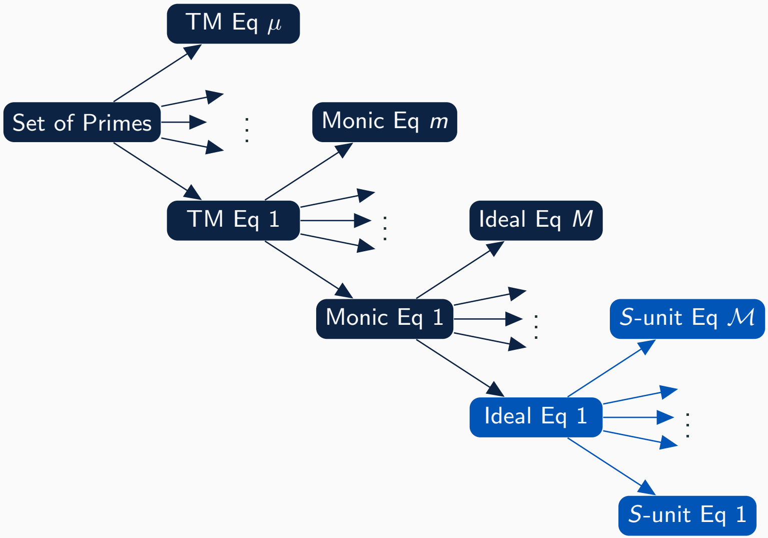
- Applying a number of principalization tests, we are left with a set of *S-unit equations*

$$x - y\theta = \alpha\varepsilon_1^{a_1}\cdots\varepsilon_r^{a_r}\gamma_1^{n_1}\cdots\gamma_\nu^{n_\nu}$$

  - $\alpha, \varepsilon_i, \gamma_i$ are computed directly from the ideal equation
  - x, y, a_i, n_i are unknown

Overview


TM Eq μ
Set of Primes
Monic Eq m
TM Eq 1
Ideal Eq M
Monic Eq 1
S-unit Eq $\mathcal{M}$
Ideal Eq 1
S-unit Eq 1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Example: the number of S-unit equations

- A nice example

$$7x^3 + 12xy^2 + 14y^3 = 7^{z_1} \cdot 11^{z_2} \cdot 37^{z_3}$$

Number of ideal equations: 16
Number of principalization tests: 11,664
Number of S-unit equations: 1296
Total time: 21.5 seconds

Example: the number of S-unit equations

- A less nice example

$$2x^3 + 20x^2y - 14xy^2 + 37y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 11^{z_4} \cdot 13^{z_5} \cdot 17^{z_6}$$

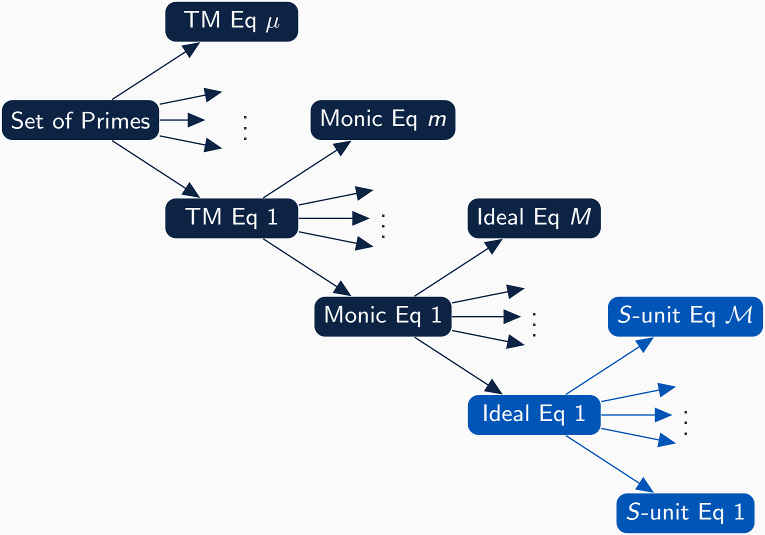
Number of ideal equations: 448
Number of principalization tests: 7,560,000
Number of S-unit equations: 139,264
Total time: 4 hours

Example: the number of S-unit equations

- A really, really bad example

$$14x^3 + 20x^2y + 24xy^2 + 15y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3} \cdot 37^{z_4} \cdot 53^{z_5}$$

Number of ideal equations: 64
Number of principalization tests: 113,848,416
Number of S-unit equations: ????
Total time: ????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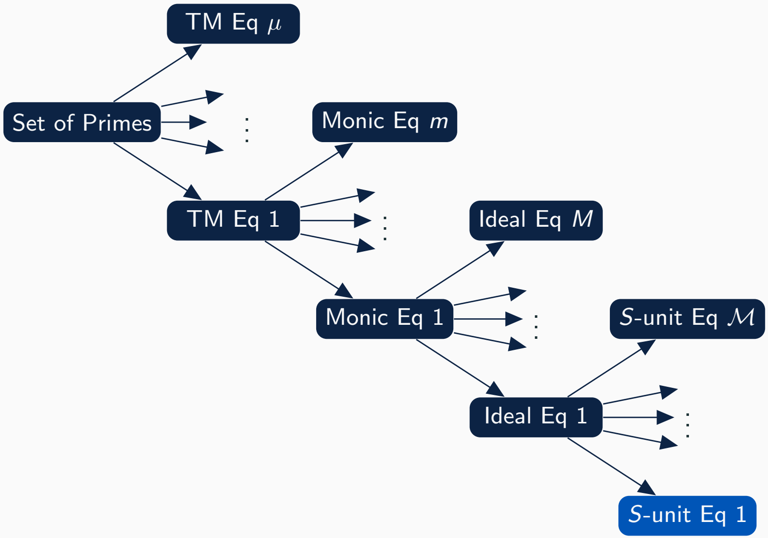
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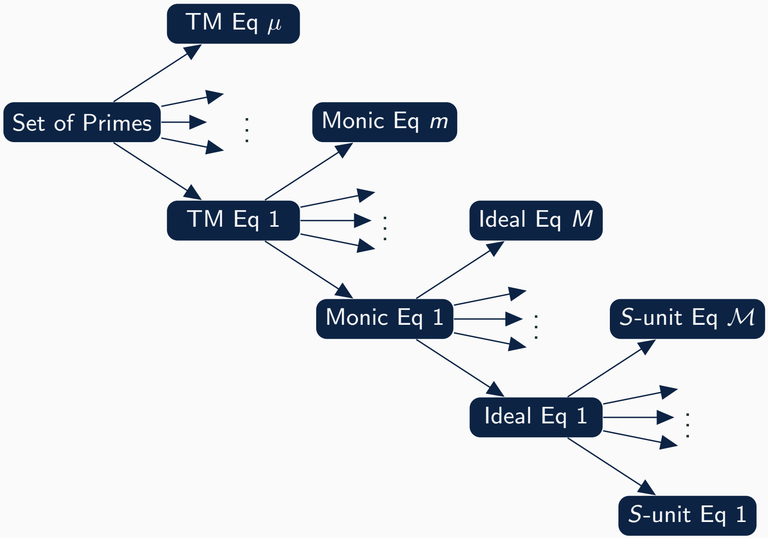
For each S-unit equation...

- Generate a very large upper bound on the solutions
- Reduce this bound
  - Compute all short vectors in a lattice
- Search below this reduced bound

Refined algorithms for solving Thue-Mahler equations

Overview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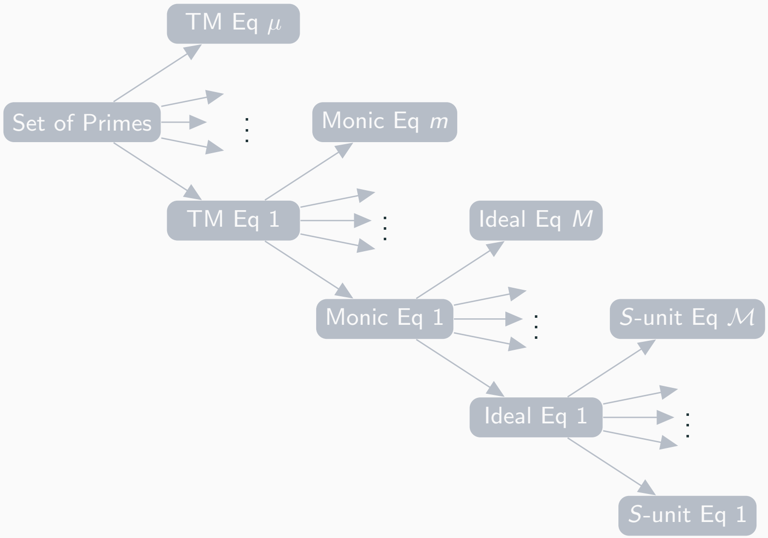

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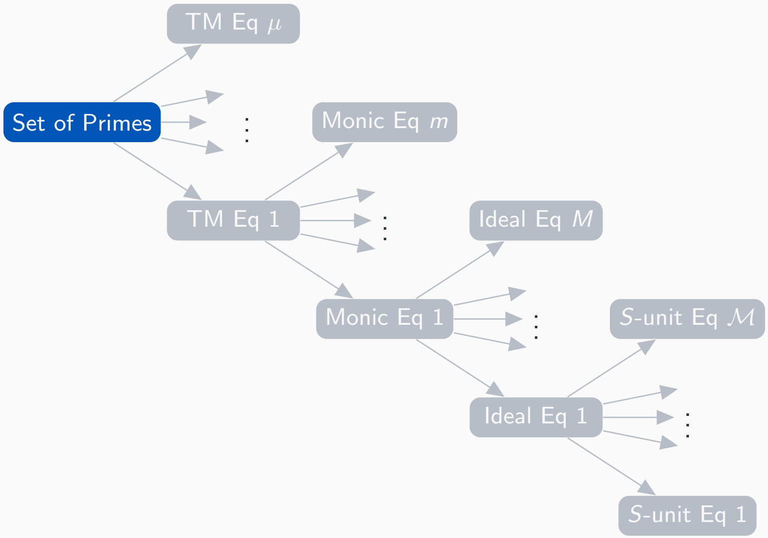

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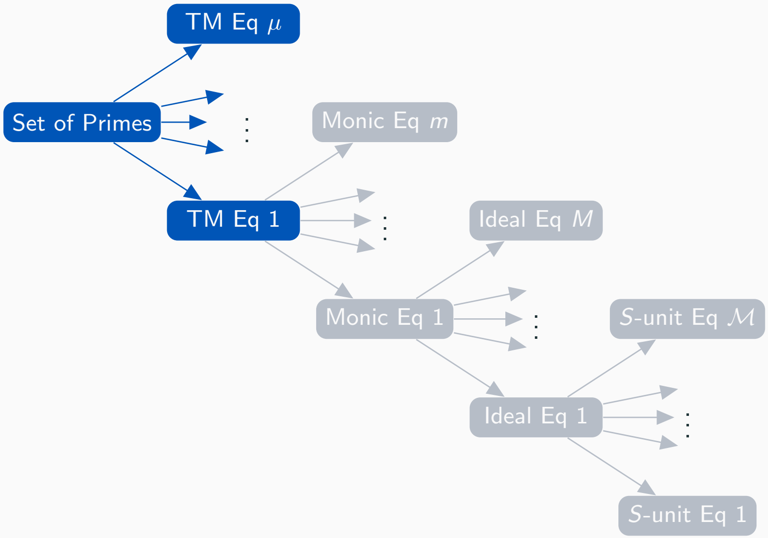
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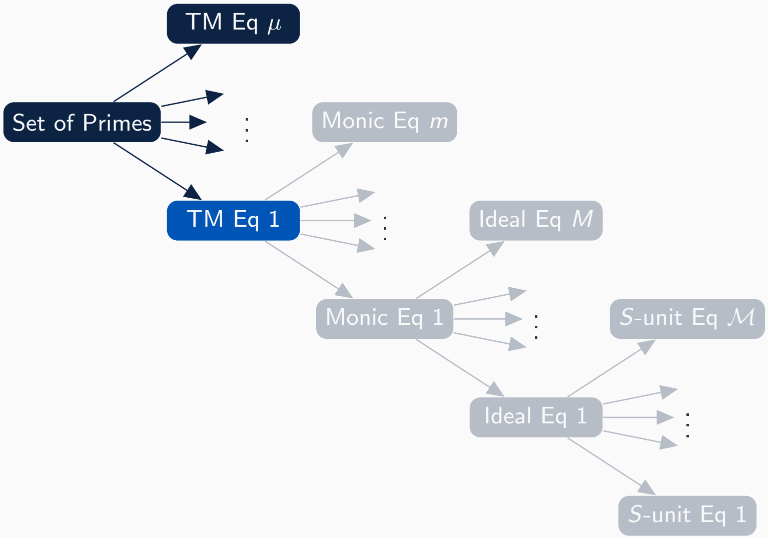

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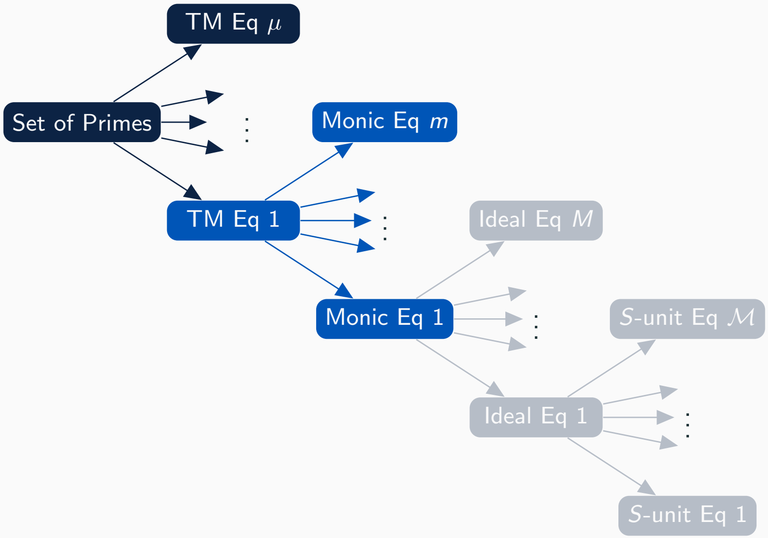
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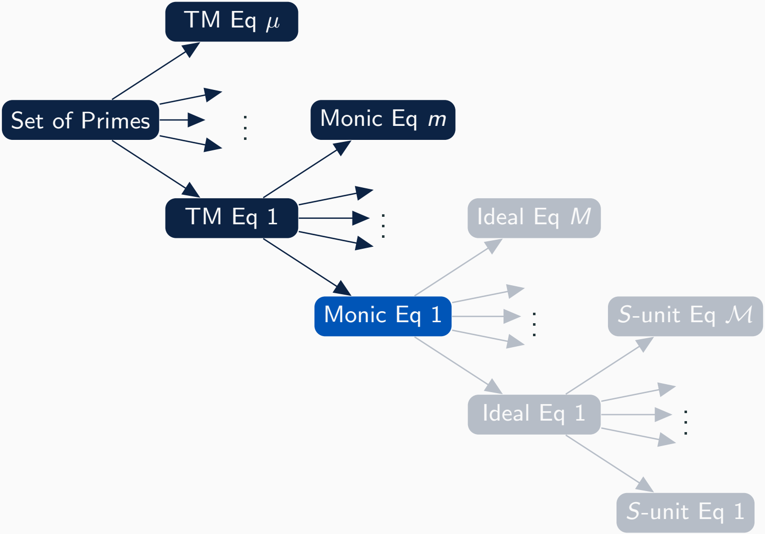
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Our refinements

- Fewer ideal equations

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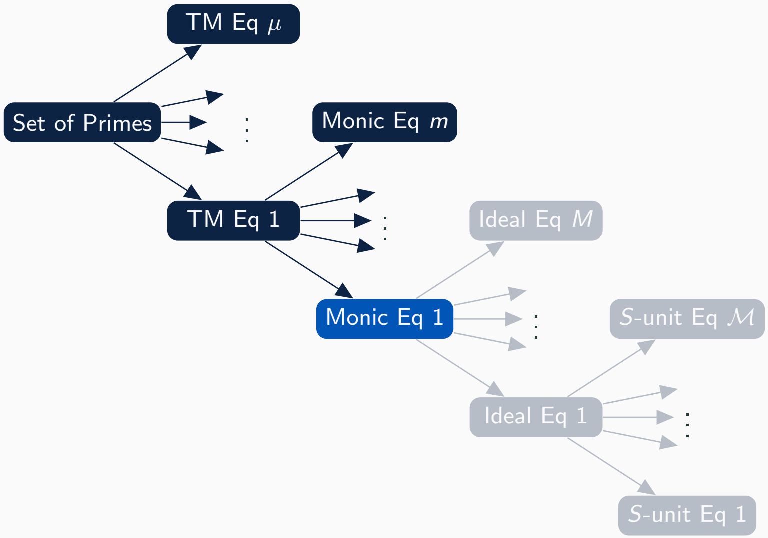
    - Yields only 1 S-unit equation for each ideal equation
- Faster bound reduction for each S-unit equation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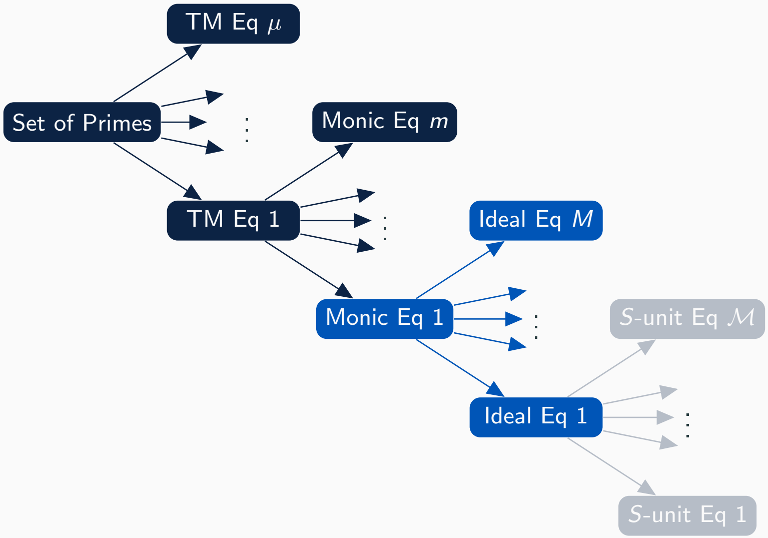
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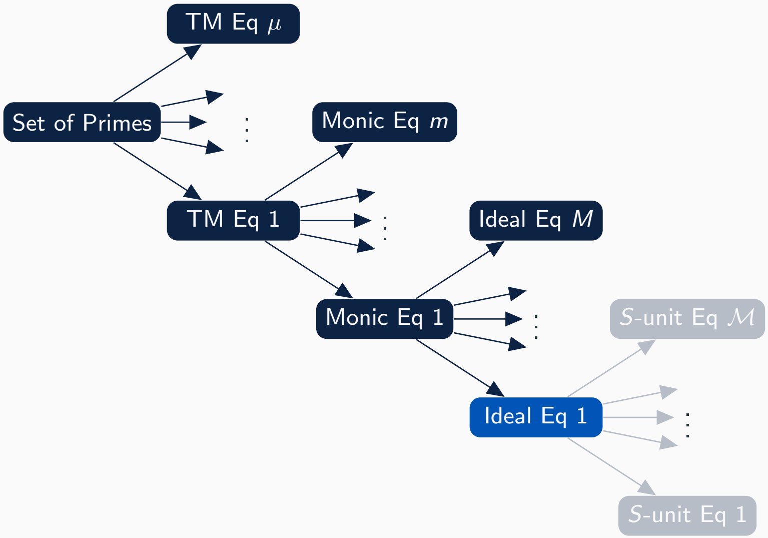
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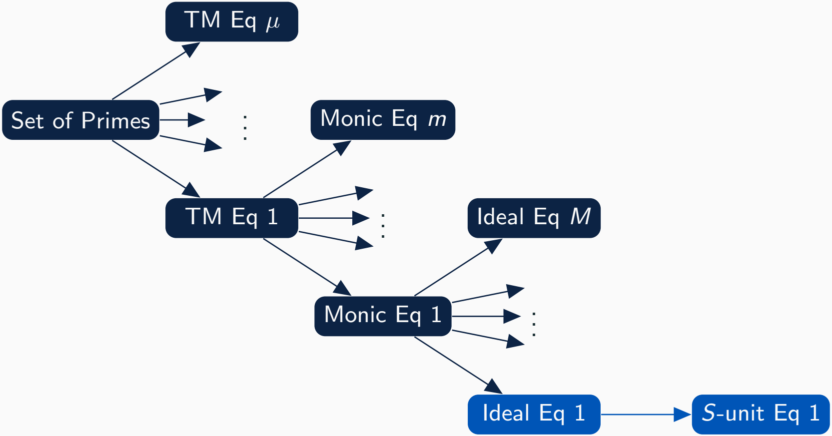
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Number of ideal equations: 16

Now: 8

Number of principalization tests: 11,664

Now: 8

Number of S-unit equations: 1296

Now: 8

Total time: 21.5 seconds

Now: 0.16 seconds

Example: The number of S-unit equations

- A less nice example

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Number of ideal equations: 448

Now: 64

Number of principalization tests: 7,560,000

Now: 64

Number of S-unit equations: 139,264

Now: 60

Total time: 4 hours

Now: 0.45 seconds

Example: The number of S-unit equations

- This god-forsaken example

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Number of ideal equations: 64

Now: 64

Number of principalization tests: 113,848,416

Now: 64

Number of S-unit equations: ????

Now: 60

Total time: ????

Now: 0.5 seconds

Current Ideas and Future Work

Our next steps

- Refine and optimize the Thue-Mahler solver at each S-unit equation

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- Refine and optimize the Thue-Mahler solver at each S-unit equation
- Compute all elliptic curves of conductor $N < 10^6$

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- Refine and optimize the Thue-Mahler solver at each S-unit equation
- Compute all elliptic curves of conductor $N < 10^6$
- Generalize the Thue-Mahler solver over number fields

Thank You