Thue equations

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Abstract

In this dissertation, we are mainly interested in effective methods to solve parametrized Thue equations. After briefly talking about the different effective methods, two parametrized families of cubic Thue equations are completely solved by using Padé approximation and linear forms in logarithms. The Thue inequality

$$\left|x^3 + pxy^2 + qy^3\right| \leq k,$$

is studied by using Bombieri's method. We find all solutions under some conditions on k, p and q. As an application of Thue equations, we find the integral points on the Mordell curves $Y^2 = X^3 + k$ for all nonzero integers k with $|k| \leq 10^7$. Our approach uses a classical connection between these equations and cubic Thue equations.

Preface

Chapter 4 is a joint work with Prof. Bennett, under the title 'Mordell equations: a classic approach', submitted for publication .

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Chapter 1

Introduction

1.1 Introduction

In 1909, Axel Thue [81] had proved the following theorem:

Theorem 1.1. *(Thue) Let $F(x,y) \in \mathbf{Z}[x,y]$ be an irreducible homogeneous polynomial of degree $d \geq 3$ and m a nonzero integer, then the equation*

$$F(x,y) = m, \tag{1.1.1}$$

has only finitely many solutions.

An equation of the form (1.1.1) is named a Thue equation in honour of Axel Thue. Thue's proof of the finiteness of the number of solutions is based on an improvement of Liouville's theorem of rational approximation to algebraic numbers. To understand Thue's proof, consider this simple observation. Let

$$F(x,y) = a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d = m$$

be a Thue equation of degree d and $\alpha_1, \cdots \alpha_n$ the roots of $f(x,1) = 0$ in a splitting field . Then we have

$$a_0(x - \alpha_1 y) \cdots (x - \alpha_d y) = m$$

$$\left|\frac{x}{y} - \alpha_1\right| \cdots \left|\frac{x}{y} - \alpha_d\right| = \left|\frac{m}{y^d a_0}\right|.$$

Suppose (x,y) is a solution for $f(x,y) = m$, and j is an index, such that

$$\left|\frac{x}{y} - \alpha_j\right| = \min_{1 \leq i \leq d} \left|\frac{x}{y} - \alpha_i\right|.$$

We call (x,y) a solution of type j, and for this solution we have

$$\left|\frac{x}{y} - \alpha_i\right| \geq \frac{1}{2} \min_{i \neq j} |\alpha_i - \alpha_j|.$$

Therefore

$$\left|\frac{x}{y} - \alpha_j\right| \le \frac{c}{|y|^d},$$

for some constant c. Now if there exists a constant $C(\alpha)$, such that

$$\left|\frac{x}{y} - \alpha_j\right| \ge \frac{C(\alpha)}{|y|^\mu} \quad \text{for some} \quad \mu < d,$$

for all $x \in \mathbb{Z}$ and $y \in \mathbb{N}$, then we can find a bound on the y value of solutions of type j and therefore the Thue equation has finitely many solutions. The first result of this type is due to Thue; it is a general sharpening of Liouville's theorem.

Theorem 1.2 (Thue [81]). *If θ is an algebraic number of degree at least 3 and $\varepsilon \ge 0$, then there is a constant $C(\theta, \varepsilon)$, such that*

$$\left|\theta - \frac{p}{q}\right| > \frac{C(\theta, \varepsilon)}{q^{\frac{d}{2}+1+\varepsilon}}, \tag{1.1.2}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Definition 1.1. *Let α be an irrational number. We call μ an irrationality measure for α, if for every $\epsilon > 0$, there exists a constant $c(\alpha, \epsilon)$, such that*

$$\left|\alpha - \frac{p}{q}\right| > c(\alpha, \epsilon) q^{-\mu}$$

for all integers $p, q \ge 1$. If $c(\alpha, \epsilon)$ is effectively computable, then μ is an effective measure.

An obvious upper bound for the irrationality measure is obtained by Liouville, who showed that every algebraic number of degree n has an effective irrationality measure n. According to Theorem 1.1.1, every algebraic number of degree n has irrationality measure $\frac{n}{2}+1$. Siegel [72] improved the exponent to $2\sqrt{n}$. Gel'fond and Dyson [33],[25] independently obtained a better bound, $\sqrt{2n}$, and finally Roth proved a sharp bound of 2 for the irrationality measure of all algebraic numbers. Unfortunately, none of these measures are effective in the sense that they don't supply an upper bound $C = C(F, m)$ such that $max(|x|, |y|) < C$. However, we can obtain some upper bounds on the number of solutions. Siegel proved the first important result in this direction, for the case $d = 3$ and binomial case $F(x, y) = ax^d - by^d = m$. He asked whether it is possible to find a bound that depends on d (the degree of F) and m but is independent of F. In 1983, J.H. Evertse answered this question. Let $N_{d,m}$ be the number of solutions for equation (1.1.1). Evertse [29] obtained the bound

$$N_{d,m} < 7^{15\left(\binom{d}{3}+1\right)^2} + 6 \cdot 7^{2\binom{d}{3}(\omega(m)+1)}$$

for the number of primitive solutions of 1.1.1 , where $\omega(m)$ is the number of distinct prime factors of m. Bombieri and Schmidt proved the following result

Theorem 1.3. *If F is an irreducible binary form of degreed d , the number of solutions to*

$$F(x, y) = 1$$

is bound above by $c_0 d$, where c_0 is an absolute constant, and it has at most $215d$ solutions if d is large enough [13].

Remark 1.1. *Consider the Thue equation*

$$F(x, y) = x^d + c(x - y)(x - 2y)(x - dy) = 1.$$

It has at least d solutions $(1, 1) \cdots (1, d)$. Thus the upper bound $c_1 d$ is the best possible, except for the determination of c_1 [13].

Bombieri and Schmidt[13] showed that if N_d is an upper bound for number of solutions of $F(x, y) = 1$, then $N_{d,m} < d^v N_d$, where v is the number of distinct prime factors of m. Therefore the number of primitive solutions for $F(x, y) = m$ does not exceed

$$C_1 d^{1+v}.$$

Later, some improvements were made on the upper bound for the number of solutions for Thue equations and Thue inequalities. We will mention two of them.

Theorem 1.4. *(Stewart [76]). Let F be an irreducible binary form of degree d, and m a given integer . The number of solutions of the Thue inequality*

$$|F(x, y)| \le m$$

is at most

$$dm^{2/d}(1 + \log m^{1/d}).$$

Theorem 1.5. *(Akhtari [1]). The Thue equation*

$$|F(x, y)| = 1$$

has at most $11d - 2$ *solutions provided that disc*(F) *is large in terms of* d.

In case of cubic, which is the focus of our interest, better bounds are given by several authors . The first result is due Siegel. He proved that the number of solutions for $F(x, y) = 1$ is at most 18, provided that the discriminant is sufficiently large . Later on, Gel'man [22] showed that 18 can be replaced by 10. Nagell [66] and Delone [23] showed that if $F(x, y)$ is an irreducible binary form with negative discriminant, then $F(x, y) = 1$ has at most 5 solutions . This bound is sharp, since the Thue equation

$$x^3 - xy^2 + y^3 = 1$$

has discriminant -23 and 5 solutions $(1, 0), (0, 1), (-1, 1), (1, 1), (4, -3)$. Cubic Thue equations with positive discriminant have been treated in several papers . Evertse [28] proved that the number of solutions of the cubic Thue equation $F(x, y) = 1$ with positive discriminant (not necessary large) is at most 12. Bennett [9] improved the result to 10. They used Padé approximation to find a gap between solutions of certain type. This result is almost sharp, since the cubic form

$$F(x, y) = x^3 + x^2y - 2xy^2 - y^3 = 1$$

has 9 solutions.

$$(1, 0), (0, 1), (-1, 1), (-1, -1), (2, 1), (-1, -2), (5, -4), (4, 9), (-9, -5).$$

But presumably, the "truth" of matter is the following conjecture by Pethő [69].

Conjecture 1. *If* F *is a binary cubic form with* $D_F > 0$, *then* $N_F = 9$ *if* $D_F = 49, N_F = 6$ *if* $D_F \in \{81, 229, 257, 361\}$ *and* $N_f \leq 5$, *otherwise .*

Finally, Okazaki [67] proved that when discriminant of F is greater than $5 \cdot 10^{65}$, the Thue equation $F(x, y) = 1$ has at most 7 solutions. Nevertheless, all of these results are ineffective in the sense that they don't provide an algorithm to compute all the solutions of (1.1.1) , or give an upper bound to the height of solutions. In this thesis we are mainly interested in effective methods of solving families of Thue equations . In Chapter 2, we introduce the basic definitions and preliminaries of effective methods for solving families of Thue equations and a survey on the previous results and families that have been solved will be given. In Chapter 3, two families of cubic Thue equations will be completely solved using the methods introduced in Chapter 2. As far as we know, the only algorithmic method to find integer points on elliptic curves is to reduce them to finitely many Thue equations. In Chapter 4, we will follow[22] ,[65] to give an algorithmic approach to finding all solutions for small values of k up to a bound for the Mordell's equations $y^2 = x^3 + k$. We solve the Mordell's equation with $0 < |k| < 10^7$. The numerical result in Chapter 4 includes an estimation for Hall's conjecture, distribution of number of solutions and also curves with large integer points . In Chapter 5, we will closely follow [14] , to explain elements of Bombieri's method for finding the irrationality measure of algebraic numbers of degree 3. We will explicitly find the constants in section 9 of [14] to solve inequality

$$\left|x^3 + pxy^2 + qy^3\right| \leq k. \tag{1.1.3}$$

Chapter 2

Background

2.1 Effective methods

There are three methods to improve Liouvill's bound effectively or solve a Thue equation . The first method is the most general and is based on Baker's theory of linear forms in logarithms of algebraic numbers [4]. The second method is Padé approximation method. It is originally due Thue [82] . Baker was the first to use this method to obtain an effective irrationality measure of some algebraic numbers [2]. This method is only applicable to certain types of equations . The third method has been developed by Bombieri [11] and Bombieri and Mueller [12] . They combined elements of the noneffective methods of Thue and Siegel with an improvement of Dyson's lemma [25]. In this chapter we will discuss the basic elements of these effective methods.

2.1.1 Linear forms in logarithms

In 1968, Baker [3] has shown the first general improvement of Liouville's bound for approximating algebraic numbers . He succeeded in showing an effective upper bound for the solutions of any Thue equations.

Theorem 2.1 (Baker[3])**.** *Let $k > d+1$ and (x, y) be an integer solution of (1.1.1), then*

$$\max\{|x|, |y|\} < Ce^{\log^k |m|},$$

where C is an effectively computable constant depending only on n, k and the coefficients of $F(x, y)$.

The bounds that we find using Baker's theorem are large, but later refinements have led to the construction of algorithms for solving the Thue equation [42],[85]. Using the work of Baker, a theoretical algorithm could be given to find the solutions. Later on, some equations had been solved by applying Baker's method in an efficient way. In 1989, Tzanakis and de Weger produced, for the first time, a general practical algorithm to solve any Thue equation. In this thesis we are mainly interested in solving parametrized

Thue equations. Since the techniques of solving a parametrized Thue equation are similar to those of solving a single Thue equation, we first give an overview of the latter (see [85],[75]). Before explaining the algorithm we will briefly talk about the lower bound for linear forms in logarithms. By a linear form in logarithms, we mean a linear form

$$\lambda = b_1 \log \gamma_1 + \cdots + b_n \log \gamma_n$$

where $\gamma_1, \gamma_2, \cdots, \gamma_n$ are algebraic numbers, not zero or one, and b_i's belong to $\mathbb{Z}$. Baker proved that if $\lambda \neq 0$ then $|\lambda|$ cannot get arbitrarily close to zero.

Lower bound of Linear form in logarithms

In this section we briefly give some results on lower bounds for linear forms in logarithms.

Definition 2.1. *For an algebraic number α with minimal polynomial*

$$\sum_{i=0}^{d} a_i x^i$$

*and conjugates $\alpha_1, \cdots, \alpha_d$ we define the **Weil height** of α by*

$$h(\alpha) = \frac{1}{d} \log \left(a_d \prod_{i=1}^{d} \max(1, |\alpha_i|) \right).$$

The first classic result that can be used in the general case is:

Theorem 2.2 (Baker-Wüstholz [5])**.** *Let $\gamma_1, \cdots, \gamma_n$ be algebraic numbers, not 0 or 1. Let $d = [\mathbf{Q}(\gamma_1, \cdots, \gamma_n) : \mathbf{Q}]$ and set:*

$$\lambda = b_1 \log \gamma_1 + \cdots + b_n \log \gamma_n \neq 0, \qquad b_i \in \mathbf{Z}.$$

If $B \geq \max\{|b_1|, \cdots, |b_n|\}$ then

$$\log|\lambda| \geq -c(n,d) h_1 \cdots h_n \log B,$$

where

$$c(n,d) = 18(n+1)! n^{n+1} (32d)^{n+2} \log(2nd).$$

One of the best results on forms in three logarithms is the following:

Theorem 2.3 (Matveev [55]). *Let $\lambda_1, \lambda_2, \lambda_3$ be logarithms of nonzero algebraic numbers linearly independent over* $\mathbf{Q}$. *Let b_1, b_2, b_3 be rational integers with $b_1 \neq 0$ set:*

$$\lambda = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3.$$

Define $\alpha_j = e^{\lambda_j}$ for $j = 1, 2, 3$ and set $D = [\mathbf{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbf{Q}]\,, \chi = [\mathbf{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbf{R}]$. Let A_1, A_2, A_3 to be positive real numbers which satisfy

$$A_j \geq \max\left\{Dh(\alpha_j), |\lambda_j|\,, 0.16\right\} \qquad (1 \leq j \leq 3)\,.$$

Assume that

$$B \geq \max\left\{1, |b_1|\frac{A_1}{A_3}, |b_2|\frac{A_2}{A_3}, |b_3|\frac{A_3}{A_3}\right\}.$$

Define also

$$C_1 = \frac{5 \cdot 16^5}{6\chi} e^3\,(7 + 2\chi)\left(\frac{3e}{2}\right)^{\chi}\left(20.2 + \log\left(3^{5.5}D^2\log(eD)\right)\right).$$

Then

$$\log|\lambda| > -C_1 D^2 A_1 A_2 A_3 \log\left(1.5eDB\log(eD)\right).$$

Sometimes if we are lucky enough, we can reduce the number of logarithms to two, and get much better bounds. As an example we can mention a result by Laurent, Mignotte and Nesterenko:

Theorem 2.4 (Laurent-Mignotte-Nesterenko; [50]). *Let γ_1, γ_2 be multiplicatively independent positive algebraic numbers, set:*

$$\lambda = b_2\log\gamma_2 - b_1\log\gamma_1, \qquad b_1, b_2 \in \mathbf{N}.$$

Let $D = [\mathbf{Q}(\gamma_1, \gamma_2) : \mathbf{Q}]$ and for $i = 1, 2$ define:

$$h_i \geq \max\left\{h(\gamma_i), \frac{\log\gamma_i}{D}, \frac{1}{D}\right\}.$$

Also:

$$b' \geq \frac{b_1}{Dh_2} + \frac{b_2}{Dh_1}.$$

If $\log|\lambda| \neq 0$ then

$$\log|\lambda| \geq (-24.34)h_1h_2D^4\left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2.$$

Mignotte [58] in an unpublished paper provides a refined method to find lower bounds for forms in three logarithms. In this method, he first finds a lower bound for linear form in all cases except the cases that he called "degenerate", which means there exists a linear relation between the coefficients. This makes them related by coefficients that are not too large, whereby we can reduce our form to two logarithms.

Lemma 2.1. *Assume $\alpha_1, \alpha_2, \alpha_3 \in (1, +\infty)$ are multiplicatively independent real numbers. Set*

$$D = \frac{[\mathbf{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbf{Q}]}{[\mathbf{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbf{R}]}$$

and consider the linear form:

$$\lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where $b_1, b_2, b_3 \in \mathbf{N}$ are co-prime. Put

$$d_1 = gcd(b_1, b_2), \qquad d_3 = gcd(b_3, b_2), \qquad b_2 = d_1 b_2' = d_3 b_2''.$$

Let a_1, a_2 and a_3 be real numbers such that:

$$a_i \geq \max\{4, 5.296\ell_i - \log|\alpha_i| + 2Dh(\alpha_i)\}, \qquad \ell_i = |\log \alpha_i|$$
$$a_1 a_2 a_3 \geq 100.$$

Put

$$\begin{aligned} \Omega &= a_1 a_2 a_3 \\ b' &= \left(\frac{b_1'}{a_2} + \frac{b_2'}{a-1}\right)\left(\frac{b_3''}{a_2} + \frac{b_2''}{a_3}\right) \\ \log B &= \max\left\{0.882 + \log b', \frac{10}{D}\right\}, \end{aligned}$$

then either

$$\begin{aligned} \log|\lambda| &> (-790.95)\Omega D^2 \log B \\ &> (-307187) D^5 \log^2 B \prod_{i=1}^{3} \max\left\{0.55, h_i, \frac{\ell_i}{D}\right\} \end{aligned}$$

or one of the following two holds:

- *there are nonzero integers r_0, s_0 such that $r_0 b_2 = s_0 b_1$ with:*

$$\begin{aligned} |r_0| &\leq 5.61 a_2 \sqrt[3]{D \log B} \\ |s_0| &\leq 5.61 a_1 \sqrt[3]{D \log B}. \end{aligned}$$

- *there are* $r_1, s_1, t_1, t_2 \in \mathbf{Z}$ *such that:*

$$\begin{aligned} r_1 s_1 &\neq 0 \\ (t_1 b_1 + r_1 b_3)\, s_1 &= r_1 b_2 t_2, \\ gcd(r_1, t_1) &= gcd(s_2, t_2) = 1, \end{aligned}$$

which also satisfy

$$\begin{aligned} |r_1 s_1| &\leq (5.61)\delta a_3 \sqrt[3]{D \log B}, \\ |s_1 t_1| &\leq (5.61)\delta a_1 \sqrt[3]{D \log B}, \\ |r_1 t_2| &\leq (5.61)\delta a_2 \sqrt[3]{D \log B}, \end{aligned}$$

where $\delta = gcd(r_1, s_1)$. *Moreover, when* $t_1 = 0$ *we can take* $r_1 = 1$ *and when* $t_2 = 0$ *we can take* $s_1 = 1$.

Algorithm

In this subsection we will explain the algorithmic approach to solving a single Thue equation first introduced by Tzanakis and de Weger [85]. Following [75], Assume

$$F(x, y) = m \tag{2.1.1}$$

is a Thue equation of degree $n \geq 3$. First, we consider a simple case in which $F(X, 1)$ has no real roots.

Lemma 2.2. *Suppose* $F(x, 1)$ *has no real roots , then any solutions* (X, Y) *of equation* (2.1.1) *satisfy*

$$|Y| \leq \frac{|m|}{\min\limits_{1 \leq i \leq n} |Img(\alpha_i)|}.$$

Proof. Suppose (X, Y) is a solution and α_i is chosen so that $|X - \alpha_i Y|$ is less than m , then we have

$$|Img(\alpha_i)Y| \leq |X - \alpha_i Y| \leq |m|,$$

and the result follows. □

Therefore we assume $F(X, 1)$ has s real roots ($s \leq 1$) and t pairs of complex conjugate roots, so $n = s + 2t$. Let α be a root of $F(X, 1)$ and $K = \mathbb{Q}(\alpha)$. We order the roots in the standard way.

$$\begin{aligned} \alpha_i &\in \mathbb{R} && \text{if } 1 \leq i \leq s, \\ \alpha_i &= \overline{\alpha_{i+t}} && \text{if } s + 1 \leq i \leq s + t. \end{aligned}$$

We set
$$\beta_i = X - y\theta_i$$
For any solution (X, Y) of (2.1.1), let i_0 be the index such that
$$\left|\frac{x}{y} - \alpha_{j_0}\right| = \min_{1\leq i\leq n}\left|\frac{x}{y} - \alpha_i\right|.$$
Or equivalently, $|\beta_i| = \min_{1\leq i\leq n}|\beta_i|$. From now on we will refer to these solutions as solutions of type i_0. Let $r = s + t - 1$ be the rank of units of $K = \mathbb{Q}(\alpha_1)$. Let $\epsilon_{1,1}, \cdots, \epsilon_{1,r}$ be a system of fundamental units of K. Denote by $\epsilon_{i,1} \cdots, \epsilon_{i,r}$ the conjugates of $\epsilon_{1,1}, \cdots \epsilon_{1,r}$ In Galois closure L of K, there exist elements μ_i in $\mathbb{Q}(\alpha_i)$ of norm m , and integers $u_1, \cdots u_r$ such that
$$\beta_i = \mu_i \epsilon_{i,1}^{u_1} \cdots \epsilon_{i,r}^{u_r}.$$
For a solution of type i, we take distinct indices labelled by i, j, k. The indices j and k are arbitrary. We obtain three linear equations,
$$X - \alpha_i Y = \beta_i,$$
$$X - \alpha_j Y = \beta_j,$$
$$X - \alpha_k Y = \beta_k$$
Using this, we can obtain
$$\beta_i(\alpha_j - \alpha_k) + \beta_j(\alpha_k - \alpha_i) + \beta_k(\alpha_i - \alpha_j) = 0.$$
Dividing both sides by $\beta_j(\alpha_k - \alpha_i)$, we get
$$\eta_1\tau_1 + \eta_2\tau_2 + 1 = 0,$$
where
$$\eta_1 = \frac{\mu_i(\alpha_j - \alpha_k)}{\mu_j(\alpha_k - \alpha_i)}, \qquad \eta_2 = \frac{\mu_k(\alpha_i - \alpha_j)}{\mu_j(\alpha_k - \alpha_i)},$$
$$\tau_1 = \prod_{l=1}^{l=r}\left(\frac{\epsilon_{i,l}}{\epsilon_{j,l}}\right)^{u_l}, \qquad \tau_2 = \prod_{l=1}^{l=r}\left(\frac{\epsilon_{k,l}}{\epsilon_{j,l}}\right)^{u_l}.$$
Suppose that $|\eta_1\tau_1|$ is very small; then $|\eta_2\tau_2|$ have to be very close to 1. Thus we have a small value for
$$\lambda = \log(\eta_2\tau_2),$$

by taking the principal value of the logarithm, we can write

$$\lambda = \log(-\eta_2) + \sum_{l=1}^{r} u_l \log\left(\frac{\epsilon_{k,l}}{\epsilon_{j,l}}\right) + u_0 2\pi\sqrt{-1},$$

for some $u_0 \in \mathbb{Z}$, we set

$$U = \max_{i\neq 0}(|u_i|). \tag{2.1.2}$$

If we can find a bound (hopefully small) on U then we can loop through all possible u_i's and solve our problem. First we can see

$$\begin{aligned} |u_0 2\pi| &= \left| arg(-\eta_2\tau_2) - arg(-\eta_2) - \sum_{i=1}^{r} u_i arg\left(\frac{\epsilon_{k,l}}{\epsilon_{j,l}}\right) \right| \\ &\leq \pi\left(2 + rU\right). \end{aligned}$$

Therefore for $U > 2$, $|u_0| \leq \frac{(r+1)U}{2}$. Next we find the equation:

$$|y|\,|\alpha_j - \alpha_i| \leq |x - \alpha_j y| + |x - \alpha_i y| \leq 2\,|x - \alpha_j y|\,, = 2\,|\beta_j|\,,$$

which implies:

$$\frac{1}{|x - \alpha_i y|} \leq \frac{2}{|y|\,|\alpha_i - \alpha_{j_0}|}.$$

Now one can write:

$$\begin{aligned} |x - \alpha_i y| &= \prod_{i\neq i} \frac{1}{|x - \alpha_j y|} \\ &\leq \prod_{i\neq i} \frac{2}{|y|\,|\alpha_j - \alpha_i|} \\ &\leq \frac{2^{n-1}}{|y|^{n-1}} \left(\prod_{i\neq j_0} \frac{1}{|\alpha_j - \alpha_i|} \right) \\ &\leq \frac{c_1}{|y|^{n-1}} < c_1. \end{aligned}$$

Remark 2.1. *The constant c_1 that appears above is effective and only depends on $\mathbf{Q}(\alpha_1)$.*

Remark 2.2. *Using Lagrange's theorem for $y > (2c_1)^{\frac{1}{n-2}}$ (which is usually a small number) we find that $\frac{x}{y}$ is a convergent of α_j.*

By setting

$$c_2 = \max_{l_1 \neq l_2 \neq l_3 \neq l_1} \left| \frac{\alpha_{l_2} - \alpha_{l_3}}{\alpha_{l_3} - \alpha_{l_1}} \right|,$$

we have

$$|\eta_1 \tau_1| \leq c_1 c_2 |\beta_i| = c_3 |\beta_i|.$$

Therefore, to make sure $\eta_1 \tau_1$ is small , we need to find a bound on β_i. Regarding this matter we recall the following lemma, which relates the size of a particular conjugate of a unit to the maximum exponent of a unit .

Lemma 2.3. *Let K be a number field with r fundamental units $\epsilon_i \in K$ and let $u_i \in \mathbb{Z}$ for $1 \leq i \leq r$. Define $U = max\,|u_i|$ and*

$$\xi = \prod_{i=1}^{r} \epsilon_i^{u_i}$$

Let $I = \{i_1, \cdots, i_r\} \subset \{1, \cdots, s+t\}$ denote any set of r distinct indices; then, the following matrix is invertible

$$U_I = \begin{pmatrix} \log(\epsilon_{i_{1,1}}) & \dots & \log(\epsilon_{i_{1,r}}) \\ \vdots & & \vdots \\ \log(\epsilon_{i_{r,1}}) & \dots & \log(\epsilon_{i_{r,r}}) \end{pmatrix}$$

and there exists an index $t \in I$ such that

$$|\log|\xi_t|| = \max_{1 \leq l \leq r+1} |\log|\xi_l||$$

then

$$|\log|\xi_t|| \geq U / \left\| U_I^{-1} \right\|_\infty,$$

where $\|.\|_\infty$ denotes the infinity norm of a matrix [75].

Set ξ_i as β_i/μ_i , and c_4 to be the maximum of $\left|\left|U_I^{-1}\right|\right|_\infty$ over all possible subsets I . Applying this lemma, there exists an index t such that

$$|\log|\xi_t|| \geq c_4 U.$$

Then choose a constant c_5 to be a positive real number with $c_5 < c_4/(n-1)$. There are 3 different cases

- Case I: $|\beta_i| > e^{-c_5 U}$ and $|\xi_t| \leq e^{c_4 U}$. We have inequalities

$$|\beta_t| < |m| \prod_{l \neq t} |\beta_l|^{-1} < |m| \, |\beta_i|^{-(n-1)} < |m| \, e^{(n-1)c_5 U}.$$

Thus we obtain the inequality

$$e^{c_4 U} \leq |\xi_t| = |\beta_i| \, / \, |\mu_i| \leq c_6 \, |m| \, e^{(n-1)c_5 U},$$

where $c_6 = \max_{1 \leq l \leq n} |\mu_l|^{-1}$. Taking the logarithm from both sides, we have

$$U \leq \frac{\log(c_6 \, |m|)}{c_4 - (n-1)c_5} = U_1.$$

- Case II: $|\beta_i| > e^{-c_5 U}$ and $|\xi_t| \leq e^{-c_4 U}$. We have inequalities

$$e^{-c_4 U} \geq |\xi_t| = |\beta_i| \, / \, |\mu_i| \geq c_7 \, |m| \, e^{-c_5 U},$$

where $c_7 = \min_{1 \leq l \leq n} |\mu_l|^{-1}$. Taking the logarithm from both sides, we have

$$U \leq \frac{\log(c_7)}{c_4 - c_5} = U_2.$$

- Case III $|\beta_i| \leq e^{-c_5 U}$. Usually, since β_i is chosen to be the smallest conjugate of β, it can be very small . In practice we normally deal with this case. We have

$$\left| e^{\lambda} - 1 \right| = |\eta_1 \tau_1| \leq c_3 e^{-c_5 U}.$$

For any complex number z if $z - 1 \leq \frac{1}{2}$, then $|\log z| \leq 2 \, |z-1|$. Therefore we have either

$$U \leq \frac{log(2c_3)}{c_5} = U_3,$$

or

$$\left| e^{\lambda} - 1 \right| = |\eta_1 \tau_1| \leq 1/2.$$

Which means that

$$\lambda \leq 2c_3 e^{-c_5 U}. \tag{2.1.3}$$

We can use theory of linear forms in logarithms as in section 2.1.1, to find a lower bound for λ and determine a constant c_8 such that for $U \geq 3$

$$\log|\lambda| > -c_8 \log((r+1)U/2).$$

Comparing this lower bound with the upper bound (2.1.3), we have

$$U < \frac{c_8 \log U + \log(2c_3) + c_8 \log((r+1)/2)}{c_5}.$$

By the lemma of Pethő and de Weger [85] we can obtain a better bound

$$U \leq U_4 = \frac{2}{c_5}\left(\log(2c_3) + c_8 \log((r+1)/2) + c_8 \log\left(\frac{c_8}{c_5}\right)\right).$$

The bound U_4 could be rather large. We can use techniques such as the LLL algorithm [75] to reduce it to a more manageable size, such as U_5. Therefore we have

$$U \leq \max(3, U_1, U_2, U_3, U_5).$$

As we had shown, all of these constants are effectively computable. Therefore one can obtain an effective bound on U and so on Y, to solve the Thue equation.

2.1.2 Padé approximation method

Unlike Baker's method, when using Padé approximation we don't need to know features of extension, fundamental units or even the roots. And when Padé approximation is applicable (which is not always the case) it typically gives better results than linear forms in logarithms.

In order to know what we mean by Padé approximation, we provide this elementary observation: Given a power series

$$f(x) = \sum_{i=0}^{\infty} r_i x^i, \qquad r_i \in \mathbf{Q},$$

for any fixed n there exist nonzero polynomials $P_n(x), Q_n(x) \in \mathbf{Q}[x]$, of degree at most n, such that

$$P_n(x) - Q_n(x)f(x) = c_n x^{2n+1} + \cdots.$$

We call the pairs $(P_n(x), Q_n(x))$ Padé approximations to $f(x)$. By the above argument, the Padé approximation to any such function always exists. But in practice, it is necessary to know some features of the polynomials such as the size of denominators of the coefficients and bounds on the heights of polynomials P_n and Q_n.

An Example by Thue and Siegel

The idea of using Padé approximation for approximating irrational numbers is originally due to Thue and Siegel. Thue was interested in solving:

$$\left|(a+1)x^3 - ay^3\right| = 1,$$

which can be rewritten as:

$$\left|\left(1+\frac{1}{a}\right)x^3 - y^3\right| = \frac{1}{|a|},$$

which in turn is equivalent to:

$$\left|\sqrt[3]{1+\frac{1}{a}} - \frac{y}{x}\right| = \frac{1}{|a||x|g(x,y)},$$

where $g(x,y)$ is a degree two polynomial in x, y and $\sqrt[3]{1+\frac{1}{a}}$. Thus one sees that solving the equation is intimately related to the question of approximating irrational numbers. More general, Padé approximations to

$$f(x) = \sqrt[d]{1+x}$$

are given by:

$$\begin{aligned} P_n(x) &= F\left(-\frac{1}{d} - n, -n, -2n; -x\right), \\ Q_n(x) &= F\left(\frac{1}{d} - n, -n, -2n; -x\right), \end{aligned}$$

where F is a hypergeometric polynomial. Plugging in $\frac{1}{a}$ for x and clearing denominators, and multiplying by a^n we obtain

$$p_n - q_n\sqrt[d]{1+\frac{1}{a}} = c_n'\left(\frac{1}{a}\right)^{n+1} + \cdots \qquad p_n, q_n \in \mathbf{Z}.$$

It means we can find a sequence of integers p_n and q_n such that:

$$\frac{p_n}{q_n} \longrightarrow \sqrt[d]{1+\frac{1}{a}}.$$

Using Padé approximation we have the following lemma.

Lemma 2.4. *Let $\theta \in \mathbf{R}$. Suppose that there are real numbers $k_0, \ell_0 > 0$ and $E, Q > 1$ such that for all $n \in \mathbf{N}$ there exist rational integers p_n and q_n with $|q_n| < k_0 Q^n$ and $|q_n\theta - p_n| \leq \ell_0 E^{-n}$ satisfying $p_n q_{n+1} \neq p_{n+1} q_n$. Then for $p, q \in \mathbf{Z}$ with $|q| \geq \frac{1}{2\ell_0}$, we have:*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{c\,|q|^{\kappa+1}},$$

where:

$$c = 2k_0 Q\,(2\ell_0 E)^{\kappa} \qquad \text{and} \qquad \kappa = \frac{\log Q}{\log E}.$$

Remark 2.3. *Although the sequences p_n, q_n obtained by Padé approximation converge to θ, they are not as strong as principle convergents. However, using lemma 2.4 we still get an effective rational approximation.*

Remark 2.4. *From Lemma 2.4 it is clear that the smaller the constant Q, the better the result. Finding better bounds on the common denominator improves the irrationality measure.*

As an example in our first equation, the Padé approximation for $\sqrt[3]{1+x}$ has the following properties

- $|P_n(z)| < 4^n$, $|Q_n(z)| < 4^n$.
- $|E_n(z)| < 4^n\big(1 - |z|\big)^{-\frac{1}{2}(2n+1)}$.
- If $z \neq 0$ then $P_n(z)Q_n(z) \neq P_{n+1}(z)Q_{n+1}(z)$.

Using the same method with better approximations, Baker proved an irrationality measure of 2.955 for irrationality measure of $\sqrt[3]{2}$ with effective constant 10^{-6} (see [2]). Subsequently he used this to solve Thue equation $x^3 - 2y^3 = n$. Using better bounds on denominators Chudnovsky improved the measure to 2.42971 (see [16]). Bennett [8] proved an irrationality measure of 2.5 and constant $\frac{1}{4}$. The following proposition, which is a version of Thue's "Fundamental theorem", is very important in order to give Padé approximation for more general Thue equations rather than binomial ones

Proposition 2.1. *Let $P \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 2$, and U a quadratic polynomial in $\mathbb{Q}[x]$ with disc$(U) \neq 0$ such that*

$$UP'' - (n-1)U'P' + \frac{n(n-1)}{2}U''P = 0$$

holds. Let $Y_1 = 2UP' - nU'P$ *and* $\lambda = \frac{disc(U)}{4}$*, and define the following polynomials :*

$$a = \frac{n^2-1}{6}\left(\sqrt{\lambda}U' + 2\lambda\right), \qquad c = \frac{n^2-1}{6}\left(\sqrt{\lambda}(U'X - 2U) + 2\lambda X\right),$$

$$b = \frac{n^2-1}{6}\left(\sqrt{\lambda}U' - 2\lambda\right), \qquad d = \frac{n^2-1}{6}\left(\sqrt{\lambda}(U'X - 2U) - 2\lambda X\right),$$

$$z = \frac{1}{2}\left(\frac{Y_1}{2n\sqrt{\lambda}} + P\right), \qquad u = z - P, \qquad w = \frac{z}{u}.$$

Then for $r \in \mathbb{N}$ *there are rational polynomials* $A_r, B_r \in \mathbb{Q}[X]$ *given by*

$$\left(\sqrt{\lambda}\right)^r A_r = a\chi^*_{n,r}(z,u) - b\chi^*_{n,r}(u,z),$$

$$\left(\sqrt{\lambda}\right)^r B_r = c\chi^*_{n,r}(z,u) - d\chi^*_{n,r}(u,z),$$

such that for any root β *of* P *the polynomial*

$$C_r = \beta A_r - B_r$$

is divisible by $(X-\beta)^{2r+1}$ *[52] .*

2.1.3 Bombieri's method

The proof of the finiteness theorem of Thue equation is based on the idea that if there is a rational number which is an exceptionally good approximation to a given real number θ then all other approximations cannot be too close from some point onward. In essence, Thue has proven the following:

Theorem 2.5. *(Thue) Let* α *be an algebraic number of degree* r *and let* h, k *be given positive numbers. There is an effectively computable constant* $G_0 = G(\alpha, h, k)$ *such that if there exist* p_0, q_0 *with*

$$\left|\alpha - \frac{p_0}{q_0}\right| < q_0^{-r/2-1-k}, \quad q_0 > G_0$$

then we can effectively determine a $G = G(\alpha, h, k.q_0)$ *such that*

$$\left|\alpha - \frac{p}{q}\right| > q^{-\frac{r}{2(k+1)}-1-h}$$

for all $q > G$*.*

The problem is the existence of such a good approximation. The constant G_0 in the Thue theorem is far too large and thus no pair $(\alpha, p_0/q_0)$ has been found to satisfy the theorem's hypothesis [11]. By using a significant result obtained by Dyson, Bombieri [25] succeeded to remove the requirement that q_0 should be large. The main idea is to construct auxiliary polynomial $P(x, y)$ which vanishes at the point (θ_1, θ_2) to high order, P vanishes at $(p_0/q_0, p/q)$ only to a low order and $h(p)$ is not too large. One way to ensure that P vanishes at (p_0, q_0) to a low order is to make the unpleasant restriction that q_0 is sufficiently large . An alternative way is to use Dyson's lemma to find vanishing order of $(p_0/q_0, p/q)$ by using information on the degree of P and the vanishing degree at (θ_1, θ_2).

To express Dyson's lemma, we first define a notation [11]. Let K be an algebraically closed field and $\xi_\mu = (\xi_{\mu 1}, \cdots \xi_{\mu n})$ be m points in K^n. We assume that for $i = 1, 2 \cdots n$, the m numbers $\xi_{1i}, \xi_{2i}, \cdots \xi_{mi}$ are distinct. In this case the set of points ξ_μ will be called admissible. Let $v_i > 0 \;\; i = 1, \cdots n$ be real numbers, and $d_1, \cdots d_n$ be positive numbers, and let $t_\mu \;\; \mu = 1, \cdots m$ be real numbers. We define

$$\rho(d, v; t_1, \cdots, t_m | \xi_1, \cdots, \xi_m),$$

and abbreviate $\rho(d; t_\mu)$ to be the vector space of all polynomials $P = P(x_1, \cdots x_n)$ in n variables, satisfying

$$deg_{x_i} P \leq d_i,$$

for $i = 1, .., n$

$$\Delta^I P(\xi_\mu) = 0,$$

for

$$\Delta^I = \frac{\partial^{i_1 + \dots i_n}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$$

and all indices $I = (i_1, \cdots \imath_n)$ with

$$v_1 \frac{i_1}{d_1} + \cdots v_n \frac{i_n}{d_n} < t_\mu,$$

we also define

$$\varphi_n(t) = \underset{v_1 x_1 + \cdots + v_n x_n \leq t}{\int_0^1 \cdots \int_0^1} dx_1 \cdots dx_n.$$

According to the above notation, following is Bombieri's version of Dyson's lemma

Lemma 2.5. *Let $n = 2$ and $\xi_1, \cdots \xi_m$ be admissible then if $\rho(d; t_\mu) \neq 0$, we have*
$$\sum_\mu \varphi_2(t_\mu) \leq 1 + \max\left(\frac{m-2}{2}, 0\right) \frac{d_2}{d_1}.$$

Bombieri [11] and Bombieri and Mueller [12] describe the auxiliary polynomials in a quantitative fashion. As an example, assume $k(\alpha_1) = k(\alpha_2) = K$ and they have degree r over k. As well, t and v satisfy

$$0 < t < \sqrt{\frac{2}{r}} \qquad t \leq v \leq t^{-1}.$$

Then there exists a polynomial $p \in k[x_1, x_2]$ with

$$deg_{x_i} p \leq d_i \quad d_2 \leq \sigma d_1$$

and

$$\frac{\partial^{i_1+i_2}}{\partial x_1^{i_1} \partial x_2^{i_2}} p(\alpha_1, \alpha_2) = 0$$

for all $(i_1, i_2)$with
$$v^{-1}\frac{i_1}{d_1} + v\frac{i_2}{d_2} < t.$$

Now let us define
$$\tau = \sqrt{2 - rt^2 + (r-1)\sigma}.$$

According to Dyson's lemma, there are (i_1^*, i_2^*) with

$$\frac{\partial^{i_1^*+i_2^*}}{\partial x_1^{i_1^*} \partial x_2^{i_2^*}} p(\beta_1, \beta_2) \neq 0$$

and
$$v^{-1}\frac{i_1^*}{d_1} + v\frac{i_2^*}{d_2} < \tau,$$

for the polynomial with these properties, using Siegel's box principal lemma, one can show that

$$\log h(p) \leq A_1 d_1 + A_2 d_2 + o(d_1 + d_2),$$

where $A_i = \frac{rt^2}{2-rt^2}\left(\log h(\alpha_i) + \frac{1}{2}\right)$.

Remark 2.5. *The sharp bound of A_1 appears to be a mystery. For algebraic numbers of the form $\alpha_1 = \sqrt[r]{\xi}$, Bombieri and Mueller [12] found a better bound for A_1 and therefore a better irrationality measure .*

The last step is to find a Thue Siegel type principle using the information we have about p. Let

$$\gamma = \frac{1}{i_1^*! i_2^*!} \frac{\partial^{i_1^*+i_2^*}}{\partial x_1^{i_1^*} \partial x_2^{i_2^*}} p(\beta_1, \beta_2).$$

So we have, $\gamma \neq 0$. We proceed to estimate $\log |\gamma|_w$ for all different valuations w of K over k in terms of $h(p)$ and height of β_1 and β_2. Using the fact $\sum_w \log |\gamma| = 0$, one can prove effective versions of Thue-Siegel's principle. As an example we can mention this result [12] :

Theorem 2.6. *Let $K = \mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_2)$ be an algebraic number field of degree $n \geq 3$, and let θ, t and τ be positive real numbers satisfying the conditions: $0 < \sqrt{2 - nt^2} < \tau < t < \sqrt{\frac{2}{n}} \leq \min(\theta, \theta^{-1})$. Then there exist effective constants c, c_0 such that one of the following inequalities hold:*

$$\left| \alpha_1 - \frac{p_1}{q_1} \right| \geq (cH(\alpha_1))^{-\frac{4}{n\theta(2-nt^2)(t-\tau)}} (|p_1| + q_1)^{-\frac{2}{\theta(t-\tau)}} ,$$

$$\left| \alpha_2 - \frac{p_2}{q_2} \right| \geq (cH(\alpha_2))^{-\frac{4}{n\theta(2-nt^2)(t-\tau)}} (|p_2| + q_2)^{-\frac{2\theta}{(t-\tau)}} ,$$

$$\left(\frac{cH(\alpha_1)}{c_0 H(\alpha_2)} \right)^{\frac{2}{n(2-nt^2)}} > q_2 (|p_1| + q_1)^{-\frac{n}{nt^2+\tau^2-2}} .$$

2.2 Parametrized families of Thue equations

Definition 2.2. *A family of Thue equations :*

$$p_n(t)x^n + p_{n-1}(t)x^{n-1}y + \ldots + p_0(t)y^n = p(t),$$

where p and each p_i are integral polynomials in the variable t is called a parametrized family of Thue equations.

The goal is to find a set of solutions expressed in terms of the parameter t, for which all the solutions of all the members of the family, or at least for values of t bigger than some bound, are expressed in terms of t. Thomas [80] formulates this as follows:

Definition 2.3. *The set $\phi : \{(g_1(t), h_1(t)), \ldots, (g_s(t), h_s(t))\}$ is called a set of solutions if there exists an effectively computable positive integer $\hat{T}$ such that for all $t > \hat{T}$, all the solutions of the family are given by the elements of ϕ. A family is stably solvable if the family has a complete solution set.*

Note that all families are not stable solvable. The study of parametrized families of Thue equations dates back to Thue [82]. He showed that the equation

$$(t+1)x^n - ty^n = 1$$

has only the solutions $x = y = 1$ if t is large enough in relation to the prime $n \geq 3$. Although the Baker method provides a general algorithm for solving a single Thue equation, it was not until 1993 that parametrized families of Thue equations were considered by this method. Thomas [79], studied the family

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \pm 1 \qquad t \in \mathbb{Z}.$$

Thomas [79] and Mignotte [56] solved this equation completely. It has only solutions $\pm(1,0), \pm(0,1) \pm (1,-1)$ for all $t \in \mathbb{Z}$, as well as several extra solutions for $t = 0, 1, 3$. Since then, several families have been considered by multiple authors. We will mention some of the results. For a survey, refer to [42] and [44].

- Mignotte, Pethő and Lemmermeyer provided a complete solution for the following family [59]

  $$\left|x^3 - (t-1)x^2y - (t+2)xy^2 - y^3\right| \leq 2t + 1.$$

- In [61] Mignotte and Tzanakis proved that the only integer solution of the family

  $$x^3 - tx^2y - (t+1)xy^2 - y^3 = x(x+y)(x-(t+1)y) - y^3 = 1$$

  are: $(1,0), (0,-1), (1,-1)(-t-1,-1), (1,-t)$ when $t \geq 3.33 \cdot 10^{23}$. Later, Mignotte [57] proved the same result for all $t \geq 3$.

- Considering the family:

  $$x(x - t^a y)(x - t^b y) \pm y^3 = 1$$

  Thomas [80] proved that if

  $$0 < a < b \qquad \text{and} \qquad t \geq \left(2 \cdot 10^6 \cdot (a+2b)\right)^{\frac{4.85}{b-a}}$$

  then there are no nontrivial solutions. In fact, this family is in a special case of families that Thomas termed split families; we will discuss them in more detail later.

- Wakabayashi [87] proved that the only integer solutions of the family:

$$x^3 - t^2xy^2 + y^3 = 1$$

are the trivial ones: $(0, 1), (1, 0), (1, t^2), (t, 1), (-t, 1)$, if $t \geq 1.35 \cdot 10^{14}$.

- Togbe [83] proved that if $t > 1$ the only integral solutions of

$$\begin{aligned} x^3 + &\left(t^8 + 2t^6 - 3t^5 + 3t^4 - 4t^3 + 5t^2 - 3t + 3\right) x^2y + \\ &+ (2 - t^3)t^2xy^2 - y^3 = \pm 1 \end{aligned}$$

are $(\pm 1, 0), (0, \pm 1)$.

- The quartic family

$$x^4 - tx^3y - x^2y^2 + txy^3 + y^4 = \pm 1$$

was solved by Pethő [70] for large values of t; Mignotte, Pethő, and Roth [60] solved it completely. The only solutions are:
$\pm\{(0, 1), (1, 0),), (1, 1), (1, -1), (a, 1), (1, -a)\}$ for $|t| \notin \{2, 4\}$

- the equation $x^4 - tx^3y - 3x^2y^2 + txy^3 + y^4 = \pm 1$ has been solved for $t \geq 9.9 \cdot 10^{27}$ by Pethő [70].

- $x^4 - tx^3y - 6x^2y^2 + txy^3 + y^4 \in \pm\{1, 4\}$ was completely solved by Lettle and Pethő [51].

- Wakabayashi [86] proved that

$$\left|x^4 - t^2x^2y^2 + y^4\right| = \left|x^2(x - t)(x + t) + y^4\right| \leq t^2 - 2$$

has only trivial solutions with $|y| \leq 1$ for $t \geq 8$.

- $$x(x^2 - y^2)(x^2 - t^2y^2) - y^5 = \pm 1$$

For $t > 3.6 \cdot 10^{19}$, all solutions have been founded by Heuberger [41].

Wakabayashi [88] used a combination of the Padé approximation method and linear forms in logarithms.
Given a parametrized family of Thue equations, one should try to perform the same steps as in the case of single Thue equation. To do so we need to express every quantity in the algorithm in terms of the parameter t. As we see in the case of a single Thue equation, we require information about

the unit group of the corresponding number field. Hanrot [37] mentions that we don't need to know the fundamental units; a suitable system of independent units is enough. In the case of cubic forms, fortunately, there is a rich theory of finding fundamental units (see [78] and [10]). Moreover, to solve parametrized families of Thue equations, we need an extra tool that is crucial for finding integral solutions: stable growth, developed by Thomas [80] . The idea is to exclude the trivial solutions. The concept of trivial solutions is vague. We call a solution trivial if the quantity U of the equation(2.1.2) is not small. In other words, the solutions that can be found with checking small values of U are called trivial. If we exclude them, we have to prove a stable growth condition for the remaining possible solutions of the family of parametrized equations .

Definition 2.4. *We say that a parametrized family of Thue equations has **stable growth** if for each nontrivial solution we have*

$$U > Ct^g \log t,$$

for some positive constant C and $g \in \mathbb{N}$.

The lower bound for U hopefully contradicts the upper bound that we find from previous estimations. As of now, we don't know of any condition that guarantees stable growth in the general case. For the cubic case, Thomas [80] proved the existence of stable growth for some families of split cubic forms. In the next chapter, while studying two families of parametrized cubic Thue equations with five solutions, we will discuss a couple of different methods to prove the stable growth condition.

Using Padé approximation method, some results on families of Thue equations were obtained. G.V. Chudnovsky [16] estimated more precisely the common denominators of hypergeometric polynomials of $\sqrt[3]{1+x}$, transforming cubic Thue equations to diagonal forms. He proved the following result

Theorem 2.7. *Let a be an integer (positive or negative) with $a \equiv -3 \pmod 9$ and let θ be the real zero of $f(x) = x^3 + ax + 1$, whose absolute value is the smallest among the three zeros of f . Then for any $\epsilon > 0, \theta$ has effective irrationality measure*

$$\frac{\log\left(\left(G_1 + \sqrt{G_1^2 + D_1}\right)^2 / |D_1|\right)}{\log\left(\left(G_1 + \sqrt{G_1^2 + D_1}\right)^2 / |D_1|\,\gamma\right)} + \epsilon,$$

where

$$D_1 = -\frac{4}{27}a^3 - 1, \qquad G_1 = \frac{2}{27}a^6 + \frac{2}{3}a^3 + 1 \qquad \gamma = e^{\sqrt{3}\pi/6}/\sqrt{3}[\mathit{16}].$$

Wakabayashi [87] Using Padé approximation, find all the solutions of $f(x) = x^3 + axy^2 + by^3 < a + b$ with the condition $a \geq 360b^4$ Using the Padé approximation method, Lettle, Pethő and Voutier [52] completely solved this family of Thue equations:

- $$F_t^3(x, y) = x^3 - tx^2y - (t+3)xy^2 - y^3,$$

- $$F_t^4(x, y) = x^4 - tx^3y - 6x^2y^2 + txy^3 + y^4,$$

- $$F_t^6(x, y) = x^6 - 2tx^5y - (5t+15)x^4y^2 - 20x^3y^3$$
  $$+5tx^2y^4 + (2t+6)xy^5 + y^6.$$

They used the proposition 2.1 with the quadratic polynomial

$$U = x^2 + x + 1.$$

These families possess special features, and are called the simple families of Thue equations. For these families, we have

- There exists an $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PGL_2(\mathbb{Q})$ such that $\psi : z \to \frac{az+b}{cz+d}$ permutes the zeros of $F(x, 1)$ transitively.
- Let ψ_A permute the roots. Then there is some $r \in \mathbb{Q}^*$ such that $F^A = rF$.
- The roots of $F(x, 1)$ generate a cyclic number field of degree $\deg(F)$. The Galois action on the roots is given by ψ_A.

Up to equivalence, the only simple forms in $\mathbb{Q}[x, y]$ are

- $$F_t^3 \ \ \textit{with} \quad t \in \mathbb{Z}.$$

- $$F_t^4 \ \ \textit{with} \ \ t \in \mathbb{Z} \setminus \{-3, 0, 3\}.$$

- $$F_t^6 \ \ with \ \ t \in \mathbb{Z} \setminus \{-8, -3, 0, 5\}.$$

Let $F(x, y)$ be an irreducible binary cubic form with integral coefficients and positive discriminant. Evertse (1983) proved that the number of solutions of the cubic Thue equation $F(x, y) = 1$ with positive discriminant is at most 12. Bennett (2000) improved the result to 10. They used Padé approximation to find a gap between solutions of a certain type. In Chapter 3 we will make some small refinements to the gap principal proved by Evertse under certain conditions. We will then use these refinements to find solutions of a certain type to some families of parametric cubic Thue equations.

According to Bombieri's method explained in section 2.1.3, effective measure of irrationality has been proved for some algebraic numbers.

Theorem 2.8. *Let $d \geq 40$ and let θ be the positive root of $x^d - ax^{d-1} + 1$, where $a \geq A(d)$ and $A(d)$ is effectively computable . Then, θ has effective irrationality measure $\mu = 39.2574$[11].*

Refining the method Bombieri and Mueller [12] proved

Theorem 2.9. *Let $d \geq 3$, and let a and b be coprime positive integers , and let $\mu = \frac{\log|a-b|}{\log b}$. Let $\theta \in \mathbb{Q}\left(\sqrt[d]{a/b}\right)$ be of degree d. If $\mu < 1 - 2/d$, then for any $\epsilon > 0$ θ has effective irrationality measure*

$$\lambda = \frac{2}{1-\mu} + 6\left(\frac{d^5 \log d}{\log b}\right)^{1/3} + \epsilon.$$

Chapter 3

Families of cubic Thue equations

3.1 Introduction

E. Thomas [80] introduced families of Thue equations of the form

$$F_t(x,y) = \prod_{i=1}^{n} (x - P_i y) - y^n = \pm 1, \qquad P_i \in \mathbf{Z}[t],$$

which he called **split families**. The trivial solutions for these families are $\{(\pm P_i, 1), (\pm 1, 0)\}$. Thomas conjectured that if

$$\begin{gathered} P_1 = 0, \\ 0 < \deg P_2 < \cdots < \deg P_n \\ \text{All } P_i \text{ are monic} \end{gathered}$$

then these families have exactly four solutions for large values of parameter t. He also proved his conjecture for $n = 3$ under some technical hypothesis [80].

Other authors (see [36], [41],[45] and [43]), proved the conjecture under different conditions. In 2007 Ziegler [90] found two split families of cubic Thue equations with at least 5 solutions for all values of t that disprove Thomas's conjecture. These two families are

$$F(x,y) = x^3 - (t^4 - t)x^2 y + (t^5 - 2t^2)xy^2 + y^3 = 1 \tag{3.1.1}$$

with solutions

$$(1,0), (0,1), (t,1), (t^4 - 2t, 1) \text{ and } (1 - t^3, t^8 - 3t^5 + 3t^2) \tag{3.1.2}$$

and

$$F'(x,y) = x^3 - (t^4 + 4t)x^2 + (t^5 + 3t^2)xy^2 + y^3 = 1$$

with solutions

$$(1,0),(0,1),(t,1),(t^4+3t,1) \text{ and } (t^9+3t^6+4t^3+1,t^8+3t^5+3t^2).$$

In this chapter we will use the techniques from Chapter 2 to completely solve these families. Basically we will prove that these families have exactly 5 solutions for all values of $t>0$.

To the forms $F(x,y)$ and $F'(x,y)$ we relatively associate polynomials $P(x)$ and $P'(x)$ which are defined by

$$P(x)=F(x,1)=x^3-(t^4-t)x^2+(t^5-2t^2)x+1$$

and

$$P'(x)=F'(x,1)=x^3-(t^4+4t)x^2+(t^5+3t^2)x+1.$$

$P(x)$ as a polynomial in x has three real roots; we denote them by θ,θ',θ'' and similarly $P'(x)$ as a polynomial in x has three real roots; we denote them by α,α',α''.

By studying the sign of $P(x)$ and $p'(x)$ we can deduce the following bounds for the roots:

$$\begin{aligned} -\frac{1}{t^5}-\frac{2}{t^8}-\frac{4}{t^{11}} <&\theta<-\frac{1}{t^5}-\frac{2}{t^8}-\frac{3}{t^{11}},\\ t+\frac{1}{t^5}+\frac{3}{t^8} <&\theta'<t+\frac{1}{t^5}+\frac{4}{t^8},\\ t^4-2t-\frac{2}{t^8} <&\theta''<t^4-2t-\frac{1}{t^8}, \end{aligned} \tag{3.1.3}$$

and

$$\begin{aligned} -\frac{1}{t^5}+\frac{3}{t^8}-\frac{8}{t^{11}} <&\alpha<-\frac{1}{t^5}+\frac{3}{t^8}-\frac{7}{t^{11}},\\ t+\frac{1}{t^5}-\frac{2}{t^8} <&\alpha'<t+\frac{1}{t^5}-\frac{1}{t^8},\\ t^4+3t-\frac{1}{t^8} <&\alpha''<t^4+3t-\frac{1}{t^9}. \end{aligned} \tag{3.1.4}$$

It is easy to check that if $|y|\le 1$ then both $F(x,y)=1$ and $F'(x,y)=1$ have no nontrivial solutions. So we may assume $|y|>1$. Let (x,y) be a solution to equation $F(x,y)=1$ or $F'(x,y)=1$ with $|y|>1$; then we categorized the solutions based on the root that they are closer to. Therefore, for the solutions of $F(x,y)=1$, by studying the sign we define these type of solutions:

$$
\begin{aligned}
&\text{solution of type I if:} && -\frac{2}{t^5}+\frac{1}{t^6} && <\frac{x}{y}<-\frac{1}{t^6},\\
&\text{solution of type II if :} && t+\frac{1}{t^6} && <\frac{x}{y}<t+\frac{2}{t^5},\\
&\text{solution of type III if:} && t^4-2t-\frac{2}{t^8} && <\frac{x}{y}<t^4-2t-\frac{1}{t^9},
\end{aligned}
\tag{3.1.5}
$$

and similarly for the solutions of $F'(x,y)=1$ let

$$
\begin{aligned}
&\text{solution of type I' if} && -\frac{2}{t^5}+\frac{2}{t^8}<\frac{x}{y}<-\frac{1}{t^6},\\
&\text{solution of type II' if} && t+\frac{1}{t^6}<\frac{x}{y}<t+\frac{2}{t^5}-\frac{1}{t^6},\\
&\text{solution of type III' if} && t^4+3t-\frac{1}{t^8}<\frac{x}{y}<t^4+3t-\frac{1}{t^9}.
\end{aligned}
\tag{3.1.6}
$$

In this chapter we assume $t>10$. We will use Padé approximation method to find all solutions of type I and II'. The remaining types of solutions are treated by Baker's method. Using these methods we will prove that the forms have exactly 5 solutions for large and intermediate values of t. Finally for the small values of t we use diophantine techniques and computer search.

3.2 Solutions of type I,II'

In this section we use Padé approximation method to prove a gap principle for solutions of a cubic Thue equation. The main result is that there exists at most one "suitably large" solution of a certain type for a cubic Thue equation. We will show that the unique "large solutions" for $F(x,y)=1$ and $F'(x,y)=1$ are relatively $(1-t^3, t^8-3t^5+3t^2)$ and $(t^9+3t^6+4t^3+1, t^8+3t^5+3t^2)$ and there is no other solution for these types. During the course of the proof it will become clear what we mean by large solution.

3.2.1 Gap principle

The key step to proving our gap principal is to reduce the problem to a diagonal form over an imaginary quadratic field, since the Padé approximation method is well suited to diagonal forms. It is a classic method originating in works of Eisenstein, Hermite, Arndt and Berwick; see Dickson [24] for references.

preliminaries of the proof For a cubic form F, Define the associated quadratic form, the Hessian $H = H_F$ and cubic form G_F by

$$H(x,y) = -\frac{1}{4}\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2\right)^2,$$

$$H(x,y) = Ax^2 + Bxy + Cy^2,$$
$$H(x,y) = y^2(A(\frac{x}{y})^2 + B(\frac{x}{y}) + C),$$
$$G_F(x,y) = \frac{\partial F}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial x}.$$

The forms H_F and G_F are related to F by the identity

$$4H(x,y)^3 = G(x,y)^2 + 27D_F F(x,y)^2.$$

Working in number field $\mathbb{Q}(\sqrt{-\Delta})$ and a fixed choice of square root, from above identity

$$\frac{G(x,y) \pm 3\sqrt{-\Delta}F(x,y)}{2}$$

are cubic forms in $M[x,y]$ with coeeficents conjugate to one another and no common factors; they are cubes of linear forms $\varepsilon(x,y)$ and $\eta(x,y)$ in $M[x,y]$ with complex conjugate coefficients that (see Evertse [28])satisfy these equations:

$$\varepsilon(x,y)^3 - \eta(x,y)^3 = 3\sqrt{-\Delta}F(x,y),$$
$$\varepsilon(x,y)^3 + \eta(x,y)^3 = G(x,y),$$
$$\varepsilon(x,y)\eta(x,y) = H(x,y),$$
$$\frac{\varepsilon(x,y)}{\varepsilon(1,0)} \quad \text{and} \quad \frac{\eta(x,y)}{\eta(0,1)} \in M[x,y],$$

For any rational pairs x_0, y_0, the numbers $\varepsilon(x_0,y_0)$ and $\eta(x_0,y_0)$ are complex algebraic integers. The pair of forms ε and δ satisfying above properties is called a pair of resolvent forms. We say that a pair of rational integers (x,y) is related to a pair of resolvent forms (ε,η) if

$$\left|1 - \frac{\eta(x,y)}{\varepsilon(x,y)}\right| = \min_{0\le k\le 2}\left|e^{\frac{2k\pi i}{3}} - \frac{\eta(x,y)}{\varepsilon(x,y)}\right|.$$

By this definition if (x,y) is related to the pair (ε,η) and $\Delta \geq 72000$ then from inequality (5.4) of [9] we have

$$\left|1 - \frac{\eta(x,y)}{\varepsilon(x,y)}\right| < \frac{1.012\sqrt{\Delta}}{|\varepsilon(x,y)|^3}. \tag{3.2.1}$$

The next lemma by Bennett [9] shows that two different solutions related to the same set of resolvent can't be very close to each other.

Lemma 3.1. *If* (x_1,y_1) *and* (x_2,y_2) *are distinct solutions related to* (ε,η),*with*

$$|\varepsilon(x_2,y_2)| \geq |\varepsilon(x_1,y_1)| \geq \frac{1}{\sqrt{2}}\Delta^{\frac{1}{4}}$$

and $\Delta \geq 72000$, *then*

$$|\varepsilon(x_2,y_2)| > 0.987\,|\varepsilon(x_1,y_1)|^2 .$$

For our purpose we need a stronger gap between solutions. Under some stronger conditions we have the following lemma:

Lemma 3.2. *with the same hypothesis as above lemma if* $|x_1y_2 - x_2y_1| \geq 2$ *then*

$$|\varepsilon(x_2,y_2)| > 1.975\,|\varepsilon(x_1,y_1)|^2 .$$

Proof. By inequality (3.2.1) for $\Delta > 72000$, $\left|1 - \frac{\eta(x,y)}{\varepsilon(x,y)}\right| < \frac{1.012\sqrt{\Delta}}{|\varepsilon(x,y)|^3}$ Also from inequality $\varepsilon_2\eta_1 - \varepsilon_1\eta_2 = \pm\sqrt{-\Delta}(x_1y_2 - x_2y_1)$, using lemma (3.1), we have

$$\sqrt{\Delta} \leq \frac{|\varepsilon_2\eta_1 - \varepsilon_1\eta_2|}{2} \leq \frac{1}{2}\,|\varepsilon_1|\,|\varepsilon_2|\left(\left|1 - \frac{\eta_1}{\varepsilon_1}\right| + \left|1 - \frac{\eta_2}{\varepsilon_2}\right|\right).$$

So

$$2\sqrt{\Delta} < |\varepsilon_1|\,|\varepsilon_2|\left(1.012\sqrt{\Delta}\right)\left(|\varepsilon_1|^{-3} + |\varepsilon_2|^{-3}\right).$$

It follows that

$$\begin{aligned} &|\varepsilon_1|^{-3} + |\varepsilon_2|^{-3} > 1.976\,|\varepsilon_1|^{-1}\,|\varepsilon_2|^{-1} \\ &\Longrightarrow |\varepsilon_1|^3 + |\varepsilon_2|^3 > 1.976\,|\varepsilon_1|^2\,|\varepsilon_2|^2 . \end{aligned}$$

Let $|\varepsilon_2| = k\,|\varepsilon_1|^2$ then

$$|\varepsilon_1|^3 + k^3\,|\varepsilon_1|^6 > 1.976k^2\,|\varepsilon_1|^6 .$$

Therefore

$$\left(1.976k^2 - k^3\right)|\varepsilon_1|^3 < 1.$$

Since

$$|\varepsilon| \geq \frac{1}{\sqrt{2}}\Delta^{\frac{1}{4}} > 11.58,$$

the above inequality holds for $k > 1.975$ So

$$|\varepsilon_2| > 1.975\,|\varepsilon_1|^2$$

as desired. □

Using the above notation we have these two lemmas:

Lemma 3.3. *Assume that the Thue equation $F(x, y) = 1$ with $\Delta_F > 72000$ has a solution (x_1, y_1) , such that $\epsilon_1 = \epsilon(x_1, y_1)$ satisfy $\frac{\Delta^2}{\epsilon_1^3} < \frac{1}{22}$. Let (x_2, y_2) be another solution related to the same resolvent form with $\epsilon_2 > \epsilon_1$ then $\epsilon_2 > (2.4\Delta)^{-15}\epsilon_1^{47}$.*

Lemma 3.4. ***Main Lemma(Gap Principal)*** *Assume that the Thue equation $F(x, y) = 1$ with $\Delta_F > 72000$ has a solution (x_1, y_1) , such that $\epsilon_1 = \epsilon(x_1, y_1)$ satisfy $\frac{\Delta^2}{\epsilon_1^3} < \frac{1}{22\Delta^{2/45}}$. Let (x_2, y_2) be another solution related to the same resolvent form with $\epsilon_2 > \epsilon_1$ then $|x_2y_1 - x_1y_2| = 1$.*

In the rest of this subsection we will prove these two lemmas.

Auxiliary polynomials To prove the lemmas 3.3 and the main lemma 3.4 we apply arguments due to Siegel [73] with refinements by Gel'man [22] and Evertse [28]. A hypergeometric function is a power series of the form

$$F(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)n!} z^n.$$

Following (Bennett [9] ,Evertse [28]) we define

$$A_{r,g}(z) = \sum_{m=0}^{r} \binom{r-g+\frac{1}{3}}{m}\binom{2r-g-m}{r-g}(-z)^m$$

and

$$B_{r,g}(z) = \sum_{m=0}^{r-g} \binom{r-\frac{1}{3}}{m}\binom{2r-g-m}{r-g}(-z)^m.$$

For $r \in \mathbb{N}$ and $g \in \{0,1\}$ we have

$$A_{r,g} = \binom{2r-g}{r} F\left(-\frac{1}{3} - r + g, -r, -2r+g, z\right),$$

$$B_{r,g} = \binom{2r-g}{r} F\left(\frac{1}{3} - r, -r+g, -2r+g, z\right),$$

where F is the standard hypergeometric function. Define $C_{r,g}$ to be the greatest common divisor of the numerators of the coefficients of $A_{r,g}$

We recall a lemma by Evertse [28].

Lemma 3.5. *Let r, g be integers with $r \geq 1$, $g \in \{0,1\}$:*
(i) There exist a power series $F_{r,g}(z)$ such that for all complex numbers z with $|z| < 1$

$$A_{r,g}(z) - (1-z)^{\frac{1}{3}} B_{r,g}(z) = z^{2r+1-g} F_{r,g}(z)$$

(Where $(1-z)^{\frac{1}{3}} = \sum_{m=0}^{\infty} \binom{\frac{1}{3}}{m} (-z)^m$) and if $|z| < 1$

$$|F_{r,g}(z)| \leq \frac{\binom{r-g+\frac{1}{3}}{r+1-g}\binom{r-\frac{1}{3}}{r}}{\binom{2r+1-g}{r}} (1-|z|)^{-\frac{1}{2}(2r+1-g)}.$$

(ii) If $|1-z| \leq 1$, then

$$|A_{r,g}(z)| \leq \binom{2r-g}{r}.$$

Let $z_1 = 1 - \frac{\eta^3}{\varepsilon^3}$ then

$$|z_1| = \frac{3\sqrt{\Delta}}{|\varepsilon_1|^3} < (10)^{-9}. \tag{3.2.2}$$

Consider the complex sequences of $\Sigma_{r,g}$ by

$$\Sigma_{r,g} = \frac{\eta_2}{\varepsilon_2} A_{r,g}(z_1) - \frac{\eta_1}{\varepsilon_1} B_{r,g}(z_1).$$

Using inequality (3.2.2) and lemma (3.5) we conclude the following lemma See Bennett, [9]. It shows that the nonvanishing of $\Sigma_{r,g}$ enables us to find a relation between the size of ϵ_2 and ϵ_1:

Lemma 3.6. *If $\Sigma_{r,g} \neq 0$ then*

$$c_1(r,g)\Delta^{\frac{g}{3}} |\varepsilon_1|^{3r+1-g} |\varepsilon_2|^{-2} + c_2(r,g)\Delta^{r-\frac{g}{6}} |\varepsilon_2| |\varepsilon_1|^{-3r-2(1-g)} > 1 \quad (3.2.3)$$

where we may take

$$c_1(r,g) = \frac{1}{\sqrt{r}}4^r \text{ and } c_2(r,g) = \frac{1}{\sqrt{r}}(2.252)^r$$

for $r \geq 9$, and as follows , for $1 \leq r \leq 8$.

(r,g)	$c_1(r,g)$	$c_2(r,g)$	(r,g)	$c_1(r,g)$	$c_2(r,g)$
$(1,1)$	2.6	1.3	$(6,1)$	42.1	2.4
$(1,0)$	1.1	0.7	$(6,0)$	66.8	2.7
$(2,0)$	6.1	2.4	$(7,1)$	547.0	16.9
$(3,0)$	1.1	0.3	$(7,0)$	39.5	0.9
$(4,0)$	14.2	1.8	$(8,1)$	745.9	13.0
$(5,0)$	9.2	0.7	$(8,0)$	236.9	3.1

To use this key inequality we need to prove nonvanishing of $\Sigma_{r,g}$ for some values of r, g:

Lemma 3.7. $\Sigma_{r,g} \neq 0$ *for* $(r,g) = (1,1)$, *and* $(r,0)$ *for* $2 \leq r \leq 15$.

Proof. This is lemma 6.3 of [9] for $(1,1),(1,0),(2,0),(3,0),(4,0),(5,0)$; for remaining cases we mimic Bennett's proof (see appendix A). □

Also by lemma (6.4)of [9] we have

Lemma 3.8. *If $r \in \mathbb{N}$ and $h \in \{0,1\}$, then at least one of*

$$\{\Sigma_{r,0}, \Sigma_{r+h,1}\}$$

is nonzero.

3.2.2 Proof of lemmas 3.3, 3.4

Proof of Lemma 3.3

Proof. We prove a more general formula; if the hypothesis of lemma 3.3 satisfies, then

$$|\varepsilon_2| > (2.4\Delta)^{-r} |\varepsilon|^{3r+2},$$

for all $1 \le r \le 15$.
By lemma 3.7 , $\Sigma_{r,0} = 0$ for all $1 \le r \le 15$ we can apply lemma 3.2.3 with g=0 for these values of r. Since $|x_2y_1 - x_1y_2| > 1$ from lemma 3.2, we have

$$|\varepsilon_2| > 1.975\,|\varepsilon_1|^2$$

and so since $c_1(1,0) = 1.1$ we obtain

$$c_1(1,0)\left|\varepsilon_1^4\right||\varepsilon_2|^{-2} < 1.1(\frac{1}{1.975})^2 < 0.29.$$

Also since $\Sigma_{1,0} \neq 0$, we can apply lemma (3.6) to conclude that

$$c_2(1,0)\Delta\varepsilon_2\,|\varepsilon_1|^{-5} > 0.71$$

$$\implies |\varepsilon_2| > 1.014\Delta^{-1}\,|\varepsilon_1|^5\,.$$

Now we use nonvanishing of $\Sigma_{2,0}$; since $c_2(1,0) = 6.1$, we have

$$\begin{aligned} c_1(2,0)\left|\varepsilon_1^7\right||\varepsilon_2|^{-2} &< 6.1\left(\frac{\Delta^2}{\varepsilon^3}\right)\left(\frac{1}{1.014}\right)^2 \\ &< 6.1\cdot\frac{1}{22}\left(\frac{1}{1.014}\right)^2 < 0.27. \end{aligned}$$

By lemma 3.2.3 since $\Sigma_{2,0} = 0$ we have

$$c_2(2,0)\Delta^{-2}\varepsilon_2\,|\varepsilon_1|^{-8} > 0.73$$
$$\implies |\varepsilon_2| > 0.3\Delta^{-2}\,|\varepsilon_1|^8\,.$$

Arguing similarly for $(r,0), r \le 9$ we have

$$\varepsilon_2 > c_3(r,g)\Delta^{-r}\,|\varepsilon|^{3r+2}\,,$$

Where $c_3(r,g)$ is

(r,g)	$c_3(r,g)$	(r,g)	$c_3(r,g)$
$(1,0)$	1.014	$(6,0)$	0.37
$(2,0)$	0.3	$(7,0)$	1.11
$(3,0)$	3.23	$(8,0)$	0.322
$(4,0)$	0.555	$(9,0)$	0.002
$(5,0)$	1.422		

Note that for all these values we have $c_3(r,g) > (2.4)^{-r}$. Now we use induction to prove the lemma for $9 \leq r \leq 15$. Suppose it is true for some $9 \leq r < 15$, then

$$\begin{aligned} c_1(r+1,0)\left|\varepsilon_1\right|^{3r+4}\left|\varepsilon_2\right|^{-2} &< \frac{4^{r+1}}{\sqrt{r+1}}(2.4\Delta)^{2r}\left|\varepsilon_1\right|^{-3r} \\ &< \frac{4}{\sqrt{r+1}}(21.16)^r\left(\frac{1}{22}\right)^r < 0.9. \end{aligned}$$

Since $\Sigma_{r+1,0} \neq 0$, By lemma (3.6)

$$\begin{aligned} &\left|\varepsilon_2\right| > \frac{0.1}{c_2(r+1,0)}\Delta^{-r-1}\left|\varepsilon_1\right|^{3r+5} \\ &\Longrightarrow \left|\varepsilon_2\right| > (2.4\Delta)^{-r-1}\left|\varepsilon_1\right|^{3r+5}. \end{aligned}$$

So the result holds for $r+1$ and it completes the proof. □

The next lemma shows that if the hypothesis of lemma 3.4 satisfies then $|\varepsilon_2|$ is arbitrarily large in relation to $|\varepsilon_1|$.

Lemma 3.9. *Assume the Thue equation $F(x,y) = 1$ with $\Delta_F > 72000$ has a solution (x_1,y_1), such that $\varepsilon_1 = \varepsilon(x_1,y_1)$ satisfy $\frac{\Delta^2}{\varepsilon_1^3} < \frac{1}{22\Delta^{2/15}}$ and for any other solutions (x_2,y_2) related to the same resolvent forms with $\varepsilon_2 > \varepsilon_1$ we have $|x_2y_1 - x_1y_2| > 1$; then*

$$\left|\varepsilon_2\right| > (2.3\Delta)^{-r}\left|\varepsilon_1\right|^{3r+2}\Delta^{\frac{-1}{6}}, \tag{3.2.4}$$

for all $r \geq 1$.

Proof. We use the same steps as the proof of lemma 3.3 From lemma (3.2), since $c_1(1,0) = 1.1$ we obtain

$$c_1(1,0)\left|\varepsilon_1^4\right|\left|\varepsilon_2\right|^{-2} < 1.1\left(\frac{1}{1.975}\right)^2 < 0.29.$$

Also since $\Sigma_{1,0} \neq 0$, we can apply lemma (3.6) to conclude that

$$\begin{aligned} &c_2(1,0)\Delta\varepsilon_2\left|\varepsilon_1\right|^{-5} > 0.71 \\ &\Longrightarrow \left|\varepsilon_2\right| > 1.014\Delta^{-1}\left|\varepsilon_1\right|^5. \end{aligned}$$

Now we use nonvanishing of $\Sigma_{2,0}$, and since $c_2(1,0) = 6.1$ we have

$$\begin{aligned} c_1(2,0)\left|\varepsilon_1^7\right|\left|\varepsilon_2\right|^{-2} &< 6.1\left(\frac{\Delta^2}{\varepsilon^3}\right)\left(\frac{1}{1.014}\right)^2 \\ &< 6.1\cdot\frac{1}{22\Delta^{2/15}}\left(\frac{1}{1.014}\right)^2 < 0.07. \end{aligned}$$

By lemma 3.2.3, since $\Sigma_{2,0} = 0$, we have

$$\begin{gathered} c_2(2,0)\Delta^{-2}\varepsilon_2\left|\varepsilon_1\right|^{-8} > 0.93 \\ \Longrightarrow \left|\varepsilon_2\right| > 0.38\Delta^{-2}\left|\varepsilon_1\right|^8. \end{gathered}$$

Arguing similarly for $(r,0), r \leq 9$, we have, for $r \leq 9$

$$\left|\varepsilon_2\right| > c_3(r,g)\Delta^{-r}\left|\varepsilon_1\right|^{3r+2}.$$

Where $c_3(r,g)$ is:

(r,g)	$c_3(r,g)$	(r,g)	$c_3(r,g)$
$(1,0)$	1.014	$(6,0)$	0.37
$(2,0)$	0.38	$(7,0)$	1.11
$(3,0)$	3.33	$(8,0)$	0.322
$(4,0)$	0.555	$(9,0)$	0.002.
$(5,0)$	1.427		

Note that for all these values, we have $c_3(r,g) > (2.3)^{-r}$. To prove the lemma for $r > 9$ we use induction in two steps: For $r \leq 15$, we will show that

$$\left|\varepsilon_2\right| > (2.3\Delta)^{-r}\left|\varepsilon_1\right|^{3r+2}.$$

The inequality holds for $r \leq 9$. Supposing it is true for some $9 \leq r < 15$, then

$$\begin{aligned} c_1(r+1,0)\left|\varepsilon_1\right|^{3r+4}\left|\varepsilon_2\right|^{-2} &< \frac{4^{r+1}}{\sqrt{r+1}}(2.3\Delta)^{2r}\left|\varepsilon_1\right|^{-3r} \\ &< \frac{4}{\sqrt{r+1}}(21.16)^r\left(\frac{1}{22\Delta^{\frac{2}{15}}}\right)^r < 0.001. \end{aligned}$$

Since $\Sigma_{r+1,0} \neq 0$, by lemma (3.6)

$$|\varepsilon_2| > \frac{0.999}{c_2(r+1,0)}\Delta^{-r-1}|\varepsilon_1|^{3r+5}$$
$$\implies |\varepsilon_2| > (2.3\Delta)^{-r-1}|\varepsilon_1|^{3r+5}.$$

Now assume (3.2.4) holds for some $r \geq 15$, then:

$$\begin{aligned} c_1(r+1,0)|\varepsilon_1|^{3r+4}|\varepsilon_2|^{-2} &< \frac{4^{r+1}}{\sqrt{r+1}}(2.3\Delta)^{2r}|\varepsilon_1|^{-3r}\Delta^{\frac{1}{3}} \\ &< \frac{4}{\sqrt{r+1}}(21.16)^r\left(\frac{1}{22\Delta^{\frac{2}{15}}}\right)^r\Delta^{\frac{1}{3}} \\ (\text{Since } \ r \geq 15) \quad &< \left(\frac{21.16}{22}\right)^{15}\Delta^{-5/3} < 0.001. \end{aligned}$$

If $\Sigma_{r+1,0} \neq 0$, then by lemma (3.6)

$$|\varepsilon_2| > \frac{0.999}{c_2(r+1,0)}\Delta^{-r-1}|\varepsilon_1|^{3r+5}$$
$$\implies |\varepsilon_2| > (2.3\Delta)^{-r-1}|\varepsilon_1|^{3r+5}.$$

Otherwise if $\Sigma_{r+1,0} = 0$ then both $\Sigma_{r+1,1}, \Sigma_{r+2,1}$ are nonzero.Using same process for $\Sigma_{r+1,1}$

$$\begin{aligned} c_1(r+1,1)\Delta^{\frac{1}{3}}|\varepsilon_1|^{3r+4}|\varepsilon_2|^{-2} &< \frac{4^{r+1}}{\sqrt{r+1}}(2.3\Delta)^{2r}|\varepsilon_1|^{-3r}\frac{\Delta^{\frac{2}{3}}}{|\varepsilon|} \\ &< \frac{4}{\sqrt{r+1}}(21.16)^r\left(\frac{1}{22\Delta^{\frac{2}{15}}}\right)^r\frac{\Delta^{\frac{2}{3}}}{|\varepsilon|} < 0.001. \end{aligned}$$

Using this bound for ε_2 and lemma 3.2.3 for $\Sigma_{r+2,1}$ we can conclude

$$|\varepsilon_2| > \frac{0.999\sqrt{r+1}}{(2.252)^{r+1}}\Delta^{-r-\frac{5}{6}}|\varepsilon_1|^{3r+3}.$$

It follows

$$
\begin{aligned}
c_1(r+2,1)\Delta^{1/3}\left|\varepsilon_1\right|^{3r+6}\left|\varepsilon_2\right|^{-2} &< \frac{4^{r+2}}{\sqrt{r+2}}\frac{(2.252)^{2r+2}}{(r+1)(0.999)^2}(\frac{\Delta^2}{\varepsilon_1^3})^r\Delta^2 \\
&< \frac{81.31}{(r+1)\sqrt{r+2}}(20.29)^r(\frac{1}{22\Delta^{2/15}})^r\Delta^2 \\
\text{(Since } \ r \geq 15) \quad &< 0.37.
\end{aligned}
$$

And finally applying lemma 3.6 we conclude

$$
|\varepsilon| > \frac{0.73}{c_2(r+2,1)}\Delta^{-r-11/6}\left|\varepsilon\right|^{3r+6}.
$$

Since $|\varepsilon|^3 > \Delta^2$ we obtain

$$
|\varepsilon_2| > (2.3\Delta)^{-r-1}\left|\varepsilon_1\right|^{3r+5}\Delta^{\frac{-1}{6}}
$$

as desired. $\square$

Corollary 3.1. *The main lemma (lemma 3.4) is the immediate consequence of this lemma. Assume the conditions of lemma 3.9 satisfy. Since* $\left|\varepsilon_1^3\right| > 2.3\Delta$, *there is no upper bound for* ε_2. *Therefore* $F(x,y)=1$ *can not have a solution* (x_2,y_2) *such that* $\varepsilon_2 > \varepsilon_1$ *and* $|x_2y_1 - x_1y_2| > 1$.

3.2.3 Applying the main lemma

In this section we will prove that $X_1 = (x_1,y_1) = (1-t^3, t^8-3t^5+3t^2)$ is the only solution of type I for $F(x,y)=1$ for all values of $t > 8586$, and $X_1' = (x_1', y_1') = (t^9+3t^6+4t^3+1, t^8+3t^5+3t^2)$ is the unique solution of type II' for $F'(x,y)=1$. for all values of $t > 8586$. The outline of the proof for solutions of type I is as follows: (the proof for type II' is completely the same). First we will show that any solution of type I belongs to the same resolvent as X_1.Then we will show that X_1 satisfies the conditions of the lemma 3.4. Next step is to show that if there exist any other solutions $X_2 = (x_2,y_2)$ of type I, we have $\varepsilon_{X_2} > \varepsilon_{X_1}$. Moreover, we have $|x_2y_1 - x_1y_2| \geq 2$, which contradicts lemma 3.4.

3.2.4 Solution of type I

Note that

$$
H(x,y) = Ax^2 + Bxy + Cy^2
$$

is a quadratic. For the solution $(1-t^3, t^8-3t^5+3t^2)$ we have

$$\begin{aligned} A &= t^8 - 5t^5 + 7t^2, \\ B &= -t^9 + 3t^6 - 2t^3 + 6, \\ C &= t^{10} - 4t^7 + 7t^4 - 3t, \\ D_F &= t^{18} - 10t^{15} + 41t^{12} - 90t^9 + 102t^6 - 40t^3 - 27 \\ \Delta_F &= 3t^{18} - 30t^{15} + 123t^{12} - 270t^9 + 306t^6 - 120t^3 - 81. \end{aligned} \tag{3.2.5}$$

Also using estimates (3.1.5) for solutions of type I

$$y^2(t^{10}-4t^7+9t^4) > H(x,y) > y^2(t^{10}-4t^7+7t^4) \geq 4(t^{10}-4t^7+7t^4). \tag{3.2.6}$$

Moreover,

$$H(0,1) = t^{10} - 4t^7 + 7t^4 - 3t.$$

The following lemma relates the arguments made in the previous section to solutions of type I.

Lemma 3.10. *Assume the solution* $(1-t^3, t^8-3t^5+3t^2)$ *is related to pair of resolvents* (ε, η). *Then any solution* (x, y) *of type I and the solution (0,1) are also related to same pair of resolvents.*

Proof. First we can observe

$$\begin{aligned} H(1-t^3, t^8-3t^5+3t^2) =& t^{26} - 10t^{23} + 47t^{20} - 130t^{17} + \\ & + 225t^{14} - 249t^{11} + 180t^8 - 88t^5 + 25t^2. \end{aligned} \tag{3.2.7}$$

Let (ε, η) be the pair of resolvent forms that $(x_1, y_1) = (1-t^3, t^8 - 3t^5 + 3t^2)$ is related to; then from inequality (3.2.1), and considering $\Delta_F > 72000$ for all $t \geq 2$, we conclude

$$\left|1 - \frac{\eta_1(x,y)}{\varepsilon_1(x,y)}\right| < \frac{1.012\sqrt{\Delta_F}}{|\varepsilon(x,y)|^3} < t^{-27}.$$

If (x_2, y_2) is any other solution of type I then by estimations (3.1.3)

$$\frac{x_2}{y_2} - \frac{x_1}{y_1} = \frac{x_2 y_1 - x_1 y_2}{y_1 y_2} < 2t^{-5}$$

$$\Longrightarrow x_2 y_1 - x_1 y_2 < 2t^{-5} y_1 y_2.$$

Also

$$\frac{\eta_2}{\varepsilon_2} - \frac{\eta_1}{\varepsilon_1} = \frac{\eta_2\varepsilon_1 - \varepsilon_2\eta_1}{\varepsilon_1\varepsilon_2} = \frac{\pm\sqrt{\Delta_F}(x_1y_2 - x_2y_1)}{\varepsilon_1\varepsilon_2}$$
$$\implies \left|\frac{\eta_2}{\varepsilon_2} - \frac{\eta_1}{\varepsilon_1}\right| < \frac{2\sqrt{\Delta_F}t^{-5}y_1y_2}{|\varepsilon_1|\,|\varepsilon_2|}.$$

$$\begin{aligned}|\varepsilon_1|\,|\varepsilon_2| =& \left|\sqrt{H_1}\right|\left|\sqrt{H_2}\right| \\ =& y_1\sqrt{A\left(\frac{x}{y}\right)^2 + B\left(\frac{x}{y}\right) + C}\; y_2\sqrt{A\left(\frac{x}{y}\right)^2 + B\left(\frac{x}{y}\right) + C}.\end{aligned}$$

$$\begin{aligned}\implies \left|\frac{\eta_2}{\varepsilon_2} - \frac{\eta_1}{\varepsilon_1}\right| &< 2\frac{\sqrt{\Delta_F}}{t^5\sqrt{A\left(\frac{x}{y}\right)^2 + B\left(\frac{x}{y}\right) + C}\sqrt{A\left(\frac{x}{y}\right)^2 + B\left(\frac{x}{y}\right) + C}} \\ &< \frac{4t^9}{t^5t^5t^4} < \frac{4}{t^4}.\end{aligned}$$

So

$$\left|1 - \frac{\eta(x_2,y_2)}{\varepsilon(x_2,y_2)}\right| < \left|1 - \frac{\eta(x_1,y_1)}{\varepsilon(x_1,y_1)}\right| + \left|\frac{\eta_2}{\varepsilon_2} - \frac{\eta_1}{\varepsilon_1}\right| < \frac{1}{t^{27}} + \frac{4}{t^4} < \frac{\pi}{3}$$

for all $t \geq 2 \implies (x_2, y_2)$ is also related to pair (ε, η) for all $t \geq 2$.
For solution $(x_2, y_2) = (0, 1)$

$$\left|\frac{\eta_2}{\varepsilon_2} - \frac{\eta_1}{\varepsilon_1}\right| = \frac{\eta_2\varepsilon_1 - \varepsilon_2\eta_1}{|\varepsilon_1|\,|\varepsilon_2|} < \frac{2t^9t^3}{t^{12}t^4} < \frac{2}{t^4},$$

which means $(0, 1)$ is also related to same pair of resolvent forms. □

Lemma 3.11. *If (x_2, y_2) is a solution of type I different from $(x_1, y_1) = (1 - t^3, t^8 - 3t^5 + 3t^2)$ then $|\varepsilon(x_2, y_2)| > |\varepsilon(x_1, y_1)|$ and $|x_1y_2 - x_2y_1| \geq 2$.*

Proof. Let (x_2, y_2) be a solution of type I such that $(x_2, y_2) \neq (x_1, y_1) = (1 - t^3, t^8 - 3t^5 + 3t^2)$. First assume $|\varepsilon(x_2, y_2)| < |\varepsilon(x_1, y_1)|$. From the estimates (3.2.6) for solutions of type I we have:

$$\begin{aligned}H(x,y) > 4(t^{10} - 4t^7 + 7t^4) &\implies H(x_2,y_2) > H(0,1) \\ &\implies |\varepsilon(x_2,y_2)| > |\varepsilon(1,0)|.\end{aligned}$$

By using lemma (3.1),

$$
\begin{aligned}
&|\varepsilon(x_2, y_2)| > 0.987\,|\varepsilon(0,1)|^2 \\
&|\varepsilon(x_1, y_1)| > 0.987\,|\varepsilon(x_2, y_2)|^2 \\
\Longrightarrow\ &|\varepsilon(x_1, y_1)| > (0.987)^3\,|\varepsilon(0,1)|^4 .
\end{aligned}
$$

But from inequality (3.2.7) $(0.987)^3\,|\varepsilon(0,1)|^4 > |\varepsilon(x_1, y_1)|$. This contradiction leads us to conclude that $|\varepsilon(x_2, y_2)| > |\varepsilon(x_1, y_1)|$ which by using lemma (3.1)

$$
\begin{aligned}
|\varepsilon(x_2, y_2)| &> 0.987\,|\varepsilon(x_1, y_1)|^2 \\
|H(x_2, y_2)| &> (0.987)^2\,|H(x_1, y_1)|^2
\end{aligned}
\tag{3.2.8}
$$

$$
\Longrightarrow y_2^2(A(\frac{x}{2})^2 + B(\frac{x}{y}) + C) > 10^{48}.
$$

Using estimations (3.1.5)for $\frac{x}{y}$, we get

$$
|y| > t^{19}. \tag{3.2.9}
$$

On the other hand, if (x_2, y_2) is an integral solution for $F(x, y) = 1$, then

$$
x_2^3 - (t^4 - t)x_2^2 y_2 + (t^5 - 2t^2)x_2 y_2^2 + y_2^3 = 1,
$$

Multiplying both sides by x_1^3:

$$
(x_1x_2)^3 - (t^4 - t)(x_2x_1)^2(x_1y_2) + (t^5 - 2t^2)(x_2x_1)(x_1y_2)^2 = x_1^3 - (x_1y_2)^3
$$

Since x_i, y_i and t are integers $x_2 | x_1^3 - (x_1y_2)^3$.
If $x_1y_2 - x_2y_1 = 1$ then $x_1y_2 = 1 + x_2y_1$ using value of x_1

$$
x_2 \big| (t^9 - 3t^6 + 3t^3) \Longrightarrow |x_2| < t^9.
$$

Therefore from estimations (3.1.5) for solutions of type I, $|y_2| < t^{14}$. Similarly if $x_1y_2 - x_2y_1 = -1$ then $x_1y_2 = -1 + x_2y_1$

$$
x_2 \big| (t^9 - 3t^6 + 3t^3 - 2) \Longrightarrow |x_2| < t^9.
$$

Again using estimations (3.1.5), $|y_2| < t^{14}$, but by (3.2.9) $|y| > t^{19}$. This contradiction completes the proof. □

Lemma 3.12. *For $t > 8586$ there is no integral solution of type I, other than $(1-t^3, t^8-3t^5+3t^2)$.*

Proof. By (3.2.5) and (3.2.7) we can see for $t > 8586$, $\frac{\Delta_F^2}{\varepsilon_1^3} < \frac{1}{22\Delta_F^{2/15}}$. and also $\Delta_F > 72000$. By lemma 3.11, if there exist any other solution (x_2, y_2) of type l, we have $|\varepsilon(x_2, y_2)| > |\varepsilon(x_1, y_1)|$. Therefore by the lemma 3.4(main lemma), we have $|x_2y_1 - x_1y_2| = 1$ which contradicts lemma 3.11. □

3.2.5 Solutions of type II'

For solutions of type II' we prove this lemma:

Lemma 3.13. *For $t > 8586$ there is no integral solution of type II', other than $(t^9+3t^6+4t^3+1, t^8+3t^5+3t^2)$.*

For the proof we will follow exactly the same steps as in the previous section. Using the same notation as we used in 3.2.1 for the solution $(t^9 + 3t^6 + 4t^3 + 1, t^8 + 3t^5 + 3t^2)$ we have

$$H(x,y) = Ax^2 + Bxy + Cy^2,$$

where

$$\begin{aligned}
A &= t^8 + 5t^5 + 7t^2, \\
B &= -t^9 - 7t^6 - 12t^3 - 9, \\
C &= t^{10} + 6t^7 + 12t^4 + 12t, \\
D_{F'} &= t^{18} + 10t^{15} + 41t^{12} + 90t^9 + 102t^6 + 40t^3 - 27, \\
\Delta_{F'} &= 3t^{18} + 30t^{15} + 123t^{12} + 270t^9 + 306t^6 + 120t^3 - 81.
\end{aligned} \tag{3.2.10}$$

Also using estimates (3.1.6) for solutions of type II'

$$y^2(t^{10}+4t^7+9t^4) > H(x,y) > y^2(t^{10}+4t^7+7t^4) \geq 4(t^{10}-4t^7+7t^4) \tag{3.2.11}$$

$$H(t,1) = t^{10} + 4t^7 + 7t^4 + 3t,$$

$$\begin{aligned}
H(t^9 + 3t^6 + 4t^3 + 1, t^8 + 3t^5 + 3t^2) = &t^{26} + 10t^{23} + 47t^{20} + 130t^{17} \\
&+ 225t^{14} + 234t^{11} + 120t^8.
\end{aligned} \tag{3.2.12}$$

Similar to lemma (3.10) we have

Lemma 3.14. *Assume the solution* $(x_1, y_1) = (t^9+3t^6+4t^3+1, t^8+3t^5+3t^2)$ *is related to a pair of resolvents* (ε, η)*, then any solution* (x_2, y_2) *of type II' is also related to the same pair of resolvents.*

Proof. Since

$$\frac{x_2}{y_2} - \frac{x_1}{y_1} = \frac{x_2y_1 - x_1y_2}{y_1y_2} < 2t^{-5}.$$

The same argument as proof of lemma (3.10)is valid. □

Lemma 3.15. *If* (x_2, y_2) *is a solution of type II', different from* $(x_1, y_1) = (t^9+3t^6+4t^3+1, t^8+3t^5+3t^2)$ *then* $|\varepsilon(x_2, y_2)| > |\varepsilon(x_1, y_1)|$ *and* $|x_1y_2 - x_2y_1| \geq 2$.

Proof. Let (x_2, y_2) be a solution of type II' such that $(x_2, y_2) \neq (x_1, y_1) = (t^9+3t^6+4t^3+1, t^8+3t^5+3t^2)$. First assume $|\varepsilon(x_2, y_2)| < |\varepsilon(x_1, y_1)|$. From the estimates (3.2.11) for solutions of type II' we have:

$$H(x, y) > 4(t^{10}+4t^7+7t^4) \Longrightarrow H(x_2, y_2) > H(t, 1) \Longrightarrow |\varepsilon(x_2, y_2)| > |\varepsilon(t, 1)|$$

and by using lemma (3.1)

$$\begin{aligned} |\varepsilon(x_2, y_2)| &> 0.987\,|\varepsilon(t, 1)|^2\,, \\ |\varepsilon(x_1, y_1)| &> 0.987\,|\varepsilon(x_2, y_2)|^2\,, \\ \Longrightarrow |\varepsilon(x_1, y_1)| &> (0.987)^3\,|\varepsilon(0, 1)|^4\,. \end{aligned}$$

But by inequality (3.2.12) $(0.987)^3\,|\varepsilon(t, 1)|^4 > |\varepsilon(x_1, y_1)|$. This contradiction leads us to conclude that $|\varepsilon(x_2, y_2)| > |\varepsilon(x_1, y_1)|$ which by using lemma (3.1)

$$\begin{aligned} |\varepsilon(x_2, y_2)| &> 0.987\,|\varepsilon(x_1, y_1)|^2 \\ |H(x_2, y_2)| &> (0.987)^2\,|H(x_1, y_1)|^2 \\ \Longrightarrow y_2^2(A(\frac{x}{2})^2 + B(\frac{x}{y}) + C) &> 10^{48}. \end{aligned} \tag{3.2.13}$$

Using estimations (3.1.5)for $\frac{x}{y}$, we get

$$|y_2| > t^{19} \Longrightarrow x_2 > t^{20}. \tag{3.2.14}$$

If (x_2, y_2) is an integral solution for $F'(x, y) = 1$ then

$$x_2^3 - (t^4 + 4t)x_2^2y_2 + (t^5 + 3t^2)x_2y_2^2 + y_2^3 = 1.$$

Multiplying both sides by x_1^3:

$$(x_1x_2)^3 - (t^4+4t)(x_2x_1)^2(x_1y_2) + (t^5+3t^2)(x_2x_1)(x_1y_2)^2 = x_1^3 - (x_1y_2)^3.$$

Since x_i, y_i and t are integers

$$x_2 \big| \left(x_1^3 - (x_1y_2)^3 - (t^5+3t^2)(x_2x_1)(x_1y_2)\right).$$

If $x_1y_2 - x_2y_1 = 1$ then $x_1y_2 = 1 + x_2y_1$ we have

$$x_2^2 \big| x_1^3 - \left(3y_1 + (t^5+3t^2)y_1\right)x_2 - 1.$$

By plugging the values of x_1 and y_1 it is easy to check that

$$x_1^3 - \left(3y_1 + (t^5+3t^2)y_1\right)x_2 - 1 \neq 0.$$

Therefore we have

$$\left|x_2^2\right| \leq \left|x_1^3 - \left(3y_1 + (t^5+3t^2)y_1\right)x_2 - 1\right|. \tag{3.2.15}$$

On the other hand since $t > 10$,$x_1 < t^{10}$, $y_1 < t^9$. Hence from inequality (3.2.14), we have:

$$x_1^3 < \frac{x_2^2}{3}, \quad \left|\left(3y_1 + (t^5+3t^2)y_1\right)\right| < \frac{x_2}{3}$$
$$\Longrightarrow \left|x_1^3 - \left(3y_1 + (t^5+3t^2)y_1\right)x_2 - 1\right| < \frac{2x_2^2}{3},$$

which contradicts (3.2.15). Similarly if $x_1y_2 - x_2y_1 = -1$ then $x_1y_2 = -1 + x_2y_1$

$$x_2^2 \big| x_1^3 + \left(3y_1 + (t^5+3t^2)y_1\right)x_2 - 1.$$

By same argument, this cannot happen. □

Proof of theorem 3.13

Proof. By (3.2.10) and (3.2.12) for $t > 8586$, $\frac{\Delta_{F'}^2}{\varepsilon_1^3} < \frac{1}{22\Delta_{F'}^{2/15}}$. and also $\Delta_{F'} > 72000$. By lemma (3.15) if there exist any other solutions (x_2, y_2) of type II', we have $|\varepsilon(x_2, y_2)| > |\varepsilon(x_1, y_1)|$. Therefore by the lemma 3.4(main lemma), we have $|x_2y_1 - x_1y_2| = 1$ which contradicts lemma 3.15. □

3.3 Solutions of type II,I'

In this section we use linear forms in logarithms to find all solutions of type II and I' . As we mentioned in Chapter 2, for this purpose we need to have a system of fundamental units and the crucial step is to prove a "stable growth " condition. Fortunately Thomas [80] introduced a system of fundamental units for the split familes of Thue equations. By Lemma 4.11 of [80], $t-\theta$ and θ form a pair of fundamental units in $\mathbb{Q}(\theta)$ and similarly $t-\alpha$ and α form a pair of fundamental units in $\mathbb{Q}(\alpha)$. If (x,y) is a solution of a Thue equation $F(x,y)=1$ then we have

$$(x-y\theta)(x-y\theta')(x-y\theta'')=1.$$

Therefore $x-y\theta$ is a unit in $\mathbb{Q}(\theta)$. And it means that there exist $m,n\in\mathbb{Z}$ such that $x-y\theta=(t-\theta)^m\theta^n$. Similarly for the solution (x,y) of $F'(x,y)=1$ there exists a probably different pair m,n such that $(x-y\alpha)=(t-\alpha)^m\alpha^n$. Moreover we have

$$y^3(\frac{x}{y}-\theta)(\frac{x}{y}-\theta')(\frac{x}{y}-\theta'')=1 \tag{3.3.1}$$

and

$$y^3(\frac{x}{y}-\alpha)(\frac{x}{y}-\alpha')(\frac{x}{y}-\alpha'')=1. \tag{3.3.2}$$

The main idea in this section to prove the "stable growth" condition is to use some congruence conditions. Let (x,y) be a solution for $F(x,y)=1$ then

$$x-y\theta=\pm(t-\theta)^n\theta^{-m} \quad \text{for } m,n\in\mathbb{Z}. \tag{3.3.3}$$

$$=(-1)^n\left(\theta^2-\binom{n}{1}t\theta+\binom{n}{2}t^2-\binom{n}{3}t^3\theta^{-1}+\cdots\right)\theta^{(n-m-2)}.$$

The idea is to find a congruence condition for both sides of the above equation in different rings say $\mathbb{Z}/t\mathbb{Z}\theta$,$\mathbb{Z}/t^3\mathbb{Z}\theta$ and $\mathbb{Z}/t^5\mathbb{Z}\theta$.

3.3.1 Solution of type II

For solutions of type II we will prove that

Lemma 3.16. *If* $t>1119749$, *then there are no non-trivial solutions of type II.*

Let (x, y) be a solution of type II of $F(x, y) = 1$ so $t + \frac{1}{t^6} < \frac{x}{y} < t + \frac{2}{t^5}$. First we will show that $y > 0$. We rewrite the equation (3.3.1) as $y^2(\frac{x}{y} - \theta)(\frac{x}{y} - \theta'')(x - y\theta') = 1$. So by (3.1.3) and (3.1.5) We can conclude that $x - y\theta' < 0$.
Consider the equation

$$\frac{x - y\theta'}{x - y\theta''} = \left(\frac{t - \theta'}{t - \theta''}\right)^n \left(\frac{\theta'}{\theta''}\right)^{-m},$$

from estimations (3.1.3) it is clear that RHS is positive. Therefore LHS is also positive $\Longrightarrow x - y\theta'' = y\left(\frac{x}{y} - \theta''\right) < 0$,but $\frac{x}{y} - \theta'' < 0 \Longrightarrow y > 0$.
Using equation (3.3.3) and estimations (3.1.3) we can obtain

$$0 > \frac{\frac{x}{y} - \theta''}{\frac{x}{y} - \theta} = \frac{x - \theta'' y}{x - \theta y} = \left(\frac{t - \theta''}{t - \theta}\right)^n \left(\frac{\theta''}{\theta}\right)^{-m}.$$

So n , m have different parities mod (2). Taking absolute values of both sides we have:

$$\frac{\theta'' - \frac{x}{y}}{\frac{x}{y} - \theta} = \left(\frac{\theta'' - t}{t - \theta}\right)^n \left(\frac{\theta''}{|\theta|}\right)^{-m}.$$

Taking logarithms from both sides and using estimations (3.1.3) we obtain that :

$$\begin{aligned} \log(t^3 - 4) < n \log\left(\frac{\theta'' - t}{t - \theta}\right) - m \log\left(\frac{\theta''}{|\theta|}\right) < \log(t^3 - 3), \\ t^3 - 4 < \frac{\theta'' - t}{t - \theta} < t^3 - 3, \quad t^9 - 3t^6 < \frac{\theta''}{|\theta|} < t^9 - 2t^6. \end{aligned} \tag{3.3.4}$$

Considering (3.3.4) , if $m = 0$ then $n = 1$ which with positive sign gives us the solution $(t, 1)$. If $m < 0$ then $n < 0$ so $|x - y\theta'| = |(t - \theta')^n(\theta')^{-m}| > 1$. On the other hand from (3.3.1) we have $|x - y\theta'| = \frac{1}{\left|y^2(\frac{x}{y} - \theta)(\frac{x}{y} - \theta'')\right|}$, hence by the estimations (3.1.5) we have $|x - y\theta'| < 1$. This contradiction leads us to conclude that $m > 0 \Rightarrow n > 0$ and from first inequality of (3.3.4) $n > m$; moreover:

$$\begin{aligned} n - m &> \left(\frac{\log\left(\frac{\theta''}{|\theta|}\right)}{\log\left(\frac{\theta'' - t}{t - \theta}\right)} - 1\right) > m\left(\frac{\log t^6(t^3 - 3)}{\log(t^3 - 3)} - 1\right) \\ &> m\frac{\log t^6}{\log(t^3 - 3)} > 2m. \end{aligned}$$

SO $n > 3m$, $n - m - 2 > 0 \Rightarrow n - m - 2 = 3k + i$ with $i \in \{0, 1, 2\}$and $k > 0$.

Linear form in logarithms

From Siegel's identity:

$$(\theta' - \theta'')(x - \theta y) + (\theta'' - \theta)(x - \theta' y) + (\theta - \theta')(x - \theta'' y) = 0,$$

using this, corresponding to the solutions of type II we define a linear form in logarithms as follows:

$$\Lambda = \log \frac{\theta'' - \theta'}{\theta' - \theta} + nlog \left| \frac{t - \theta}{t - \theta''} \right| + m \log \left| \frac{\theta''}{\theta} \right| \tag{3.3.5}$$

$$= \log \left(1 + \frac{(\theta - \theta'')(x - \theta' y)}{(\theta' - \theta)(x - \theta'' y)} \right), \tag{3.3.6}$$

where $\frac{(\theta-\theta'')(x-\theta' y)}{(\theta'-\theta)(x-\theta'' y)} < 0$.

We will use the (3.1.3) to find an upper bound for $|\lambda|$.

Upper bound for Λ

$$\begin{aligned}
\frac{(\theta'' - \theta)(x - \theta' y)}{(\theta' - \theta)(x - \theta'' y)} =& \frac{\theta'' - \theta}{\theta' - \theta} \left(\frac{t - \theta'}{t - \theta''} \right)^n \left(\frac{\theta''}{\theta'} \right)^m \\
(\text{since } m < \frac{n}{3}), <& \frac{\theta'' - \theta}{\theta' - \theta} \left(\frac{t - \theta'}{t - \theta''} \right)^n \left(\frac{\theta''}{\theta'} \right)^{\frac{n}{3}} \\
<& (t^3 - 2) \left(\left(\frac{1}{t^3 - 3} \right)^3 \right)^n (t^3 - 2)^{\frac{n}{3}} \\
<& \left(\frac{1.01}{(t^3 - 2)^3} \right)^n (t^3 - 2)^{\frac{n}{3}+1} \\
<& (t^3 - 2)^{\left(\frac{-8n}{3}+1\right)} (1.01)^n.
\end{aligned}$$

If $0 < x < \frac{1}{2}$ then $|\log(1-x)| < 2x$, So $|\Lambda| < 2(t^3-2)^{\left(\frac{-8n}{3}+1\right)}(1.01)^n$

$$\Longrightarrow \log |\Lambda| < \log(2) + n \log(1.01) + \left(\frac{-8n}{3} + 1 \right) \log(t^3 - 2). \tag{3.3.7}$$

Lower bound for A

Next step is to prove the "stable growth" condition for solutions of type II. Let $A = \max\{|n|, |m|\}$, in this case since $n > m > 0$, A is equal to n. Using some congruence arguments we find a lower bound for A. Considering equation (3.3.3),we will write RHS of (3.3.3) as linear combination of $1, \theta, \theta^2$

and calculate the coefficient of θ^2 in the rings $\mathbb{Z}/t^j\mathbb{Z}[\theta]$ for $j \in 1, 3, 5$,in which they all should be zero. Since $n-m-2 > 0 \Rightarrow n-m-2 = 3k+i$ with $i \in \{0, 1, 2\}$and $k \geq 0$. For each value of i we do the factorization in a different ring and find the lower bound for A.

$i = 0$**, Factoring in** $\mathbb{Z}/t\mathbb{Z}[\theta]$

Lemma 3.17. *For $i = 0$ there is no integral solution.*

Proof. Let $i = 0$

$$\theta^3 = (t^4 - t)\theta^2 - (t^5 - 2t^2)\theta - 1 \equiv -1 \pmod{t}.$$

Considering (3.3.3) LHS is congruent to $x - y\theta$.

$$\begin{aligned} RHS &\equiv (-1)^n \theta^2 \left(\theta^3\right)^k \theta^i \\ &\equiv (-1)^n \theta^2 (-1)^k \\ &\equiv (-1)^{n+k} \theta^2 \end{aligned}$$

$\implies (-1)^{n'+k} \equiv 0 \pmod{t}$. which is not possible so there is no solution for $i = 0$. □

$i = 1$ **Factoring in** $\mathbb{Z}/t^3\mathbb{Z}[\theta]$ In this ring we can write this list of congruences:

$$\begin{aligned} \theta^3 &\equiv -t\theta^2 + 2t^2\theta - 1, \\ t^2\theta^4 &\equiv -t^2\theta, \\ t^2\theta^5 &\equiv -t^2\theta^2. \end{aligned} \tag{3.3.8}$$

Using congruences (3.3.8), we have

$$\begin{aligned} \left(\theta^3\right)^k &\equiv (-1)^k \left(1 + kt\theta^2 - 2kt^2\theta + \binom{k}{2} t^2\theta^4\right), \\ \left(\theta^3\right)^k \theta &\equiv (-1)^k \left(\theta + kt\theta^3 - 2kt^2\theta^2 + \binom{k}{2} t^2\theta^5\right), \\ &\equiv (-1)^k \left(-\left(3k + \binom{k}{2}\right) t^2\theta^2 + \theta - kt\right). \end{aligned}$$

Let $i = 1$; then for RHS of (3.3.3), we obtain:

$$
\begin{aligned}
&RHS\\
&\equiv (-1)^n \left(\theta^2 - nt\theta + \binom{n}{2}t^2\right) \left(\theta^3\right)^k \theta,\\
&\equiv (-1)^{n+k} \left(\theta^2 - nt\theta + \binom{n}{2}t^2\right) \left(-\left(3k + \binom{k}{2}\right) t^2\theta^2 + \theta - kt\right),\\
&\equiv (-1)^{n+k} \left(-\left(3k + \binom{k}{2}\right) t^2\theta^4 + \theta^3 - (n+k)\, t\theta^2 + \left(kn + \binom{n}{2}\right) t^2\theta\right),\\
&\equiv (-1)^{n+k} \left((n+k+1)\, t^2\theta^2 + \left(kn + \binom{n}{2} + 3k + \binom{k}{2} + 2\right) t^2\theta - 1\right).
\end{aligned}
$$

Considering coefficient of θ^2, leads us to conclude that

$$(k+n+1)t \equiv 0 \pmod{t^3} \Longrightarrow k+n+1 \equiv 0 \pmod{t^2}$$

But $k+n+1 > 0$ so $k+n+1 \geq t^2$. Using value of k one can obtain $\frac{4}{3}A > t^2 \Longrightarrow A > \frac{3}{4}t^2$. Hence we have the following lemma :

Lemma 3.18. *For $i = 1$ there is no solution unless $A > \frac{3}{4}t^2$.*

Factoring in $\mathbb{Z}/t^5\mathbb{Z}[\theta]$

Lemma 3.19. *For $i = 2$ there is either no nontrivial integral solution or $A > \left(\frac{4}{5}\sqrt{2t^3 + \frac{161}{100}} + \frac{14}{75}\right)$.*

Proof. In the ring $\mathbb{Z}/t^5\mathbb{Z}[\theta]$,we have this list of congruences:

$$
\begin{aligned}
\theta^3 &\equiv (t^4 - t)\theta^2 + 2t^2\theta - 1,\\
t^3\theta^3 &\equiv -t^4\theta^2 - t^3,\\
t^4\theta^5 &\equiv -t^4\theta^2.
\end{aligned}
\tag{3.3.9}
$$

Finally using these congruences we have:

$$
\begin{aligned}
&\left(\theta^3\right)^k\\
&\equiv (-1)^k \left(Mt^4 + \left(\binom{k}{3} + 2k(k-1) + \binom{k}{2}\right) t^3 - \left(\binom{k}{2} + 2k\right) t^2\theta + kt\theta^2 + 1\right),
\end{aligned}
$$

for some $M \in \mathbb{Z}$.

Let $i = 2$ then $n - m - 2 = 3k + 2$ with $k \geq 0$. By equation (3.3.3)we can write:

$$\text{RHS} \equiv \left(\theta^4 - nt\theta^3 + \binom{n}{2}t^2\theta^2 - \binom{n}{3}t^3\theta + \binom{n}{4}t^4\right)\left(\theta^3\right)^k.$$

Using above calculation and congruences (3.3.9) one can see the coefficient of θ^2 in $(t-\theta)^n\,(\theta)^{-m}$ is $\left(n + nk + 4k + 3 + \binom{n}{2} + \binom{k}{2}\right)t^2$

$$\implies \left(4k + \binom{k}{2} + kn + \binom{n}{2} + n + 3\right) \equiv 0 \pmod{t^3}.$$

Since LHS is positive

$$\implies \left(4k + \binom{k}{2} + kn + \binom{n}{2} + n + 3\right) > t^3$$

$$\implies (k+n)^2 + (k+n) + 6k + 6 \geq 2t^3.$$

From inequalities (3.3.4) it is easy to see $\frac{n}{4} < m$, so we can obtain

$$\frac{25}{16}A^2 - \frac{7}{12}A > 2t^3 + \frac{14}{9} \implies A > \left(\frac{4}{5}\sqrt{2t^3 + \frac{161}{100}} + \frac{14}{75}\right).$$

□

Lemma 3.20. *There is no solution of type II other than the trivial ones unless*

$$A > \left(\frac{4}{5}\sqrt{2t^3 + \frac{161}{100}} + \frac{14}{75}\right).$$

Proof. Follows from lemmas (3.17), (3.18), (3.19). □

Large values of t

Lower bound for Λ in type II By lemma 3.20, if there exists a non-trivial solution of type II, then we have:

$$A > \left(\frac{4}{5}\sqrt{2t^3 + \frac{161}{100}} + \frac{14}{75}\right).$$

Recall that:

$$\Lambda = \log\frac{\theta'' - \theta'}{\theta' - \theta} + n\log\left|\frac{t-\theta}{t-\theta''}\right| + m\log\left|\frac{\theta''}{\theta}\right|$$

We will apply theorem (2.3) to find a lower bond for Λ. Put $\alpha_1 = \frac{\theta''-\theta'}{\theta'-\theta}, \alpha_2 = \left|\frac{t-\theta}{t-\theta''}\right|, \alpha_3 = \left|\frac{\theta''}{\theta}\right|, b_1 = 1, b_2 = n, b_3 = m, D = 6$.

First, using (3.1.3) we find an upper bound for the weil heights of α_i's:

$$\begin{aligned} h\left(\frac{\theta''-\theta'}{\theta''-\theta}\right) &\leq 2h\left(\theta''-\theta'\right) = \frac{2}{3}\log\big((\theta''-\theta')(\theta''-\theta)(\theta'-\theta)\big) < 6\log t \\ h\left(\frac{\theta''}{\theta}\right) &= \frac{1}{6}\log\left(\frac{\theta''}{\theta}\right)^2 < 3\log t \\ h\left(\frac{t-\theta}{t-\theta''}\right) &= \frac{1}{6}\log\left(\frac{t-\theta''}{t-\theta'}\right)^2 < 3\log t. \end{aligned} \tag{3.3.10}$$

Therefore we can take $A_1 = 36\log(t), A_2 = 18\log(t), A_3 = 18\log(t)$ and $B = n$; by the above calculation we have:

$$\log|\Lambda| > -8.344 \cdot (10)^{15}\log(t)^3\log(68.3n).$$

Comparing this with (3.3.7) :

$$\begin{aligned} &-8.344(10)^{15}\log(t)^3\log(68.3n) \\ &\qquad < \log(2) + n\log(1.01) + \left(\frac{-8n}{3}+1\right)\log(t^3-2). \end{aligned} \tag{3.3.11}$$

Since

$$A > \left(\frac{4}{5}\sqrt{2t^3+\frac{161}{100}}+\frac{14}{75}\right),$$

the above inequality is only valid for $t < (1.197)\cdot(10)^{13}$. We can summarize the argument for solutions of type II in this lemma

Lemma 3.21. *For $t > 1.197\cdot(10)^{13}$ there is no solution of type II other than the trivial ones.*

Intermediate values of t

As we have seen there is no nontrivial solution of type II, if $t < 1.197\cdot(10)^{13}$. This lower bound for t is still huge and treating the smaller value of t is very time consuming, thus we should try to find a better lower bound for t. The idea is to reduce the linear form in three logarithms used in the previous section to a linear form in two logarithms with the aim of getting a better bound for intermediate values of t. To do so, in this section we assume $t < 1.197 \cdot (10)^{13}$ and $A > \left(\frac{4}{5}\sqrt{2t^3 + \frac{161}{100}} + \frac{14}{75}\right)$. Moreover, from inequality (3.3.11) one can obtain

$$n < 4.8 \cdot 10^{19}, \quad m < 1.6 \cdot 10^{19}.$$

Lemma 3.22. *For $t > 10$, if there exists a non-trivial solution of type II then the following inequality holds:*

$$\frac{1}{m}\left(n - 3m - 1\right) < \frac{\log\left(1 + \frac{11}{t^3}\right)}{\log(t^3 - 4)}.$$

Proof. We have

$$n \log\left(\frac{\theta'' - t}{t - \theta}\right) - \log\left(\frac{\theta'' - \frac{x}{y}}{\frac{x}{y} - \theta}\right) = m \log\left(\frac{\theta''}{|\theta|}\right).$$

Since

$$\frac{\theta'' - \frac{x}{y}}{\frac{x}{y} - \theta} < \frac{\theta'' - t}{t - \theta},$$

we can conclude

$$(n-1)\log\left(\frac{\theta'' - t}{t - \theta}\right) < m \log\left(\frac{\theta''}{|\theta|}\right).$$

And from estimates (3.1.3)

$$(n-1)\log(t^3 - 4) < m \log(t^9 - 2t^6). \tag{3.3.12}$$

On the other hand:

$$(t^9 - 2t^6) < \left(1 + \frac{11}{t^3}\right)\left(t^3 - 4\right)^3. \tag{3.3.13}$$

Now combining (3.3.12) and (3.3.13) we get:

$$(n-1)\log(t^3 - 4) < m \log\left(\left(1 + \frac{11}{t^3}\right)\left(t^3 - 4\right)^3\right),$$

which is equivalent to

$$n - 3m - 1 < m\left(\frac{\log\left(1 + \frac{11}{t^3}\right)}{\log(t^3 - 4)}\right).$$

□

Lemma 3.23. *Let $t > 1119749$. If there exists a non-trivial solution of type II, then $n = 3m + k$ with $k \leq 3$.*

Proof. Substituting $3m + k$ for n in lemma 3.22, we have the following:

$$\frac{k-1}{m} < \frac{\log\left(1+\frac{11}{t^3}\right)}{\log(t^3-4)}.$$

We know $m < 1.6 \times 10^{19}$; now, if $k \geqslant 4$, these two give us:

$$\frac{3}{1.6 \times 10^{19}} < \frac{\log\left(1+\frac{11}{t^3}\right)}{\log(t^3-4)}.$$

which implies

$$3.125 \times 10^{-20} < \frac{\log\left(1+\frac{11}{t^3}\right)}{\log(t^3-4)}.$$

The right hand side is a strictly decreasing function in the variable t and so we easily deduce that $t < 1119749$. So for $t > 1119749$ we have $k \leq 3$. □

Lemma 3.24. *If there exists a non-trivial solution such that $n = 3m + k$ with $k \leq 3$ then $t < 1119749$.*

Proof. We have:

$$\begin{aligned}
\Lambda &= \log\frac{\theta''-\theta'}{\theta'-\theta} + n\log\left|\frac{t-\theta}{t-\theta''}\right| + m\log\left|\frac{\theta''}{\theta}\right| \\
&= \log\frac{\theta''-\theta'}{\theta'-\theta} + (3m+k)\log\left|\frac{t-\theta}{t-\theta''}\right| + m\log\left|\frac{\theta''}{\theta}\right| \\
&= \left[\log\frac{\theta''-\theta'}{\theta'-\theta} + k\log\left|\frac{t-\theta}{t-\theta''}\right|\right] + \left[3m\log\left|\frac{t-\theta}{t-\theta''}\right| + m\log\left|\frac{\theta''}{\theta}\right|\right] \\
&= \log\left(\frac{\theta''-\theta'}{\theta'-\theta}\left|\frac{t-\theta}{t-\theta''}\right|^k\right) + m\log\left(\left|\frac{t-\theta}{t-\theta''}\right|^3\left|\frac{\theta''}{\theta}\right|\right) \\
&= -\log\left(\underbrace{\left(\frac{\theta'-\theta}{\theta''-\theta'}\right)\left|\frac{t-\theta''}{t-\theta}\right|^k}_{\alpha_1}\right) + m\log\left(\underbrace{\left|\frac{t-\theta}{t-\theta''}\right|^3\left|\frac{\theta''}{\theta}\right|}_{\alpha_2}\right) \\
&= m\log\alpha_2 - \log\alpha_1,
\end{aligned}$$

where $\alpha_1, \alpha_2 > 1$. To find a lower bound for this new form of λ we use the lemma 2.4

To apply the lemma for $\log|\Lambda|$ we take $b_2 = m, b_1 = 1, D = 6$, using (3.1.3), (3.5.8)

$$|\log \alpha_1| / D < h(\alpha_1) < (3k+6)\log t < 15 \log t$$
$$|\log \alpha_2| / D < h(\alpha_2) < 12 \log t.$$

Thus we can take the values of A_1, A_2 of the lemma 2.4 to be real numbers such that $\log A_1 = 18 \log t, \log A_2 = 12 \log t$. Then $\log b' + 0.21 > 30/D = 5$ for $t > 3384$; from lemma 2.4 we obtain:

$$\log|\Lambda| \geq -5.02 \times 10^6 (\log(t))^2 \left(\max\left\{\log b' + 0.38, 5\right\}\right)^2.$$

Comparing this with (3.3.7) :

$$\begin{aligned}-4.176 \times 10^6 (\log(t))^2 &\left(\max\left\{\log b' + 0.38, 5\right\}\right)^2 \\ &< \log(2) + n\log(1.01) + \left(\frac{-8n}{3} + 1\right)\log(t^3 - 2).\end{aligned}$$

Since $A > \left(\frac{4}{5}\sqrt{2t^3 + \frac{161}{100}} + \frac{14}{75}\right)$, the above inequality is not valid for $t > 1063847$; this completes the proof. □

From lemmas 3.23, 3.24 we conclude:

Corollary 3.2. *If $t > 1119749$ then there are no non-trivial solutions of type II.*

3.3.2 Solutions of type I'

In this section for solutions of type I' we will prove the following lemma:

Lemma 3.25. *If $t > 712080$ then there are no non-trivial solutions of type II'.*

The proof is quite similar to the proof of solutions of type II, therefore we can leave out some very similar details. Recall that if (x, y) is a solution of type I' of $F'(x, y) = 1$ then $-\frac{1}{t^5} + \frac{2}{t^8} < \frac{x}{y} < -\frac{1}{t^6}$. Therefore by (3.1.4) and (3.3.3) we conclude

$$0 < \frac{\frac{x}{y} - \alpha''}{\frac{x}{y} - \alpha'} = \frac{x - \alpha'' y}{x - \alpha' y} = \left(\frac{t - \alpha''}{t - \alpha'}\right)^n \left(\frac{\alpha''}{\alpha'}\right)^{-m},$$

$$\log \frac{\frac{x}{y} - \alpha''}{\frac{x}{y} - \alpha'} = n \log \left(\frac{t - \alpha''}{t - \alpha'} \right) - m \log \left(\frac{\alpha''}{\alpha'} \right). \tag{3.3.14}$$

Also from estimates (3.1.4) we have :

$$t^3 + 2 < \frac{\frac{x}{y} - \alpha''}{\frac{x}{y} - \alpha'} < t^3 + 3, \tag{3.3.15}$$
$$t^3 + 2 < \frac{\alpha''}{\alpha'} < t^3 + 3,$$
$$t^9 + 3t^6 < \left(\frac{t - \alpha''}{t - \alpha'} \right) < t^9 + 5t^6.$$

In equation (3.3.14) if $n = 0$ then $m = -1$, which implies $x = 0$, $y = 1$. If $n > 0$ then $m > 0$ so $\left| x - y\alpha' \right| > 1$. On the other hand with the same argument as the solutions of type II, $\left| x - y\alpha' \right| < 1$. By (3.3.2)and (3.3.3), we conclude that $n < 0$,$m < 0$. Let $n' = n, m' = m$ be positive numbers, then we can rewrite (3.3.14)as:

$$\log \frac{\frac{x}{y} - \alpha''}{\frac{x}{y} - \alpha'} = m' \log \left(\frac{\alpha''}{\alpha'} \right) - n' \log \left(\frac{t - \alpha''}{t - \alpha'} \right). \tag{3.3.16}$$

Using above equation we have

$$n' \left(\frac{\log \left(\frac{t-\alpha''}{t-\alpha'} \right)}{\log \left(\frac{\alpha''}{\alpha'} \right)} \right) < m' \Longrightarrow \frac{8}{3} n' < m'.$$

Also since $\frac{\frac{x}{y}-\alpha''}{\frac{x}{y}-\alpha'} < \left(\frac{\alpha''}{\alpha'} \right)$

$$\big(m' - 1 \big) \log \left(\frac{\alpha''}{\alpha'} \right) < n' \left(\frac{\log \left(\frac{t-\alpha''}{t-\alpha'} \right)}{\log \left(\frac{\alpha''}{\alpha'} \right)} \right).$$

By estimates (3.3.15)

$$\big(m' - 1 \big) < n' \left(\frac{\log \left(\frac{t-\alpha''}{t-\alpha'} \right)}{\log \left(\frac{\alpha''}{\alpha'} \right)} \right) < 3n'.$$

To summarize, we have

$$\left(\frac{8}{3} \right) n' < m' \leq 3n'.$$

Linear form in logarithms

From Siegel's identity:

$$(\alpha' - \alpha'')(x - \alpha y) + (\alpha'' - \alpha)(x - \alpha' y) + (\alpha - \alpha')(x - \alpha'' y) = 0,$$

we obtain that :

$$\begin{aligned} \lambda &= \log \frac{\alpha - \alpha''}{\alpha - \alpha'} + n \log \frac{t - \alpha'}{t - \alpha''} + m \log \frac{\alpha''}{\alpha'} && (3.3.17) \\ &= \log \left(1 + \frac{(\alpha' - \alpha'')(x - \alpha y)}{(\alpha - \alpha')(x - \alpha'' y)} \right). && (3.3.18) \end{aligned}$$

Moreover , we have :

$$\begin{aligned} \left| \frac{(\alpha' - \alpha'')(x - \alpha y)}{(\alpha - \alpha')(x - \alpha'' y)} \right| &= \frac{\alpha'' - \alpha'}{\alpha' - \alpha} \left(\frac{\alpha'' - t}{t - \alpha} \right)^{n'} \left(\frac{\alpha}{\alpha''} \right)^{m'} \\ \left(\text{Since } \; n' < \frac{3}{8} m' \right) &< \frac{\alpha'' - \alpha'}{\alpha' - \alpha} \left(\frac{\alpha'' - t}{t - \alpha} \right)^{\frac{3}{8} m'} \left(\frac{\alpha}{\alpha''} \right)^{m'} \\ &< \left(t^3 + 2 \right) \left(t^3 + 2 \right)^{\frac{3}{8} m'} \left(\left(\frac{1}{t^3 + 1} \right)^3 \right)^{m'} \\ &< \left(t^3 + 2 \right)^{\frac{3}{8} m' + 1} \left(\frac{1.01}{(t^3 + 2)^3} \right)^{m'} \\ &< (1.01)^{m'} \left(t^3 + 2 \right)^{\frac{-21}{8} m' + 1}. \end{aligned}$$

If $0 < |x| < \frac{1}{2}$ then $|\log(1+x)| < 2x$, so $|\lambda| < 2 \, (1.01)^{m'} \left(t^3 + 2 \right)^{\frac{-21}{3} m' + 1}$

$$\Longrightarrow \log |\lambda| < \log(2) + m' \log(1.01) + \left(\frac{-21 m'}{8} + 1 \right) \log(t^3 + 2). \qquad (3.3.19)$$

Lower bound for A in type I'

Let $A = \max \left\{ |n|, |m| \right\}$; in this case since $m < n < 0$, A is equal to $-m$. Using some congruence arguments we find a lower bound for A. We write both sides of equation (3.3.3) as a linear combination of 1,α and α^2 and calculate the corresponding coefficients in the rings $\left(\mathbb{Z} / t^i \mathbb{Z} \right) [\alpha]$ for $i = 1, 2, 3$.

$$x - y\alpha = \pm (t - \alpha)^n \alpha^{-m} \quad \text{for } m, n \in \mathbb{Z}.$$

We consider the positive sign with m even (considering the negative sign gives the same results)

$$\frac{1}{x-y\alpha} = (t-\alpha)^{n'}\alpha^m \quad \text{where } n' = -n$$
$$\Longrightarrow (x-y\alpha')(x-y\alpha'') = (t-\alpha)^{n'}\alpha^m.$$

$$\Longrightarrow xy(\alpha)^2 + (x^2-(t^4+4t)xy)(\alpha) - y^2 = (t-\alpha)^{n'}\alpha^{m'} \tag{3.3.20}$$
$$=(-1)^{n'}\left(\alpha^2 - \binom{n'}{1}t\alpha + \binom{n'}{2}t^2 - \binom{n'}{3}t^3\alpha^{-1} + \cdots\right)\alpha^{(n'+m'-2)}. \tag{3.3.21}$$

In above equation $n' = -n$,$m' = m+1$, $n' > 0$ and $m' < -1$ since if $m' = -1$ then $m = -2$, $n = -1$, and one can check there is no solution with this condition.So $n' + m' - 2 < 0$.

Factoring in $(\mathbb{Z}/t\mathbb{Z})[\alpha]$

Consider the equation (3.3.20), LHS is congruent to $xy(\alpha)^2 + x^2(\alpha) - y^2$ and RHS is congruent to $(-1)^{n'}(\alpha^2)(\alpha^{(n'+m'-2)})$. We put

$$n' + m' - 2 = -3k + i \text{ for } i \in \{0,1,2\}, \text{ and } k \in \mathbb{N},$$

since $n' + m' - 2 < 0$

$$\alpha^3 = (t^4+4t)\alpha^2 - (t^5+3t^2)\alpha - 1 \equiv -1 \pmod{t}.$$

$$\begin{aligned}
&\alpha^{-1} = -\alpha^2 + (t^4+4t)\alpha - (t^5+3t) \\
\Longrightarrow&\alpha^{-1} \equiv -\alpha^2 && \pmod{t} \\
\Longrightarrow&(\alpha^{-1})^3 \equiv -\alpha^6 \equiv -(\alpha^3)^2 && \pmod{t} \\
\Longrightarrow&(\alpha^{-1})^3 \equiv -1 && \pmod{t}.
\end{aligned}$$

So we can conclude

$$\text{RHS} \equiv (-1)^{n'}\alpha^2\alpha^{(n'+m'-2)} \equiv (-1)^{n'}((\alpha^{-1})^3)^k\alpha^i \equiv (-1)^{n'}\alpha^2(-1)^k\alpha^i.$$

We have three cases:

$i = 0$

Lemma 3.26. *For $i = 0$ there is no integral solution.*

Proof. Comparing coefficients of $1,\alpha$, α^2 in RHS and LHS give

$$\begin{cases} xy \equiv 1 \pmod t \\ x^2 \equiv 0 \pmod t \\ y^2 \equiv 0 \pmod t \end{cases} \qquad \text{not possible.} \tag{3.3.22}$$

So there is no solution in this case. $\square$

$i = 1$ In this case k is odd, so RHS $\equiv (-1)^{n'}\alpha^2(-1)\alpha \equiv (-1)^{n'+1}\alpha^3 \equiv (-1)^{n'+2}$. Again by comparing coefficients of powers of α we obtain

$$\begin{cases} xy \equiv 0 & \pmod t \\ x^2 \equiv 0 & \pmod t \\ -y^2 \equiv (-1)^{n'} & \pmod t \end{cases} \Rightarrow \begin{cases} x \equiv 0 \\ y^2 \equiv (-1)^{n'+1} \\ y^3 \equiv 1 \end{cases} \Rightarrow \begin{cases} x \equiv 0 \pmod t \\ y \equiv 1 \pmod t \end{cases}. \tag{3.3.23}$$

$i = 2$ In this case k is even. Therefore RHS $\equiv (-1)^{n'}\alpha^2(-1)^k\alpha^2 \equiv (-1)^{n'}\alpha^4 \equiv \alpha$

$$\begin{cases} xy \equiv 0 & \pmod t \\ x^2 \equiv 1 & \pmod t \\ -y^2 \equiv 0 & \pmod t \end{cases} \Rightarrow \begin{cases} x^2 \equiv 1 & \pmod t \\ x^3 \equiv 1 & \pmod t \\ y \equiv 0 & \pmod t \end{cases} \Rightarrow \begin{cases} x \equiv 1 \pmod t \\ y \equiv 0 \pmod t \end{cases}. \tag{3.3.24}$$

Factoring in $(\mathbb{Z}/t^2\mathbb{Z})[\alpha]$

In this ring we have this list of congruences:

$$\begin{aligned} \alpha^3 &\equiv (4t\alpha^2 - 1), \\ \alpha^6 &\equiv -8t\alpha^2 + 1, \\ \alpha^4 &\equiv 4t\alpha^3 - \alpha \equiv 4t(4t\alpha^2 - 1) - \alpha \equiv -4t - \alpha, \\ t\alpha^5 &\equiv (t\alpha^3)\alpha^2 \equiv -t\alpha^2, \\ \alpha^{-1} &\equiv -\alpha^2 + 4t\alpha, \\ (\alpha^{-1})^3 &\equiv -\alpha^6 + 12t\alpha^5 \equiv -4t\alpha^2 - 1. \end{aligned} \tag{3.3.25}$$

In equation (3.3.20) LHS is congruent to $xy(\alpha^2) + (x^2 - 4txy)\alpha - y^2$. Also by (3.3.25) for RHS we can obtain

$$\begin{aligned}
\text{RHS} &\equiv (-1)^{n'}(\alpha^2 - n't\alpha)(-(1+4t\alpha^2))^k\alpha^i \\
&\equiv (-1)^{(n'+k)}(\alpha^2 - n't\alpha)(1+4kt\alpha^2)\alpha^i \\
&\equiv (-1)^{(n'+k)}(4kt\alpha^4 + \alpha^2 - n't\alpha)\alpha^i \\
&\equiv (-1)^{(n'+k)}(\alpha^2 - (n'+4k)t\alpha)\alpha^i.
\end{aligned}$$

let $i = 1$ **then**

$$\begin{aligned}
RHS &\equiv (\alpha^3 - (n'+4k)t\alpha^2) \pmod{t^2} \\
&\equiv -(n'+4k-4)t\alpha^2 - 1 \pmod{t^2}.
\end{aligned}$$

Therefore we have:

$$\begin{cases}
xy \equiv -(4k+n'-4)t & \pmod{t^2} \\
x^2 - 4txy \equiv 0 & \pmod{t^2} \\
y^2 \equiv 1 & \pmod{t^2}.
\end{cases} \tag{3.3.26}$$

In regards to the results we found, $\pmod{t}$, we can conclude

$$\begin{cases}
x \equiv -(4k+n'-4)t & \pmod{t^2} \\
y \equiv 1 & \pmod{t^2}.
\end{cases} \tag{3.3.27}$$

$i = 2$. In this case k is even. Therefore using (3.3.25) we have

$$\begin{aligned}
\text{RHS} &\equiv (-1)^{(n'+k)}(\alpha^4 - (n'+4k)t\alpha^3) \pmod{t^2} \\
&\equiv (-1)^{(n'+k)}(4t\alpha^3 - \alpha + (n'+4k)t\alpha^3) \pmod{t^2} \\
&\equiv (-1)^{(n'+k)}((n'+4k+4)t\alpha^3 - \alpha) \pmod{t^2} \\
&\equiv (n'+4k+4)t + \alpha \pmod{t^2}.
\end{aligned}$$

By comparing the coefficients of both sides of (3.3.20) we have:

$$\begin{cases}
xy \equiv 0 & \pmod{t^2} \\
x^2 - 4txy \equiv 1 & \pmod{t^2} \\
y^2 \equiv -(n'+4k+4)t & \pmod{t^2}
\end{cases} \Longrightarrow \begin{cases}
x \equiv 1 & \pmod{t^2} \\
y \equiv 0 & \pmod{t^2}.
\end{cases} \tag{3.3.28}$$

Factoring in $(\mathbb{Z}/t^3\mathbb{Z})[\alpha]$

In this ring we can write this list of congruences:

$$\begin{aligned}
\alpha^3 &\equiv 4t\alpha^2 - 3t^2\alpha - 1,\\
\alpha^4 &\equiv 4t\alpha^3 - 3t^2\alpha^2 - \alpha,\\
t^2\alpha^4 &\equiv -t^2\alpha,\\
t\alpha^4 &\equiv 4t^2\alpha^3 - t\alpha \equiv -4t^2 - t\alpha,\\
\alpha^6 &\equiv 16t^2\alpha^4 - 8t\alpha^2 + 6t^2\alpha + 1 \equiv -8t\alpha^2 - 10t^2\alpha + 1,\\
\alpha^{-1} &\equiv -\alpha^2 + 4t\alpha - 3t^2,\\
\left(\alpha^{-1}\right)^3 &\equiv -\alpha^6 + 12t\alpha^5 - 9t^2\alpha^4 - 48t^2\alpha^4,\\
&\equiv -4t\alpha^2 + 19t^2\alpha - 1,\\
\left((\alpha^{-1})^3\right)^k &\equiv (-1)^k\left(4kt\alpha^2 - 19kt^2\alpha + 16\binom{k}{2}t^2\alpha^4 + 1\right),\\
&\equiv (-1)^k\left(4kt\alpha^2 - k(8k+11)t^2\alpha + 1\right).
\end{aligned} \tag{3.3.29}$$

$i = 2$**:**

Lemma 3.27. *For $i = 2$ there is no integral solution unless $A > \frac{3}{4}\left(t^2 - 4\right)$.*

Proof.

$$\text{LHS} \equiv xy\alpha^2 + \left(x^2 - 4txy\right)\alpha - y^2,$$

using congruencies (3.3.30) we have:

$$\begin{aligned}
&\text{RHS}\\
&\equiv (-1)^{n'}\left(\alpha^2 - n't\alpha + \binom{n'}{2}t^2\right)\left((\alpha^{-1})^3\right)^k\alpha^2\\
&\equiv (-1)^{(n'+k)}\left(\alpha^2 - n't\alpha + \binom{n'}{2}t^2\right)\left(4kt\alpha^4 - kt^2\alpha^3(8k+11) + \alpha^2\right)\\
&\equiv 4kt\alpha^6 - (8k+11+4n')kt^2\alpha^5 + \alpha^4 - n't\alpha^3 + \binom{n'}{2}t^2\alpha^2\\
&\equiv -16kt^2\alpha^2 - 4kt\alpha^3 + (8k+11+4n')kt^2\alpha^2 + \alpha^4 - n't\alpha^3 + \binom{n'}{2}t^2\alpha^2\\
&\equiv -(n'+4k-4)t\alpha^3 + \left(8k^2 - 5k + 4n'k + \binom{n'}{2} - 3\right)t^2\alpha^2 - \alpha.
\end{aligned}$$

By comparing coefficients of constant term on both sides of (3.3.20) we obtain:

$$y^2 \equiv -\left(n' + 4k + 4\right) t \pmod{t^3}.$$

Using congruences (3.3.28) $y^2 \equiv 0 \pmod{t^3}$
$\Longrightarrow n' + 4k + 4 \equiv 0 \mod t^2$, But $n' + 4k + 4 > 0$ so $n' + 4k + 4 \geq t^2$ by the value of k we have

$$A > \frac{3}{4}(t^2 + 4).$$

□

Factoring in $(\mathbb{Z}/t^5\mathbb{Z})[\alpha]$

In this ring we can write this list of congruences:

$$\begin{aligned}
\alpha^3 &\equiv (t^4 + 4t)\alpha^2 - 3t^2\alpha - 1, \\
\alpha^4 &\equiv (t^4 + 4t)\alpha^3 - 3t^2\alpha^2 - \alpha, \\
t^2\alpha^4 &\equiv 13t^4\alpha^2 - t^2\alpha - 4t^3, \\
t\alpha^4 &\equiv 13t^3\alpha^2 - (12t^4 + t)\alpha - 4t^2, \\
\alpha^6 &\equiv 119t^4\alpha^2 - 8t\alpha^2 - 10t^2\alpha - 40t^3 + 1, \\
\alpha^{-1} &\equiv -\alpha^2 + (t^4 + 4t)\alpha - (t^5 + 3t^2), \\
\left(\alpha^{-1}\right)^3 &\equiv -4t\alpha^2 + 19t^2\alpha - 24t^3 - 1 - 10t^4\alpha^2, \\
\left((\alpha^{-1})^3\right)^k &\equiv (-1)^k \left(Lt^4\alpha^2 + 4kt\alpha^2 - k(8k + 11)t^2\alpha + Mt^3 + 1\right).
\end{aligned} \tag{3.3.30}$$

where

$$L = 464 + 130k - 512\binom{k}{3} + 287\binom{k}{2} + 304k\binom{k-1}{2}$$

$$M = -26\binom{k}{2} - 64\binom{k}{3} + 24k.$$

$i = 1$

Lemma 3.28. *For $i = 1$ there is no integral solution unless*

$$A > \frac{24}{11}\left(\sqrt{2t^3 + \frac{10025}{4356}} + \frac{19}{22}\right).$$

Proof.

$$\text{LHS} \equiv xy\alpha^2 + \left(x^2 - 4txy\right)\alpha - y^2.$$

Using congruences (3.3.30) One can check that the coefficient of α^2 in above is congruent to $(n'+4k-4)t \pmod{t^4}$ where the coefficient of α is congruent to $\left(8k^2 - 5k + 4n'k + \binom{n'}{2} - 3\right)t^2 \pmod{t^5}$, and the constant coefficient is congruent to 1 (mod t^3). By comparing coefficients of both sides of (3.3.20)

$$\begin{cases} xy \equiv -(n'+4k-4)t + Ct^4 & \pmod{t^5}, \\ x^2 - 4txy \equiv \left(8k^2 - 5k + 4n'k + \binom{n'}{2} - 3\right)t^2 & \pmod{t^5}, \\ y^2 \equiv 1 + C't^3 + C''t^4 & \pmod{t^5}. \end{cases} \tag{3.3.31}$$

$$\Longrightarrow x^2 \equiv -(n'+4k-4)^2 t^2 \pmod{t^5}.$$

Using the congruency we found for x from lost equivalency in (3.3.31) we have

$$(n'+4k-4)^2 + 4(n'+4k-4) \equiv \left(8k^2 - 5k + 4n'k + \binom{n'}{2} - 3\right) \pmod{t^3}.$$

Simplifying both sides, we get

$$\frac{n'^2 + 16k^2 + 8kn' - 22k - 7n' + 6}{2} \equiv 0 \pmod{t^3}$$

$$\Longrightarrow \frac{\left(n' + 4k - \frac{7}{2}\right)^2 + 6k - \frac{25}{4}}{2} \equiv 0 \pmod{t^3}.$$

But $\left(n' + 4k - \frac{7}{2}\right)^2 + 6k - \frac{25}{4} > 0$ so

$$\left(n' + 4k - \frac{7}{2}\right)^2 + 6k - \frac{25}{4} \geq 2t^3.$$

Plug in the maximum value of k based on n we have:

$$\left(\frac{11}{9}n' - \frac{19}{22}\right)^2 - \frac{10025}{4356} > 2t^3$$

$$\Longrightarrow n > \frac{9}{11}\left(\sqrt{2t^3 + \frac{10025}{4356}} + \frac{19}{22}\right)$$

$$\Longrightarrow A = m' > \frac{24}{11}\left(\sqrt{2t^3 + \frac{10025}{4356}} + \frac{19}{22}\right).$$

□

Corollary 3.3. *If $A < \frac{24}{11}\left(\sqrt{2t^3+\frac{10025}{4356}}+\frac{19}{22}\right)$ there is no solution of type I' other than the five trivial solutions.*

Proof. Obvious by lemmas (3.26),(3.27), (3.28). □

Lower Bound for λ in type I'

During this section we assume $A > \frac{24}{11}\left(\sqrt{2t^3+\frac{10025}{4356}}+\frac{19}{22}\right)$.

$$\text{Remember} \quad \lambda = \log\frac{\alpha-\alpha''}{\alpha-\alpha'} + n\log\frac{t-\alpha'}{t-\alpha''} + m\log\frac{\alpha''}{\alpha'}.$$

Put $\alpha_1 = \frac{\alpha''-\alpha}{\alpha'-\alpha}, \alpha_2 = \left|\frac{t-\alpha'}{t-\alpha''}\right|, \alpha_3 = \left|\frac{\alpha''}{\alpha'}\right|, b_1 = 1, b_2 = n, b_3 = m, D = 6.$ Using estimations (3.1.4)

$$\begin{aligned} h\left(\frac{\alpha''-\alpha}{\alpha'-\alpha}\right) \leq 2h\left(\alpha''-\alpha'\right) =& \frac{2}{3}\log\big((\alpha''-\alpha')(\alpha''-\alpha)(\alpha'-\alpha)\big) \quad (3.3.32)\\ <& \frac{2}{3}\log(t^9+6t^6),\\ h\left(\frac{\alpha''}{\alpha'}\right) = \frac{1}{6}\log\left(\frac{\alpha''}{\alpha}\right)^2 <& \frac{1}{3}\log(t^9+7t^6),\\ h\left(\frac{t-\alpha'}{t-\alpha''}\right) = \frac{1}{6}\log\left(\frac{t-\alpha''}{t-\alpha'}\right)^2 <& \frac{1}{3}\log(t^9+5t^6). \end{aligned}$$

Therefore we can take real numbers A_i as:

$$\begin{aligned} A_1 =& 4\log\left(t^9+6t^6\right),\\ A_2 =& 2\log\left(t^9+5t^6\right),\\ A_3 =& 2\log\left(t^9+7t^6\right),\\ B =& m'\frac{\log\left(t^9+7t^6\right)}{\log\left(t^9+5t^6\right)} > (1.0007)\, m' \end{aligned}$$

We apply theorem (2.3) to find a lower bound for $\log|\Lambda|$.

$$\log|\Lambda| > -1.1446\cdot 10^{13}\log A_1A_2A_3\log\left(68.3m'\right).$$

Comparing with (3.3.19):

$$-1.1446 \cdot 10^{13} \log\left(t^9+6t^6\right)\log\left(t^9+5t^6\right)\log\left(t^9+7t^6\right)\log\left(68.3m'\right) \tag{3.3.33}$$
$$< \log(2) + m'\log(1.01) + \left(\frac{-21m'}{8}+1\right)\log(t^3+2).$$

Since $A > \frac{24}{11}\left(\sqrt{2t^3+\frac{10025}{4356}}+\frac{19}{22}\right)$,the above inequality is not valid for $t > 2.981 \cdot 10^{12}$. So we can conclude that

Lemma 3.29. *for $t > 6.005 \cdot 10^{12}$ there is no solution of type I' other than the trivial ones.*

Intermediate values of t

In this section we assume $t < 2.981 \cdot 10^{12}$ and $A > \frac{24}{11}\left(\sqrt{2t^3+\frac{10025}{4356}}+\frac{19}{22}\right)$. From (3.3.33), we can see $m' < 4.54 \cdot 10^{19}$ and $n' < 1.703 \cdot 10^{19}$.

Lemma 3.30. *For $t > 10$, if there exists a non-trivial solution of type II' then the following inequality holds:*

$$\Longrightarrow 3n' - m + 1 < \left(n'+\frac{1}{3}\right)\frac{\log\left(\frac{t^3}{t^3-6}\right)}{\log\left(t^3+3\right)}.$$

Proof. Using (3.3.16)and (3.3.15) we have

$$\begin{aligned}
n'\log\left(\frac{t-\alpha''}{t-\alpha'}\right)+\log\frac{\frac{x}{y}-\alpha''}{\frac{x}{y}-\alpha'} &= m'\log\left(\frac{\alpha''}{\alpha'}\right)\\
\Longrightarrow \left(n'+\frac{1}{3}\right)\log\left(\frac{t-\alpha''}{t-\alpha'}\right) &< m'\log\left(\frac{\alpha''}{\alpha'}\right)\\
\Longrightarrow \left(n'+\frac{1}{3}\right)\frac{\log\left(\frac{t-\alpha''}{t-\alpha'}\right)}{\log\left(\frac{\alpha''}{\alpha'}\right)} &< m'\\
\Longrightarrow \left(n'+\frac{1}{3}\right)\frac{\log t^9+3t^6}{\log t^3+2} &< m'.
\end{aligned}$$

On the other hand

$$t^9 + 3t^6 > \left(t^3 + 2\right)^3 \left(1 - \frac{6}{t^3}\right)$$

$$\implies \left(n' + \frac{1}{3}\right)\left(3 + \frac{\log\left(1 - \frac{6}{t^3}\right)}{\log\left(t^3 + 3\right)}\right) < m'$$

$$\implies 3n' - m' + 1 < \left(n' + \frac{1}{3}\right)\frac{\log\left(\frac{t^3}{t^3-6}\right)}{\log\left(t^3 + 3\right)}.$$

□

Lemma 3.31. *For $t > 712080$, if there exists a non-trivial solution of type II' then $m' = 3n' - k$ with $k \leq 5$.*

Proof. Substituting $3n' - k$ for m' in Lemma (3.30) we have the following:

$$\frac{k+1}{\left(n' + \frac{1}{3}\right)} < \frac{\log\left(\frac{t^3}{t^3-6}\right)}{\log\left(t^3 + 3\right)}.$$

We know $n' < 1.703 \cdot 10^{19}$; now, if $k \geq 7$ these two would give us:

$$\frac{7}{1.703 \cdot 10^{19}} < \frac{\log\left(\frac{t^3}{t^3-6}\right)}{\log\left(t^3 + 3\right)}.$$

The right hand side is a strictly decreasing function in the variable t and so we easily deduce that $t < 712080$. □

Lemma 3.32. *If there exists a non-trivial solution such that $m' = 3n' - k$ with $k \leq 5$ then $t < 685450$.*

Proof.

$$\begin{aligned}
\Lambda &= \log\frac{\alpha''-\alpha}{\alpha'-\alpha} + n'\log\frac{t-\alpha''}{t-\alpha'} + m'\log\frac{\alpha'}{\alpha''}\\
&= \log\frac{\alpha''-\alpha}{\alpha'-\alpha} + n'\log\frac{t-\alpha''}{t-\alpha'} + \left(3n'-k\right)\log\frac{\alpha'}{\alpha''}\\
&= \left[\log\frac{\alpha''-\alpha}{\alpha'-\alpha} + k\log\frac{\alpha''}{\alpha'}\right] + \left[n'\log\frac{t-\alpha''}{t-\alpha'} + 3n'\log\frac{\alpha'}{\alpha''}\right]\\
&= \log\left(\frac{\alpha''-\alpha}{\alpha'-\alpha}\left(\frac{\alpha''}{\alpha'}\right)^k\right) + n'\log\frac{t-\alpha''}{t-\alpha'}\left(\frac{\alpha'}{\alpha''}\right)^3\\
&= \log\left(\underbrace{\left(\frac{\alpha''-\alpha}{\alpha'-\alpha}\right)\left(\frac{\alpha''}{\alpha'}\right)^k}_{\alpha_2}\right) - n'\log\left(\underbrace{\left(\frac{t-\alpha'}{t-\alpha''}\right)\left(\frac{\alpha''}{\alpha'}\right)^3}_{\alpha_1}\right)\\
&= \log\alpha_2 - n'\log\alpha_1.
\end{aligned}$$

Where $\alpha_1, \alpha_2 > 1$. We apply lemma (2.4) for $\log|\Lambda|$, with $b_2 = 1, b_1 = n', D = 6$, using (3.1.4), (3.5.16) we have :

$$\begin{aligned}
\frac{|\log\alpha_1|}{D} < h(\alpha_1) &< \frac{1}{3}\log\left(t^9+5t^6\right) + 3\left(\frac{1}{3}\right)\log\left(t^9+7t^6\right)\\
&< \frac{4}{3}\log\left(t^9+7t^6\right),\\
\frac{|\log\alpha_2|}{D} < h(\alpha_2) &< \frac{2}{3}\log\left(t^9+6t^6\right) + k\left(\frac{1}{3}\right)\log\left(t^9+7t^6\right)\\
&< \frac{7}{3}\log\left(t^9+7t^6\right).
\end{aligned}$$

We can take A_1, A_2 as real numbers bigger than 1, such that, $\log A_1 = \frac{4}{3}\log\left(t^9+7t^6\right)$ and $\log A_2 = \frac{7}{3}\log\left(t^9+7t^6\right)$. then $b' = \frac{4n'+7}{56\log(t^9+7t^6)}$. So $\log b' + 0.38 > 30/D = 5$ for $t > 1910$; from lemma (2.4) we obtain:

$$\log|\Lambda| \geq -7.21728\times 10^4\left(\log(t^9+7t^6)\right)^2\left(\max\left\{\log b' + 0.38, 5\right\}\right)^2.$$

Comparing this with (3.3.19) :

$$\begin{aligned}
&-7.21728\times 10^4\left(\log(t^9+7t^6)\right)^2\left(\max\left\{\log b'+0.38,5\right\}\right)^2\\
&\qquad < \log(2) + m'\log(1.01) + \left(\frac{-21m'}{8}+1\right)\log(t^3+2).
\end{aligned}$$

Since $A > \frac{24}{11}\left(\sqrt{2t^3 + \frac{10025}{4356}} + \frac{19}{22}\right)$, the above inequality is not valid for $t > 685450$. This completes the proof. $\square$

The above lemmas (3.31)and (3.32) lead us to conclude:

Corollary 3.4. *If $t > 712080$ then there are no non-trivial solutions of type II'.*

3.4 Solutions of type III,III'

For this type of solution we again use Baker's method. Proving the "stable growth" condition is much easier in this case. Let (x, y) be an integral solution of type III . By the same argument as in the beginning of the section 3.3, there exist some integers m, n such that

$$x - y\theta'' = (t - \theta'')^n (\theta'')^{-m}.$$

First we will show that if there exists a nontrivial solution of type III then $n \neq m$ and n and m have the same parity . It helps us to prove a strong "stable growth" condition just by using a simple calculus argument. The argument for solutions of type III' is completely similar.

3.4.1 Solutions of type III

During this section we will prove that for $t > 361165$ there is no integral solution of type III other than the trivial ones.
Recall that if (x, y) is a nontrivial solution of type III then $t^4 - 2t - \frac{2}{t^3} < \frac{x}{y} < t^4 - 2t - \frac{1}{t^9}$. First we will show that for such a solution we have $y > 0$; we will rewrite the equation (3.3.1) as

$$y^2(\frac{x}{y} - \theta)(\frac{x}{y} - \theta')(x - y\theta'') = 1.$$

From the estimation (3.1.3) and (3.1.5) we have $y^2(\frac{x}{y} - \theta)(\frac{x}{y} - \theta') > 0$, therefore We can conclude that $x - y\theta'' > 0$.

Consider the equation

$$\frac{x - y\theta'}{x - y\theta''} = \left(\frac{t - \theta'}{t - \theta''}\right)^n \left(\frac{\theta'}{\theta''}\right)^{-m}.$$

By estimations (3.1.3), it is easy to check that RHS is positive. Therefore LHS is also positive $\Longrightarrow x - y\theta' = y\left(\frac{x}{y} - \theta'\right) > 0$. On the other hand, $\frac{x}{y} - \theta' > 0 \Longrightarrow y > 0$.

Using equation (3.3.3) and estimations (3.1.3), we can obtain that

$$0 < \frac{\frac{x}{y} - \theta}{\frac{x}{y} - \theta'} = \frac{x - \theta y}{x - \theta' y} = \left(\frac{t - \theta}{t - \theta'}\right)^n \left(\frac{\theta}{\theta'}\right)^{-m}$$

Since both $\left(\frac{t-\theta}{t-\theta'}\right)$ and $\left(\frac{\theta}{\theta'}\right)$ are negative So n , m have same parities $\pmod 2$. Taking absolute values of both sides we have

$$\frac{\frac{x}{y} - \theta}{\frac{x}{y} - \theta'} = \left(\frac{t - \theta}{\theta' - t}\right)^n \left(\frac{|\theta|}{\theta'}\right)^{-m}.$$

Taking logarithms from both sides and using estimations 3.1.3, we get

$$\log(1 + \frac{1}{t^3}) < n \log\left(\frac{t - \theta}{\theta' - t}\right) + m \log\left(\frac{\theta'}{|\theta|}\right) < \log(1 + \frac{2}{t^3}). \tag{3.4.1}$$

Also we have:

$$\begin{aligned} t^6 - 4t^3 + 16 < \frac{t - \theta}{\theta' - t} &< t^6 - 3t^3 + 10, \\ t^6 - 2t^3 + 1 < \frac{\theta'}{|\theta|} &< t^6 - 2t^3 + 3, \\ 1 + \frac{1}{t^3} < \frac{\frac{\theta'}{|\theta|}}{\frac{t-\theta}{\theta'-t}} &< 1 + \frac{2}{t^3}. \end{aligned} \tag{3.4.2}$$

Considering (3.4.1), we can deduce $n \neq 0, m \neq 0$ Also if $m < 0$ then $n > 0$ and if $m > 0$ then $n < 0$.

First assume $m < 0, n > 0 \Longrightarrow |x - y\theta''| > 1$ but

$$\left|x - y\theta''\right| = \frac{1}{y^2 \left|\frac{x}{y} - \theta\right| \left|\frac{x}{y} - \theta'\right|} < 1.$$

So $m > 0, n < 0$. let $n' = -n$ from the second inequality of (3.3.8)

$$m \log\left(\frac{\theta'}{|\theta|}\right) < n' \log\left(\frac{t - \theta}{\theta' - t}\right) + \log(1 + \frac{2}{t^3})$$

$$\Longrightarrow m \log\left(\frac{\theta'}{|\theta|}\right) < (n' + 1) \log\left(\frac{t - \theta}{\theta' - t}\right) \Longrightarrow m < (n' + 1).$$

Since they have same parity $\pmod 2$, it means $m = n'$ or we have $m + 2 \leq n'$. First assume $m = n'$, then we have

$$
\begin{aligned}
m \log\left(\frac{\theta'}{|\theta|}\right) - m \log\left(\frac{t-\theta}{\theta'-t}\right) &< \log(1+\frac{2}{t^3}) \\
\Longrightarrow m \log\left(\frac{\frac{\theta'}{|\theta|}}{\frac{t-\theta}{\theta'-t}}\right) &< \log(1+\frac{2}{t^3}).
\end{aligned}
$$

Using estimates of (3.4.2)

$$
m \log(1+\frac{1}{t^3}) < \log(1+\frac{2}{t^3}) \Longrightarrow m < 2.
$$

Therefore $m = 1, n = -1$ which with a minus sign corresponds to solution $(t^4 - 2t, 1)$.
Next we assume $n' \geq m + 2$. From first inequality of (3.3.8), we have

$$
\begin{aligned}
n' \log\left(\frac{t-\theta}{\theta'-t}\right) + \log\left(1+\frac{1}{t^3}\right) &< m \log\left(\frac{\theta'}{|\theta|}\right) \\
\Longrightarrow (m+2) \log\left(\frac{t-\theta}{\theta'-t}\right) + \log\left(1+\frac{1}{t^3}\right) &< m \log\left(\frac{\theta'}{|\theta|}\right) \\
\Longrightarrow \log\left(\left(\frac{t-\theta}{\theta'-t}\right)^2 \cdot \left(1+\frac{1}{t^3}\right)\right) &< m \log\left(\frac{\frac{\theta'}{|\theta|}}{\frac{t-\theta}{\theta'-t}}\right).
\end{aligned}
$$

Using estimations (3.4.2) we can conclude

$$
\begin{aligned}
\log\left(t^{12} - 7t^9\right) &< m \log\left(1+\frac{2}{t^3}\right) \\
\Longrightarrow \frac{\log\left(t^{12}-7t^9\right)}{\log\left(1+\frac{2}{t^3}\right)} &< m \leq A - 2 \\
\Longrightarrow \frac{\log\left(t^{12}-7t^9\right)}{\log\left(1+\frac{2}{t^3}\right)} &+ 2 < A.
\end{aligned} \tag{3.4.3}
$$

Linear form in logarithms

From Siegel's identity:

$$
(\theta' - \theta'')(x - \theta y) + (\theta'' - \theta)(x - \theta' y) + (\theta - \theta')(x - \theta'' y) = 0,
$$

we can find the linear form in logarithms corresponding to solutions of type III:

$$\Lambda = \log \frac{\theta'' - \theta'}{\theta'' - \theta} + n \log \left| \frac{t-\theta}{t-\theta'} \right| + m \log \left| \frac{\theta'}{\theta} \right| \tag{3.4.4}$$

$$= \log \left(1 + \frac{(\theta - \theta')(x - \theta'' y)}{(\theta'' - \theta)(x - \theta' y)} \right), \tag{3.4.5}$$

where $\frac{(\theta-\theta')(x-\theta'' y)}{(\theta''-\theta)(x-\theta' y)} < 0$, so $\Lambda < 0$.

$$\begin{aligned}
\frac{(\theta' - \theta)(x - \theta'' y)}{(\theta'' - \theta)(x - \theta' y)} &= \frac{\theta' - \theta}{\theta'' - \theta} \left(\frac{t - \theta''}{t - \theta'} \right)^n \left(\frac{\theta'}{\theta''} \right)^m \\
(\text{Since } n < 0, m > 0) &< \left(\frac{1}{t^3 - 3} \right) \left(\left(\frac{1}{t^3 - 3} \right)^3 \right)^{n'} \\
&< \left(\frac{1}{t^3 - 3} \right)^{(3n'+1)}.
\end{aligned}$$

If $0 < x < \frac{1}{2}$ then $|\log(1-x)| < 2x$, So $|\Lambda| < 2 \cdot \left(\frac{1}{t^3-3} \right)^{(3n'+1).}$

$$\Longrightarrow \log |\Lambda| < \log(2) - (3n'+1) \left(\log \left(t^3 - 3 \right) \right). \tag{3.4.6}$$

Lower bound for Λ in type III

So up to this point we have proved that if there exists a nontrivial solution of type III, for $F(x,y) = 1$ then we have

$$\begin{aligned}
&n' > \frac{\log \left(t^{12} - 7t^9 \right)}{\log \left(1 + \frac{2}{t^3} \right)} + 2, \\
&n' \geqslant m + 2.
\end{aligned}$$

We will apply lemma (2.3) to find a lower bound for

$$\Lambda = \log \frac{\theta'' - \theta'}{\theta'' - \theta} + n \log \left| \frac{t-\theta}{t-\theta'} \right| + m \log \left| \frac{\theta'}{\theta} \right|.$$

Let

$$b_1 = 1, b_2 = n, b_3 = m, n = 3, \gamma_1 = \frac{\theta'' - \theta'}{\theta'' - \theta}, \gamma_2 = |\frac{t-\theta}{t-\theta'}|, \gamma_3 = |\frac{\theta'}{\theta}|.$$

Now:

$$h\left(\frac{\theta''-\theta'}{\theta''-\theta}\right) \le 2h\left(\theta''-\theta'\right) = \frac{2}{3}\log\left((\theta''-\theta')(\theta''-\theta)(\theta'-\theta)\right) < 6\log t$$
$$h\left(\frac{\theta''}{\theta}\right) = \frac{1}{6}\log\left(\frac{\theta''}{\theta}\right)^2 < 3\log t$$
$$h\left(\frac{t-\theta}{t-\theta''}\right) = \frac{1}{6}\log\left(\frac{t-\theta''}{t-\theta'}\right)^2 < 3\log t$$

$$\Longrightarrow \log|\Lambda| > -8.344(10)^{15}\log(t)^3\log(68.3n').$$

Comparing with (3.4.6), we conclude

$$-8.344(10)^{15}\log(t)^3\log(68.3n') < \log(2) - (3n'+1)\left(\log\left(t^3-3\right)\right). \quad (3.4.7)$$

Which leads to contradiction for $t > 361165$; therefore we have

Lemma 3.33. *For $t > 361165$, there is no integral solution of type III other than the trivial ones.*

Remark 3.1. *From above inequality for $t < 361166$ we obtain $n' < 7.5 \times 10^{18}$.*

3.4.2 Solutions of type III'

In this section we will prove that

Lemma 3.34. *For $t > 361167$, there is no integral solution of type III' other than the trivial ones.*

The proof is completely the same as solutions of type III. First we will show that if (x, y) is an integral solution of type III' then $y > 0$. Using equation $(x-y\alpha)(x-y\alpha')(x-y\alpha'') = 1$, we conclude that $x - y\alpha'' > 0$. Consider the equation

$$\frac{x-y\alpha'}{x-y\alpha''} = \left(\frac{t-\alpha'}{t-\alpha''}\right)^n\left(\frac{\alpha'}{\alpha''}\right)^{-m}.$$

Since RHS is positive, LHS is also positive $\Longrightarrow x - y\alpha' = y\left(\frac{x}{y}-\alpha'\right) > 0$,but $\frac{x}{y}-\alpha' > 0 \Longrightarrow y > 0$. Let $t^4+3t-\frac{1}{t^8} < \frac{x}{y} < t^4+3t-\frac{1}{t^9}$. Using equation (3.3.3) and estimations (3.1.4), we can obtain :

$$0 < \frac{\frac{x}{y}-\alpha}{\frac{x}{y}-\alpha'} = \frac{x-\alpha y}{x-\alpha' y} = \left(\frac{t-\alpha}{t-\alpha'}\right)^n\left(\frac{\alpha}{\alpha'}\right)^{-m}.$$

So n , m have same parities $\pmod 2$. Taking absolute values of both sides we get:

$$\frac{\frac{x}{y}-\alpha}{\frac{x}{y}-\alpha'} = \left(\frac{t-\alpha}{\alpha'-t}\right)^n \left(\frac{|\alpha|}{\alpha'}\right)^{-m}.$$

Taking logarithms from both sides and using estimations 3.1.4 we obtain:

$$\log(1+\frac{1}{t^3}-\frac{2}{t^6}) < n\log\left(\frac{t-\alpha}{\alpha'-t}\right)+m\log\left(\frac{\alpha'}{|\alpha|}\right) < \log(1+\frac{1}{t^3}-\frac{1}{t^6}). \quad (3.4.8)$$

Also we have:

$$\begin{aligned} t^6+t^3+1 < &\frac{t-\alpha}{\alpha'-t} < t^6+2t^3+5, \\ t^6+3t^3+1 < &\frac{\alpha'}{|\alpha|} < t^6+3t^3+3, \\ 1+\frac{1}{t^3}-\frac{1}{t^5} < &\frac{\frac{\alpha'}{|\alpha|}}{\frac{t-\alpha}{\alpha'-t}} < 1+\frac{1}{t^3}. \end{aligned} \quad (3.4.9)$$

Considering (3.4.8), we can deduce that $n \neq 0, m \neq 0$. Also if $m < 0$ then $n > 0$ and if $m > 0$ then $n < 0$.
First assume $m < 0, n > 0 \Longrightarrow |x - y\alpha''| > 1$but by (3.3.2) $|x - y\alpha''| < 1$. so $m > 0, n < 0$. let $n' = -n$ from the second inequality of (3.3.30) we have:

$$m\log\left(\frac{\alpha'}{|\alpha|}\right) < n'\log\left(\frac{t-\alpha}{\alpha'-t}\right)+\log(1+\frac{1}{t^3}-\frac{1}{t^6})$$

$$\Longrightarrow m\log\left(\frac{\alpha'}{|\alpha|}\right) < (n'+1)\log\left(\frac{t-\alpha}{\alpha'-t}\right) \Longrightarrow m < (n'+1).$$

since they have same parity $\pmod 2$, it means $m = n'$or $m+2 \leq n'$. First assume $m = n'$, then we have

$$m\log\left(\frac{\alpha'}{|\alpha|}\right) - m\log\left(\frac{t-\alpha}{\alpha'-t}\right) < \log(1+1/t^3-\frac{1}{t^6})$$
$$\Longrightarrow m\log\left(\frac{\frac{\alpha'}{|\alpha|}}{\frac{t-\alpha}{\alpha'-t}}\right) < \log(1+\frac{1}{t^3}-\frac{1}{t^6}).$$

Using estimates of (3.4.9) we obtain:

$$m\log(1+\frac{1}{t^3}-\frac{1}{t^5}) < \log(1+\frac{1}{t^3}-\frac{1}{t^6}) \Longrightarrow m < 2.$$

Therefore $m = 1, n = -1$ which with a minus sign corresponds to solution $(t^4 + 3t, 1)$.
Next we assume $n' \geq m + 2$. From first inequality of (3.3.30), we have:

$$n' \log\left(\frac{t-\alpha}{\alpha'-t}\right) + \log(1 + \frac{1}{t^3} - \frac{2}{t^6}) < m \log\left(\frac{\alpha'}{|\alpha|}\right),$$
$$\Longrightarrow (m+2) \log\left(\frac{t-\alpha}{\alpha'-t}\right) + \log(1 + \frac{1}{t^3} - \frac{2}{t^6}) < m \log\left(\frac{\alpha'}{|\alpha|}\right),$$
$$\Longrightarrow \log\left(\left(\frac{t-\alpha}{\alpha'-t}\right)^2 \cdot (1 + \frac{1}{t^3} - \frac{2}{t^6})\right) < m \log\left(\frac{\frac{\alpha'}{|\alpha|}}{\frac{t-\alpha}{\alpha'-t}}\right).$$

Using estimations (3.4.9) we can conclude that

$$\begin{aligned} \log(t^{12} + 3t^9 + 3t^6) &< m \log(1 + \frac{1}{t^3}) \\ \Longrightarrow \frac{\log\left(t^{12} + 3t^9 + 3t^6\right)}{\log(1 + \frac{1}{t^3})} &< m \leq A - 2 \\ \Longrightarrow \frac{\log\left(t^{12} + 3t^9 + 3t^6\right)}{\log(1 + \frac{1}{t^3})} &+ 2 < A. \end{aligned} \tag{3.4.10}$$

Linear form in logarithms

From Siegel's identity :

$$(\alpha' - \alpha'')(x - \alpha y) + (\alpha'' - \alpha)(x - \alpha' y) + (\alpha - \alpha')(x - \alpha'' y) = 0,$$

we obtain

$$\lambda = \log \frac{\alpha'' - \alpha'}{\alpha'' - \alpha} + nlog\left|\frac{t-\alpha}{t-\alpha'}\right| + m \log\left|\frac{\alpha'}{\alpha}\right| \tag{3.4.11}$$
$$= \log\left(1 + \frac{(\alpha - \alpha')(x - \alpha'' y)}{(\alpha'' - \alpha)(x - \alpha' y)}\right), \tag{3.4.12}$$

where $\frac{(\alpha-\alpha')(x-\alpha'' y)}{(\alpha''-\alpha)(x-\alpha' y)} < 0$. Moreover , we have:

$$\begin{aligned} \frac{(\alpha' - \alpha)(x - \alpha'' y)}{(\alpha'' - \alpha)(x - \alpha' y)} &= \frac{\alpha' - \alpha}{\alpha'' - \alpha}\left(\frac{t - \alpha''}{t - \alpha'}\right)^n \left(\frac{\alpha'}{\alpha''}\right)^m \\ (\text{Since } n < 0, m > 0)\,, &< \left(\frac{1}{t^3+1}\right)\left(\left(\frac{1}{t^3+1}\right)^3\right)^{n'} \\ &< \left(\frac{1}{t^3+1}\right)^{(3n'+1)}. \end{aligned}$$

If $0 < x < \frac{1}{2}$ then $|\log(1-x)| < 2x$, So $|\Lambda| < 2\left(\frac{1}{t^3+1}\right)^{(3n'+1)}$

$$\Longrightarrow \log|\Lambda| < \log(2) - (3n'+1)\left(\log\left(t^3+1\right)\right). \tag{3.4.13}$$

Lower bound for λ in type III'

In this section we assume $A > \frac{\log\left(t^{12}+3t^9+3t^6\right)}{\log(1+\frac{1}{t^3})} + 2$ and $n' \geq m+2$.To find a lower bound for $\log|\Lambda|$ we apply lemma (2.3) . Recall that :

$$\Lambda = \log\frac{\alpha''-\alpha'}{\alpha''-\alpha} + n\log\left|\frac{t-\alpha}{t-\alpha'}\right| + m\log\left|\frac{\alpha'}{\alpha}\right|.$$

Put

$$b_1 = 1, b_2 = n, b_3 = m, n = 3, \gamma_1 = \frac{\alpha''-\alpha'}{\alpha''-\alpha}, \gamma_2 = \left|\frac{t-\alpha}{t-\alpha'}\right|, \gamma_3 = \left|\frac{\alpha'}{\alpha}\right|.$$

We have:

$$\begin{aligned}
h\left(\frac{\alpha''-\alpha}{\alpha'-\alpha}\right) &\leq 2h\left(\alpha''-\alpha'\right) = \frac{2}{3}\log\left((\alpha''-\alpha')(\alpha''-\alpha)(\alpha'-\alpha)\right) \\
&< \frac{2}{3}\log(t^9+6t^6), \\
h\left(\frac{\alpha''}{\alpha'}\right) &= \frac{1}{6}\log\left(\frac{\alpha''}{\alpha}\right)^2 < \frac{1}{3}\log(t^9+7t^6), \\
h\left(\frac{t-\alpha'}{t-\alpha''}\right) &= \frac{1}{6}\log\left(\frac{t-\alpha''}{t-\alpha'}\right)^2 < \frac{1}{3}\log(t^9+5t^6).
\end{aligned}$$

Therefore we can take real numbers A_i as:

$$\begin{aligned}
A_1 =& 4\log\left(t^9+6t^6\right), \\
A_2 =& 2\log\left(t^9+5t^6\right), \\
A_3 =& 2\log\left(t^9+7t^6\right), \\
B =& n'.
\end{aligned}$$

$$\Longrightarrow \log|\Lambda| > -1.1446\cdot 10^{13} A_1\cdot A_2\cdot A_3\log\left(68.3n'\right).$$

Comparing with (3.4.13) we conclude that:

$$\begin{aligned}
-1.1446\cdot 10^{13}\log\left(t^9+6t^6\right)\log\left(t^9+5t^6\right)\log\left(t^9+7t^6\right)\log\left(68.3n'\right)& \\
< \log(2) - (3n'+1)\left(\log\left(t^3+1\right)\right).&
\end{aligned} \tag{3.4.14}$$

Which leads to contradiction for $t > 361167$.

3.5 Small values of t

For the remaining small values of t in each type of solution, we will use some diophantine techniques and run a computer search.

3.5.1 Application of diophantine approximation

The following lemma of Mignotte [57], which is a variant of a result by Baker-Davenport, is essential for our computer search.

Lemma 3.35. *Let $\Lambda = \mu\alpha + \nu\beta + \delta$, where α, β and δ are nonzero real numbers and where μ, ν are rational integers, with $|\mu| < A$. Let $Q > 0$ be a real number; suppose that θ_1 and θ_2 satisfy*

$$\left|\theta_1 - \frac{\alpha}{\beta}\right| < \frac{1}{100Q^2} \quad \textit{and} \quad \left|\theta_2 - \frac{\alpha}{\beta}\right| < \frac{1}{Q^2}.$$

Let $\frac{p}{q}$ be a rational number with $1 \leq q \leq Q$ and $\left|\theta_1 - \frac{p}{q}\right| < \frac{1}{q^2}$, and suppose $q\,||q\theta_2|| \geq 1.01A + 2$,(where $||\cdot||$ denote the distance to nearest integer) then

$$|\Lambda| > \frac{|\beta|}{Q^2}. \tag{3.5.1}$$

The idea is to apply the above lemma to the corresponding linear form in logarithms to each type of solution. We have all the data to apply the lemma to all types of solutions other than solutions of type I and II'. However since we used the Padé approximation for solutions of type I and II', applying the lemma requires some extra work.

3.5.2 Solutions of type I

During this section we assume $10 \leq t < 8586$ and let (x, y) be a solution of type I. We have $x - y\theta = (t-\theta)^n\theta^{-m}$. The corresponding linear form in logarithms for this solution is

$$\begin{aligned}\lambda &= \log\frac{\theta - \theta''}{\theta - \theta'} + n\log\frac{t-\theta'}{t-\theta''} + m\log\frac{\theta''}{\theta'} \\ &= \log\left(1 + \frac{(\theta' - \theta'')(x - \theta y)}{(\theta - \theta')(x - \theta'' y)}\right).\end{aligned} \tag{3.5.2}$$

Let $A = \max\{1, m, n\}$. First, for all remaining values of t we find an absolute upper bound for A by linear forms in logarithms; then, applying the lemma 3.3, we find a lower bound for y value of nontrivial solution which leads us to find a lower bound for A. The bounds that we obtained in this way help us treat the small values of t with the lemma 3.35.

Finding upper bound for A using linear forms in logarithms

From (3.3.3) we conclude

$$0 < \frac{\frac{x}{y} - \theta''}{\frac{x}{y} - \theta'} = \frac{x - \theta'' y}{x - \theta' y} = \left(\frac{t - \theta''}{t - \theta'}\right)^n \left(\frac{\theta''}{\theta'}\right)^{-m}.$$

Then by using estimations (3.1.3), we obtain:

$$\log(t^3 - 3) < n \log\left(\frac{t - \theta''}{t - \theta'}\right) - m \log\left(\frac{\theta''}{\theta'}\right) < \log(t^3). \tag{3.5.3}$$

By estimates (3.1.3) we have :

$$t^3 + 2 < \frac{\frac{x}{y} - \theta''}{\frac{x}{y} - \theta'} < t^3 + 3, \tag{3.5.4}$$
$$t^3 + 2 < \frac{\theta''}{\theta'} < t^3 + 3,$$
$$t^9 + 3t^6 < \left(\frac{t - \theta''}{t - \theta'}\right) < t^9 + 5t^6.$$

in equation (3.5.3) if $n = 0$ then $m = -1$, which implies $x = 0$, $y = 1$. If $n > 0$ then $m > 0$, so $\left|x - y\theta\right| > 1$. But it is easy to check that,$\left|x - y\theta\right| < 1$. Therefore we conclude that $n < 0$ and $m < 0$. Let $n' = n, m' = m$ be positive numbers; then we can rewrite (3.5.3)as:

$$\log \frac{\frac{x}{y} - \theta''}{\frac{x}{y} - \theta'} = m' \log\left(\frac{\theta''}{\theta'}\right) - n' \log\left(\frac{t - \theta''}{t - \theta'}\right). \tag{3.5.5}$$

Using above equation, we have

$$n' \left(\frac{\log\left(\frac{t-\theta''}{t-\theta'}\right)}{\log\left(\frac{\theta''}{\theta'}\right)}\right) < m' \Longrightarrow \frac{8}{3} n' < m'.$$

From Siegel's identity :

$$(\theta' - \theta'')(x - \theta y) + (\theta'' - \theta)(x - \theta' y) + (\theta - \theta')(x - \theta'' y) = 0,$$

we obtain

$$\begin{aligned} \lambda &= \log \frac{\theta - \theta''}{\theta - \theta'} + n \log \frac{t - \theta'}{t - \theta''} + m \log \frac{\theta''}{\theta'} \\ &= \log\left(1 + \frac{(\theta' - \theta'')(x - \theta y)}{(\theta - \theta')(x - \theta'' y)}\right). \end{aligned} \tag{3.5.6}$$

Moreover, we have:

$$\begin{aligned}\left|\frac{(\theta'-\theta'')(x-\theta y)}{(\theta-\theta')(x-\theta'' y)}\right| &= \frac{\theta''-\theta'}{\theta'-\theta}\left(\frac{\theta''-t}{t-\theta}\right)^{n'}\left(\frac{\theta}{\theta''}\right)^{m'}\\ \left(\text{Since } \; n'<\frac{3}{8}m'\right) &< \frac{\theta''-\theta'}{\theta'-\theta}\left(\frac{\theta''-t}{t-\theta}\right)^{\frac{3}{8}m'}\left(\frac{\theta}{\theta''}\right)^{m'}\\ &< \left(t^3-3\right)\left(t^3-3\right)^{\frac{3}{8}m'}\left(\left(\frac{1}{t^3-2}\right)^3\right)^{m'}\\ &< \left(t^3-3\right)^{\frac{3}{8}m'+1}\left(\frac{1}{(t^3-3)^3}\right)^{m'}\\ &< \left(t^3-3\right)^{\frac{-21}{8}m'+1}.\end{aligned}$$

If $0<|x|<\frac{1}{2}$ then $|\log(1+x)|<2x$, So $|\lambda|<2\left(t^3-3\right)^{\frac{-21}{3}m'+1}$

$$\Longrightarrow \log|\lambda| < \log(2) + \left(\frac{-21m'}{8}+1\right)\log(t^3-3). \tag{3.5.7}$$

Recall

$$\Lambda = \log\frac{\theta-\theta''}{\theta'-\theta} + nlog\left|\frac{t-\theta'}{t-\theta''}\right| + mlog\left|\frac{\theta''}{\theta'}\right|.$$

To find a lower bound for the linear form in logarithms, we use theorem 2.3.

Let $\alpha_1=\frac{\theta-\theta''}{\theta-\theta'}, \alpha_2=\left|\frac{t-\theta'}{t-\theta''}\right|, \alpha_3=\left|\frac{\theta''}{\theta'}\right|, b_1=1, b_2=n, b_3=m, D=6.$ Using estimations (3.1.3), we have:

$$h\left(\frac{\theta-\theta''}{\theta-\theta'}\right) \le 2h\left(\theta''-\theta'\right) = \frac{2}{3}\log\left((\theta''-\theta')(\theta''-\theta)(\theta'-\theta)\right) < 6\log t, \tag{3.5.8}$$

$$h\left(\frac{\theta''}{\theta'}\right) = \frac{1}{6}\log\left(\frac{\theta''}{\theta}\right)^2 < 3\log t,$$

$$h\left(\frac{t-\theta'}{t-\theta''}\right) = \frac{1}{6}\log\left(\frac{t-\theta''}{t-\theta'}\right)^2 < 3\log t.$$

Therefore we can take $A_1=36\log(t), A_2=18\log(t), A_3=18\log(t)$ and $B=m'$. By above calculation we have:

$$\log|\Lambda| > -8.344\cdot(10)^{15}\log(t)^3\log(68.3m').$$

Comparing this with (3.5.7) :

$$8.344(10)^{15}\log(t)^3\log(68.3m') < \log(2)+\left(\frac{-8m'}{3}+1\right)\log(t^3-3). \tag{3.5.9}$$

Since $t < 8586$, from above inequality we can conclude $A < (4.03)\cdot 10^{18}$.

Finding lower bound for A using gap principal

Since $t \geq 10$, by (3.2.5) and (3.2.7) we can see $\frac{\Delta_F^2}{\varepsilon_1^3} < \frac{1}{22}, \Delta_F > 72000$. These inequalities, together with lemma (3.11), satisfy the conditions of lemma 3.3. So if $F(x,y) = 1$ have another solution (x_2, y_2) of type I other than $(x_1, y_1) = (1-t^3, t^8-3t^5+3t^2)$ we have

$$\begin{aligned} & |\varepsilon_2| > 0.001\Delta_F^{-15}\,|\varepsilon_1|^{47} \\ \text{(By (3.2.5) and (3.2.7), since } t > 10) \quad & > t^{-3}\left(t^{\frac{37}{2}}\right)^{-15}\left(t^{\frac{25}{2}}\right)^{47} > t^{310} \\ \Longrightarrow \quad & H(x_2, y_2) > t^{620}. \end{aligned}$$

On the other hand by (3.2.6):

$$y^2(t^{10}) > H(x,y)$$

$$\Longrightarrow y^2 > t^{610}.$$

Now using equation $(x-y\theta)(x-y\theta')(x-y\theta'') = 1$ and(3.3.3) we obtain:

$$\left|(t-\theta)^n\,\theta^{-m}\right| = \left|\frac{1}{y^2\left(\frac{x}{y}-\theta'\right)\left(\frac{x}{y}-\theta''\right)}\right|.$$

By (3.1.3), (3.1.5) and the lower bound we found for y^2

$$\left|\frac{1}{y^2\left(\frac{x}{y}-\theta'\right)\left(\frac{x}{y}-\theta''\right)}\right| < \frac{1}{t^{610}\cdot t\cdot t^3} < t^{-614}.$$

As for the left hand side,

$$\begin{aligned}
\left|(t-\theta)^n\,\theta^{-m}\right| &= \left|\frac{1}{(t-\theta)}\right|^{n'}\left|\theta^{m'}\right| \\
\left(\text{since}\frac{1}{(t-\theta)}<1 \quad \text{and} \quad n'<\frac{3}{8}m'\right) \quad &> \left|\frac{1}{(t-\theta)}\right|^{\frac{3}{8}m'}\left|\theta^{m'}\right| \\
(\text{using estimates}(3.1.3)) \quad &> \left(\frac{1}{t^{\frac{4}{3}}}\right)^{\frac{3}{8}m'}\left(\frac{1}{t^5}\right)^{m'} = t^{-\frac{11}{2}m'}.
\end{aligned}$$

$$\Longrightarrow 614 < \frac{11}{2}m' \Longrightarrow A = m' > 111. \tag{3.5.10}$$

Applying lemma 3.35

We apply lemma 3.35 to the linear form :

$$\lambda = \log\frac{\theta-\theta''}{\theta-\theta'} + n\log\frac{t-\theta'}{t-\theta''} + m\log\frac{\theta''}{\theta'}$$

correspond to a nontrivial solution of type I.

$$\text{Put} \quad \alpha = \log\left|\frac{\theta''}{\theta'}\right|, \beta = \log\left|\frac{t-\theta'}{t-\theta''}\right|, \delta = \log\frac{\theta-\theta''}{\theta-\theta'}, \mu = m, \nu = n,$$

From(3.5.7)and (3.5.10), since $t > 10$, if the inequality (3.5.1)holds then $Q > 10^{870}$. We apply lemma 3.35 with the smaller value of Q being $Q = 10^{60}$. Find θ_1, θ_2 with sufficient precision. Compute the denominators of principal convergents of $q < Q$ and check if the hypothesis of the lemma is satisfied with the smaller value of Q. Indeed this is the case for all $10 < t < 8586$, and this contradiction completes the proof for small values of t. Using a code written in Pari/GP version, the verification takes less than half an hour.

3.5.3 Solutions of type II”

During this section we assume $10 \leq t < 8586$. Let (x, y) be a solution of type I. We have $x - y\alpha = (t-\alpha)^n\alpha^{-m}$. The corresponding linear form in logarithms for this solution is

$$\begin{aligned}
\lambda &= \log\frac{\alpha-\alpha''}{\alpha-\alpha'} + n\log\frac{t-\alpha'}{t-\alpha''} + m\log\frac{\alpha''}{\alpha'} \\
&= \log\left(1+\frac{(\alpha'-\alpha'')(x-\alpha y)}{(\alpha-\alpha')(x-\alpha'' y)}\right).
\end{aligned} \tag{3.5.11}$$

Let $A = \max\{1, m, n\}$. First for all remaining values of t we find an absolute upper bound for A with linear forms in logarithms; then applying the lemma 3.3 we find a lower bound for y value of a nontrivial solution which leads us to find a lower bound for A. The bounds that we obtained in this way help us treat the small values of t by the lemma 3.5.1.

Finding upper bound for A using linear forms in logarithms

First we will show that if (x, y) is an integral solution of this type then $y > 0$, Using equation $(x-y\alpha)(x-y\alpha')(x-y\alpha'') = 1$, we conclude that $x - y\alpha < 0$. Consider the equation

$$\frac{x - y\alpha'}{x - y\alpha''} = \left(\frac{t-\alpha'}{t-\alpha''}\right)^n \left(\frac{\alpha'}{\alpha''}\right)^{-m}.$$

Since RHS is positive, LHS is also positive $\Longrightarrow x - y\alpha'' = y\left(\frac{x}{y} - \alpha''\right) < 0$,but $\frac{x}{y} - \alpha'' < 0 \Longrightarrow y > 0$.
Using equation (3.3.3) and estimations (3.1.4) we can obtain

$$0 > \frac{\frac{x}{y} - \alpha''}{\frac{x}{y} - \alpha} = \frac{x - \alpha'' y}{x - \alpha y} = \left(\frac{t-\alpha''}{t-\alpha}\right)^n \left(\frac{\alpha''}{\alpha}\right)^{-m}.$$

So n , m have different parities $\pmod 2$. Taking absolute values of both sides yields :

$$\frac{\alpha'' - \frac{x}{y}}{\frac{x}{y} - \alpha} = \left(\frac{\alpha''-t}{t-\alpha}\right)^n \left(\frac{\alpha''}{|\alpha|}\right)^{-m}.$$

Taking logarithms from both sides and using estimations 3.1.4 we have:

$$\begin{gathered}\log(t^3+1) < n\log\left(\frac{\alpha''-t}{t-\alpha}\right) - m\log\left(\frac{\alpha''}{|\alpha|}\right) < \log(t^3+2), \\ t^3+1 < \frac{\alpha''-t}{t-\alpha} < t^3+2, \quad t^9+6t^6 < \frac{\alpha''}{|\alpha|} < t^9+7t^6+6.\end{gathered} \tag{3.5.12}$$

Considering 3.3.4 , if $m = 0$ then $n = 1$ which, with positive sign, give us the solution $(t, 1)$, if $m < 0$ then $n < 0$ so $|x - y\alpha'| = |(t-\alpha')^n(\alpha')^{-m}| > 1$, but from(3.3.2) we have
$|x - y\alpha'| < 1$. This contradiction leads us to conclude that $m > 0 \Rightarrow n > 0$ and from first inequality of (3.5.12) $n > m$; Moreover,

$$m\frac{\log\left(\frac{\alpha''}{|\alpha|}\right)}{\log\left(\frac{\alpha''-t}{t-\alpha}\right)} < n \Longrightarrow \frac{8}{3}m < n.$$

From Siegel's identity :

$$(\alpha' - \alpha'')(x - \alpha y) + (\alpha'' - \alpha)(x - \alpha' y) + (\alpha - \alpha')(x - \alpha'' y) = 0,$$

we obtain

$$\Lambda = \log \frac{\alpha'' - \alpha'}{\alpha' - \alpha} + nlog \left| \frac{t - \alpha}{t - \alpha''} \right| + mlog \left| \frac{\alpha''}{\alpha} \right| \tag{3.5.13}$$

$$= \log \left(1 + \frac{(\alpha - \alpha'')(x - \alpha' y)}{(\alpha' - \alpha)(x - \alpha'' y)} \right), \tag{3.5.14}$$

where $\frac{(\alpha-\alpha'')(x-\alpha' y)}{(\alpha'-\alpha)(x-\alpha'' y)} < 0$ Using (3.1.3), we have:

$$\begin{aligned}
\frac{(\alpha'' - \alpha)(x - \alpha' y)}{(\alpha' - \alpha)(x - \alpha'' y)} &= \frac{\alpha'' - \alpha}{\alpha' - \alpha} \left(\frac{t - \alpha'}{t - \alpha''} \right)^n \left(\frac{\alpha''}{\alpha'} \right)^m \\
(\text{since } m < \frac{3n}{8}), &< \frac{\alpha'' - \alpha}{\alpha' - \alpha} \left(\frac{t - \alpha'}{t - \alpha''} \right)^n \left(\frac{\alpha''}{\alpha'} \right)^{\frac{3n}{8}} \\
&< (t^3 + 3) \left(\left(\frac{1}{t^3 + 1} \right)^3 \right)^n (t^3 + 3)^{\frac{3n}{8}} \\
&< \left(\frac{1.01}{(t^3 + 3)^3} \right)^n (t^3 + 3)^{\frac{3n}{8} + 1} \\
&< (t^3 + 3)^{\left(\frac{-21n}{8} + 1 \right)} (1.01)^n.
\end{aligned}$$

If $0 < x < \frac{1}{2}$ then $|\log(1 - x)| < 2x$, So $|\Lambda| < 2(t^3 - 2)^{\left(\frac{-8n}{3} + 1 \right)} (1.01)^n$

$$\Longrightarrow \log |\Lambda| < \log(2) + n \log(1.01) + \left(\frac{-21n}{8} + 1 \right) \log(t^3 + 3) \tag{3.5.15}$$

Recall

$$\Lambda = \log \frac{\alpha'' - \alpha'}{\alpha' - \alpha} + nlog \left| \frac{t - \alpha}{t - \alpha''} \right| + mlog \left| \frac{\alpha''}{\alpha} \right|.$$

We can apply theorem (2.3) to find a lower bound for Λ. Put $\alpha_1 = \frac{\alpha'' - \alpha'}{\alpha' - \alpha}$, $\alpha_2 = \left| \frac{t - \alpha}{t - \alpha''} \right|, \alpha_3 = \left| \frac{\alpha''}{\alpha} \right|, b_1 = 1, b_2 = n, b_3 = m, D = 6$.

Using estimations (3.1.4)

$$\begin{aligned} h\left(\frac{\alpha''-\alpha'}{\alpha'-\alpha}\right) &\leq 2h\left(\alpha''-\alpha'\right)=\frac{2}{3}\log\left((\alpha''-\alpha')(\alpha''-\alpha)(\alpha'-\alpha)\right), \\ &< \frac{2}{3}\log(t^9+6t^6), \\ h\left|\frac{t-\alpha}{t-\alpha''}\right| &= \frac{1}{6}\log\left(\frac{\alpha''}{\alpha}\right)^2 < \frac{1}{3}\log(t^9+7t^6) \qquad (3.5.16) \\ h\left|\frac{\alpha''}{\alpha}\right| &= \frac{1}{6}\log\left(\frac{t-\alpha''}{t-\alpha'}\right)^2 < \frac{1}{3}\log(t^9+5t^6). \end{aligned}$$

Therefore we can take real numbers A_i as:

$$\begin{aligned} A_1 =& 4\log\left(t^9+6t^6\right), \\ A_2 =& 2\log\left(t^9+5t^6\right), \\ A_3 =& 2\log\left(t^9+7t^6\right), \\ B =& n. \end{aligned}$$

We apply theorem (2.3) to find a lower bound for $\log|\Lambda|$.

$$\log|\Lambda| > -1.1446\cdot 10^{13}\cdot A_1\cdot A_2\cdot A_3\log\left(68.3n\right).$$

Comparing with (3.5.15) we obtain

$$\begin{aligned} &-1.1446\cdot 10^{13}\log\left(t^9+6t^6\right)\log\left(t^9+5t^6\right)\log\left(t^9+7t^6\right)\log\left(68.3n\right) \\ &\qquad < \log(2)+n\log(1.01)+\left(\frac{-21n}{8}+1\right)\log(t^3+3). \end{aligned}$$

Since $n < 8586$ we obtain

$$A = n < 4.09\cdot 10^{18}.$$

Lower bound for A using gap principal

Since $t \geq 10$, by (3.2.10) and (3.2.12), we can see for $t > 10$, $\frac{\Delta_{F'}^2}{\varepsilon_1^3} < \frac{1}{22}, \Delta_{F'} > 72000$. These inequalities, together with lemma 3.15, satisfy the conditions of the lemma 3.3. So if $F'(x,y)=1$ has another solution (x_2,y_2) of type II' other than$(x_1,y_1)=(t^9+3t^6+4t^3+1, t^8-3t^5+3t^2)$ we have

$$\begin{aligned} & |\varepsilon_2| > 0.001\Delta^{-15}\left|\varepsilon_1\right|^{47} \\ \text{(By (3.2.10) and (3.2.12), since } t>10) \quad & > t^{-3}\left(t^{\frac{37}{2}}\right)^{-15}\left(t^{\frac{25}{2}}\right)^{47} > t^{307} \\ \Longrightarrow \quad & H(x_2,y_2) > t^{614}. \end{aligned}$$

On the other hand we have

$$y^2(t^{10}+4t^7+9t^4) > H(x,y)$$
$$\Longrightarrow y^2 > t^{604}.$$

Now using equation $(x-y\alpha)(x-y\alpha')(x-y\alpha'')=1$ and(3.3.3) we obtain:

$$\left|\left(t-\alpha'\right)^n \alpha'^{-m}\right| = \left|\frac{1}{y^2\left(\frac{x}{y}-\alpha\right)\left(\frac{x}{y}-\alpha''\right)}\right|.$$

By (3.1.3), (3.1.5) and the lower bound we found for y^2

$$\left|\frac{1}{y^2\left(\frac{x}{y}-\alpha\right)\left(\frac{x}{y}-\alpha''\right)}\right| < \frac{1}{t^{604}\cdot t\cdot t^4} < t^{-609}.$$

As for the left hand side we have:

$$\begin{aligned}
\left|\left(t-\alpha'\right)^n \alpha'^{-m}\right| &= \left|\left(t-\alpha'\right)\right|^n \left|\frac{1}{\alpha'}\right|^m \\
\left(\text{since}\frac{1}{\alpha'}<1 \quad \text{and} \quad m<\frac{3}{8}n\right) \quad &> \left|\left(t-\alpha'\right)\right|^n \left|\frac{1}{\alpha'}\right|^{\frac{3n}{8}} \\
\text{(using estimates(3.1.4))} \quad &> \left(\frac{1}{t^{\frac{4}{3}}}\right)^n \left(\frac{1}{t^{\frac{16}{3}}}\right)^{\frac{3n}{8}} = t^{-\frac{-10}{3}n}.
\end{aligned}$$

$$\Longrightarrow 609 < \frac{10}{3}n \Longrightarrow A = n > 182. \tag{3.5.17}$$

Applying lemma 3.35

We apply lemma 3.35 to the linear form :

$$\lambda = \log\frac{\alpha-\alpha''}{\alpha-\alpha'} + n\log\frac{t-\alpha'}{t-\alpha''} + m\log\frac{\alpha''}{\alpha'}$$

corresponding to a nontrivial solution of type II'.

Put $\alpha = \log\left|\frac{\alpha''}{\alpha'}\right|, \beta = \log\left|\frac{t-\alpha'}{t-\alpha''}\right|, \delta = \log\frac{\alpha-\alpha''}{\alpha-\alpha'}, \mu = m, \nu = n,$

From(3.5.7)and (3.5.17), since $t>10$,if the inequality (3.5.1) holds we have $Q>10^{1421}$. We apply lemma 3.35 with a smaller value of Q, $Q=10^{60}$. Find

θ_1, θ_2 with sufficient precision, then computing the denominators of principal convergents of $q < Q$ and check if the hypothesis of lemma is satisfied for this smaller value of Q. This is the case for all $10 < t < 8586$, and this contradiction completes the proof for small values of t. Using a code written in Pari/GP version, the verification takes less than half an hour.

3.5.4 Solutions of type II

We prove that there is no nontrivial solution of type II, for $10 \leq t \leq 1119749$. We apply the lemma 3.35 for the corresponding linear forms in logarithms :

$$\Lambda = \log \frac{\theta'' - \theta'}{\theta' - \theta} + nlog \left| \frac{t - \theta}{t - \theta''} \right| + mlog \left| \frac{\theta''}{\theta} \right|.$$

Let $A = \max\{m, n\}$. By inequality (3.20). Since $t \geq 10$ we have $A > 35$. From inequality (3.3.11) since $t \leq 1119744$ we have $A < 1.35 \cdot 10^9$. And finally from (3.3.7) since $A > 35$ if the the inequality (3.5.1) happens we have $Q > 10^{287}$. we apply lemma 3.35 with the smaller value of Q, $Q = 10^{60}$. We checked the hypothesis of lemma for this smaller value of Q to get a contradiction. Using a code written in Pari/GP version,it takes less than 6 hours to get contradiction for all values of $10 \leq t \leq 1119744$.

3.5.5 Solution of type I'

To prove there is no nontrivial solution of type I' for $10 \leq t \leq 712018$. We perform exactly the same steps as the section 3.5.4 . The corresponding linear form is

$$\lambda = \log \frac{\alpha - \alpha''}{\alpha - \alpha'} + n \log \frac{t - \alpha'}{t - \alpha''} + m \log \frac{\alpha''}{\alpha'}.$$

Let $A = \max\{m, n\}$. By inequality (3.3). Since $t \geq 10$ we have $A > 99$. From inequality (3.3.33) since $t \leq 712018$ we have $A < 1.85 \cdot 10^9$. And finally from (3.3.19) since $A > 99$, if the the inequality (3.5.1) happens then $Q > 10^{772}$. we apply lemma 3.35 with the smaller value $Q = 10^{60}$. Using a code written in Pari/GP version,it takes less than 5 hours to got contradiction for all values of $10 \leq t \leq 722018$.

3.5.6 Solutions of type III

To prove there is no nontrivial solution of type III for $10 \leq t \leq 361165$. We perform exactly the same steps as the section 3.5.4. The corresponding linear form is:

$$\Lambda = \log \frac{\theta'' - \theta'}{\theta'' - \theta} + n \log \left| \frac{t-\theta}{t-\theta'} \right| + m \log \left| \frac{\theta'}{\theta} \right| .$$

Let $A = \max\{m, n\}$. By inequality (3.4.3). Since $t \geq 10$ we have $A > 13827$. From inequality (3.4.7) since $t \leq 361165$ we have $A < 3.62 \cdot 10^{18}$. And finally from (3.4.6) since $A > 13827$ if the the inequality (3.5.1) happens we have $Q > 10^{12439}$. we apply lemma 3.35 with $Q = 10^{60}$. Using a code written in Pari/GP version,it takes less than 4 hours to got contradiction for all values of $10 \leq t \leq 361165$.

3.5.7 Solutions of type III'

To prove there is no nontrivial solution of type III' for $10 \leq t \leq 361167$. We perform exactly the same steps as the section 3.5.4. The corresponding linear form is:

$$\Lambda = \log \frac{\alpha'' - \alpha'}{\alpha'' - \alpha} + n \log \left| \frac{t-\alpha}{t-\alpha'} \right| + m \log \left| \frac{\alpha'}{\alpha} \right| .$$

Let $A = \max\{m, n\}$. By lemma 3.4.10. Since $t \geq 10$ we have $A > 27649$. From inequality (3.4.14) since $t \leq 361167$ we have $A < 7.23 \cdot 10^{18}$. And finally from (3.4.13) since $A > 27649$, if the inequality (3.5.1) happens we have $Q > 10^{24873}$. we apply lemma 3.35 with a smaller value of Q being $Q = 10^{60}$. Using a code written in Pari/GP version,it takes less than 4 hours to got contradiction for all values of $10 \leq t \leq 361167$.

Chapter 4

Mordell curves

4.1 Introduction

If k is a nonzero integer, then the equation

$$Y^2 = X^3 + k \tag{4.1.1}$$

defines an elliptic curve over $\mathbb{Q}$. Such Diophantine equations have a long history, dating back (at least) to work of Bachet in the 17th century, and are nowadays termed *Mordell equations*, honouring the substantial contributions of L. J. Mordell to their study. Indeed the statement that, for a given $k \neq 0$, equation (4.1.1) has at most finitely many integral solutions is implicit in work of Mordell [63] (via application of a result of Thue [81]), and explicitly stated in [64].

Historically, the earliest approaches to equation (4.1.1) for certain special values of k appealed to simple local arguments; references to such work may be found in Dickson [24]. More generally, working in either $\mathbb{Q}(\sqrt{-k})$ or $\mathbb{Q}(\sqrt[3]{k})$, one is led to consider a finite number of Thue equations of the shape $F(x,y) = m$, where the m are nonzero integers and the F are, respectively, binary cubic or quartic forms with rational integer coefficients. Via classical arguments of Lagrange (see e.g. page 673 of Dickson [24]), these in turn correspond to a finite (though typically larger) collection of Thue equations of the shape $G(x,y) = 1$. Here, again, the G are binary cubic or quartic forms with integer coefficients. In case k is positive, one encounters cubic forms of negative discriminant which may typically be treated rather easily via Skolem's p-adic method (as the corresponding cubic fields have a single fundamental unit). For negative values of k, one is led to cubic or quartic fields with a pair of fundamental units, which may sometimes be treated by similar if rather more complicated methods; see e.g. [30], [38] and [53].

There are alternative approaches for finding the integral points on a given model of an elliptic curve. The most commonly used currently proceeds via appeal to lower bounds for linear forms in elliptic logarithms, the idea for which dates back to work of Lang [49] and Zagier [89] (though the bounds required to make such arguments explicit are found in work of David and of Hirata-Kohno, see e.g. [21]). Using these bounds, Gebel, Pethő and Zimmer [31], Smart [74] and Stroeker, Tzanakis [77] obtained, independently, a "practical" method to find integral points on elliptic curves. Applying this method, in 1998, Gebel, Pethő, Zimmer [32] solved equation (4.1.1) for all integers $|k| < 10^4$ and partially extended the computation to $|k| < 10^5$. As a byproduct of their calculation, they obtained a variety of interesting information about the corresponding elliptic curves, such as their ranks, generators of their Mordell-Weil groups, and information on their Tate-Shafarevic groups.

The only obvious disadvantage of this approach is its dependence upon knowledge of the Mordell-Weil basis over $\mathbb{Q}$ of the given Mordell curve (indeed, it is this dependence that ensures, with current technology at least, that this method is not strictly speaking algorithmic). For curves of large rank, in practical terms, this means that the method cannot be guaranteed to solve equation (4.1.1).

Our goal in this paper is to present (and demonstrate the results of) a practical algorithm for solving Mordell equations with values of k in a somewhat larger range. At its heart are lower bounds for linear forms in complex logarithms, stemming from the work of Baker [3]. These were first applied in the context of explicitly solving equation (4.1.1), for fixed k, by Ellison et al [27]. To handle values of k in a relatively large range, we will appeal to classical invariant theory and, in particular, to the reduction theory of binary cubic forms (where we have available very accessible, algorithmic work of Belabas [6], Belabas and Cohen [7] and Cremona [17]). This approach has previously been outlined by Delone and Fadeev (see Section 78 of [22]) and Mordell (see e.g. [65]; its origins lie in [62]). We use it to solve Mordell's equation (4.1.1) for all k with $0 < |k| \leq 10^7$. We should emphasize that our algorithm does not provide *a priori* information upon, say, the ranks of the corresponding elliptic curves, though values of k for which (4.1.1) has many solutions necessarily (as long as k is 6-th power free) provide curves with at least moderately large rank (see [34]).

We proceed as follows. In Section 4.2, we will discuss the precise correspondence that exists between integer solutions to (4.1.1) and integer solutions to cubic Thue equations of the shape $F(x, y) = 1$, for certain binary cubic forms of discriminant $-108k$. In Section 4.3, we indicate a method to choose representatives from equivalence classes of forms of a given discriminant. Section 4.4 contains a brief discussion of our computation, while Section 4.5 is devoted to presenting a summary of our data, including information on both the number of solutions, and on their heights.

4.2 Preliminaries

In this section, we begin by outlining a correspondence between integer solutions to the equation $Y^2 = X^3 + k$ and solutions to certain cubic Thue equations of the form $F(x, y) = 1$, where F is a binary cubic form of discriminant $-108k$. As noted earlier, this approach is very classical. To make it computationally efficient, however, there are a number of details that we must treat rather carefully.

Let us suppose that a, b, c and d are integers, and consider the binary cubic form

$$F(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3, \tag{4.2.1}$$

with discriminant

$$D = D_F = -27\left(a^2d^2 - 6abcd - 3b^2c^2 + 4ac^3 + 4b^3d\right).$$

It is important to observe (as a short calculation reveals) that the set of forms of the shape (4.2.1) is closed within the larger set of binary cubic forms in $\mathbb{Z}[x, y]$, under the action of both $SL_2(\mathbb{Z})$ and $GL_2(\mathbb{Z})$. To such a form we associate covariants, namely the Hessian $H = H_F(x, y)$ given by

$$H = H_F(x, y) = -\frac{1}{4}\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2\right)$$

and the Jacobian determinant of F and H, a cubic form $G = G_F$ defined as

$$G = G_F(x, y) = \frac{\partial F}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial x}.$$

Note that, explicitly, we have

$$H/9 = (b^2 - ac)x^2 + (bc - ad)xy + (c^2 - bd)y^2$$

and

$$G/27 = a_1x^3 + 3b_1x^2y + 3c_1xy^2 + d_1y^3,$$

where

$$a_1 = -a^2d + 3abc - 2b^3,\ b_1 = -b^2c - abd + 2ac^2,\ c_1 = bc^2 - 2b^2d + acd$$

and $d_1 = -3bcd + 2c^3 + ad^2$.

Crucially for our arguments, these covariants satisfy the syzygy

$$4H(x, y)^3 = G(x, y)^2 + 27DF(x, y)^2.$$

Defining $D_1 = D/27$, $H_1 = H/9$ and $G_1 = G/27$, we thus have

$$4H_1(x,y)^3 = G_1(x,y)^2 + D_1 F(x,y)^2.$$

If (x_0, y_0) satisfies the equation $F(x_0, y_0) = 1$ and $D_1 \equiv 0 \pmod 4$ (i.e. if $ad \equiv bc \pmod 2$), then necessarily $G_1(x_0, y_0) \equiv 0 \pmod 2$. We may therefore conclude that $Y^2 = X^3 + k$, where

$$X = H_1(x_0, y_0), \quad Y = \frac{G_1(x_0, y_0)}{2} \quad \text{and} \quad k = -\frac{D_1}{4} = -\frac{D}{108}.$$

It follows that, to a given triple (F, x_0, y_0), where F is a cubic form as in (4.2.1) with discriminant $-108k$, and x_0, y_0 are integers for which $F(x_0, y_0) = 1$, we can associate an integral point on the Mordell curve $Y^2 = X^3 + k$.

Conversely suppose, for a given integer k, that (X, Y) satisfies equation (4.1.1). To the pair (X, Y), we associate the cubic form

$$F(x,y) = x^3 - 3Xxy^2 + 2Yy^3.$$

Such a form F is of the shape (4.2.1), with discriminant

$$D_F = -108Y^2 + 108X^3 = -108k$$

and covariants satisfying

$$X = \frac{G_1(1,0)}{2} = \frac{G(1,0)}{54} \quad \text{and} \quad Y = H_1(1,0) = \frac{H(1,0)}{9}.$$

In summary, there exists a correspondence between the set of integral solutions

$$S_k = \{(X_1, Y_1), \ldots, (X_{N_k}, Y_{N_k})\}$$

to the Mordell equation $Y^2 = X^3 + k$ and the set T_k of triples (F, x, y), where each F is a binary cubic form of the shape (4.2.1), with discriminant $-108k$, and the integers x and y satisfy $F(x, y) = 1$. Note that the forms F under consideration here need not be irreducible.

In the remainder of this section, we will show that there is, in fact, a bijection between T_k under $SL_2(\mathbb{Z})$-equivalence, and the set S_k. We begin by demonstrating the following pair of lemmata.

Lemma 4.2.1. *Let k be a nonzero integer and suppose that F_1 and F_2 are $SL_2(\mathbb{Z})$-inequivalent binary cubic forms of the shape (4.2.1), each with discriminant $-108k$, and that x_1, y_1, x_2 and y_2 are integers such that (F_1, x_1, y_1) and (F_2, x_2, y_2) are in T_k. Then the tuples (F_1, x_1, y_1) and (F_2, x_2, y_2) correspond to distinct elements of S_k.*

Proof. Suppose that the tuples (F_1, x_1, y_1) and (F_2, x_2, y_2) are in T_k, so that

$$F_1(x_1, y_1) = F_2(x_2, y_2) = 1.$$

Since, for each i, x_i and y_i are necessarily coprime, we can find integers m_i and n_i such that $m_i x_i - n_i y_i = 1$, for $i = 1, 2$. Writing $\tau_i = \left(\begin{smallmatrix} x_i & n_i \\ y_i & m_i \end{smallmatrix}\right)$, we thus have

$$F_i \circ \tau_i(x, y) = x^3 + 3b_i x^2 y + 3c_i xy^2 + d_i y^3,$$

for integers b_i, c_i and d_i, and hence, under the further action of $\gamma_i = \left(\begin{smallmatrix} 1 & -b_i \\ 1 & 0 \end{smallmatrix}\right)$, we observe that F_i is $SL_2(\mathbb{Z})$-equivalent to

$$x^3 - 3p_i xy^2 + 2q_i y^3,$$

where

$$p_i = \frac{G_{F_i}(x_i, y_i)}{54} \quad \text{and} \quad q_i = \frac{H_{F_i}(x_i, y_i)}{9}.$$

If the two tuples correspond to the same element of S_k, necessarily

$$G_{F_1}(x_1, y_1) = G_{F_2}(x_2, y_2) \quad \text{and} \quad H_{F_1}(x_1, y_1) = H_{F_2}(x_2, y_2),$$

contradicting our assumption that F_1 and F_2 are $SL_2(\mathbb{Z})$-inequivalent. □

Lemma 4.2.2. *Suppose that k is a nonzero integer, that F is a binary cubic form of the shape (4.2.1) and discriminant $-108k$, and that $F(x_0, y_0) = F(x_1, y_1) = 1$ where (x_0, y_0) and (x_1, y_1) are distinct pairs of integers. Then the tuples (F, x_0, y_0) and (F, x_1, y_1) correspond to distinct elements of S_k.*

Proof. Via $SL_2(\mathbb{Z})$-action, we may suppose, without loss of generality, that $F(x, y) = x^3 + 3bx^2 y + 3cxy^2 + dy^3$ and that $(x_0, y_0) = (1, 0)$. If the triples $(F, 1, 0)$ and (F, x_1, y_1) correspond to the same element of S_k, necessarily

$$G_F(1, 0) = G_F(x_1, y_1) \quad \text{and} \quad H_F(1, 0) = H_F(x_1, y_1),$$

whereby

$$x_1^3 + 3bx_1^2 y_1 + 3cx_1 y_1^2 + dy_1^3 = 1, \tag{4.2.2}$$

$$(b^2 - c)x_1^2 + (bc - d)x_1 y_1 + (c^2 - bd)y_1^2 = (b^2 - c) \tag{4.2.3}$$

and

$$a_1 x_1^3 + 3b_1 x_1^2 y_1 + 3c_1 x_1 y_1^2 + d_1 y_1^3 = a_1. \tag{4.2.4}$$

It follows that

$$3(ba_1 - b_1)x_1^2 y_1 + 3(ca_1 - c_1)x_1 y_1^2 + (da_1 - d_1)y_1^3 = 0.$$

Since $(x_1, y_1) \neq (1, 0)$, we have that $y_1 \neq 0$ and so

$$3(ba_1 - b_1)x_1^2 + 3(ca_1 - c_1)x_1y_1 + (da_1 - d_1)y_1^2 = 0,$$

i.e.

$$-3(b^2-c)^2x_1^2 + 3(b^2-c)(d-bc)x_1y_1 + (3bcd - b^3d - c^3 - d^2)y_1^2 = 0. \quad (4.2.5)$$

If $b^2 = c$, it follows that

$$3bcd - b^3d - c^3 - d^2 = -(d-b^3)^2 = 0,$$

so that $d = b^3$ and $F(x, y) = (x + by)^3$, contradicting our assumption that $D_F \neq 0$. We thus have

$$(b^2 - c)x_1^2 + (bc - d)x_1y_1 = \frac{(3bcd - b^3d - c^3 - d^2)}{3(b^2 - c)}y_1^2$$

and so

$$\frac{(3bcd - b^3d - c^3 - d^2)}{3(b^2 - c)}y_1^2 = (bd - c^2)y_1^2 + b^2 - c,$$

i.e.

$$\left(-d^2 + 6bcd - 4b^3d - 4c^3 + 3b^2c^2\right) y_1^2 = 3(b^2 - c)^2$$

and so

$$D_F\, y_1^2 = 81(b^2 - c)^2.$$

Since $D_F = -108k$, it follows that $k = -3m^2$ for some integer m, where $2my_1 = b^2 - c$. From (4.2.5), we thus have

$$-12m^2x_1^2 + 6(d - bc)mx_1 + 3bcd - b^3d - c^3 - d^2 = 0,$$

whereby

$$x_1 = \frac{1}{4m}\left(d - bc \pm 2\right).$$

Substituting the expressions for x_1 and y_1 into equation (4.2.5), we conclude, from $b^2 \neq c$, that

$$\frac{-3}{4}(d - bc \pm 2)^2 + \frac{3}{2}(d - bc)(d - bc \pm 2) + 3bcd - b^3d - c^3 - d^2 = 0,$$

whence, since the left-hand-side of this expression is just $D_F/27 - 12$, we find that $D_F = 324$. It follows that the triple $(F, 1, 0)$, say, corresponds to an integral solution to the Mordell equation $Y^2 = X^3 - 3$. Adding 4 to both sides of this equation, however, we observe that necessarily $X^2 - X + 1 \equiv 3 \pmod 4$, contradicting the fact that it divides the sum of two squares $Y^2 + 4$. The lemma thus follows as stated. $\square$

To conclude as desired, we have only to note that, for any $\gamma \in SL_2(\mathbb{Z})$, covariance implies that $H_{F\circ\gamma} = H_F \circ \gamma$ and $G_{F\circ\gamma} = G_F \circ \gamma$, and hence triples

$$(F, x_0, y_0) \text{ and } (F \circ \gamma, \gamma(x_0), \gamma(y_0))$$

in T_k necessarily correspond to the same solution to (4.1.1) in S_k.

Remark 4.1. *Instead of working with $SL_2(\mathbb{Z})$-equivalence, we can instead consider $GL_2(\mathbb{Z})$-equivalence classes (and, as we shall see in the next section, this equivalence is arguably a more natural one with which to work). Since $H(x,y)$ and $G^2(x,y)$ are $GL_2(\mathbb{Z})$-covariant, if two forms are equivalent under the action of $GL_2(\mathbb{Z})$, but not under $SL_2(\mathbb{Z})$, then we have*

$$H_{F\circ\gamma} = H_F \circ \gamma \quad \textit{and} \quad G_{F\circ\gamma} = -G_F \circ \gamma.$$

It follows that, in order to determine all pairs of integers (X, Y) satisfying equation (4.1.1), it is sufficient to find a representative for each $GL_2(\mathbb{Z})$-equivalence class of forms of shape (4.2.1) and discriminant $-108k$ and, for each such form, solve the corresponding Thue equation. A pair of integers (x_0, y_0) for which $F(x_0, y_0) = 1$ now leads to a pair of solutions $(X, \pm Y)$ to $Y^2 = X^3 + k$, where

$$X = H_1(x_0, y_0) \quad \textit{and} \quad Y = G_1(x_0, y_0)/2,$$

at least provided $G_1(x_0, y_0) \neq 0$.

4.3 Finding representative forms

As we have demonstrated in the previous section, to solve Mordell's equation for a given integer k, it suffices to determine a set of representatives for $SL_2(\mathbb{Z})$-equivalence classes (or, if we prefer, $GL_2(\mathbb{Z})$-equivalence classes) of binary cubic forms of the shape (4.2.1), with discriminant $-108k$, and then solve the corresponding Thue equations $F(x, y) = 1$. In this section, we will describe how to find distinguished representatives for equivalence classes of cubic forms with a given discriminant. In all cases, the various notions of *reduction* arise from associating to a given cubic form a particular definite quadratic form – in case of positive discriminant, the Hessian defined earlier works well. In what follows, we will state our definitions of reduction solely in terms of the coefficients of the given cubic form, keeping the role of the associated quadratic form hidden from view.

4.3.1 Forms of positive discriminant

In case of positive discriminant forms (i.e. those corresponding to negative values of k), there is a well-developed classical reduction theory, dating back to work of Hermite [39], [40] and later applied to great effect by Davenport (see e.g. [18], [19] and [20]). This procedure allows us to determine distinguished *reduced* elements within each equivalence class of forms. We can, in fact, apply this reduction procedure to both irreducible and reducible forms; initially we will assume the forms we are treating are irreducible, for reasons which will become apparent. We will follow work of Belabas [6] (see also Belabas and Cohen [7] and Cremona [17]), in essence a modern treatment and refinement of Hermite's method.

Definition 4.1. *An irreducible binary integral cubic form*

$$F(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

of positive discriminant is called reduced if we have

- $|bc - ad| \leq b^2 - ac \leq c^2 - bd$,
- $a > 0, b \geq 0$, *where* $d < 0$ *whenever* $b = 0$,
- *if* $bc = ad, d < 0$,
- *if* $b^2 - ac = bc - ad$, $b < |a - b|$, *and*
- *if* $b^2 - ac = c^2 - bd$, $a \leq |d|$ *and* $b < |c|$.

The main value of this notion of reduction is apparent in the following result (Corollary 3.3 of [6]).

Proposition 4.1. *Any irreducible cubic form of the shape (4.2.1) with positive discriminant is* $GL_2(\mathbb{Z})$*-equivalent to a unique reduced one.*

To determine equivalence classes of reduced cubic forms with bounded discriminant, we will appeal to the following result (immediate from Lemma 3.5 of Belabas [6]).

Lemma 4.3.1. *Let* K *be a positive real number and*

$$F(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

be a reduced form whose discriminant lies in $(0, K]$*. Then we have*

$$1 \leq a \leq \frac{2K^{1/4}}{3\sqrt{3}}$$

and

$$0 \leq b \leq \frac{a}{2} + \frac{1}{3}\left(\sqrt{K} - \frac{27a^2}{4}\right)^{1/2}.$$

If we denote by P_2 the unique positive real solution of the equation

$$-4P_2^3 + (3a+6b)^2 P_2^2 + 27a^2K = 0,$$

then

$$\frac{9b^2 - P_2}{9a} \leq c \leq b - a.$$

In practice, to avoid particularly large loops on the coefficients a, b, c and d, we will instead employ a slight refinement of this lemma to treat forms with discriminants in the interval $(K_0, K]$ for given positive reals $K_0 < K$. It is easy to check that we have

$$\frac{9b^2 - P_2}{9a} \leq c \leq \min\left\{\frac{3b^2 - (a^2K_0/4)^{1/3}}{3a}, b - a\right\}.$$

To bound d, we note that the definition of reduction implies that

$$\frac{(a+b)c - b^2}{a} \leq d \leq \frac{(a-b)c - b^2}{a}.$$

The further assumption that $K_0 < D_F \leq K$ leads us to a quadratic equation in d which we can solve to determine a second interval for d. Intersecting these intervals provides us with (for values of K that are not too large) a reasonable search space for d.

4.3.2 Forms of negative discriminant

In case of negative discriminant, we require a different notion of reduction, as the Hessian is no longer a definite form. We will instead, following Belabas [6], appeal to an idea of Berwick and Mathews [54]. We take as our definition of a reduced form an alternative characterization due to Belabas (Lemma 4.2 of [6]).

Definition 4.2. *An irreducible binary integral cubic form*

$$F(x,y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

of negative discriminant is called reduced if we have

- $d^2 - a^2 > 3(bd - ac)$,

- $-(a-3b)^2 - 3ac < 3(ad-bc) < (a+3b)^2 + 3ac,$
- $a > 0,\ b \geq 0$ *and* $d > 0$ *whenever* $b = 0.$

Analogous to Proposition 4.1, we have, as a consequence of Lemma 4.3 of [6] :

Proposition 4.2. *Any irreducible cubic form of the shape (4.2.1) with negative discriminant is* $GL_2(\mathbb{Z})$*-equivalent to a unique reduced one.*

To count the number of reduced cubic forms in this case we appeal to Lemma 4.4 of Belabas [6] :

Lemma 4.3.2. *Let* K *be a positive real number and*

$$F(x,y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

be a reduced form whose discriminant lies in $[-K, 0)$*. Then we have*

$$1 \leq a \leq \left(\frac{16K}{27}\right)^{1/4}$$

$$0 \leq b \leq \frac{a}{2} + \frac{1}{3}\left(\sqrt{K/3} - \frac{3a^2}{4}\right)^{1/2}$$

$$1 - 3b \leq 3c \leq \left(\frac{K}{4a}\right)^{1/3} + \begin{cases} 3b^2/a & \text{if } a \geq 2b, \\ 3b - 3a/4 & \text{otherwise.} \end{cases}$$

As in the case of forms of positive discriminant, from a computational viewpoint it is often useful to restrict our attention to forms with discriminant Δ with $-\Delta \in (K_0, K]$ for given $0 < K_0 < K$. Also as previously, the loop over d is specified by the inequalities defining reduced forms and by the definition of discriminant.

It is worth noting here that a somewhat different notion of reduction for cubic forms of negative discriminant is described in Cremona [17], based on classical work of Julia [48]. Under this definition, one encounters rather shorter loops for the coefficient a – it appears that this leads to a slight improvement in the expected complexity of this approach (though the number of tuples (a, b, c, d) considered is still linear in K).

4.3.3 Reducible forms

Suppose finally that F is a reducible cubic form of discriminant $-108k$, as in (4.2.1), for which $F(x_0, y_0) = 1$ for some pair of integers x_0 and y_0. Then, under $SL_2(\mathbb{Z})$-action, F is necessarily equivalent to

$$f(x,y) = x(x^2 + 3Bxy + 3Cy^2), \tag{4.3.1}$$

for certain integers B and C. We thus have

$$D_f = D_F = 27C^2(3B^2 - 4C) \tag{4.3.2}$$

(so that necessarily $BC \equiv 0 \pmod 2$). Almost immediate from (4.3.2), we have

Lemma 4.3.3. *Let $K > 0$ be a real number and suppose that f is a cubic form as in (4.3.1). If we have $0 < D_f \le K$ then*

$$-\left(\frac{K}{108}\right)^{1/3} \le C \le \left(\frac{K}{27}\right)^{1/2}, \quad C \ne 0$$

and

$$\max\left\{0, \left(\frac{4C+1}{3}\right)^{1/2}\right\} \le B \le \left(\frac{K + 108C^3}{81C^2}\right)^{1/2}.$$

If, on the other hand, $-K \le D_f < 0$ then

$$1 \le C \le \left(\frac{K}{27}\right)^{1/2}$$

and

$$\max\left\{0, \left(\frac{-K + 108C^3}{81C^2}\right)^{1/2}\right\} \le B \le \left(\frac{4C-1}{3}\right)^{1/2}.$$

One technical detail that remains for us, in the case of reducible forms, is that of identifying $SL_2(\mathbb{Z})$-equivalent forms. If we have

$$f_1(x,y) = x(x^2 + 3B_1xy + 3C_1y^2) \text{ and } f_2(x,y) = x(x^2 + 3B_2xy + 3C_2y^2),$$

with

$$f_1 \circ \tau(x,y) = f_2(x,y) \text{ and } \tau(1,0) = (1,0), \tag{4.3.3}$$

where $\tau \in SL_2(\mathbb{Z})$, then necessarily $\tau = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, for some integer n, so that

$$B_2 = B_1 + n, \; C_2 = C_1 + 2B_1n + n^2 \text{ and } 3C_1n + 3B_1n^2 + n^3 = 0.$$

Assuming $n \neq 0$, it follows that $f_1(x,1)$ factors completely over $\mathbb{Z}[x]$, whereby, in particular, $C_1 = 3C_0$ for $C_0 \in \mathbb{Z}$ and $B_1^2 - 4C_0$ is a perfect square, say $B_1^2 - 4C_0 = D_0^2$ (where $D_0 \neq 0$ if we assume that $D_{f_1} \neq 0$). We may thus write

$$n = \frac{-3B_1 \pm 3D_0}{2},$$

whence there are precisely three pairs (B_2, C_2) satisfying (4.3.3), namely

$$(B_2, C_2) = (B_1, C_1), \ \left(\frac{-B_1 + 3D_0}{2}, \frac{3}{2} D_0 (D_0 - B_1) \right)$$

and

$$\left(\frac{-B_1 - 3D_0}{2}, \frac{3}{2} D_0 (D_0 + B_1) \right).$$

Let us define a notion of reduction for forms of the shape (4.3.1) as follows :

Definition 4.3. *A irreducible binary integral cubic form*

$$F(x,y) = x(x^2 + 3bxy + 3cy^2)$$

of nonzero discriminant D_F is called reduced if we have either

- *D_F is not the square of an integer, or*
- *D_F is the square of an integer, and b and c are positive.*

From the preceding discussion, it follows that such reduced forms are unique in their $SL_2(\mathbb{Z})$-class. Note that the solutions to the equation

$$F(x,y) = x(x^2 + 3bxy + 3cy^2) = 1$$

are precisely those given by $(x,y) = (1,0)$ and, if $c \mid b$, $(x,y) = (1, -b/c)$.

4.4 Running the algorithm

We implement the algorithm implicit in the preceding sections for finding the integral solutions to equations of the shape (4.1.1) with $|k| \leq K$, for given $K > 0$. The number of cubic Thue equations $F(x,y) = 1$ which we are required to solve is of order K. To handle these equations, we appeal to by now well-known arguments of Tzanakis and de Weger [85] (which, as noted previously, are based upon lower bounds for linear forms in complex logarithms, together with lattice basis reduction); these are implemented in several computer algebra packages, including Magma and Pari (Sage). We used the former despite concerns over its reliance on closed-source code, primarily due to its stability for longer runs. The main computational bottleneck in this approach is typically that of computing the fundamental units in the corresponding cubic fields; for computations with $K = 10^7$, we encountered no difficulties with any of the Thue equations arising (in particular, the fundamental units occurring can be certified without reliance upon the Generalized Riemann Hypothesis). We are unaware of a computational complexity analysis, heuristic or otherwise, of known algorithms for solving Thue equations and hence it is not by any means obvious how timings for this approach should compare to that based upon lower bounds for linear forms in elliptic logarithms, as in [31]. Using a "stock" version of Magma, our approach does have the apparent advantage of not crashing for particular values of k in the range under consideration (i.e with $K = 10^7$).

4.5 Numerical results

The full output for our computation is available at the weblink
`http://www.math.ubc.ca/~bennett/BeGa-data.html`,
which documents the results of a month-long run on a MacBookPro. Realistically, with sufficient perseverance and suitably many machines, one should be able to readily extend the results described here to something like $K = 10^{10}$. In the remainder of this section, we will briefly summarize our data.

4.5.1 Number of solutions

In what follows, we tabulate the number of curves encountered for which equation (4.1.1) has a given number N_k of integer solutions, with $|k| \leq 10^7$. We include results for all values of k, and also those obtained by restricting to 6-th power free k (the two most natural restrictions here are, in our opinion, this one or the restriction to solutions with $\gcd(X, Y) = 1$). In the range under consideration, the maximum number of solutions encountered for positive values of k was 58, corresponding to the case $k = 3470400$. For negative values of k, the largest number of solutions we found was 66, for $k = -9754975$.

Regarding the largest value of N_k known (where, to avoid trivialities, we can, for example, consider only 6-th power free k), Noam Elkies kindly provided the following example, found in October of 2009 :

$$k = 509142596247656696242225 = 5^2 \cdot 7^3 \cdot 11^2 \cdot 19^2 \cdot 149 \cdot 587 \cdot 15541336441.$$

This value of k corresponds to an elliptic curve of rank (at least) 12, with (at least) 125 pairs of integral points (i.e. $N_k \geq 250$), with X-coordinates ranging from -79822305 to 801153865351455.

Table 4.1 Number of Mordell curves with N_k integral points for positive values of k with $0 < k \leq 10^7$

N_k	# of curves	N_k	# of curves	N_k	# of curves
0	8667066	12	3890	32	33
1	108	14	2186	34	18
2	1103303	16	1187	36	28
3	34	17	1	38	17
4	145142	18	589	40	11
5	33	20	347	42	6
6	55518	22	197	44	3
7	8	24	148	46	5
8	13595	26	91	48	2
9	6	28	63	56	1
10	6308	30	55	58	1

Table 4.2 Number of Mordell curves with N_k integral points for negative values of k with $|k| \leq 10^7$

N_k	# of curves	N_k	# of curves	N_k	# of curves
0	9165396	11	4	32	8
1	167	12	1351	34	8
2	729968	14	655	36	2
3	23	16	340	38	1
4	67639	18	238	40	1
5	10	20	160	42	1
6	23531	22	107	44	2
7	3	24	71	46	2
8	7318	26	37	48	1
9	6	28	20	50	2
10	2912	30	15	66	1

Table 4.3 Number of Mordell curves with N_k integral points for positive $k \leq 10^7$ 6-th power free

N_k	# of curves	N_k	# of curves	N_k	# of curves
0	8545578	12	3575	32	29
1	79	14	1998	34	18
2	1067023	16	1055	36	22
3	24	17	1	38	15
4	139090	18	506	40	9
5	10	20	278	42	6
6	51721	22	161	44	1
7	2	24	112	46	5
8	12271	26	76	48	2
9	1	28	58	56	1
10	5756	30	44		

Table 4.4 Number of Mordell curves with N_k integral points for negative k, $|k| \leq 10^7$ 6-th power free

N_k	# of curves	N_k	# of curves	N_k	# of curves
0	9026739	11	1	32	8
1	109	12	1174	34	8
2	705268	14	562	36	2
3	12	16	291	38	1
4	63685	18	197	42	1
5	5	20	138	44	2
6	21883	22	96	46	2
7	3	24	68	48	1
8	6644	26	31	50	1
9	3	28	17	66	1
10	2561	30	13		

4.5.2 Number of solutions by rank

From a result of Gross and Silverman [34], if k is a 6-th power free integer, then the number of integral solutions N_k to equation (4.1.1) is bounded above by a constant $N(r)$ that depends only on the Mordell-Weil rank over $\mathbb{Q}$ of the corresponding elliptic curve. It is easy to show that we have $N(0) = 6$, corresponding to $k = 1$. For larger ranks, we have that $N(1) \geq 12$ (where the only example of a rank 1 curve we know with $N_k = 12$ corresponds to

$k = 100$), $N(2) \geq 26$ (where $N_{225} = 26$), $N(3) \geq 46$ (with $N_{1334025} = 46$), $N(4) \geq 56$ (with $N_{5472225} = 56$), $N(5) \geq 50$ (with $N_{-9257031} = 50$) and $N(6) \geq 66$ (where $N_{-9754975} = 66$). The techniques of Ingram [46] might enable one to prove that indeed one has $N(1) = 12$.

4.5.3 Hall's conjecture and large integral points

Sharp upper bounds for the heights of integer solutions to equation (4.1.1) are intimately connected to the ABC-conjecture of Masser and Oesterle. In this particular context, we have the following conjecture of Marshall Hall :

Conjecture 2. *(Hall) Given $\epsilon > 0$, there exists a positive constant C_ϵ so that, if k is a nonzero integer, then the inequality*

$$|X| < C_\epsilon |k|^{2+\epsilon}$$

holds for all solutions in integers (X, Y) to equation (4.1.1).

The original statement of this conjecture, in [35], actually predicts that a like inequality holds for $\epsilon = 0$. The current thinking is that such a result is unlikely to be true (though it has not been disproved).

We next list all the Mordell curves encountered with Hall Measure $X^{1/2}/|k|$ exceeding 1; in each case we round this measure to the second decimal place. In the range under consideration, we found no new examples to supplement those previously known and recorded in Elkies [26] (note that the case with $k = -852135$ was omitted from this paper due to a transcription error) and in work of Jiménez Calvo, Herranz and Sáez [47].

Table 4.5 Hall's conjecture extrema for $|k| \leq 10^7$

k	X	$X^{1/2}/\|k\|$	k	X	$X^{1/2}/\|k\|$
-1641843	5853886516781223	46.60	1	2	1.41
1090	28187351	4.87	14668	384242766	1.34
17	5234	4.26	14857	390620082	1.33
225	720114	3.77	-2767769	12438517260105	1.27
24	8158	3.76	8569	110781386	1.23
-307	939787	3.16	-5190544	35495694227489	1.15
-207	367806	2.93	-852135	952764389446	1.15
28024	3790689201	2.20	11492	154319269	1.08
117073	65589428378	2.19	618	421351	1.05
4401169	53197086958290	1.66	297	93844	1.03

Our final table lists all the values of k in the range under consideration for which equation (4.1.1) has a sufficiently large solution :

Table 4.6 Solutions to (4.1.1) with $X > 10^{12}$ for $|k| \leq 10^7$

k	N_k	X	k	N_k	X
-1641843	6	5853886516781223	-1923767	2	2434890738626
4401169	6	53197086958290	-2860984	4	2115366915022
-5190544	4	35495694227489	2383593	10	1854521158546
-4090263	2	16544006443618	2381192	10	1852119707102
-4203905	2	15972973971249	-4024909	4	1569699004069
-2767769	4	12438517260105	-9218431	6	1183858050175
7008155	2	9137950007869	5066001	8	1026067837540
-9698283	2	7067107221619	5059001	8	1024245337460
2214289	4	4608439927403	3537071	6	1007988055117

In particular, by way of example, the equation

$$Y^2 = X^3 - 4090263$$

thus has the feature that its smallest (indeed only) solution in positive integers is given by

$$(X, Y) = (16544006443618, 67291628068556097113).$$

Chapter 5

Bombieri method

5.1 Introduction

In 1996, Bombieri, van der Poorten and Vaaller [14] applied Bombieri's method to algebraic number of degree 3, and under a certain assumption, obtained an effective irrationality measure for some set of algebraic numbers. Their method originated in the method introduced in section 2.1.3, using Dyson's lemma in an essential way. They also introduce new technical devices. In this chapter we will closely follow [14] , and explicitly find the constants in section 9 of [14] to solve inequality

$$\left|x^3 + pxy^2 + qy^3\right| \leq k, \tag{5.1.1}$$

under some restrictions on p and q.
The outline of this method is as follows. Let K be a cubic extension of k and v some valuation on k. Identify K with an embedding of K in k_v. Fix an embedding of the Galois closure L of K/k, then determine a faithful projective representation $\sigma \to P_\sigma$ of $G = Gal(L/k)$ into $PGL(2, \mathbb{Q})$. Next step is to define a subset $\Lambda(p) \subseteq \mathbb{P}^1(L)$ by

Definition 5.1.

$$\Lambda(P) = \{\lambda \in \mathbb{P}^1(L) : \sigma^{-1} = P_\sigma \lambda \qquad \textit{for all } \sigma \textit{ in } G\}.$$

For the points λ in $\Lambda(P)$, we will show a new version of the Thue-Siegel principle, by introducing new metric δ_v on $\mathbb{P}^1(\bar{k}_v)$. We see that the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{P}^1(L)$, cannot have two excellent approximations λ_1 and λ_2 in $\Lambda(P)$. for which both $H(\lambda_1)$ and $H(\lambda_2)$ are large. This will be expressed as an inequality in which all the constants are explicitly given. It will be an effective version of Thue-Siegel's principal. We use this to find effective irrationality measure for all generators of the cubic extension K/k. Finally, for the the irreducible polynomial $x^3 + px + q$. we use the roots of f to construct a point λ_1 in $\Lambda(P)$, which is a good approximation for the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. According to the Thue-Siegel's principle we find an effective irrationality measure for the smallest root of f and solve the Thue inequality (5.1.1).

5.2 Preliminiaries

Let k be a number field ,and $d = [K : Q]$. For every place v of K define $d_v = [K_v : Q_v]$. We will use two absolute values $|\ \ |_v$ and $\|\ \ \|_v$; in particular, we have $|x|_v = \|x\|_v^{d_v/d}$. The absolute value $|\ \ |_v$ is normalized as

- if $v|p$ then

$$|p|_v = p^{\frac{-[k_v:Q_v]}{k:Q}},$$

- if $v|\infty$ and v is real then

$$|x|_v = |x|^{1/d},$$

- if $v|\infty$ and v is complex then

$$|x|_v = |x|^{2/d}.$$

Let M_k denote the set of places of K. In view of normalization, we have the product formula

$$\prod_{v \in M_k} |x|_v = 1.$$

We also define the absolute height of an algebraic number $x \in K$ by

$$H(x) = \prod_{v \in M_k} \max(1, |x|_v),$$

so that

$$h(x) = \log H(x) = \sum_{v \in M_k} \log^+ |x|_v,$$

where $\log^+ x = \max(\log(x), 0)$. These absolute values have a unique extension to Ω_v , the completion of an algebraic closure k_v. We extend $|\ \ |_v$ to a norm on finite dimensional vector spaces over Ω_v . If

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{pmatrix}$$

is a column vector in Ω_v^N, we have

$$\|x\|_v = \begin{cases} \max\{\|x_n\|_v : 1 \le n \le N\} & \text{if } v \nmid \infty \\ \left\{\sum_{n=1}^N \|x\|_v^2\right\}^{1/2} & \text{if } v \mid \infty \end{cases} \tag{5.2.1}$$

and $|x|_v = \|x\|_v^{d_v/d}$ in both cases.

We identify elements of $Hom\left(\Omega_v^N, \Omega_v^M\right)$ with $M \times N$ matrices over Ω_v . Then we extend $|\ \ |_v$ to such matrices A by setting

$$|A|_v = sup\left\{|Ax| : x \in \Omega_v^N, |x| \le 1\right\}.$$

Lemma 5.1. *([84]) Let $A = (a_{mn})$ be an $M \times N$ matrix over Ω_v If $v \nmid \infty$ Then*

$$|A|_v = \max\{|a_{mn}| : 1 \le m \le M, 1 \le n \le N\}.$$

If $v \mid \infty$ let A^* denote the complex conjugate transpose of A and

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$$

denote the eigenvalues of the positive semi-definite matrix A^*A , then

$$|A|_v = \lambda_N^{d_v/2d}.$$

Proof. Assume $v \nmid \infty$ and $\vec{x} \in \Omega_v^N$ such that $|\vec{x}| \le 1$, then

$$\begin{aligned} |A\vec{x}|_v &= \max_{1 \le m \le M} \left\{ \left| \sum_{n=1}^N a_{mn} x_n \right|_v \right\} \\ &\le \max_{1 \le m \le M} \left\{ \max_{1 \le n \le N} |a_{mn}|_v \right\}. \end{aligned}$$

By choosing a suitable $\vec{x}$ we have equality in this inequality.

If $v \mid \infty$ then A^*A is a positive definite hermitian matrix. Thus there exists an orthogonal basis $\vec{\xi_1}, \vec{\xi_2}, \cdots, \vec{\xi_n}$ for Ω_v^N such that $\vec{\xi_i}$ is an eigenvector of A^*A with eigenvalues λ_n and $\left|\vec{\xi_i}\right| = 1$ Let $\vec{x} \in \Omega_v^N$. we have

$$\vec{x} = \sum_{i=1}^N \alpha_i \vec{\xi_i}, \quad \alpha_i \in \Omega_v,$$

Then

$$
\begin{aligned}
\|\vec{x}\|_v^2 =& \| \sum_{i=1}^{N} \alpha_i \vec{\xi_i} \|_v^2 \\
=& \sum_{i=1}^{N} \|\alpha_i \vec{\xi_i}\|_v^2 \\
=& \sum_{i=1}^{N} \|\alpha_i\|_v^2.
\end{aligned}
$$

And since the vectors $\vec{\xi_1}, \vec{\xi_2}, \cdots, \vec{\xi_n}$ form an orthogonal basis we have

$$
\begin{aligned}
\|A\vec{x}\|_v^2 =& \vec{x}^* A^* A \vec{x} \\
=& \left(\sum_{i=1}^{N} \alpha_i \vec{\xi_i} \right)^* \left(\sum_{n=1}^{N} \alpha_i A^* A \vec{\xi_i} \right) \\
=& \left(\sum_{i=1}^{N} \alpha_i \vec{\xi_i} \right)^* \left(\sum_{i=1}^{N} \alpha_i \lambda_i \vec{\xi_i} \right) \\
=& \sum_{i=1}^{N} \lambda_i \|\alpha_i\|_v^2 \\
\geq& \lambda_N \sum_{i=1}^{N} \|\alpha_i\|_v^2 \\
=& \lambda_N \|\vec{x}\|_v^2.
\end{aligned}
$$

Hence

$$
|A\vec{x}| \leq \lambda_N^{d_v/2d} \left|\vec{x}\right|_v . \tag{5.2.2}
$$

And by choosing $\vec{x} = \vec{\xi_n}$ we get the equality

$$
\left|A\right|_v = \lambda_N^{d_v/2d}
$$

as desired.

□

Lemma 5.2. *Let A be an $N \times N$ matrix over,Ω_v , rank$(A) = N$, and let $\vec{x} \in \Omega_v^N$, then*

$$
\left|A^{-1}\right|_v^{-1} \left|\vec{x}\right|_v \leq \left|A\vec{x}\right|_v \leq \left|A\right|_v \left|\vec{x}\right|_v \text{ [84]}.
$$

Proof. If $v \mid \infty$ then by (5.2.2) we have

$$\begin{aligned} |A\vec{x}|_v \leq& \lambda_N^{d_v/2d} |\vec{x}|_v \\ =& |A|_v |\vec{x}|_v . \end{aligned}$$

If $v \mid \infty$, let $A = (a_{mn})\,, 1 \leq m, n \leq N$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$, then

$$\begin{aligned} |A\vec{x}|_v &= \max_{1\leq m\leq N} \left\{ |\sum_{n=1}^{N} a_{mn}x_n|_v \right\} \\ &\leq \max_{1\leq m\leq N} \left\{ \max_{1\leq n\leq N} \{|a_{mn}|_v|\vec{x}|_v\} \right\} \\ &\leq |A|_v |\vec{x}|_v . \end{aligned}$$

This completes the proof of the right hand side inequality; To prove the other inequality for any place v we have

$$\begin{aligned} |\vec{x}|_v =& \left|A^{-1}A\vec{x}\right|_v \\ \leq& \left|A^{-1}\right|_v |A\vec{x}|_v . \end{aligned}$$

And so we have

$$\left|A^{-1}\right|_v^{-1} |\vec{x}|_v \leq |A\vec{x}|_v .$$

□

The following corollary is the immediate conclusion of this lemma:

Corollary 5.1. *Let A, B be $N \times N$ nonsingular matrices over Ω_v . Then*

$$|AB|_v \leq |A|_v |B|_v .$$

Define the map

$$\eta_v : GL(N, \Omega_v) \to [1, \infty)$$

by $\eta_v(A) = |A|_v|A^{-1}|_v$. Since $\eta_v(aA) = \eta_v(A)$ for all $a \neq 0$ in Ω_v , η_v is well defined as a map

$$\eta_v : PGL(N, \Omega_v) \to [1, \infty).$$

Moreover let $A \in GL(N, \Omega_v)$ for any $\vec{x} \in \Omega_v^N$ by corollary 5.1 we have

$$\begin{aligned}
|\vec{x}| &= \left|AA^{-1}\vec{x}\right|_v \\
&\leq |A|_v \left|A^{-1}\vec{x}\right|_v \\
&\leq |A|_v \left|A^{-1}\right|_v |\vec{x}|_v .
\end{aligned}$$

By dividing both sides by $|\vec{x}|_v$ we can conclude that $|A|_v \left|A^{-1}\right|_v \geq 1$ and so we have $\eta_v(A) \geq 1$ for all $A \in GL(N, \Omega_v)$.
In the case of $N = 2$ we have a closed simple formula for $\eta_v(A)$.

Lemma 5.3.

$$\eta_v(A) = |detA|_v^{-1} |A|_v^2 \, [84].$$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then $A^{-1} = \frac{1}{detA} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ Therefore

$$\begin{aligned}
\left|A^{-1}\right|_v &= \left|\frac{1}{detA} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right|_v \\
&= \frac{1}{|detA|_v} |A|_v .
\end{aligned}$$

So

$$\begin{aligned}
\eta_v(A) &= |A|_v \left|A^{-1}\right|_v \\
&= |A|_v \frac{|A|_v}{|detA|_v} \\
&= \frac{|A|_v^2}{|detA|_v}.
\end{aligned}$$

□

Let $e_1, e_2, \cdots e_N$ be the standard basis vectors in Ω_v^n. For each subset $I \subseteq \{1, 2, \cdots, N\}$, let e_I be the corresponding standard basis vector in the exterior algebra $\wedge(\Omega_v^N)$. This is the graded algebra

$$\wedge(\Omega_v^N) = \sum_{n=0}^{\infty} \wedge_n(\Omega_v^N)$$

in which each subspace $\wedge_n(\Omega_v^N)$ has dimension $\binom{N}{n}$ and basis vector $\{e_I : |I| = n\}$. By applying (5.2.1) to the basis $\{e_I : I \subseteq \{1, 2, \cdots N\}$ we can extend $|\ \ |$ to $\wedge(\Omega_v^N)$. We define a second map

$$\delta_v : \Omega_v^N \times \Omega_v^N \to [0, 1]$$

by

$$\delta_v(x,y) = \frac{|x \wedge y|_v}{|x|_v\,|y|_v}.$$

Since $\delta_v(ax,by) = \delta_v(x,y)$ for all $a \neq 0$ and $b \neq 0$ in Ω_v, it follows that δ_v is well defined as map

$$\delta_v : \mathbb{P}^{N-1} \times \mathbb{P}^{N-1} \to [0,1]$$

One can show that δ_v is metric on $\mathbb{P}^{N-1}(\Omega_v)$[71].

Lemma 5.4. *In the case $N = 2$ the maps η_v and σ_v are related to each other by inequality*

$$\eta_v(A)^{-1}\sigma_v(x,y) \leq \sigma_v(Ax,Ay) \leq \eta_v(A)\sigma_v(x,y) \tag{5.2.3}$$

for all A in $PGL(2,\Omega_v)$ and x, y in $\mathbb{P}^1(\Omega_v)$.

Proof. Note that

$$\begin{aligned}
\sigma_v(Ax,Ay) =& |Ax|_v^{-1}\,|Ay|_v^{-1}\,|detA|_v\,|x \wedge y|_v \\
\geq& |A|_v^{-2}\,|detA|_v\,|x|_v^{-1}\,|y|^{-1}\,|x \wedge y|_v \quad \text{by corolarry 5.1} \\
=& \eta_v(A)^{-1}\sigma_v(x,y) \quad \text{by lemma 5.3.}
\end{aligned}$$

This proves the left hand side inequality. The right hand side follows, since $\eta_v(A^{-1}) = \eta_v(A)$. □

Remark 5.1. *Let $\alpha = \begin{bmatrix}\alpha \\ 1\end{bmatrix}$ in $\mathbb{P}^1(k)$, $\beta = \begin{bmatrix}\beta \\ 1\end{bmatrix}$ in $\mathbb{P}^1(k)$ Since $\delta_v(a,b) \leq |\alpha - \beta|_v$. To find the lower bound for $|\alpha - \beta|_v$ we can actually find a lower bound for $\delta_v(a,b)$.*

If β occurs in k^n , We define its heights by

$$H(\beta) = \prod_v |\beta|_v\,.$$

Next we will prove Liouville's lower bound for the projective metric. Suppose that $\alpha = \begin{bmatrix}\alpha \\ 1\end{bmatrix}$ in homogenous coordinates with α algebraic over k of degree $r \geq 2$. As $K = k(\alpha)$ is embedded in k_v, there exists a place $\hat{w}$ of K with $\hat{w} \mid v$, $[K_{\hat{w}} : k_v] = 1$, and $|\alpha|_{\hat{w}}^r = |\alpha|_v$. It follows that

$$\delta_v(\alpha,\beta) = \delta_{\hat{w}}(\alpha,\beta)^r \geq \left\{\prod_w \delta_w(\alpha,\beta)\right\}^r,$$

Since $\beta \neq \alpha$, we obtain

$$\begin{aligned} \delta_v(\alpha, \beta) &\geq \left\{ \prod_w |\alpha|_w^{-1} \, |\beta|_w^{-1} \, |\alpha \wedge \beta|_w \right\}^r \\ &= \{H(\alpha)H(\beta)\}^{-r} \end{aligned} \tag{5.2.4}$$

and this is the Liouville's lower bound for the projective metric. If $r = 2$, then with $N = 2$ this result is sharp [15] .

5.3 The Thue-Siegel inequality

Let $\alpha \in k_v$ be an algebraic number of degree 3 over k, so that $K = k(\alpha) \subseteq k_v$. α', α'' are conjugates of α in $\bar{k}_v$ and $L = K(\alpha, \alpha', \alpha'') \subseteq k_v$ denotes the Galois closure of the extension K/k. We denote the points $\alpha, \alpha', \alpha''$ by the corresponding points in $\mathbb{P}^1(L)$ as

$$\alpha = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}, \quad \alpha' = \begin{bmatrix} \alpha' \\ 1 \end{bmatrix}, \quad \alpha'' = \begin{bmatrix} \alpha'' \\ 1 \end{bmatrix}.$$

For each σ in $G = Gal(L/k)$ there exists a unique element Q_σ in $PGL(2, L)$ such that

$$\sigma(\alpha) = Q_\sigma \alpha, \quad \sigma(\alpha') = Q_\sigma \alpha', \quad \sigma(\alpha'') = Q_\sigma \alpha''.$$

Hence we can obtain a faithful projective representation of G in $PGL(2, L)$: $\sigma \to Q_\sigma$. We will define another representation of G in $PGL(2, \mathbb{Q})$. Let Φ be the unique element in $PGL(2, L)$ such that

$$\Phi\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Phi\alpha' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Phi\alpha'' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.3.1}$$

Then the conjugate representation $\sigma' \to \Phi Q_\sigma \Phi^{-1}$ is a faithful projective representation of G in $PGL(2, \mathbb{Q})$, which permutes the elements of the set $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Remark 5.2. *If G is noncyclic then*

$$p_\sigma \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

and if G is cyclic then

$$p_\sigma \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

In the next lemma we will find the set $\Lambda(p_\sigma)$ of definition 5.1 for this representation .

Lemma 5.5. *Let λ belong to $\mathbb{P}^1(L)$. Then the identity*

$$\sigma^{-1}(\lambda) = P_\sigma(\lambda)$$

holds for all σ in G if and only if $\Phi^{-1}\lambda$ occurs in $\mathbb{P}^1(k)$ [14].

Proof. For any σ in G we have

$$\begin{aligned}\sigma^{-1}(\Phi)\alpha &= \sigma^{-1}(\Phi)\sigma^{-1}(\sigma(\alpha)) = \sigma^{-1}\left\{\Phi\sigma(\alpha)\right\} \\ &= \sigma^{-1}\left\{\Phi Q_\sigma \Phi^{-1}\Phi\alpha\right\} = \sigma^{-1}\left\{P_\sigma\Phi\alpha\right\}.\end{aligned}$$

Since $P_\sigma\Phi\alpha$ occurs in the set $\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$, it is fixed by σ^{-1} and therefore $\sigma^{-1}(\Phi)\alpha = P_\sigma\Phi\alpha$. By similar argument we have $\sigma^{-1}(\Phi)\alpha' = P_\sigma\Phi\alpha'$ and $\sigma^{-1}(\Phi)\alpha'' = P_\sigma\Phi\alpha''$. It means that $\sigma^{-1}(\Phi) = P_\sigma\Phi$. as elements of $PGL(2, L)$ for each σ in G. Assume $\sigma^{-1}(\lambda) = P_\sigma\lambda$ for all σ in G; then we have

$$\sigma^{-1}(\Phi^{-1}\lambda) = (\sigma^{-1}(\Phi))^{-1}\sigma^{-1}(\lambda) = (P_\sigma\Phi)^{-1}P_\sigma\lambda = \Phi^{-1}\lambda.$$

Since $\Phi^{-1}\lambda$ is fixed by all elements of $Gal(L/k)$, it belongs to $\mathbb{P}^1(k)$ and $\lambda \in \Phi\mathbb{P}^1(k)$. Conversely assume $\Phi^{-1}\lambda$ occurs in $\mathbb{P}^1(k)$; then $\lambda = \Phi\beta$ for some β in $\mathbb{P}^1(k)$. Then we have

$$\sigma^{-1}(\lambda) = \sigma^{-1}(\Phi)\beta = P_\sigma\Phi\beta = P_\sigma\lambda,$$

for all σ in G. This completes the proof and it means that

$$\Lambda(P) = \{\lambda \in \mathbb{P}^1(L) : \sigma^{-1} = P_\sigma\lambda \quad \text{for all } \sigma \text{ in } G\} = \left\{\Phi\beta : \beta \in \mathbb{P}^1(k)\right\}.$$

□

The only points in $\mathbb{P}^1(k)$ fixed by each element of P_σ are $\begin{bmatrix}\zeta_6\\1\end{bmatrix}$ and $\begin{bmatrix}\zeta_6^5\\1\end{bmatrix}$ where ζ_6 is primitive sixth root of unity. These are fixed points only if G is cyclic therefore $\Lambda(P)$ does not intersect $\mathbb{P}^1(\mathbb{Q})$ [14]. The main problem is giving an effective lower bound for $\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, \lambda\right)$. For this matter, we define exponent approximation $e_v(\lambda)$ for each Λ in $\lambda(P)$ by

$$e_v(\lambda) = \frac{-\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, \lambda\right)}{\log\left\{8H(\lambda)\right\}},$$

so that

$$\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda\right) = 8H(\lambda)^{-e_v(\lambda)}.$$

We also set

$$\eta_w(P) = \max\{\eta_w(P_\sigma) : \sigma \in G\}$$

for each place w of L and

$$\eta(P) = \prod_w \eta_w(P).$$

By remark 5.2 we can see that $\eta(P) = \frac{1}{2}\left(3 + 5^{1/2}\right)$. If $[L : k] = 6$ and α', α'' occurs in k_v; then by Liouville's bound for projective metric we have $e_v(\lambda) < 6$. Moreover in [14] it has been shown that

Theorem 5.1. *For each point λ in $\Lambda(P)$ we have $0 \leq e_v(\lambda) < 3$.*

For the points λ_1 and λ_2 in $\Lambda(P)$ define

$$r(\lambda_1,\lambda_2) = \min\left\{\frac{\log\{8H(\lambda_1)\}}{\log\{8H(\lambda_2)\}}, \frac{\log\{8H(\lambda_2)\}}{\log\{8H(\lambda_1)\}}\right\}$$

and

$$s(\lambda_1,\lambda_2) = \left(\log\{8H(\lambda_1)\}\right)^{-1} + \left(\log\{8H(\lambda_2)\}\right)^{-1}.$$

By this definition, following is the Thue-Siegel principle for cubic extension

Theorem 5.2. *If λ_1 and λ_2 are points in $\Lambda(P)$ then*

$$e_v(\lambda_1)e_v(\lambda_2) \leq 6 + 19s(\lambda_1,\lambda_2)^{1/3} + 24r(\lambda_1,\lambda_2)^{1/2}[14].$$

So in the case that both $r(\lambda_1,\lambda_2)$ and $s(\lambda_1,\lambda_2)$ are small , we will get a much better inequality for $e_v(\lambda_1)e_v(\lambda_2)$ than theorem 5.1. Assuming the Thue-Siegel inequality (theorem5.2), we can find an effective irrationality measure for generators of the cubic extension . Let

$$\mu_v^*(\lambda_1) = e_v(\lambda_1)^{-1}\left\{6 + 19\left(\log\{8H(\lambda_1)\}\right)^{-1/3}\right\}. \tag{5.3.2}$$

Assume we can find a point λ_1 in $\Lambda(P)$ such that $\mu_v^*(\lambda_1)$ is less than 3 . Then for all generators α of the cubic extension $K = k(\alpha) \subseteq k_v$,we have the following theorem [14].

Theorem 5.3. *Assume that* λ_1 *in* $\Lambda(P)$ *satisfies* $\mu_v^*(\lambda) < 3$ *. Then*

$$-\log \sigma_v(\alpha, \beta) \leq \mu_v^*(\lambda_1) \log\{8H(\beta)\} + O\left\{(\log 8H(\beta))^{1/2}\right\}$$

for all β *in* $\mathbb{P}^1(k)$ *. The implied constant depends on* λ_1 *and* α *.*

Proof. First note that if $e_v(\lambda_1) < 2$ since $e_v(\lambda_2) < 3$ theorems 5.2 and 5.3 follow immediately. So we assume $2 \leq e_V(\lambda) \leq 3$. By mean value theorem we have

$$\left(x^{-1} + y^{-1}\right)^{1/3} - x^{-1/3} \leq \frac{1}{3} x^{2/3} y^{-1}, \tag{5.3.3}$$

where x and y are positive real numbers. Let $x = \log\{8H(\lambda_1)\}$ and $y = \log\{8H(\lambda_2)\}$ where λ_2 belongs to $\Lambda(P)$. Therefore from the above inequality, since $2 \leq e_v(\lambda_1)$, we have:

$$\begin{aligned} e_v(\lambda_1)^{-1}\left\{6 + 19s(\lambda_1, \lambda_2)^{1/3}\right\} -\mu_v^*(\lambda_1) \leq \\ 4\left(\log\{8H(\lambda_1)\}\right)^{2/3}\left(\log\{8H(\lambda_2)\}\right)^{-1} \end{aligned} \tag{5.3.4}$$

and from theorem 5.2 we obtain that:

$$\begin{aligned} e_v(\lambda_2) \leq& \mu_v^*(\lambda_1) + 4\left(\log\{8H(\lambda_1)\}\right)^{2/3}\left(\log\{8H(\lambda_2)\}\right)^{-1} \\ &+ 12\left(\log\{8H(\lambda_1)\}\right)^{1/2}\left(\log\{8H(\lambda_2)\}\right)^{-1/2}. \end{aligned} \tag{5.3.5}$$

Now let Φ be an element of $PGL(2, L)$ as defined in 5.3.1 . From lemma 5.3 we have

$$\eta(\Phi) = \prod_w \eta_w(\Phi) = \prod_w det(\phi)^{-1}|\Phi|_w^2 = \prod_w |\Phi|_w^2.$$

Where the last equation follows from product rule. So for β in $\mathbb{P}^1(k)$ we have

$$H(\Phi\beta) \leq \prod_w |\Phi|_w |\beta|_w = \eta(\Phi)^{1/2} H(\beta). \tag{5.3.6}$$

From inequality (5.2.3) we conclude that

$$\begin{aligned}\sigma_v(\alpha,\beta) =&\sigma_v\left(\Phi^{-1}\begin{bmatrix}0\\1\end{bmatrix},\Phi^{-1}\Phi\beta\right)\\ &\geq \eta_v(\Phi)^{-1}\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\Phi\beta\right)\\ &\geq \eta(\Phi)^{-1}\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\Phi\beta\right).\end{aligned} \tag{5.3.7}$$

By lemma 5.5 $\Phi\beta$ occurs in $\Lambda(P)$ so the inequalities (5.3.5) and (5.3.7) can be combined to let us conclude that

$$\begin{aligned}-\log\sigma_v(\alpha,\beta) \leq&\mu_v^*\log\{8H(\Phi\beta)\}+(\log\{8H(\lambda_1)\})^{1/2}(\log\{8H(\Phi\beta)\})^{1/2}\\ &4(\log\{8H(\lambda_1)\})^{2/3}+\log\eta(\Phi).\end{aligned} \tag{5.3.8}$$

Finally using the upper bound (5.3.6) for $H(\Phi\beta)$ completes the proof.

□

5.4 Dyson's lemma, proof of Thue-Siegel's principle

In this section we will will closely follow [14] to show how we can construct the auxiliary polynomials using a projective version of Dyson's lemma and how we can use it to prove Thue-Siegel' s principle (Theorem 5.2).

5.4.1 Preliminaries

Assume that M and N are nonnegative integers, w is a place of the number field L and Ω_w is the completion of an algebraic closure $\bar{L}_w$. The vector space $E_w = \Omega_w^{(M+1)(N+1)}$ can be identified with bihomogeneous polynomials

$$F(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} f_{m,n}x_0^m x_1^{M-m} y_0^n y_1^{N-n}.$$

We define two norms on E_w. The first one is

$$|F|_w = sup\left\{|F(x,y)|_w : x\in\Omega_w^2, y\in\Omega_w^2, |x|_w\leq 1 \text{ and } |y|_w\leq 1\right\}.$$

We define a representation $(A,B) \to \rho_{(A,B)}$ of $GL(2,\Omega_w) \times GL(2,\Omega_w)$ in $GL(E_w)$ by

$$(\rho_{(A,B)}F)(x,y) = F(A^{-1}x, B^{-1}y),$$

so that $\rho_{(A,B)} o \rho_{(C,D)} = \rho_{(AC,BD)}$. To define the second norm, first we set

$$\epsilon_w = \begin{cases} 0 & \text{if } w \nmid \infty \\ [L_w : \mathbb{Q}_w]\,[L : \mathbb{Q}]^{-1} & \text{if } w \mid \infty. \end{cases} \tag{5.4.1}$$

Definition 5.2. *If $w \nmid \infty$, we define*

$$[F]_w = \max\left\{|f_{m,n}|_w : 0 \le m \le M \ \text{and}\ 0 \le n \le N\right\},$$

and if $w \mid \infty$ we define

$$[F]_w = \left\{\sum_{m=0}^{M}\sum_{n=0}^{N}\binom{M}{m}^{-1}\binom{N}{n}^{-1}||f_{m,n}||_w^2\right\}^{\epsilon_w/2}.$$

Suppose $N = 0$; then F is simply a homogenous polynomial in $\omega_w[x]$ so we can write

$$F(x) = \sum_{m=0}^{M} f_m x_0^m x_1^{M-m}. \tag{5.4.2}$$

If F is not identically zero then there exist nonzero vectors $\xi_1, \xi_2, \cdots, \xi_M$ in Ω_w^2 such that,

$$F(x) = \prod_{m=1}^{M}(x \wedge \xi_m). \tag{5.4.3}$$

The next lemma relates both of the norms we defined for this case:

Lemma 5.6. *Let $F(x)$ in E_w be given by* (5.4.2) *and* (5.4.3) *. If $w \mid \infty$ then*

$$\log|f|_w = \log[f]_w = \sum_{m=1}^{M}\log|\xi_m|_w \tag{5.4.4}$$

and if $w \mid \infty$ then

$$\begin{aligned} \log|F|_w \le \log[F]_w \le & \sum_{m=1}^{M}\log|\xi_m|_w \\ & \le \frac{\epsilon_w}{2}\{M - \log(M+1)\} + \log[f]_w \\ & \le \frac{\epsilon_w}{2}M + \log|F|_w[14]. \end{aligned} \tag{5.4.5}$$

We also define

$$\begin{aligned}\Gamma =&\Gamma(M,N,\theta,\tau)\\ =&\left\{(m,n)\in\mathbb{Z}^2 : 0\le m\le M, 0\le n\le N, \text{ and } 1\le \frac{m}{\theta M}+\frac{n}{\tau N}\right\}.\end{aligned}$$

Also set

$$\mathcal{Y}_w=\mathcal{Y}_w(\theta,\tau)\subseteq E_w$$

to be the set of polynomials $F(x,y)$ in E_w such that

$$f_{m,n}=0 \text{ whenever } \frac{m}{\theta M}+\frac{n}{\tau N}<1.$$

Let F in E_w and $(A,B)\in GL(2,\Omega_w)\times GL(2,\Omega_w)$. Then $\rho_{(A,B)}$ acts on both F and the vector of coefficients $(m,n)\to f_{m,n}$

$$\begin{aligned}(\rho_{(A,B)}F))(x,y) =&F(A^{-1}x,B^{-1}y),\\ &\sum_{m=0}^{M}\sum_{n=0}^{N}(\rho_{(A,B)}f)_{m,n}x_0^m x_1^{M-m}y_0^n y_1^{N-n},\end{aligned}$$

So that $(m,n)\to(\rho_{(A,B)}f)_{m,n}$ is the vector of the coefficients of $\rho_{(A,B)}F$. Then we define

$$\text{Index}(F,A,B,\theta M,\tau N)=\min\left\{\frac{m}{\theta M}+\frac{n}{\tau N}:(\rho_{(A,B)}F)_{m,n}\neq 0\right\}.$$

If F is identically zero we set the index equal to ∞

The index function has the following simple properties:

Proposition 5.1. *1. F occurs in $\mathcal{Y}_w$ if and only if*

$$\textit{Index }(F,I_2,I_2,\theta M,\tau N)\ge 1$$

Where I_2 is the identity matrix, $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

2. For $\kappa>0$ we have

$$\kappa^{-1}\textit{ Index }(F;A,B,\theta M,\tau N)=\textit{Index }(F;A,B,\kappa\theta M,\kappa\tau N).$$

3. For the representation ρ

$$\textit{Index}(\rho_{(C,D)}F;A,B,\theta M,\tau N)=\textit{Index }(F;AC,BD,\theta M,\tau N).$$

4. For nonzero elements a and b in Ω_w we have

$$\textit{Index}\ (F; aA, bB, \kappa\theta M, \kappa\tau N) = \textit{Index}\ (F; A, B, \kappa\theta M, \kappa\tau N).$$

From the last property, we can see that the map

$$(A, B) \to \text{Index}\ (F; A, B, \kappa\theta M, \kappa\tau N)$$

is well defined for (A, B) in $PGL(2, \Omega_w) \times PGL(2, \Omega_w)$. The next theorem gives a local estimate for the norm of F [14].

Theorem 5.4. *Let F be a polynomial in E_w , $x \in \mathbb{P}^1(\Omega_w)$ and $y \in \mathbb{P}^1(\Omega_w)$. Assume that*

$$1 \leq \textit{Index}\ (F; A, B, \theta M, \tau N)$$

for some pair (A, B) in $PGL(2, \Omega_w) \times PGL(2, \Omega_w)$; then we have

$$|x_w|^{-M} |y|_w^{-N} |F(x, y)|_w \leq \left\{(M+1)(N+1)2^{M+N}\right\}^{\epsilon/2} \eta_w(A)^m \eta_w(B)^n$$
$$|F|_w \max \left\{ \sigma_w \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, x \right)^{\theta M}, \sigma_w \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, y \right)^{\tau N} \right\}.$$

Now we try to find a global estimate for the norm. As previously, we assume L/k is a Galois extension of an algebraic number field, with $G = Gal(L/k)$ and $\sigma \to \psi_\sigma$ is a faithful projective representation of G in $PGL(2, k)$ at each place v of k. As previously, we assume L/k is a Galois extension of an algebraic number field, with $G = Gal(L/k)$ and $\sigma \to \psi_\sigma$ is a faithful projective representation of G in $PGL(2, k)$. At each place v of k we define

$$\eta_v(\psi) = \max \{\eta_v(\psi_\sigma) : \sigma \in G\}$$

And with the normalization, we define

$$\eta_v(\psi) = \prod_{w|v} \left(\max \{\eta_w(\psi_\sigma) : \sigma \in G\}\right).$$

We also define

$$\eta(\psi) = \prod_v \eta_v(\psi).$$

Since $\eta_v(\psi) = 1$ for almost all places of k, it is therefore well defined. Next we define

$$\Lambda(\psi) = \left\{\lambda \in \mathbb{P}^1(L) : \sigma^{-1}(\lambda) = \psi_\sigma \lambda \quad \text{for all } \sigma \text{ in } G\right\}.$$

For any v fixed place of k, the group G acts on places w of L such that $w \mid v$. If σ is in G then σw is the place of L determined by

$$|x|_{\sigma w} = \left|\sigma^{-1}(x)\right|_w \quad \text{for all } x \text{ in } L$$

for all σ in G.
Let $E = L^{(M+1)(N+1)}$ denote the L-vector space of bihomgenous polynomials

$$F(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} f_{m,n} x_0^m x_1^{M-m} y_0^n y_1^{N-n}$$

in $L(x,y)$ having bidegree (M,N).

We can apply theorem 5.4 at each place w of L. Taking the product to all of the places, we can obtain a global estimate for the norm

Theorem 5.5. *Let F be a polynomial in E and let λ_1 and λ_2 be points in $\Lambda(\psi)$. Assume that $F(\lambda_1,\lambda_2) \neq 0$ and that*

$$1 \leq \textit{Index}\ (F; \psi_\sigma, \psi_\sigma, \theta M, \tau N)$$

for all σ in G . Then for any place v of k and any embedding of L into $\bar{K}_v$ we have

$$\begin{aligned}
0 \leq\ & M \log\left\{2^{1/2}\eta(\psi)H(\lambda_1)\right\} + N\log\left\{2^{1/2}\eta(\psi)H(\lambda_2)\right\} \\
&+ \frac{1}{2}\log\left\{(M+1)(N+1)\right\} + \sum_w \log|F|_w \\
&+ \max\left\{\theta M \log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, x\right)^{\theta M}, \tau N\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, y\right)^{\tau N}\right\} [14].
\end{aligned}$$

5.4.2 Dyson's lemma

Assume Ω is an algebraically closed file of characteristic zero and, like in the previous sections, $F(x,y)$ is a bihomogeneous polynomial of bidegree (M,N) with coefficients in Ω. First we try to generalize the Index definition for points in $\mathbb{P}^1(\Omega)\times\mathbb{P}^1(\Omega)$ .We will define stabilizer of $\begin{bmatrix}0\\1\end{bmatrix}$ in $PGL(2,\Omega)$

$$\begin{aligned}
\text{stab}\ \left\{\begin{bmatrix}0\\1\end{bmatrix}\right\} &= \left\{U \in PGL(2,\Omega) : U\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}\right\} \\
&\left\{\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix} \in PGL(2,\Omega) : u_{12} = 0\right\}.
\end{aligned}$$

For any elements U and V in stab $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ we have

$$\text{Index }(F;A,B,\theta M,\tau N)=\text{Index }(F;UA,VB,\theta M,\tau N)$$

so the map

$$(A,B)\to\text{Index }(F;A,B,\theta M,\tau N)$$

is constant on all right cosets of stab $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}\times$ stab $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$. For the point (α,β) in $\mathbb{P}^1(\Omega)\times\mathbb{P}^1(\Omega)$ we select (A,B) so that $A^{-1}\begin{bmatrix}0\\1\end{bmatrix}=\alpha$ and $B^{-1}\begin{bmatrix}0\\1\end{bmatrix}=\beta$ then we set

$$\text{Index }(F;\alpha,\beta,\theta M,\tau N)=\text{Index }(F;A,B,\theta M,\tau N).$$

According to the previous argument, it is well defined. We now describe the projective form of Dyson's lemma introduced in the section 2.1.3 . For this let $\alpha_1,\alpha_2,\cdots,\alpha_J$ be distinct points in $\mathbb{P}^1(\Omega)$ and let $\beta_1,\beta_2,\cdots,\beta_j$ be distinct points in $\mathbb{P}^1(\Omega)$ and also $0<\theta_j<1, 0<\tau_j<1$ for $j=1,2,\cdots,J$ The projective formulation of lemma 2.5 will be as following [14] :

Lemma 5.7. *Let F be a bihomogeneous polynomial of bidegree (M,N) in $\Omega[x,y]$ which is not identically zero. If*

$$\textit{Index }(F;\alpha_j,\beta_j,\theta_j M,\tau_j N)\geq 1$$

for each $j=1,2,\cdots,J$ then

$$\frac{1}{2}\sum_{j=1}^{J}\theta_j\tau_j\leq 1+\left(\frac{J-2}{2}\right)\min\left\{\frac{M}{N},\frac{N}{M}\right\}.$$

Corollary 5.2. *Let $F\neq 0$ be a bihomgeneous polynomial of bidegree (M,N) in $\Omega[x,y]$ then*

$$\frac{1}{2}\sum_{j=1}^{J}\left\{\frac{\tau_j}{\theta_j},\frac{\theta_j}{\tau_j},\theta_j\tau_j\left(\textit{Index }(F;\alpha_j,\beta_j,\theta_j M,\tau_j N)\right)^2\right\}$$
$$1+\left(\frac{J-2}{2}\right)\min\left\{\frac{M}{N},\frac{N}{M}\right\}[14].$$

Proof. Let $\kappa_j, j = 1, 2 \cdots, J$ be chosen such that

$$0 < \kappa_j < \min \left\{ \theta_j^{-1}, \tau_j^{-1}, \text{Index } (F; \alpha_j, \beta_j, \theta_j M, \tau_j N) \right\}$$

Therefore we have $0 < \kappa_j \theta_j < 1, 0 < \kappa_j \tau_j < 1$ and

$$1 \le \text{Index } (F; \alpha_j, \beta_j, \theta_j M, \tau_j N) = \text{Index } (F; \alpha_j, \beta_j, \kappa_j \theta_j M, \kappa_j \tau_j N).$$

From Dyson's lemma

$$\frac{1}{2} \sum_{j=1}^{J} \theta_j \tau_j \kappa_j^2 \le 1 + \binom{J-2}{2} \min \left\{ \frac{M}{N}, \frac{N}{M} \right\}$$

By taking supremum on the left hand side over all values of K and considering the way we choose κ, the corollary follows. □

To apply Dyson's lemma, it is important to know what happens to the index when we apply a differential operator to F. For any bihomogeneous polynomial of bidegree (R, S) in $\Omega[x, y]$ there exists a corresponding linear partial differential operator

$$T(D) = \sum_{r=1}^{R} \sum_{s=1}^{S} t_{r,s} \frac{1}{r!} \left(\frac{\partial}{\partial x_0} \right)^r \frac{1}{(R-r)!} \left(\frac{\partial}{\partial x_1} \right)^{R-r} \frac{1}{s!} \left(\frac{\partial}{\partial y_0} \right)^s \frac{1}{(S-s)!} \left(\frac{\partial}{\partial y_1} \right)^{S-s}. \tag{5.4.6}$$

We have

$$\begin{aligned} \text{Index } (T(D)F; &\alpha, \beta, \theta M, \tau N) \ge \text{Index } (F; \alpha, \beta, \theta M, \tau N) \\ &+ \text{Index } (T; I_2\alpha, I_2\beta, \theta M, \tau N) - \left(\frac{R}{\theta M} + \frac{S}{\tau N} \right), \end{aligned} \tag{5.4.7}$$

where $I_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let $A^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \alpha$ and $B^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \beta$. We call the pair of integers (R, S) critical with respect to the index at (α, β) if

$$\begin{aligned} \text{Index } (F; \alpha, \beta, \theta M, \tau N) &= \min \left\{ \frac{m}{\theta M} + \frac{n}{\tau N} : (\rho_{(A,B)} f)_{m,n} \ne 0 \right\} \\ &= \frac{R}{\theta M} + \frac{S}{\tau N}. \end{aligned} \tag{5.4.8}$$

Remark 5.3. *Assume that (R, S) is critical with respect to (α, β), then by 5.4.7 and 5.4.8 we have*

$$\textit{Index}\ (T(D)F; \alpha, \beta, \theta M, \tau N) \Longleftrightarrow \textit{Index}\ (T; I_2\alpha, I_2\beta, \theta M, \tau N).$$

And

$$\{T(D)F\}(\alpha, \beta) \neq 0 \Longleftrightarrow T(I_2\alpha, I_2\beta) \neq 0$$

5.4.3 The auxiliary polynomials

During this section we assume that $0 < \theta < 1, 0 < \tau < 1$ and also that $\theta\tau < \frac{2}{3}$. We define

$$\Gamma' = \Gamma'(M, N, \theta, \tau)$$
$$\left\{(m, n)\mathbb{Z}^2 : 0 \leq m \leq M, 0 \leq n \leq N, \text{ and } 1 \leq \frac{m}{\theta M} + \frac{n}{\tau N} \leq \frac{1}{\theta} + \frac{1}{\tau} - 1\right\}$$

and

$$\Gamma'' = \Gamma''(M, N, \theta, \tau)$$
$$\left\{(m, n)\mathbb{Z}^2 : 0 \leq m \leq M, 0 \leq n \leq N, \text{ and } \frac{m}{\theta M} + \frac{n}{\tau N} < 1\right\}.$$

Let $M \to \infty$ $N \to \infty$ in such a way that

$$0 < \lim M/N < \infty \tag{5.4.9}$$

then

$$\lim M^{-1}N^{-1}\left|\Gamma'\right| = 1 - \theta\tau$$
$$\lim M^{-1}N^{-1}\left|\Gamma''\right| = \frac{1}{2}\theta\tau.$$

Therefore

$$\lim M^{-1}N^{-1}\left(\left|\Gamma'\right| - \left|\Gamma''\right|\right) = 1 - \frac{3}{2}\theta\tau.$$

So since $\theta\tau < 2/3$ we have particularly $|\Gamma'| - |\Gamma''| > 0$.

Now we plan to construct an auxiliary polynomial $F(x, y)$ such that F has a large index (bigger than 1) with respect to the points $p_1 = \left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$, $p_2 = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$ and $p_3 = \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$, and $h(P)$ is not too large . By definition of Γ', it is necessary for the coefficients $f_{m,n}$ of F to be supported on Γ' so that F has an index bigger than 1 in p_1 and p_2. Moreover F has an index bigger than 1 in P_3 if

$$\sum_{(m,n)\in\Gamma'} \binom{m}{i}\binom{n}{j} f_{m,n} = 0$$

for all pairs (i,j) in Γ''. Therefore what we need is basically a nontrivial solution in integers to $|\Gamma''|$ homogeneous linear equations in $|\Gamma'|$ variables and we choose M, N such that $|\Gamma'| - |\Gamma''| > 0$. Siegel's Box principle lemma guarantees existence of such a polynomial with bounded height. We will summarize our argument in the next theorem.

Theorem 5.6. *Let M and N be positive integers that satisfy (5.4.9) and $|\Gamma'| - |\Gamma''| > 0$. Then there exists a bihomogeneous polynomial*

$$F(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} f_{m,n} x_0^m x_1^{M-m} y_0^n y_1^{N-n}$$

of bidegree (M,N) in $\mathbb{Z}[x,y]$ such that

1. *F is not identically zero.*
2. $1 \leq$ *Index* $\left(F; \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \theta M, \tau N\right)$.
3. $1 \leq$ *Index* $\left(F; \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \theta M, \tau N\right)$.
4. $1 \leq$ *Index* $\left(F; \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \theta M, \tau N\right)$.
5. *For the coefficient $f_{m,n}$ of F we have*
$$\log|f_{m,n}| \leq \frac{1}{4}\left(1 - \frac{3}{2}\theta\tau\right)^{-1} (\theta M + \tau N) + o(M) + o(N).$$
6. *either*
$$F(x,y) = F\left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} x, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} y\right)$$
Or
$$F(x,y) = F\left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} x, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} y\right).[14]$$

Proof of Thue-Siegel's inequality

In this section we will prove the Thue-Siegel inequlity

$$e_v(\lambda_1)e_v(\lambda_2) \leq 6\left(1 + C_s s(\lambda_1, \lambda_2)^{1/3} + C_r r(\lambda_1, \lambda_2)^{1/2}\right),$$

where $C_s = 2^{-3}3^{7/2} + 2^{-3}3^{4/3} = 3.1522...$ and $C_r = (3/4)7^{1/2} + 2^{-1/2}7^{1/2}$ $= 3.8551...$; thus it is slightly stronger than theorem (5.2) First note that, since for elements of $\Lambda(P)$ we have $e_v(\lambda) < 3$ if $e_v(\lambda_1)$ or $e_v(\lambda_2)$ is less than 2, then the theorem follows immediately. So we may assume

$$2 < e_v(\lambda_1) < 3, \quad 3 < e_v(\lambda_2) < 3$$

and by similar argument we assume

$$C_s s(\lambda_1, \lambda_2)^{1/3} + C_r r(\lambda_1, \lambda_2)^{1/2} < 1/2. \tag{5.4.10}$$

Let $\mathcal{G}$ be the subgroup in the remark 5.2; then $\mathcal{G}$ acts on $\mathbb{P}^1(\bar{k}_v)$. Every orbit has six elements with only three exceptions:

$$\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}, and \left\{\begin{bmatrix}\zeta_6\\1\end{bmatrix}, \begin{bmatrix}\zeta_6^5\\1\end{bmatrix}\right\}.$$

For all of these points we have $e_v(\lambda) < 2$. Hence we assume that the orbit of λ_1 and λ_2 under the action of $\mathcal{G}$ has six points.
Let $0 < \theta < 1$ and $\frac{1}{2} < \theta\tau < \frac{2}{3}$. We chose positive integers M and N such that

$$\frac{1}{2}r(\lambda_1, \lambda_2) \leq \min\left\{\frac{M}{N}, \frac{N}{M}\right\} \leq 2r(\lambda_1, \lambda_2).$$

Therefore by (5.4.10)

$$1 + \frac{7}{2}\min\left\{\frac{M}{N}, \frac{N}{M}\right\} < \frac{9}{2}\theta\tau. \tag{5.4.11}$$

Let F be a bihomogenous polynomial of bidegree (M, N) in $\mathbb{Z}[x]$ that satisfies the hypothesis of theorem 5.6. Assume $G = Gal(L/k)$ is noncyclic. Then $\mathcal{G} = P_\sigma : \sigma \in G$ therefore $P_\sigma\lambda_i = \sigma^{-1}\lambda_i$ and it means that for any $A \in \mathcal{G}$ the index at (λ_1, λ_2) is the same as the index at $(A\lambda_1, A\lambda_2)$. If G is cyclic then Let $I_2 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$,

$$\mathcal{G} = \{p_\sigma : \sigma \in G\} \cup \{I_2 p_\sigma : \sigma \in G\}$$

Then we have

$$\begin{aligned}
&\operatorname{Index}\left(F; I_2 p_\sigma \lambda_1, I_2 p_\sigma \lambda_2, \theta M, \tau N\right) \\
&\quad = \operatorname{Index}\left((\rho_{I_2,I_2})F; p_\sigma \lambda_1, p_\sigma \lambda_2, \theta M, \tau N\right), \quad \text{By (3) of proposition 5.1} \\
&\quad = \operatorname{Index}\left(F; p_\sigma \lambda_1, p_\sigma \lambda_2, \theta M, \tau N\right), \quad \text{By (6) of theorem 5.6} \\
&\quad = \operatorname{Index}\left(F; \sigma^{-1}\lambda_1, \sigma^{-1}\lambda_2, \theta M, \tau N\right), \quad \text{Since } \lambda_1 \in \Lambda(p) \\
&\quad = \operatorname{Index}\left(F; \lambda_1, \lambda_2, \theta M, \tau N\right) \quad \text{Since } F(x,y) \in \mathbb{Z}[x]\,.
\end{aligned}$$

In any case, for any $A \in \mathcal{G}$ the map

$$A \to \operatorname{Index}\left(F; \lambda_1, \lambda_2, \theta M, \tau N\right)$$

is constant. So let us set

$$\kappa = \operatorname{Index}\left(F; \lambda_1, \lambda_2, \theta M, \tau N\right).$$

Assume $\kappa \geq 1$. Then by applying Dyson's lemma 5.7 at the nine points

$$\{(A\lambda_1, A\lambda_2) : A \in \mathcal{G}\} \cup \left\{ \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right\}$$

in $\mathbb{P}^1(\bar{k}_v) \times \mathbb{P}^1(\bar{k}_v)$, we obtain that

$$\frac{9}{2}\theta\tau \leq 1 + \frac{7}{2}\min\left\{\frac{M}{N}, \frac{N}{M}\right\},$$

Which contradicts (5.4.11) . Therefore $0 \leq \kappa < 1$ and F vanishes at (λ, λ_2) with low order . Next use the corollary 5.2 of Dyson's lemma to get the inequality

$$\frac{3}{2}\theta\tau + 3\theta\tau\kappa^2 \leq 1 + \frac{7}{2}\min\left\{\frac{M}{N}, \frac{N}{M}\right\}.$$

Now we suppose that the pair of nonnegative integers (R,S) is critical with respect to the index at (λ_1, λ_2) . We chose a bihomogeneous polynomial $T(x,y)$ in $\mathbb{Z}[x,y]$ of bidegree (R,S) such that $T(I_2\lambda_1, I_2\lambda_2) \neq 0$, for example $T(x,y) = x_0^r x_1^{R-r} y_0^s y_1^{S-s}$, where $0 \leq r \leq R$ and $0 \leq s \leq S$. By the remark 5.3 we have

$$\{T(D)F\}(\lambda_1, \lambda_2) \neq 0$$

And from equation (5.4.8)

$$1 - \kappa \leq \operatorname{Index}\left(T(D)F; \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \theta M, \tau N\right)$$

and similarly at $\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$ and $\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)$. So we may apply global estimate theorem 5.5 to $\{T(D)F\}(x,y)$ with (M,N) replaced by $(M-R,N-S)$, θ replaced by $(1-\kappa)(M-R)^{-1}\theta M$, and τ replaced by $(1-\kappa)(N-S)^{-1}\tau N$ so we obtain:

$$\begin{aligned}0 \leq &(M-R)\log\left\{2^{1/2}\eta(p)H(\lambda_1)\right\} + (N-S)\log\left\{2^{1/2}\eta(p)H(\lambda_2)\right\}\\ &+\frac{1}{2}\log\{(M+1)(N+1)\} + \sum_w \log|T(D)F|_w\\ &+(1-\kappa)\max\left\{\theta M\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},x\right)^{\theta M}, \tau N\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},y\right)^{\tau N}\right\}.\end{aligned} \tag{5.4.12}$$

Without loss of generality, assume $T(D)F$ has relatively prime integer coefficients. Considering the expansion of $T(D)F$ and estimate (5) in theorem 5.6, for a coefficient of $T(D)F$ we obtain

$$\begin{aligned}&\log\left|f_{m+r,n+s}\binom{m+r}{r}\binom{M-m-r}{R-r}\binom{n+s}{s}\binom{N-n-s}{S-s}\right|\\ &\leq |f_{m+r,n+s}| + (m+r)\psi\left(\frac{r}{m+r}\right) + (M-m-r)\psi\left(\frac{R-r}{M-m-r}\right)\\ &+(n+s)\psi\left(\frac{s}{n+s}\right) + (N-n-s)\psi\left(\frac{S-s}{N-n-s}\right)\\ &\frac{1}{4}\left(1-\frac{3}{2}\theta\tau\right)^{-1}(\theta M+\tau N) + o(M) + o(N) + M\psi\left(\frac{R}{M}\right) + N\psi\left(\frac{s}{n}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_w \log|T(D)F|_w \leq \sum_w \log[T(D)F]_w\\ &\frac{1}{4}\left(1-\frac{3}{2}\theta\tau\right)^{-1}(\theta M+\tau N) + (M+N)\log 2 + o(M) + o(N).\end{aligned} \tag{5.4.13}$$

By 5.4.12 and 5.4.13 we have:

$$\begin{aligned} 0 \leq & M \log\{8H(\lambda_1)\} + N \log\{8H(\lambda_2)\} + \frac{1}{4}\left(1 - \frac{3}{2}\theta\tau\right)^{-1}(\theta M + \tau N) \\ & + (1-\kappa)\max\left\{\theta M \log \sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, x\right)^{\theta M}, \tau N \log \sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, y\right)^{\tau N}\right\} \\ & + o(M) + o(N). \end{aligned} \tag{5.4.14}$$

Now let $0 < t < \frac{1}{4}$; we can choose θ and τ

$$\theta = \left\{\frac{2(1-t)e_v(\lambda_2)}{3e_v(\lambda_1)}\right\}^{1/2} \quad , \quad \tau = \left\{\frac{2(1-t)e_v(\lambda_1)}{3e_v(\lambda_2)}\right\}^{1/2}.$$

Then $0 < \theta < 1$ and $0 < \tau < 1$ and since $\theta\tau = \frac{2}{3}(1-t)$ we have $\frac{1}{2}\theta\tau < \frac{2}{3}$. And we let $M \to \infty$ and $N \to \infty$ in such a way that

$$\frac{M}{N} \to \frac{\log\{8H(\lambda_1)\}}{\log\{8H(\lambda_2)\}}.$$

By compactness we can restrict (M, N) to a suitable subsequence along with $\kappa \to \kappa*$ with

$$(1-t) + 2(1-t)(\kappa^*)^2 \leq 1 + \frac{7}{2}r(\lambda_1, \lambda_2)$$

With our choice of θ, τ , M and N and from inequality (5.4.14)

$$(1-\kappa^*)\left\{\frac{2}{3}(1-t)e_v(\lambda_1)e_v(\lambda_2)\right\}^{1/2} \leq 2 + (4t)^{-1}s(\lambda_1, \lambda_2)$$

and from these two latter inequalities we obtain

$$\begin{aligned} \{e_v(\lambda_1)e_v\lambda_2\} \leq & 6^{1/2}\left\{\frac{1 + (8t)^{-1}s(\lambda_1,\lambda_2)}{(1-t)^{1/2} - \left(\frac{1}{2}t + \frac{7}{4}r(\lambda_1,\lambda_2)\right)^{1/2}}\right\} \\ \leq & 6^{1/2}\left\{\frac{1 + (8t)^{-1}s(\lambda_1,\lambda_2)}{1 - \left(\frac{3}{2}t + \frac{7}{4}r(\lambda_1,\lambda_2)\right)^{1/2}}\right\}. \end{aligned} \tag{5.4.15}$$

On the square $0 \leq x \leq b < 1$, and $0 \leq y \leq b < 1$ we have the simple inequality

$$\left(\frac{1+x}{1-y}\right)^2 \leq 1+2(1-b)^{-2}(x+y). \tag{5.4.16}$$

Let $t=(24)^{-1/3}s(\lambda_1,\lambda_2)^{2/3}$, take $x=\frac{1}{4}\left(3s(\lambda_1,\lambda_2)\right)^{1/3}$ and $y=\left\{\frac{1}{4}\left(3s(\lambda_1,\lambda_2)\right)^{2/3}+\frac{7}{4}r(\lambda_1,\lambda_2)\right\}^{1/2}$ so by inequality (5.4.10) we can take $b=3-2^{3/2}$ and all conditions of inequality (5.4.16) is satisfied. Thus we can obtain

$$\begin{aligned}
&e_v(\lambda_1)e_v(\lambda_2)\\
&\leq 6\left\{1+(1-b)^{-2}\left\{\frac{1}{4}\left(3s(\lambda_1,\lambda_2)\right)^{1/3}+\left\{\frac{1}{4}\left(3s(\lambda_1,\lambda_2)\right)^{2/3}+\frac{7}{4}r(\lambda_1,\lambda_2)\right\}^{1/2}\right\}\right\}\\
&\leq 6\left\{1+(1-b)^{-2}\left\{\frac{3}{2}\left(3s(\lambda_1,\lambda_2)\right)^{1/3}+\left(7r(\lambda_1,\lambda_2)\right)^{1/2}\right\}\right\}\\
&=6\left\{1+c_s(\lambda_1,\lambda_2)^{1/3}+c_r r(\lambda_1,\lambda_2)^{1/2}\right\}.
\end{aligned}$$

5.5 The Thue inequality $\left|x^3+pxy^2+qy^3\right|<k$

Wakabayashi [87], studied the Thue inequality $\left|x^3+pxy^2+qy^3\right|<k$ by using Padé approximation method , and proved under some condition that this inequality only has nontrivial solutions. More precisely, he proved the family of cubic Thue inequality

$$\left|x^3+pxy^2+qy^3\right| \leq k,$$

when p is positive and greater than $360q^4$,and $k=p+|q|+1$ has only trivial solutions, namely

$$(0,0),(\pm1,0),(\pm1,1),(-q/d,p/d) \quad \text{where } d=gcd(p,q)$$

In this section we will use the machinery introduced in previous sections (Bombieri method) to solve the same Thue inequality

$$\left|x^3+pxy^2+qy^3\right|<k \tag{5.5.1}$$

under some stronger restrictions on p and q. Since the proof is related to heights of the Thue polynomials, the restriction that we assumed is stronger than the one assumed by Wakabayashi.

Let $f(x) = x^3 + px + q$ be an irreducible polynomial in $\mathbb{Q}[x]$. And let L be the splitting field of f where

$$f(x) = (x - \theta)(x - \theta')(x - \theta'')$$

in $L[x]$. Assume $|\theta| < |\theta'| < |\theta''|$. The unique element of $PGL(2, L)$, defined in (5.3.1) is

$$\Phi = \begin{bmatrix} (\theta' - \theta)^{-1} & -\theta(\theta' - \theta)^{-1} \\ -(\theta'' - \theta')^{-1} & \theta''(\theta'' - \theta')^{-1} \end{bmatrix}. \qquad (5.5.2)$$

We have the following theorems:

Theorem 5.7. *Assume* $p \geq 360q^6$*; then for any rational number* $\beta = \frac{x}{y}$ *We have*

$$\left|\theta - \frac{x}{y}\right| > \frac{1}{(8My)^{\mu} \, . \, (8My)^{\frac{9.01\sqrt{\log(p+15/12)}}{\sqrt{2}\sqrt{\log 8My}}} . p^{\frac{1.7}{\sqrt[3]{\log p/2+15/12}}} . e^{\frac{2.1}{\sqrt[3]{\log p+15/12}}} . M^2},$$

where $\eta(\phi) = \prod_{w} \eta_w(\phi)$ and the product is over all the valuations of L.

$$\mu \leq \frac{(3\log p + 15) + 5.76(3\log p + 15)^{2/3}}{\frac{3}{2}\log p - \log q - 0.19}$$

and

$$M = \left(\sqrt{p^2 + 5\,|q| + 5} + \sqrt{8\,|q| + 1}\right)\sqrt{2}.$$

Let (x, y) be a solution for (5.5.1); then the transformation $(x, y) \to (-x, -y)$ gives another solution of (5.5.1), so without lose of generality we can assume for any solution we have $y \geq 0$.

Theorem 5.8. *Assume the hypothesis of theorem 5.7 is satisfied and* $\log p > 8012$. *For any solutions of Thue inequality* (5.5.1) *we have*

$$|y| < p^{518}k.$$

Theorem 5.9. *With the same hypothesis as theorem 5.7, the only primitive solutions with* $y \geq 0$, *of the Thue inequality*

$$\left|x^3 + px^2y + qy^3\right| \leq p + |q| + 1$$

are $(0, 0), (\pm 1, 0), (\pm 1, 1), (-q/d, p/d)$, *where* $d = gcd(p, q)$.

5.5.1 Irrationality measure

For the proof of theorem 5.7, we use (5.3.8) to find an effective irrationality measure of θ. Set

$$\Delta = (\theta - \theta')(\theta' - \theta'')(\theta'' - \theta).$$

The discriminant of F is $\Delta^2 = -4p^3 - 27q^2$. As previously, we define

$$\theta = \begin{bmatrix}\theta \\ 1\end{bmatrix}, \quad \theta' = \begin{bmatrix}\theta' \\ 1\end{bmatrix}, \quad \theta'' = \begin{bmatrix}\theta'' \\ 1\end{bmatrix}$$

be points in $\mathbb{P}^1(L)$. To apply Theorem 5.3 we need to find a point in $\Lambda(p)$ that satisfies the hypothesis of theorem 5.3. We find that

$$\Phi \begin{bmatrix}3q \\ -2p\end{bmatrix} = \begin{bmatrix}-\theta \\ \theta''\end{bmatrix} = \lambda_1. \tag{5.5.3}$$

Therefore λ belongs to $\Lambda(p)$. Moreover we have to show that λ is actually a good approximation for the point $\begin{bmatrix}0 \\ 1\end{bmatrix}$. The first step toward this goal is to find an upper bound for $\log H(\lambda_1)$ in terms of p and q. We introduce polynomials

$$\begin{aligned}
r_1(x) &= \left(x + \frac{\theta'}{\theta}\right)\left(x + \frac{\theta''}{\theta'}\right)\left(x + \frac{\theta'}{\theta''}\right) \\
&= \left(x^3 - \frac{3}{2}x^2 - \frac{3}{2}x + 1\right) + \frac{\Delta}{2q}(x^2 - x) \\
r_2(x) &= \left(x + \frac{\theta}{\theta'}\right)\left(x + \frac{\theta'}{\theta''}\right)\left(x + \frac{\theta''}{\theta}\right) \\
&= \left(x^3 - \frac{3}{2}x^2 - \frac{3}{2}x + 1\right) - \frac{\Delta}{2q}(x^2 - x)
\end{aligned}$$

and their homogenization

$$R_1(x) = x_1^3 r_1\left(\frac{x_0}{x_1}\right), \quad R_2(x) = x_1^3 r_2\left(\frac{x_0}{x_1}\right).$$

Lemma 5.8. *Let w be a place of L. If $w \nmid \infty$ then*

$$\log[R_1]_w = \log[R_2]_w = \frac{1}{2}\log^+\left|\frac{p^3}{q^2}\right|_w$$

If $w \mid \infty$ then

$$\begin{aligned}\log[R_1]_w =& \log[R_2]_w \\ =& \frac{\epsilon_w}{2}\log\left\{\frac{7}{2}+\frac{2}{3}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|_w\right\}\end{aligned}$$

Where ϵ_w *is as* (5.4.1). *Moreover, the polynomials* $r_1(x)$ *and* $r_2(x)$ *are irreducible in* $k(\Delta)[x]$*[14]*.

Proof. If $w \nmid \infty$. Then obviously we have $[R_1]_w = [R_2]_w$ and we have

$$\log[R_1]_w = \frac{1}{2}\log[R_1R_2]_w$$

We find that

$$\begin{aligned}R_1(x)R_2(x) =& x_0^6 - 3x_0^5x_1 + \left(\frac{p^3}{q^2}+6\right)x_0^4x_1^2 \\ =& -\left(2\frac{p^3}{q^2}+7\right)x_0^3x_1^3 + \left(\frac{p^3}{q^2}+6\right)x_0^2x_1^4 - 3x_0x_1^5 + x_1^6.\end{aligned}$$

By using definition (5.2), the proof for this case is completed.
If $w \mid \infty$ then by definition (5.2)

$$\begin{aligned}[R_1]_w &= \left\{1+\frac{1}{3}\left\|\frac{\Delta}{2q}-\frac{3}{2}\right\|_w^2+\frac{1}{3}\left\|\frac{\Delta}{2q}+\frac{3}{2}\right\|_w^2+1\right\}^{\epsilon_w/2} \\ &= \left\{\frac{7}{2}+\frac{2}{3}\left\|\frac{\Delta}{2q}\right\|_w^2\right\}^{\epsilon_w/2} \\ &= \left\{\frac{7}{2}+\frac{2}{3}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|_w^2\right\}^{\epsilon_w/2}.\end{aligned}$$

Similar calculations hold for R_2.
The subgroup $Gal(L/k(\Delta)) \subseteq Gal(L/k)$ is cyclic of order 3. Assume r_1 has a root in $k(\Delta)$ then since $Gal(L/k(\Delta))$ acts transitively on roots of f,it acts transitively on roots of r_1 . Therefore r_1 has a root of multiplicity 3, but the discriminant of r_1 is $p^6q^{-4} \neq 0$. This contradiction leads us to conclude that $r_1(x)$ and $r_2(x)$ are irreducible in $k(\Delta)[x]$. □

Lemma 5.9. *Let λ_1 in $\Lambda(P)$ be defined by* (5.5.3)*; then*

$$\begin{aligned}\frac{1}{6}\sum_{u\nmid\infty}\log^+\left|\frac{p^3}{q^2}\right|+\sum_{u\mid\infty}\frac{d_u}{6d}\log\left\{\frac{7}{2}+\frac{2}{3}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|_u\right\}&\\ \le\log H(\lambda_1)&\\ \le\frac{1}{2}+\frac{1}{6}\sum_{u\mid\infty}\log^+\left|\frac{p^3}{q^2}\right|_u+\sum_{u\mid\infty}\frac{d_u}{6d}\log\left\{\frac{7}{8}+\frac{1}{6}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|_u\right\}&[14].\end{aligned}$$

Proof. $r_1(x)$ is an irreducible polynomial over $k(\Delta)(x)$, thus it is the minimal polynomial of $\frac{-\theta}{\theta''}$. Since the action of $Gal(L/k(\Delta))$ is transitive, we find that the points $\begin{pmatrix}-\theta\\ \theta''\end{pmatrix}$, $\begin{pmatrix}-\theta''\\ \theta\end{pmatrix}$, *and* $\begin{pmatrix}-\theta'\\ \theta\end{pmatrix}$ have the same height. So we can apply (5.4.4) , (5.4.5) and sum over all places w of L. We obtain the inequality

$$\sum_w\log[R_1]_w\le 3\log H(\lambda_1)\le\left(\frac{3}{2}-\log 2\right)+\sum_w\log[R_1]_w.$$

The lemma follows by applying the values of $\log[R_1]_w$ from lemma 5.8 .

□

Let v be a place of k, and assume that f has a root in k_v, and we identify L with an embedding of L in $\bar{k}_v$, such that

$$|\theta|\le\left|\theta'\right|\le\left|\theta''\right|. \tag{5.5.4}$$

Lemma 5.10. *Let λ in $\Lambda(P)$ be defined as* (5.5.3)*, then if $v\nmid\infty$ we have*

$$-\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda_1\right)=\frac{1}{2}\log^+\left|\frac{p^3}{q^2}\right|$$

and if $v\mid\infty$ then

$$\begin{aligned}\frac{d_v}{d}\log\left\{\frac{7}{8}+\frac{1}{6}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|\right\}&\le-\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda_1\right)\\ &\le\frac{3d_v}{d}+\frac{d_v}{d}\log\left\{\frac{7}{8}+\frac{1}{6}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|\right\}[14].\end{aligned}$$

Proof. Our embedding of L into $\bar{k}_v$ determines a place $\hat{w}$ of L such that $\hat{w} \mid v$ and $|\ \ |_{\hat{w}} = |\ \ |_v$ on $\bar{k}_v$. If $v \mid \infty$ then

$$\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, \lambda_1\right) = \frac{|\theta|_v}{\left|\begin{bmatrix}0\\1\end{bmatrix}\right|_v \left|\begin{bmatrix}-\theta\\\theta''\end{bmatrix}\right|_v} = \frac{|\theta|_v}{|\theta''|_v}.$$

On the other hand, from lemma 5.8 and equation (5.4.4) we have

$$\frac{1}{2}\log^+\left|\frac{p^3}{q^2}\right|_{\hat{w}} = \log[R_2]_{\hat{w}} = \log^+\left|\frac{\theta}{\theta'}\right|_{\hat{w}} + \log^+\left|\frac{\theta'}{\theta''}\right|_{\hat{w}} + \log^+\left|\frac{\theta''}{\theta}\right|_{\hat{w}}.$$

Normalized with respect to v and using (5.5.4) we have:

$$\frac{1}{2}\log^+\left|\frac{p^3}{q^2}\right|_v = \log^+\left|\frac{\theta''}{\theta}\right|_v = -\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, \lambda_1\right).$$

If $v \mid \infty$ then

$$\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, \lambda_1\right) = \left\{\frac{\|\theta\|_v}{\left\|\begin{bmatrix}0\\1\end{bmatrix}\right\|_v \left\|\begin{bmatrix}-\theta\\\theta''\end{bmatrix}\right\|_v}\right\}^{d_v/2d} = \left\{\frac{\|\theta\|_v}{\|\theta^2 + \theta''^2\|_v}\right\}^{d_v/2d}.$$

On the other hand, by lemma 5.8 and inequality(5.4.5) we have:

$$\begin{aligned}
&\frac{\epsilon_{\hat{w}}}{2}\log\left\{\frac{7}{2} + \frac{2}{3}\left\|\frac{p^3}{q^2} + \frac{27}{4}\right\|_{\hat{w}}\right\} = \log[R_2]_{\hat{w}} \\
&\quad \leq \frac{\epsilon_{\hat{w}}}{2}\log\left\{1 + \|\frac{\theta}{\theta'}\|_{\hat{w}}^2\right\} + \frac{\epsilon_{\hat{w}}}{2}\log\left\{1 + \|\frac{\theta'}{\theta''}\|_{\hat{w}}^2\right\} \\
&\quad + \frac{\epsilon_{\hat{w}}}{2}\log\left\{1 + \|\frac{\theta''}{\theta}\|_{\hat{w}}^2\right\} \\
&\quad \leq \frac{3\epsilon_{\hat{w}}}{2} + \frac{\epsilon_{\hat{w}}}{2}\log\left\{\frac{7}{8} + \frac{1}{6}\left\|\frac{p^3}{q^2} + \frac{27}{4}\right\|\right\}.
\end{aligned}$$

In the above equation by (5.5.4) we have $0 \leq \log\left\{1 + \|\frac{\theta}{\theta'}\|_{\hat{w}}^2\right\} \leq 2$ and $0 \leq \log\left\{1 + \|\frac{\theta'}{\theta''}\|_{\hat{w}}^2\right\} \leq 2$. Therefore the result follows immediately. $\square$

So to summarize, lemma 5.9 and lemma 5.10 can provide us with an upper bound for irrationality measure

$$\mu_v^*(\lambda_1)=\left(\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda_1\right)\right)^{-1}\left(6\log\{8H(\lambda_1)\}+19\log\{8H(\lambda_1)\}^{2/3}\right) \tag{5.5.5}$$

in terms of p and q. In the next section we will prove theorem 5.7 using precise bounds of lemmas 5.9 and 5.10.

5.5.2 Proof of theorems 5.7, 5.8

During this section we assume v is the usual absolute value on $\mathbb{Q}$. k is the field of rational numbers. Then $k_v=\mathbb{R}$, and $\bar{k}_v=\mathbb{C}$. We also assume that $p\geq 360q^6$ and $\log p\geq 1000$.

Lemma 5.11. *Let $p\neq 0$ and q be integers and $f(x)=x^3+px+q$ be irreducible . Assume $|p|>e^{1000}$. Moreover,*

$$p\geq 360q^6. \tag{5.5.6}$$

Then F has a unique real root θ. The effective irrationality measure for θ is μ. where

$$\mu\leq\frac{(3\log p+15)+5.76(3\log p+15)^{2/3}}{\frac{3}{2}\log p-\log q-0.19}.$$

Proof. The discriminant of f is $\Delta^2=-4p^3-27q^2$. When $0\leq p$ then the discriminant is negative so f has a unique real root, say θ. So $\theta\in k_v$. The other roots of f are complex conjugates . θ' and θ'' $f(x)=(x-\theta)(x-\theta')(x-\theta'')$ we have

$$\begin{aligned}\theta=&-2a,\\ \theta'=&a-bi,\\ \theta''=&a+bi,\end{aligned} \tag{5.5.7}$$

where a,b are real numbers. Moreover

$$\begin{aligned}-3a^2+b^2&=p,\\ 2a(a^2+b^2)&=q.\end{aligned} \tag{5.5.8}$$

Identify L with an embedding of L in $\bar{k}_v=\mathbb{C}$, then $|\theta|_v=2a$,$|\theta'|_v=|\theta''|_v=\sqrt{a^2+b^2}=\sqrt{4a^2+p}$ and we have

$$|\theta|_v\leq\left|\theta'\right|_v\leq\left|\theta''\right|_v.$$

So all the conditions of lemmas 5.9 and 5.10 are satisfied. From lemma 5.9 we have

$$\begin{aligned}
&6\log\{8H(\lambda_1)\}\\
&\quad\leq 6\log 8+3+\sum_{u\nmid\infty}\log^+\left|\frac{p^3}{q^2}\right|_u+\sum_{u|\infty}\frac{d_u}{d}\log\left\{\frac{7}{8}+\frac{1}{6}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|_u\right\}\\
&\quad\leq 6\log 8+3+\sum_{u\nmid\infty}\log^+\left|\frac{p^3}{q^2}\right|_u+\sum_{u|\infty}\frac{d_u}{d}\log\left\{\frac{7}{8}+\frac{1}{6}\left|\frac{p^3}{q^2}+\frac{27}{4}\right|_u^{d/d_u}\right\}\\
&\quad 6\log 8+3+\sum_{u\nmid\infty}\log^+\left|\frac{p^3}{q^2}\right|_u+\sum_{u|\infty}\frac{d}{d_u}\left\{\frac{1}{5}\left|\frac{p^3}{q^2}\right|_u^{d/d_u}\right\}\\
&\quad\leq h\left(\frac{p^3}{q^2}\right)+15.
\end{aligned}\tag{5.5.9}$$

We use the argument in the proof of lemma 5.10 to obtain a better lowerbound for $-\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda_1\right)$. Note that

$$-\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda_1\right)=\frac{d_v}{d}\log\left\{1+\left\|\frac{\theta''}{\theta}^2_v\right\|\right\}.$$

Also from equations (5.5.7) and (5.5.8), we have $\left\|\frac{\theta'}{\theta''}\right\|=1$ and $\left\|\frac{\theta}{\theta'}\right\|<1/1000$. Therefore by the same argument as the proof of lemma 5.10 we have

$$\frac{\epsilon_{\hat{w}}}{2}\log\left\{\frac{7}{4}+\frac{1}{3}\left\|\frac{p^3}{q^2}+\frac{27}{4}\right\|_{\hat{w}}\right\}+\frac{1}{1000}<-\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda_1\right)$$

hence

$$\frac{1}{2}h_v\left(\frac{p^3}{q^2}\right)-0.19<-\log\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix},\lambda_1\right),\tag{5.5.10}$$

and the lemma is followed immediately by (5.5.5), (5.5.9) and (5.5.10).

□

Recall that

$$e_v(\lambda) = \frac{-\log \sigma_v \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda_1 \right)}{\log 8H(\lambda_1)}.$$

By (5.5.9), and (5.5.10) we have

$$\begin{aligned} e_v(\lambda_1) >& \frac{\frac{1}{2} h_v \left(\frac{p^3}{q^2} \right) - 0.19}{\frac{1}{6} h \left(\frac{p^3}{q^2} \right) + \frac{5}{2}} \\ =& \frac{\frac{3}{2} \log p - \log q - 0.19}{\frac{1}{2} \log p + \frac{5}{2}} \\ >& \frac{\frac{8}{3} \log p + \frac{\log 360}{6} - 0.19}{\frac{1}{2} \log p + \frac{5}{2}} \qquad \text{by (5.5.6)} \\ >& 2.666 \qquad (\text{ since } \log p > 6)\,. \end{aligned} \tag{5.5.11}$$

Lemma 5.12. *Let β be in $\mathbb{P}^1(\mathbb{Q})$, then we have*

$$\begin{aligned} &-\log \sigma_v(\theta, \beta) \\ &\leq \mu_v^* \log \{8H(\Phi\beta)\} + 9.01 \left(\log \{8H(\lambda_1)\} \right)^{1/2} \left(\log \{8H(\Phi\beta)\} \right)^{-1/2} \cdot \\ &2.38 \left(\log \{8H(\lambda_1)\} \right)^{2/3} + \log \eta(\Phi). \end{aligned}$$

Proof. Let λ_2 be in $\Lambda(p)$. By the theorem 5.2 we have

$$e_v(\lambda_2) \leq e_v(\lambda)^{-1} \left(6 + 19 s(\lambda, \lambda_2)^{1/3} + 24 r(\lambda, \lambda_2) \right).$$

From (5.3.3) , by letting $x = \log \{8H(\lambda)\}$ and $y = \log \{8H(\lambda_2)\}$, we can obtain that

$$s(\lambda, \lambda_2) \leq \frac{1}{3} \log \{8H(\lambda)\}^{2/3} \log \{8H(\lambda_2)\}^{-1} + \log \{8H(\lambda)\}^{1/3}.$$

Hence

$$\begin{aligned} e_v(\lambda_2) \leq e_v(\lambda)^{-1} \{ & 6 + 19 \log \{8H(\lambda)\}^{1/3} + \\ & \frac{19}{3} \log \{8H(\lambda)\}^{2/3} \log \{8H(\lambda_2)\}^{-1} + 24 r(\lambda, \lambda_2) \}. \end{aligned}$$

By the definition of (5.3.2) , we have

$$e_v(\lambda_2) \leq$$
$$\mu + e_v(\lambda)^{-1}\left(\frac{19}{3}\log\{8H(\lambda)\}^{2/3}\log\{8H(\lambda_2)\}^{-1} + 24\left(\frac{\log\{8H(\lambda)\}}{\log\{8H(\lambda_2)\}}\right)^{1/2}\right).$$

From (5.5.11) we have $e_v(\lambda_1) > 2.666$, therefore

$$e_v(\lambda_2) \leq$$
$$\mu + \left(2.38\log\{8H(\lambda)\}^{2/3}\log\{8H(\lambda_2)\}^{-1} + 9.01\left(\frac{\log\{8H(\lambda)\}}{\log\{8H(\lambda_2)\}}\right)^{1/2}\right). \tag{5.5.12}$$

Also from (5.3.7) we have

$$\sigma_v(\theta,\beta) \geq \eta(\Phi)^{-1}\sigma_v\left(\begin{bmatrix}0\\1\end{bmatrix}, \Phi\beta\right). \tag{5.5.13}$$

Since $\Phi\beta$ belongs to $\Lambda(p)$ we can combine (5.5.13) and (5.5.12) to obtain the bound:

$$\begin{aligned}-\log|\alpha-\beta|_v \leq &\mu_v^* \log\{8H(\Phi\beta)\} + 9.01\left(\log\{8H(\lambda_1)\}\right)^{1/2}\left(\log\{8H(\Phi\beta)\}\right)^{1/2}\\ &2.38\left(\log\{8H(\lambda_1)\}\right)^{2/3} + \log\eta(\Phi),\end{aligned} \tag{5.5.14}$$

as desired. □

From equation 5.3 we have $\eta_w(A) = |detA|_w^{-1}|A|_w^2$ so

$$\eta(\phi) = \prod_w \eta_w(\Phi) = \prod_w |det(\Phi)|_w^{-1}|\Phi|_w^2.$$

And by product rule

$$\eta(\phi) = \prod_w |\phi|_w^2.$$

Also

$$H(\phi\beta) \leq \prod_w |\phi|_w |\beta|_w = \eta(\phi)^{1/2}H(\beta).$$

Therefore to find the effective irrationality measure for θ we only need to find an upper for $\eta(\phi)$. Φ is an element of $PGL(2,L)$ so we rewrite $\Phi = \begin{bmatrix} (\theta'-\theta)^{-1} & -\theta(\theta'-\theta)^{-1} \\ -(\theta''-\theta')^{-1} & \theta''(\theta''-\theta')^{-1} \end{bmatrix}$ as

$$\Phi' = \begin{bmatrix} (\theta''-\theta') & -\theta(\theta''-\theta') \\ -(\theta'-\theta) & \theta''(\theta'-\theta) \end{bmatrix}.$$

Consider Φ', If $w \nmid \infty$, since all entries of ϕ', are algebraic integers , $|\eta|_w < 1$, therefore $|\eta(\Phi')| < \|\eta(\Phi')\|_2 < \sqrt{2}\|\eta(\Phi')\|_1$.
Using equations (5.5.7) we will write ϕ in terms of a, b

$$\Phi' = \begin{bmatrix} 2bi & 4abi \\ -3a+bi & 3a^2+b^2+2abi \end{bmatrix}.$$

Hence from matrix column norm and using equations 5.5.8, we obtain $\eta(\phi) < M^2$. where

$$M = \left(\sqrt{p^2+5\,|q|+5}+\sqrt{8\,|q|+1}\right)\sqrt{2}.$$

Also $\sigma_v(\alpha,\beta) \le |\alpha-\beta|_v$ so using lemma 5.12 and (5.5.9) we obtain:

$$\left|\theta-\frac{x}{y}\right| > \frac{1}{(8My)^{\mu}\,.\,(8My)^{\frac{9.01\sqrt{\log(p+15/12)}}{\sqrt{2}\sqrt{\log 8My}}}.p^{\frac{1.7}{\sqrt[3]{\log p/2+15/12}}}.e^{\frac{2.1}{\sqrt[3]{\log p+15/12}}}.M^2}.$$

This completes the proof of theorem 5.7.

Lemma 5.13. *Let* (x, y) *be a solution of* (5.5.1) *then we have*

$$\left|\theta-\frac{x}{y}\right| < \frac{k}{py^3}.$$

Proof. ([87]) We have

$$\left|\theta-\frac{x}{y}\right| \le \frac{k}{\left|\left(\theta'-\frac{x}{y}\right)\left(\theta''-\frac{x}{y}\right)\right||y|^3}.$$

Note that $\theta\theta'\theta''=-q$, $\theta^2+p=-q/\theta$ and from equations (5.5.8)

$$\begin{aligned}\left|\left(\theta'-\frac{x}{y}\right)\left(\theta''-\frac{x}{y}\right)\right| &\ge \left|\ \mathrm{Im}\ \theta'\right|^2 = \left|\theta'\right|^2-\left(\mathrm{Re}\ \theta'\right)^2 = \frac{|q|}{\theta}-\frac{\theta^2}{4} \\ &= \theta^2+p-\frac{\theta^2}{4} > p.\end{aligned}$$

□

Proof of theorem 5.8

Proof. Let (x, y) be a solution of (5.5.1) with $y \neq 0$, theorem 5.7 provide a lower bound for $\left|\theta - \frac{x}{y}\right|$ and lemma 5.13 give an upper bound for $\left|\theta - \frac{x}{y}\right|$. Let $C = \mu + \frac{9.01\sqrt{\log(p+15/12)}}{\sqrt{2}\sqrt{\log 8My}}$. Comparing these bounds, we have

$$y^{3-C} \leq \frac{M^{2+C}.8^{C}.e^{\frac{2.1}{\sqrt[3]{\log p+5/4}}}p^{\frac{1.7}{\sqrt[3]{\log p+5/4}}}}{p}k \tag{5.5.15}$$

□

Assume $\mu < 2.7$. If $p > 360q^6$ it happens whenever $\log p > 8012$ and if $q = 1$ then it happens when $\log p > 1528$. Under this condition, from the inequality (5.5.15) we obtain that

$$y < p^{518}.k.$$

This completes the proof of theorem 5.8.

5.6 Proof of theorem 5.9

To prove theorem 5.9 , we fix the value k in theorem 5.8 to be $k = p+|q|+1$. By theorem 5.8, there is no solution for the inequality (5.5.1) with $y > p^{520}$. So to complete the proof of theorem 5.9, we have to show that there is no relatively small solution of the inequality 5.5.1 other than the trivial solutions

$$(0, 0), (\pm 1, 0), (\pm 1, 1), (-q/d, p/d).$$

It is easy to see that for any solution (x, y) of inequality (5.5.1), x/y is a principal convergent of θ . So if we can find the continued faction convergence to θ we can find the solutions for (5.5.1) but since we need to find this convergence to high order there will be some technical difficulties in finding the continued fractions with integer partial quotients, so instead, following Wakabayashi [87], we introduce the continued fractions with partial quotients.

5.6.1 Continued fractions and Legendre's theorem

Let ξ be a real number. For ξ we chose a rational number k_0 satisfying

$$k_0 < \xi < k_0 + 1$$

and define ξ_1 by

$$\xi_1 = \frac{1}{\xi - k_0}$$

and then let k_1 be a rational number such that

$$k_1 < \xi_1 < k_1 + 1.$$

Since $\xi_1 > 1$, we have $k_1 > 0$ and then define ξ_2 by

$$\xi_2 = \frac{1}{\xi_1 - k_1},$$

and continue the same process for each $i \geq 1$; namely, choose k_i in each step, satisfying

$$k_i < \xi_i < k_i + 1$$

and define ξ_{i+1} by

$$\xi_{i+1} = \frac{1}{\xi_i - k_i}.$$

For each i, since $\xi_i > 1$, we have $k_i > 0$; then we have

$$\xi = [k_0, k_1, k_2, \cdots, k_n, \xi_{n+1}] = k_0 + \frac{1}{k_1 + k_2 + \cdots + k_n + \xi_{n+1}}.$$

We call this a continued fraction expansion with rational partial quotients for ξ. From the way we define this convergence it is clear that the expansion is not unique. We define convergence p_n/q_n by

$$\begin{cases} p_0 = 1, \quad p_1 = k_0, \quad p_{n+1} = k_n p_n + p_{n-1} \quad (n \geq 1) \\ q_0 = 1, \quad q_1 = 1, \quad q_{n+1} = k_n q_n + q_{n-1} \quad (n \geq 1)\,. \end{cases}$$

By this definition, for $n \geq 1$ we have

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n, \tag{5.6.1}$$

$$\xi = \frac{p_n \xi_n + p_{n-1}}{q_n \xi_n + q_{n-1}} \quad \text{and} \quad \left| \xi - \frac{p_n}{q_n} \right| = \frac{1}{q_n \left(q_n \xi_n + q_{n-1} \right)}.$$

We also choose positive rational number d_n, so that both $d_n p_n, d_n q_n \in \mathbb{Z}$.. Call d_n the common denominator of p_n and q_n. Note that d_n is not unique.

Theorem 5.10. *(Generalized Legendre Theorem [87]) Let ξ be a real number, and let p_n/q_n $(n = 1, 2, ...)$ be the convergence defined by a continued fraction expansion with partial quotients for ξ. For a fixed $n \geq 1$ assume that $q_n/d_{n-1} < q_{n+1}/d_n$. If integers p and q satisfy*

$$\frac{q_n}{d_{n-1}} \leq q \leq \frac{q_{n+1}}{d_n} \tag{5.6.2}$$

and

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{d_n\left(d_{n-1} + d_{n+1}\right) q^2}, \tag{5.6.3}$$

then

$$\frac{p}{q} = \frac{p_n}{q_n}.$$

Proof. Let us assume $p/q \neq p_n/q_n$. Let (x, y) is the solution for this linear system

$$\begin{cases} p_n x + p_{n+1} y = p \\ q_n x + q_{n+1} y = q \end{cases} \tag{5.6.4}$$

By (5.6.1) the unique solution is given by $x = (-1)^{n+1}\left(p_{n+1}q - q_{n+1}p\right)$, $y = (-1)^{n+1}\left(q_n p - p_n q\right)$. Since p and q are integers we can see that $d_n y \in \mathbb{Z}$. If $x = 0$ then $q = q_{n+1}y$ so $y > 0$ and since $d_n y \in \mathbb{N}$ we have $q = q_{n+1}y \geq q_{n+1}/d_n$ which contradicts (5.6.2) . Also if $y = 0$ then $p/q = p_n/q_n$ which contradicts our assumption, therefore $xy \neq 0$. Now we will show that $xy < 0$. Assume $xy > 0$; then since q_n, q_{n+1} are positive, if both x, y are negative, from the second equation of (5.6.4) we would have q be negative, which contradicts (5.6.2). As well, if $x > 0$ and $y > 0$ then $q > q_{n+1}y \geq q_{n+1}/d_n$ which again contradicts (5.6.2), then x and y have different signs . On the other hand $\xi - p_n/q_n$ and $\xi - p_{n+1}/q_{n+1}$ have different signs. Hence $x(\xi - p_n/q_n)$ and $y(\xi - p_{n+1}/q_{n+1})$ have the same sign and therefore

$$\begin{aligned} |\xi q - p| &= \left|x\left(\xi q_n - p_n\right) + y\left(\xi q_{n+1} - p_{n+1}\right)\right| \\ &= \left|x\left(\xi q_n - p_n\right)\right| + \left|y\left(\xi q_{n+1} - p_{n+1}\right)\right| \\ &\geq \left|x\left(\xi q_n - p_n\right)\right| \geq \left|\left(\xi q_n - p_n\right)\right| / d_{n+1}. \end{aligned}$$

Hence

$$\left|\xi - \frac{p_n}{q_n}\right| \leq \frac{d_{n+1}q}{q_n}\left|\xi - \frac{p}{q}\right|. \tag{5.6.5}$$

Then from (5.6.5) , (5.6.3) and (5.6.2) we have

$$\begin{aligned}\frac{1}{d_n q_n q} &\leq \left|\frac{p}{q} - \frac{p_n}{q_n}\right| \leq \left|\xi - \frac{p}{q}\right| + \left|\xi - \frac{p_n}{q_n}\right| \leq \left|\xi - \frac{p}{q}\right| + \frac{d_{n+1} q}{q_n}\left|\xi - \frac{p}{q}\right| \\ &< \frac{q_n + d_{n+1} q}{q_n d_n \left(d_{n-1} + d_{n+1}\right) q^2} \leq \frac{d_{n-1} q + d_{n+1} q}{q_n d_n \left(d_{n-1} + d_{n+1}\right) q^2} = \frac{1}{d_n q_n q}.\end{aligned}$$

This contradiction completes the proof. □

A solution of a Thue inequality often satisfies the assumptions of Generalized Legendre theorem. Therefore their quotient x/y is one of convergence p_n/q_n. So one way to show that a Thue inequality has no non trivial solution is to verify that the convergents do not give a solution. But since the values p_n and q_n become large for large value of n and since we want to find them to a high order, the direct verification may be very time consuming. To overcome this difficulty we use an alternative method to verify that convergents do not give solutions. It is originally due to Pethő [68] and formulated for partial rational quotients by Wakabayashi [87] .

Theorem 5.11. *Let $F(x, y)$ be a homogenous polynomial of degree $d \geq 3$ with integer coefficients , and let k be a positive integer . Let $\xi = \xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(d)}$ be the solutions of $F(x, 1) = 0$ and assume ξ is a simple root. Let (x, y) be a solution of the Thue inequality*

$$|F(x, y)| \leq k \tag{5.6.6}$$

Suppose that for a positive number A we have

$$\left|\prod_{i=2}^{d}\left(\xi_i - \frac{x}{y}\right)\right| \geq A. \tag{5.6.7}$$

Choose a continued fraction expansion with rational partial quotients $k_n \, (n \geq 0)$ for ξ , and let p_n/q_n be its convergents and let d_n be a common denominator of p_n and q_n . Moreover, assume for a fixed $n \geq 1$ the following three conditions (5.6.8) (5.6.9) *and* (5.6.10) *are satisfied*

$$\frac{q_n}{d_{n-1}} < \frac{q_n + 1}{d_n}, \tag{5.6.8}$$

$$\left(\frac{k d_n (d_{n-1} + d_n)}{A}\right)^{1/(d-2)} d_{n-1} < q_n \tag{5.6.9}$$

and

$$\begin{cases} \left(\frac{k(k_n+1+1/k_{n-1})}{A}\right)^{1/(d-2)} d_{n-1}^{d/(d-2)} \leq q_n & \textit{if } n \geq 2 \\ \left(\frac{k(k_1+1)}{A}\right)^{1/(d-2)} d_0^{d/(d-2)} \leq q_1 & \textit{if } n = 1\ . \end{cases} \tag{5.6.10}$$

Then

$$y \notin \left[\frac{q_n}{d_{n-1}}, \frac{q_{n+1}}{d_n}\right].$$

Proof. Assume

$$\frac{q_n}{d_{n-1}} \leq y \leq \frac{q_{n+1}}{d_n}. \tag{5.6.11}$$

From (5.6.6) , (5.6.7) , (5.6.11) and (5.6.9) we have

$$\begin{aligned} \left|\xi - \frac{x}{y}\right| \leq & \frac{k}{\left|\prod_{i=2}^{d}(\xi^{(i)} - x/y\right| y^{d-2}y^2} \leq \frac{k}{A(q_n/d_{n-1})^{d-2}y^2} \\ < & \frac{1}{d_n(d_{n-1}+d_{n+1})y^2}. \end{aligned}$$

So by the generalized Legendre theorem we have $x/y = p_n/q_n$. Let $n \geq 2$; then since $k_n < \xi_n < k_n + 1$ we have

$$\begin{aligned} \left|\xi - \frac{x}{y}\right| = \left|\xi - \frac{p_n}{q_n}\right| = & \frac{1}{q_n(q_n\xi_n + q_{n-1})} > \frac{1}{q_n\left(q_n(k_n+1) + q_{n-1}\right)} \\ = & \frac{1}{q_n^2\left(k_n + 1 + q_{n-1}/q_n\right)} = \frac{1}{q_n^2\left(k_n + 1 + q_n/(k_{n-1}q_{n-1} + q_{n-2})\right)} \\ \geq & \frac{1}{q_n^2\left(k_n + 1 + 1/k_{n-1}\right)}. \end{aligned}$$

Then by (5.6.6),(5.6.7) and (5.6.11) we have

$$\frac{1}{q_n^2\left(k_n+1+1/k_{n-1}\right)} < \left|\xi - \frac{x}{y}\right| \leq \frac{k}{Ay^d} \leq \frac{k}{A\left(q_n/d_{n-1}\right)^d},$$

contradicts (5.6.10). For case $n = 1$ since $q_{n-1} = q_0 = 0$ we have a similar contradiction. □

5.6.2 Applying Legendre's theorm for small values of y

As usual we will take $k = p + |q| + 1$. By using $GL2$ transformation $(x, y) \to (x, -y)$ we can assume $y \geq 0$. Moreover, instead of (5.5.1) we will consider

$$\left|x^3 + px - q\right| \leq p + q + 1. \tag{5.6.12}$$

Now let θ_1 be the real zero of (5.6.12) then $\theta_1 = -\theta$ theorems 5.7 and 5.8 hold with same value of μ. Then we assume without loss of generality that $q > 0$. It helps us to find only positive continued fractions and avoid the difficulties caused by the minus sign during our computations. For the real number θ_1 we calculated the continued fraction expansion with the rational quotients up to the order 450, by using the software Maple. Then we obtained

$$\theta_1 = [k_0, k_1, k_2, \cdots, k_{450}, ...],$$

with

$$k_0 = 0, \quad k_1 = p/q, \quad k_2 = p^2/q, \quad k_3 = p/2q \quad ....$$

The convergents p_n/q_n are given by

$$p_0 = 1, \quad p_1 = 0, \quad p_2 = 1, \quad p_3 = p^2/q, ...$$

and common denominators d_n of p_n and q_n are given by

$$d_0 = 1, \quad d_1 = 1, \quad d_2 = q, \quad d_3 = q^2, \quad d_4 = 2q^3.....$$

The full output is available at the weblink
`http://www.math.ubc.ca/~amir/fracdata.htm`.

Lemma 5.14. *Suppose $q > 0$ and*

$$p \geq (q^2 + 1)^2$$

Then the only primitive solutions (x, y) of (5.6.12), with $0 \leq y < q_3/d_2 = p^3/q^3 + 1/q$ are the trivial ones [87].

Proof. If $y = 1$ then since $p \geq (q^2 + 1)^2$, we can see that $x = 0$ or $x = \pm 1$. Hence we may assume $2 \leq y < q_3/d_2$ and $x \neq 0$. First note that $x > 0$ since otherwise we would have $\left|x^3 + px^2y - qy^3\right| \geq 1 + 4p + 8q > p + q + 1$ contradicts our choice of value k in the inequality (5.6.12). Next we will consider two different cases

Case (i) $2 \le y \le q+1$. In this case we have

$$\begin{aligned} &\left|x^3 + pxy^2 - qy^3\right| - (p+q+1) \ge 1 + py^2 - qy^3 - (p+q+1) \\ &= p(y^2-1) - q(y^3+1) \ge 3(p^2+1)^2 - q\left((q+1)^3+1\right) > 0. \end{aligned}$$

Which contradicts the inequality (5.6.12).

Case (ii) $q+2 \le y < p/q$. In this case we apply Legendre theorem 5.10 for n=1. For $n=1$. we have $1 = q_1/d_o < q_2/d_1 = p/q$ and therefore $q_1/d_o \le y \le q_2/d_1$ by lemma 5.13 and assumption for y we have

$$\left|\theta_1 - \frac{x}{y}\right| < \frac{p+q+1}{py.y^2} \le \frac{p+q+1}{p(q+2)y^2} = \frac{(p+q+1)(q+1)}{p(q+2)d_1(d_0+d_2)y^2}.$$

Moreover we have $p(q+2) - (p+q+10(q=1) = p-(q+1)^2 \ge 0$ so $\left|\theta_1 - \frac{x}{y}\right| < 1/d_1(d_0+d_2)y^2$. Hence all the hypothesis of theorem 5.10 are satisfied for $n=1$ so we have $x/y = p_1/q_1 = 0$, and $x=0$, a contradiction.

Case(iii) $p/q \le y < q_3/d_2$. For this case we apply Legendre theorem 5.10 for n=2. By similar argument we have $\left|\theta_1 - \frac{x}{y}\right| < 1/d_2(d_1+d_3)y^2$ Therefore from Theorem 5.10 we have $x/y = p_2/q_2 = p/q = (p/d)/(q/d)$ so $x = p/d$ and $y = q/d$ and it completes the proof . □

Lemma 5.15. *Suppose $q > 0$ and $p \ge 360q^8$. Then the nontrivial solutions of* (5.6.12)*, with $y \ge 0$ satisfy*

$$y > p^{520}.$$

Proof. By lemma 5.14 all the primitive solutions in the range $0 \le y \le q_3/d_2$ are the trivial ones. So we assume $y \ge q_3/d_2$. We apply theorem 5.11, for the $F(x,y) = x^3 + pxy^2 - qy^3$. We take $A = q$. Under assumption of theorem 5.8, we verify the conditions of theorem 5.11 for all the values $n = 3, 4, ...450$. From this verification we have $y \notin [q_3/d_2, q_{450}/d_{449}]$. Therefore $y > p^{530}$. □

Finally theorem 5.9 immediately follows form theorem 5.8, and lemma 5.15 .

Bibliography

[1] S. Akhtari. Representation of unity by binary forms. *Trans. Amer. Math. Soc.*, 364(4):2129–2155, 2012.

[2] A. Baker. Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers. *Quart. J. Math. Oxford Ser. (2)*, 15:375–383, 1964.

[3] A. Baker. Contributions to the theory of Diophantine equations. II. The Diophantine equation $y^2 = x^3 + k$. *Philos. Trans. Roy. Soc. London Ser. A*, 263:193–208, 1967/1968.

[4] A. Baker. *Transcendental number theory.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1990.

[5] A. Baker and G. Wüstholz. Logarithmic forms and group varieties. *J. Reine Angew. Math.*, 442:19–62, 1993.

[6] K. Belabas. A fast algorithm to compute cubic fields. *Math. Comp.*, 66(219):1213–1237, 1997.

[7] K. Belabas and H. Cohen. Binary cubic forms and cubic number fields. In *Computational perspectives on number theory (Chicago, IL, 1995)*, volume 7 of *AMS/IP Stud. Adv. Math.*, pages 191–219. Amer. Math. Soc., Providence, RI, 1998.

[8] M. A. Bennett. Effective measures of irrationality for certain algebraic numbers. *J. Austral. Math. Soc. Ser. A*, 62(3):329–344, 1997.

[9] M. A. Bennett. On the representation of unity by binary cubic forms. *Trans. Amer. Math. Soc.*, 353(4):1507–1534 (electronic), 2001.

[10] W. E. H. Berwick. Algebraic number-fields with two independent units. *Proc. London Math. Soc.*, 2(1):360–378, 1932.

[11] E. Bombieri. On the Thue-Siegel-Dyson theorem. *Acta Math.*, 148:255–296, 1982.

[12] E. Bombieri and J. Mueller. On effective measures of irrationality for $\sqrt[r]{a/b}$ and related numbers. *J. Reine Angew. Math.*, 342:173–196, 1983.

[13] E. Bombieri and W. M. Schmidt. On Thue's equation. *Invent. Math.*, 88(1):69–81, 1987.

[14] E. Bombieri, A. J. Van der Poorten, and J. D. Vaaler. Effective measures of irrationality for cubic extensions of number fields. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(2):211–248, 1996.

[15] S. Choi. *Diophantine approximation on projective spaces over number fields.* PhD thesis, University of Texas at Austin, 1996.

[16] G. V. Chudnovsky. On the method of Thue-Siegel. *Ann. of Math. (2)*, 117(2):325–382, 1983.

[17] J. E. Cremona. Reduction of binary cubic and quartic forms. *LMS J. Comput. Math.*, 2:64–94 (electronic), 1999.

[18] H. Davenport. The reduction of a binary cubic form. I. *J. London Math. Soc.*, 20:14–22, 1945.

[19] H. Davenport. The reduction of a binary cubic form. II. *J. London Math. Soc.*, 20:139–147, 1945.

[20] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. II. *Proc. Roy. Soc. London Ser. A*, 322(1551):405–420, 1971.

[21] S. David and N. Hirata-Kohno. Linear forms in elliptic logarithms. *J. Reine Angew. Math.*, 628:37–89, 2009.

[22] B. N. Delone and D. K. Faddeev. *The theory of irrationalities of the third degree.* Translation of Math. Monographs, Vol. 10. AMS, Providence, R.I., 1964.

[23] B.N. Delone. Über die darstellung der zahlen durch die binären kubischen formen von negativer diskriminante. *Math. Zeitschr*, 31(1):1–26, 1930.

[24] L. E. Dickson. *History of the theory of numbers. Vol. II: Diophantine analysis.* Chelsea Publishing Co., New York, 1966.

[25] F. J. Dyson. The approximation to algebraic numbers by rationals. *Acta Math.*, 79:225–240, 1947.

[26] N. Elkies. Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction. In *Algorithmic number theory (Leiden, 2000)*, volume 1838 of *Lecture Notes in Comput. Sci.*, pages 33–63. Springer, Berlin, 2000.

[27] W. J. Ellison, F. Ellison, J. Pesek, C. E. Stahl, and D. S. Stall. The Diophantine equation $y^2 + k = x^3$. *J. Number Theory*, 4:107–117, 1972.

[28] J.H. Evertse. On the representation of integers by binary cubic forms of positive discriminant. *Invent. Math.*, 73(1):117–138, 1983.

[29] J.H. Evertse. *Upper bounds for the numbers of solutions of Diophantine equations*, volume 168 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam, 1983.

[30] R. Finkelstein and H. London. On Mordell's equation $y^2 - k = x^3$: An interesting case of Sierpiński. *J. Number Theory*, 2:310–321, 1970.

[31] J. Gebel, A. Pethő, and H. G. Zimmer. Computing integral points on elliptic curves. *Acta Arith.*, 68(2):171–192, 1994.

[32] J. Gebel, A. Pethő, and H. G. Zimmer. On Mordell's equation. *Compositio Math.*, 110(3):335–367, 1998.

[33] A. O. Gel′fond. *Transcendental and algebraic numbers*. Translated from the first Russian edition by Leo F. Boron. Dover Publications Inc., New York, 1960.

[34] R. Gross and J. Silverman. S-integer points on elliptic curves. *Pacific J. Math.*, 167(2):263–288, 1995.

[35] M. Hall. The Diophantine equation $x^3 - y^2 = k$. In *Computers in number theory (Proc. Sci. Res. Council Atlas Sympos. No. 2, Oxford, 1969)*, pages 173–198. Academic Press, London, 1971.

[36] F. Halter-Koch, G. Lettl, A. Pethő, and R. F. Tichy. Thue equations associated with Ankeny-Brauer-Chowla number fields. *J. London Math. Soc. (2)*, 60(1):1–20, 1999.

[37] G. Hanrot. Solving Thue equations without the full unit group. *Math. Comp.*, 69(229):395–405, 2000.

[38] O. Hemer. Notes on the Diophantine equation $y^2 - k = x^3$. *Ark. Mat.*, 3:67–77, 1954.

[39] C. Hermite. Note sur la réduction des formes homogènes à coefficients entiers et à deux indéxtermineées. *J. Reine Angew. Math.*, 36:357–364, 1848.

[40] C. Hermite. *Sur la réduction des formes cubiques à deux indéterminées*, volume 48. 1859.

[41] C. Heuberger. On a family of quintic Thue equations. *J. Symbolic Comput.*, 26(2):173–185, 1998.

[42] C. Heuberger. On general families of parametrized Thue equations. In *Algebraic number theory and Diophantine analysis (Graz, 1998)*, pages 215–238. de Gruyter, Berlin, 2000.

[43] C. Heuberger. On a conjecture of E. Thomas concerning parametrized Thue equations. *Acta Arith.*, 98(4):375–394, 2001.

[44] C. Heuberger, A. Pethő, and R. F. Tichy. Complete solution of parametrized Thue equations. In *Proceedings of the 13th Czech and Slovak International Conference on Number Theory (Ostravice, 1997)*, volume 6, pages 93–114, 1998.

[45] C. Heuberger and R. F. Tichy. Effective solution of families of Thue equations containing several parameters. *Acta Arith.*, 91(2):147–163, 1999.

[46] P. Ingram. Multiples of integral points on elliptic curves. *J. Number Theory*, 129(1):182–208, 2009.

[47] I. Jiménez Calvo, J. Herranz, and G. Sáez. A new algorithm to search for small nonzero $|x^3 - y^2|$ values. *Math. Comp.*, 78(268):2435–2444, 2009.

[48] G. Julia. Etude sur les formes binaires non quadratiques a indetermindes relles ou complexes. *Mem. Acad. Sci. l'Inst. France*, 55:1–293, 1917.

[49] S. Lang. *Elliptic curves: Diophantine analysis*, volume 231 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1978.

[50] M. Laurent, M. Mignotte, and Y. Nesterenko. Formes linéaires en deux logarithmes et déterminants d'interpolation. *J. Number Theory*, 55(2):285–321, 1995.

[51] G. Lettl and A. Pethő. Complete solution of a family of quartic Thue equations. *Abh. Math. Sem. Univ. Hamburg*, 65:365–383, 1995.

[52] G. Lettl, A. Pethő, and P. Voutier. Simple families of Thue inequalities. *Trans. Amer. Math. Soc.*, 351(5):1871–1894, 1999.

[53] W. Ljunggren. The diophantine equation $y^2 = x^3 - k$. *Acta Arith.*, 8:451–465, 1961.

[54] G. B. Mathews. On the reduction of arithmetical binary cubics which have a negative determinant. *Proc. London. Math. Soc.*, 10:128–138, 1912.

[55] E. M. Matveev. An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II. *Izv. Ross. Akad. Nauk Ser. Mat.*, 64(6):125–180, 2000.

[56] M. Mignotte. Verification of a conjecture of E. Thomas. *J. Number Theory*, 44(2):172–177, 1993.

[57] M. Mignotte. Pethő's cubics. *Publ. Math. Debrecen*, 56(3-4):481–505, 2000. Dedicated to Professor Kálmán Győry on the occasion of his 60th birthday.

[58] M. Mignotte. A kit on linear forms in three logarithms. *preprint, June*, 2004.

[59] M. Mignotte, A. Pethő, and F. Lemmermeyer. On the family of Thue equations $x^3-(n-1)x^2y-(n+2)xy^2-y^3=k$. *Acta Arith.*, 76(3):245–269, 1996.

[60] M. Mignotte, A. Pethő, and R. Roth. Complete solutions of a family of quartic Thue and index form equations. *Math. Comp.*, 65(213):341–354, 1996.

[61] M. Mignotte and N. Tzanakis. On a family of cubics. *J. Number Theory*, 39(1):41–49, 1991.

[62] L. J. Mordell. The diophantine equation $y^2 - k = x^3$. *Proc. London. Math. Soc.*, 18:60–80, 1913.

[63] L. J. Mordell. Indeterminate equations of the third and fourth degrees. *Quart. J. pure and appl. Math*, 45:170–186, 1914.

[64] L. J. Mordell. A statement by fermat. *Proc. Lond. Math. Soc (2)*, 18, 1919.

[65] L. J. Mordell. *Diophantine equations*. Pure and Applied Mathematics, Vol. 30. Academic Press, London, 1969.

[66] T. Nagell. Darstellung ganzer Zahlen durch binäre kubische Formen mit negativer Diskriminante. *Math. Zeitscher.*, 28(1):10–29, 1928.

[67] R. Okazaki. Geometry of a cubic Thue equation. *Publ. Math. Debrecen*, 61(3-4):267–314, 2002.

[68] A. Pethő. On the resolution of Thue inequalities. *J. Symbolic Comput.*, 4(1):103–109, 1987.

[69] A. Pethő. On the representation of 1 by binary cubic forms with positive discriminant. In *Number theory (Ulm, 1987)*, volume 1380 of *Lecture Notes in Math.*, pages 185–196. Springer, New York, 1989.

[70] A. Pethő. Complete solutions to families of quartic Thue equations. *Math. Comp.*, 57(196):777–798, 1991.

[71] R. S. Rumely. *Capacity theory on algebraic curves*, volume 1378 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[72] C. Siegel. Approximation algebraischer Zahlen. *Math. Z.*, 10(3-4):173–213, 1921.

[73] C. Siegel. *Gesammelte Abhandlungen. Bände I, II, III*. Herausgegeben von K. Chandrasekharan und H. Maass. Springer-Verlag, Berlin, 1966.

[74] N. P. Smart. S-integral points on elliptic curves. *Math. Proc. Cambridge Philos. Soc.*, 116(3):391–399, 1994.

[75] N. P. Smart. *The algorithmic resolution of Diophantine equations*, volume 41 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1998.

[76] C. L. Stewart. On the number of solutions of polynomial congruences and Thue equations. *J. Amer. Math. Soc.*, 4(4):793–835, 1991.

[77] R. J. Stroeker and N. Tzanakis. Solving elliptic Diophantine equations by estimating linear forms in elliptic logarithms. *Acta Arith.*, 67(2):177–196, 1994.

[78] E. Thomas. Fundamental units for orders in certain cubic number fields. *J. Reine Angew. Math.*, 310:33–55, 1979.

[79] E. Thomas. Complete solutions to a family of cubic Diophantine equations. *J. Number Theory*, 34(2):235–250, 1990.

[80] E. Thomas. Solutions to certain families of Thue equations. *J. Number Theory*, 43(3):319–369, 1993.

[81] A. Thue. Über annäherungswerte algebraischer zahlen. *J. Reine Angew. Math.*, 135:284–305, 1909.

[82] A. Thue. *Berechnung aller lösungen gewisser gleichungen von der form.* Vid. Skifter I Mat.-Naturv. Klasse, 1918.

[83] A. Togbé. Complete solutions of a family of cubic Thue equations. *J. Théor. Nombres Bordeaux*, 18(1):285–298, 2006.

[84] S. M. Tyler. *The Lagrange spectrum in projective space over a local field.* PhD thesis, University of Texas at Austin, 1994.

[85] N. Tzanakis and B. M. M. de Weger. On the practical solution of the Thue equation. *J. Number Theory*, 31(2):99–132, 1989.

[86] I. Wakabayashi. On a family of quartic Thue inequalities. I. *J. Number Theory*, 66(1):70–84, 1997.

[87] I. Wakabayashi. Cubic Thue inequalities with negative discriminant. *J. Number Theory*, 97(2):222–251, 2002.

[88] I. Wakabayashi. On a family of cubic Thue equations with 5 solutions. *Acta Arith.*, 109(3):285–298, 2003.

[89] D. Zagier. Large integral points on elliptic curves. *Math. Comp.*, 48(177):425–436, 1987.

[90] V. Ziegler. Thomas' conjecture over function fields. *J. Théor. Nombres Bordeaux*, 19(1):289–309, 2007.

Appendix A

Non-vanishing of $\Sigma_{r,g}$

We will prove non-vanishing of $\Sigma_{r,g}$ for $g = 0, 6 \leq r \leq 15$. Let

$$A^{\#}_{r,g} = \frac{3^{\alpha}(r,g)}{C_{r,g}} A_{r,g}(Z),$$
$$B^{\#}_{r,g} = \frac{3^{\alpha}(r,g)}{C_{r,g}} B_{r,g}(Z),$$

which $\alpha_{r,g}$ is the smallest integer such that $A^{\#}_{r,g}$ and $B^{\#}_{r,g}$ have integral coefficients. For each $r \in \mathbb{N}$ there exist a polynomial $K_r(Z) \in \mathbb{Z}[z]$ satisfying

$$A^{\#^3}_{r,g} - (1-z)B^{\#^3}_{r,g} = z^{2r+1} K_r(z)$$

Also define A^*_r, B^*_r by the formulas

$$A^*_r = x^r A^{\#}_{r,g}$$

and

$$B^*_r = x^r B^{\#}_{r,g}.$$

If $\Sigma_{r,0} = 0$ for some r , setting $u = \varepsilon_1^3$ and $v = \varepsilon_1^3 - \gamma_1^3$,we have

$$\frac{\gamma^3}{\varepsilon^3} = \frac{(u-v)(B^*_{r,0}(u,v))^3}{u(A^*_r(u,v)^3)}$$

Using the same argument as proof of lemma 6.3 Bennett [9]. Let a_r be the integral ideal in $M = \mathbb{Q}(\sqrt{-\Delta})$ generated by $(u-v)(B^*_{r,0}(u,v))^3$and $u(A^*_r(u,v)^3)$. Moreover let $N(a_r)$ be its absolute norm, then

$$\frac{N(a_r)^{1/2} \left|v\right|^{-3r-1} \left|z_1\right|^r 3\sqrt{\Delta}}{\left|K_r(z_1)\right|} \geq 1$$

And therefore

$$\left|\varepsilon_1\right|^{3r} \leq \frac{N(a_r)^{1/2} \left|v\right|^{-3r-1} \left(3\sqrt{\Delta}\right)^{r+1}}{\left|K_r(z_1)\right|} \tag{A.0.1}$$

To find an upper bound for $N(a_r)^{1/2}\,|v|^{-3r-1}$. Define M_1 to be an extension of M where the ideal genrated by u and v is principal, say generated by w. Set $u_1 = \frac{u}{w}, v_1 = \frac{v}{w}$ and let the extension of a_r to M_1 be b_r. Further if we define

$$\tau_r = (A_r^*(u_1, v_1), B_r^*(u_1, v_1)),$$

then by [6] , formula (6.10)

$$b_r \supset w^{3r+1} B_r^*(0,1)\tau_r^3. \tag{A.0.2}$$

We use this to find $N(a_r)$, and find an upper bound for $|\varepsilon_1|$. We will show that if $\Sigma_{r,0} = 0$ for any $6 \le r \le 15$ then $|\varepsilon_1| < \Delta^{2/3}$, but we assumed $|\varepsilon_1| > \Delta^{2/3}$. This contradiction completes the proof of nonvanishing of $\Sigma_{r,0}$, for each r.

r=6

$$\begin{aligned}
A^{\#}_{6,0}(z) &= 988z^6 - 31122z^5 + 266760z^4 - 960336z^3 + 1662120z^2 - 1371249z + 433026\\
B^{\#}_{6,0}(z) &= 187z^6 - 11781z^5 + 141372z^4 - 636174z^3 + 1301265z^2 - 1226907z + 433026\\
K_{6,0}(z) &= 6539203z^6 - 278018298z^5 + 2790451188z^4 -\\
&\quad - 11179031292z^3 + 20974929426z^2 - 18462496536z + 6154165512
\end{aligned}$$

Setting:

$$\begin{aligned}
F_6(x,y) &= 79202404017x^5 - 159949407645yx^4 + 109186695090y^2x^3 -\\
&\quad - 29296172862y^3x^2 + 2680421535y^4x - 43464971y^5\\
G_6(x,y) &= 79202404017x^5 - 186350208984yx^4 + 153702897192y^2x^3\\
&\quad - 52808618772y^3x^2 + 6928019514y^4x - 229643804y^5
\end{aligned}$$

We can see that

$$\begin{aligned}
F_6(x,y)A_6^*(x,y) - G_6(x,y)B_6^*(x,y) &= -106562117376x^5y^6\\
B_5^*(x,y)A_6^*(x,y) - A_5^*(x,y)B_6^*(x,y) &= 1144y^{11}
\end{aligned}$$

From this two equation we conclude that

$$\tau_6 \supset (106562117376u_1^5v_1^6, 1144v_1^{11}) \supset 106562117376(v_1)^6$$

since $B_6(0,1) = 187$ and $K_6(z_1) > 6154165492$ formulas (A.0.2) and (A.0.1)

$$|\varepsilon_1| < 72.16\Delta^{7/36}$$

this contradiction proves non-vanishing of $\Sigma_{(6,0)}$.

r=7

$$
\begin{aligned}
A^{\#}_{7,0}(z) &= -494z^7 + 20748z^6 - 240084z^5 + 1200420z^4 - 3047220z^3 \\
&\quad + 4113747z^2 - 2814669z + 767637 \\
B^{\#}_{7,0}(z) &= -85z^7 + 7140z^6 - 115668z^5 + 722925z^4 - 2168775z^3 \\
&\quad + 3346110z^2 - 2558790z + 767637 \\
K_{7,0}(z) &= -614125z^7 + 34819841z^6 - 471884616z^5 + 2626058475z^4 \\
&\quad - 7227728820z^3 + 10412142975z^2 - 7521051447z + 2148871842
\end{aligned}
$$

Setting

$$
\begin{aligned}
F_7(x,y) &= 218954190636x^6 - 547175054385yx^5 + 500968274391y^2x^4 \\
&\quad - 206106707898y^3x^3 + 37428532188y^4x^2 - 2465893775y^5x \\
&\quad + 29807035y^6 \\
G_7(x,y) &= 218954190636x^6 - 620159784597yx^5 + 659031715782y^2x^4 \\
&\quad - 325814595210y^3x^3 + 75233403960y^4x^2 - 7055472502y^5x \\
&\quad + 173231474y^6
\end{aligned}
$$

We can see that

$$F_7(x,y)A^*_7(x,y) - G_7(x,y)B^*_7(x,y) = -40351037922y^7x^6$$

$$B^*_6(x,y)A^*_7(x,y) - A^*_6(x,y)B^*_7(x,y) = -8398y^{13}$$

From this two equation we conclude that

$$\tau_7 \supset (40351037922u_1^6v_1^7, 8398v_1^{13}) \supset 40351037922(v_1)^6$$

since $B_7(0,1) = -85$ and $K_7(z_1) > 2148871830$ formulas (A.0.2) and (A.0.1)

$$|\varepsilon_1| < 33.74\Delta^{4/21}$$

this contradiction proves non-vanishing of $\Sigma_{(7,0)}$.

r=8

$$
\begin{aligned}
A^{\#}_{8,0}(z) &= 2470z^8 - 133380z^7 + 2000700z^6 - 13204620z^5 + 45708300z^4 - \\
&\quad - 89131185z^3 + 98513415z^2 - 57572775z + 13817466 \\
B^{\#}_{8,0}(z) &= 391z^8 - 42228z^7 + 886788z^6 - 7316001z^5 + 29929095z^4 - \\
&\quad - 66699126z^3 + 82393038z^2 - 52966953z + 13817466 \\
K^{\#}_{8,0}(z) &= 59776471z^8 - 4358130075z^7 + 76570832520z^6 - 562540646655z^5 \\
&\quad + 2111528671200z^4 - 4394226699135z^3 + 5127936702525z^2 - \\
&\quad - 3139880973840z + 784970243460
\end{aligned}
$$

Setting

$$F_8(x,y) = 257842779464439x^7 - 769043822230242yx^6 + 886262102158212y^2x^5 -$$
$$- 497428263302319y^3x^4 + 140750028719955y^4x^3 - 18693133706664y^5x^2$$
$$+ 934070121522y^6x - 8748416597y^7$$
$$G_8(x,y) = 257842779464439x^7 - 854991415385055yx^6 + 1113960845183355y^2x^5 -$$
$$- 723315871395180y^3x^4 + 244946909822940y^4x^3 - 41180909742180y^5x^2 +$$
$$+ 2916341162280y^6x - 55264933490y^7$$

We can see that

$$F_8(x,y)A_8^*(x,y) - G_8(x,y)B_8^*(x,y) = -64363009411584y^8x^7$$
$$B_7^*(x,y)A_8^*(x,y) - A_7^*(x,y)B_8^*(x,y) = 16796y^{15}$$

From this two equation we conclude that

$$\tau_8 \supset \left\langle 64363009411584u_1^7v_1^8, 16796v_1^{15} \right\rangle \supset \left\langle 64363009411584v_1^7 \right\rangle.$$

Since $B_8(0,1) = 391$ and $K_8(z_1) > 784970240000$ formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 54.13\Delta^{\frac{3}{16}}$$

this contradiction proves non-vanishing of $\Sigma_{(8,0)}$.

r=9

$$A_{9,0}^{\#}(z) = -2660z^9 + 179550z^8 - 3385800z^7 + 28440720z^6 - 127983240z^5 +$$
$$335956005z^4 - 530456850z^3 + 496011600z^2 - 252965916z + 54206982$$
$$B_{9,0}^{\#}(z) = 391z^9 + 52785z^8 - 1393524z^7 + 14632002z^6 - 77815647z^5 +$$
$$233446941z^4 - 411965190z^3 + 423735624z^2 - 234896922z + 54206982$$
$$K_{9,0}^{\#}(z) = -59776471z^9 + 5448151226z^8 - 120346110612z^7 + 1125360861408z^6$$
$$- 5491758685140z^5 + 15385952831760z^4 - 25651868175360z^3 +$$
$$25132846361040z^2 - 13352805876060z + 2967290194680$$

Setting

$$\begin{aligned}F_9(x,y) &= 6766871023502262x^8 - 23475154665503151yx^7 \\ &+ 32675890549211586y^2x^6 - 23369579565193512y^3x^5 \\ &+ 9132870725085555y^4x^4 - 1913581724600637y^5x^3 \\ &+ 195321674212056y^6x^2 - 7678452802046y^7x \\ &+ 57387225227y^8 \\ G_9(x,y) &= 6766871023502262x^8 - 25730778340003905yx^7 \\ &+ 39749067546212385y^2x^6 - 32070900651078645y^3x^5 \\ &+ 14462687637429120y^4x^4 - 3616439390830530y^5x^3 \\ &+ 465907291477830y^6x^2 - 25884419856610y^7x \\ &+ 390409256020y^8\end{aligned}$$

We can see that

$$\begin{aligned}F_9(x,y)A_9^*(x,y) - G_9(x,y)B_9^*(x,y) &= -488514352385880y^9x^8 \\ B_8^*(x,y)A_9^*(x,y) - A_8^*(x,y)B_9^*(x,y) &= -74290y^{17}\end{aligned}$$

From this two equation we conclude that

$$\tau_9 \supset \left\langle 488514352385880u_1^8v_1^9, 74290v_1^{17}\right\rangle \supset \left\langle 488514352385880v_1^9\right\rangle.$$

Since $B_9(0,1) = 391$ and $K_9(z_1) > 2967290180000$ formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 43.15\Delta^{\frac{5}{27}}$$

this contradiction, proves non-vanishing of $\Sigma_{(9,0)}$.

r=10

$$\begin{aligned}A_{10,0}^{\#}(z) &= 82460z^{10} - 6802950z^9 + 157439700z^8 - 1637372880z^7 \\ &+ 9257454360z^6 - 31243908465z^5 + 65776649400z^4 \\ &- 87132704400z^3 + 70577490564z^2 - 31927912398z + 6179595948 \\ B_{10,0}^{\#}(z) &= 11339z^{10} - 1870935z^9 + 60618294z^8 - 788037822z^7 \\ &+ 5265525447z^6 - 20309883867z^5 + 47787962040z^4 \\ &- 69633887544z^3 + 61308096642z^2 - 29868047082z + 6179595948 \\ K_{10,0}^{\#}(z) &= 1457888351219z^{10} - 162413351268624z^9 + 4403229639855552z^8 \\ &- 50981179991962968z^7 + 312597353265293124z^6 \\ &- 1126073476171223496z^5 + 2503363190796325872z^4 \\ &- 3474864128904362640z^3 + 2932290444707346060z^2 \\ &- 1375716274987503600z + 275143254997500720\end{aligned}$$

Setting

$$
\begin{aligned}
F_{10}(x,y) = {} & 754050190230991251302 58x^9 - 2984438932322154029553 42yx^8 \\
& + 4874589196116315292271 88y^2x^7 - 4249705586157133600764 42y^3x^6 \\
& + 2136423441799053591181 83y^4x^5 - 624346021208519446379 13y^5x^4 \\
& + 101588851207359200471 22y^6x^3 - 8251968531653222033 88y^7x^2 \\
& + 262326728274879965 07y^8x - 1601270628125391 13y^9 \\
G_{10}(x,y) = {} & 754050190230991251302 58x^9 - 3235788995732484446654 28yx^8 \\
& + 5785618819086923163089 40y^2x^7 - 5589510633754616105478 90y^3x^6 \\
& + 3164564702153414524107 15y^4x^5 - 1065137782414409125102 80y^5x^4 \\
& + 206417218538249276040 00y^6x^3 - 21120059183163176394 00y^7x^2 \\
& + 947005731893550783 30y^8x - 1164483428831640820y^9
\end{aligned}
$$

we can see that

$$
\begin{aligned}
F_{10}(x,y)A^*_{10}(x,y) - G_{10}(x,y)B^*_{10}(x,y) &= -4500251086039221897216 0y^{10}x^9 \\
B^*_9(x,y)A^*_{10}(x,y) - A^*_9(x,y)B^*_{10}(x,y) &= -2080120y^{19}
\end{aligned}
$$

From this two equation we conclude that

$$
\begin{aligned}
\tau_{10} &\supset \left\langle 4500251086039221897216 0u_1^9v_1^{10}, 2080120v_1^{19} \right\rangle \\
&\supset \left\langle 4500251086039221897216 0v_1^{10} \right\rangle .
\end{aligned}
$$

Since $B_{10}(0,1) = 11339$ and $K_{10}(z_1) > 275143253000000000$ formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 183.83\Delta^{\frac{11}{60}}$$

this contradiction, proves non-vanishing of $\Sigma_{(10,0)}$.

r=11

$$
\begin{aligned}
A^{\#}_{11,0}(z) = & -20615z^{11} + 2040885z^{10} - 56853225z^{9} + 716350635z^{8} \\
& - 4959350550z^{7} + 20829272310z^{6} - 55910151990z^{5} \\
& + 98024292450z^{4} - 111747693393z^{3} + 79819780995z^{2} \\
& - 32442878727z + 5725213893 \\
B^{\#}_{11,0}(z) = & -2668z^{11} + 528264z^{10} - 20602296z^{9} + 324486162z^{8} \\
& - 2654886780z^{7} + 12743456544z^{6} - 38230369632z^{5} \\
& + 73729998576z^{4} - 91361085192z^{3} + 70277757840z^{2} \\
& - 30534474096z + 5725213893 \\
K^{\#}_{11,0}(z) = & -18991421632z^{11} + 2538969762665z^{10} - 82860204771729z^{9} \\
& + 1162359271452456z^{8} - 8727488325684576z^{7} + 39125898864421488z^{6} \\
& - 110904610688488764z^{5} + 203758154517092436z^{4} \\
& - 242003539343672064z^{3} + 179278783381721142z^{2} \\
& - 75300131449876410z + 13690932990886620
\end{aligned}
$$

setting

$$
\begin{aligned}
F_{11,0}(x,y) = & 10086380874316141564545x^{10} - 44867702308185202410144yx^{9} \\
& + 84177949522001285566548y^{2}x^{8} - 86728758817629067960410y^{3}x^{7} \\
& + 53544447564487777981062y^{4}x^{6} - 20286382993552760112756y^{5}x^{5} \\
& + 4642061682309232093164y^{6}x^{4} - 606506715319471273458y^{7}x^{3} \\
& + 40240887892938973440y^{8}x^{2} - 1056968121552891712y^{9}x \\
& + 5369611493366676y^{10} \\
G_{11,0}(x,y) = & 10086380874316141564545x^{10} - 48229829266290582931659yx^{9} \\
& + 98013141305361226196091y^{2}x^{8} - 110425391171850655570635y^{3}x^{7} \\
& + 75455462185942660347330y^{4}x^{6} - 32152155740359388970390y^{5}x^{5} \\
& + 8461604539337689022850y^{6}x^{4} - 1314812676605125272030y^{7}x^{3} \\
& + 109725035932977136335y^{8}x^{2} - 4059459622628640965y^{9}x \\
& + 41489707996909305y^{10}.
\end{aligned}
$$

We can see that

$$
\begin{aligned}
F_{11}(x,y)A^{*}_{11}(x,y) - G_{11}(x,y)B^{*}_{11}(x,y) &= -399028707182191791915y^{1}1x^{1}0 \\
B^{*}_{10}(x,y)A^{*}_{11}(x,y) - A^{*}_{10}(x,y)B^{*}_{11}(x,y) &= -13750205y^{2}1
\end{aligned}
$$

Since $B_{11}(0,1) = 2668$ and $K_{11}(z_1) > 13690932900000000$ formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 73.93\Delta^{\frac{2}{11}}$$

this contradiction, proves non-vanishing of $\Sigma_{(11,0)}$.

r=12

$$
\begin{aligned}
A^{\#}_{12,0}(z) &= 108965z^{12} - 12748905z^{11} + 420713865z^{10} \\
&- 6310707975z^{9} + 52427420100z^{8} - 267379842510z^{7} \\
&+ 886575267270z^{6} - 1968887931210z^{5} + 2953331896815z^{4} \\
&- 2953331896815z^{3} + 1886321663127z^{2} - 696022431849z \\
&+ 112868502462 \\
B^{\#}_{12,0}(z) &= 13340z^{12} - 3121560z^{11} + 144216072z^{10} \\
&- 2704051350z^{9} + 26548867800z^{8} - 154741972320z^{7} \\
&+ 573455544480z^{6} - 1400869972944z^{5} + 2284027129800z^{4} \\
&- 2459721524400z^{3} + 1679396075280z^{2} - 658399597695z \\
&+ 112868502462 \\
K^{\#}_{12,0}(z) &= 2373927704000z^{12} - 375089280379875z^{11} + 14501580864313725z^{10} \\
&- 242183646588189150z^{9} + 2182172532842687580z^{8} \\
&- 11879523104989091400z^{7} + 41597624578535753868z^{6} \\
&- 96807735644023943748z^{5} + 151292328990025212000z^{4} \\
&- 156915475662561817410z^{3} + 103571640742394696850z^{2} \\
&- 39375570332663090928z + 6562595055443848488,
\end{aligned}
$$

setting

$$
\begin{aligned}
F_{12}(x,y) = {} & 16564533422227754585497501869x^{11} \\
& - 8183153039260307232416775192 0yx^{10} \\
& + 17348394518686769376421726761 6y^2x^9 \\
& - 20647594631097089017635007864 8y^3x^8 \\
& + 15154612073886835556719159348 8y^4x^7 \\
& - 70954048020900642058780095024y^5x^6 \\
& + 21189749516162281303370832600y^6x^5 \\
& - 3921491602361190913545847716y^7x^4 \\
& + 421810845344399723947616178y^8x^3 \\
& - 23329066630579789398301528y^9x^2 \\
& + 515169318819791836231200y^{10}x \\
& - 2212379957408092237460y^{11} \\
G_{12}(x,y) = {} & 16564533422227754585497501869x^{11} \\
& - 87353041533345657186000252543yx^{10} \\
& + 198920618270821189584995684715y^2x^9 \\
& - 256234037837182480962076547385y^3x^8 \\
& + 205465103260720137096648512850y^4x^7 \\
& - 106368872500017696858627681270y^5x^6 \\
& + 35689548164123863423957803390y^6x^5 \\
& - 7588152966949903270265545350y^7x^4 \\
& + 969656379093261326061310275y^8x^3 \\
& - 67381836937780814496984625y^9x^2 \\
& + 2093703367638590949649005y^{10}x \\
& - 18071362972936489554335y^{11}
\end{aligned}
$$

We can see that

$$
\begin{aligned}
&F_{12}(x,y)A^*_{12}(x,y) - G_{12}(x,y)B^*_{12}(x,y) = -9140297826765396747556454 40y^{12}x^{11} \\
&B^*_{11}(x,y)A^*_{12}(x,y) - A^*_{11}(x,y)B^*_{12}(x,y) = -15714520y^{23}
\end{aligned}
$$

Since $B_{12}(0,1) = 13340$ and $K_{12}(z_1) > 6562595010000000000$ formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 173.83\Delta^{\frac{13}{72}}$$

this contradiction, proves non-vanishing of $\Sigma_{(12,0)}$.

r=13

$$
\begin{aligned}
A^{\#}_{13,0}(z) &= -11470z^{13} + 1565655z^{12} - 60389550z^{11} \\
&+ 1062856080z^{10} - 10424165400z^{9} + 63326804805z^{8} \\
&- 253307219220z^{7} + 690837870600z^{6} - 1305683575434z^{5} \\
&+ 1709823729735z^{4} - 1522294675506z^{3} + 879186229704z^{2} \\
&- 297022374900z + 44553356235 \\
B^{\#}_{13,0}(z) &= -1334z^{13} + 364182z^{12} - 19665828z^{11} \\
&+ 432648216z^{10} - 5014786140z^{9} + 34816943772z^{8} \\
&- 155652219216z^{7} + 466956657648z^{6} - 959291394516z^{5} \\
&+ 1352846838420z^{4} - 1287536991048z^{3} + 790079517234z^{2} \\
&- 282171256155z + 44553356235 \\
K^{\#}_{13,0}(z) &= -2373927704z^{13} + 437617194280z^{12} - 19776004312428z^{11} \\
&+ 387525903467742z^{10} - 4122328258070217z^{9} \\
&+ 26732502858952926z^{8} - 112925788901103168z^{7} \\
&+ 322751842740529812z^{6} - 635560579106005113z^{5} \\
&+ 863236915928990064z^{4} - 794254560542886960z^{3} \\
&+ 472640666946852420z^{2} - 164115373424072850z \\
&+ 25248518988318900,
\end{aligned}
$$

setting

$$\begin{aligned}
F_{13}(x,y) &= 4557135961912237412250339x^{12} \\
&- 2475923879665383078682816yx^{11} \\
&+ 5854904576225007862850463 6y^{2}x^{10} \\
&- 7910481196619998625140257 6y^{3}x^{9} \\
&+ 6739502843424244913613846 0y^{4}x^{8} \\
&- 3770568384309649060714555 2y^{5}x^{7} \\
&+ 1399135521825060995916662 4y^{6}x^{6} \\
&- 339908443707964163289998 4y^{7}x^{5} \\
&+ 52117373797341819085947 6y^{8}x^{4} \\
&- 4705787949646642546554 0y^{9}x^{3} \\
&+ 220531474361899018600 8y^{10}x^{2} \\
&- 4153572847652498406 8y^{11}x \\
&+ 152777968424470498y^{12} \\
G_{13}(x,y) &= 4557135961912237412250339x^{12} \\
&- 2627828411729124325757292 9yx^{11} \\
&+ 6629577692092221806719553 7y^{2}x^{10} \\
&- 9615143870966207060880345 9y^{3}x^{9} \\
&+ 8859866117503311518994019 8y^{4}x^{8} \\
&- 5411403641761284738556197 0y^{5}x^{7} \\
&+ 2218539449128760533700667 0y^{6}x^{6} \\
&- 605054745519428205839601 0y^{7}x^{5} \\
&+ 106494140895381267448765 5y^{8}x^{4} \\
&- 11413834626045527362678 5y^{9}x^{3} \\
&+ 671524308831367680670 5y^{10}x^{2} \\
&- 1778239996042932606 55y^{11}x \\
&+ 131361566553873809 0y^{12}.
\end{aligned}$$

We can see that

$$F_{13}(x,y)A_{13}^{*}(x,y) - G_{13}(x,y)B_{13}^{*}(x,y) = -695649299209411359375 0y^{1}3x^{1}2$$
$$B_{12}^{*}(x,y)A_{13}^{*}(x,y) - A_{12}^{*}(x,y)B_{13}^{*}(x,y) = -7650490y^{25}$$

Since $B_{13}(0,1) = 1334$ and $K_{13}(z_1) > 25248518824203527$ formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 47\Delta^{\frac{7}{39}}$$

this contradiction, proves non-vanishing of $\Sigma_{(13,0)}$.

r=14

$$
\begin{aligned}
A^{\#}_{14,0} = {} & 493210z^{14} - 77680575z^{13} + 3462334200z^{12} \\
& - 70631617680z^{11} + 806830401960z^{10} - 5748666613965z^{9} \\
& + 27230526066150z^{8} - 89118085307400z^{7} + 205862777060094z^{6} \\
& - 338203133741583z^{5} + 392752026280548z^{4} - 315041732310600z^{3} \\
& + 166035507569100z^{2} - 51726446588835z + 7217643710070 \\
B^{\#}_{14,0} = {} & 54694z^{14} - 17228610z^{13} + 1075065264z^{12} \\
& - 27414164232z^{11} + 370091217132z^{10} - 3013599910932z^{9} \\
& + 15954352469640z^{8} - 57435668890704z^{7} \\
& + 144213472975572z^{6} - 255146913726012z^{5} \\
& + 316734099797808z^{4} - 269943835054950z^{3} \\
& + 150397279530615z^{2} - 49320565352145z + 7217643710070 \\
K^{\#}_{14,0} = {} & 163613471287384z^{14} - 34802000734704264z^{13} + 1817391854469497292z^{12} \\
& - 41279776184184885378z^{11} + 511451389462905300249z^{10} \\
& - 3889993184036806675896z^{9} + 19459875151016939648052z^{8} \\
& - 66742985230004787962220z^{7} + 160639639553407956137505z^{6} \\
& - 273729016682091111345750z^{5} + 328513030934711203617660z^{4} \\
& - 271518908064701256378900z^{3} + 147079259156386264856850z^{2} \\
& - 46996665144124333289400z + 6713809306303476184200
\end{aligned}
$$

Setting

$$
\begin{aligned}
F_{14,0}(x,y) = {} & 16081960716183328596163062459067095x^{13} \\
& - 9531549189205689818473481599704387 9x^{12}y \\
& + 2487863202825905859826696971938137 74x^{11}y^{2} \\
& - 3763666391217443893174573411586148 24x^{10}y^{3} \\
& + 3654725656939951126536958956786011 88x^{9}y^{4} \\
& - 2383569247324620272041493388651395 88x^{8}y^{5} \\
& + 1061660258728049348402802868060518 24x^{7}y^{6} \\
& - 32202194222781556839906771562778448x^{6}y^{7} \\
& + 6515393037852325418599013184588012x^{5}y^{8} \\
& - 843345574480971712492022357472348x^{4}y^{9} \\
& + 64927370960819614086248526357888x^{3}y^{10} \\
& - 2613328852222066663267892558100x^{2}y^{11} \\
& + 42491140539608352987480589614xy^{12} \\
& - 135373722566161155430179998y^{13} \\
G_{14,0}(x,y) = {} & 16081960716183328596163062459067095x^{13} \\
& - 100676145464118007716789170150066244x^{12}y \\
& + 278771266389255848866896517808487612x^{11}y^{2} \\
& - 449697515963501801426720104029663126x^{10}y^{3} \\
& + 468507050848111861213331099073579855x^{9}y^{4} \\
& - 330282656538347376302815645153153398x^{8}y^{5} \\
& + 160503645414233690985067374601203120x^{7}y^{6} \\
& - 53754745220736691551385089149002620x^{6}y^{7} \\
& + 12201653749896332062590242813939505x^{5}y^{8} \\
& - 1811811131106426461897599110642000x^{4}y^{9} \\
& + 165466932299883388917261787036860x^{3}y^{10} \\
& - 8355918025805974118330433499530x^{2}y^{11} \\
& + 190901137541784504341875653735xy^{12} \\
& - 1220749510126455250479377570y^{1}3
\end{aligned}
$$

we can see that

$$
\begin{aligned}
& F_{14}(x,y)A^{*}_{14}(x,y) - G_{14}(x,y)B^{*}_{14}(x,y) \\
& = -2764569626769982477475860070400000y^{14}x^{13} \\
& B^{*}_{13}(x,y)A^{*}_{14}(x,y) - A^{*}_{13}(x,y)B^{*}_{14}(x,y) = 30601960y^{27}
\end{aligned}
$$

Since $B_{14}(0,1) = 5469$ and $K_{14}(z_1) > 671380925930681104007 6$,formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 167.35\Delta^{\frac{5}{28}}$$

this contradiction, proves non-vanishing of $\Sigma_{(14,0)}$.

r=15

$$\begin{aligned}
A^{\#}_{15,0}(z) = &-246605z^{15} + 44388900z^{14} - 2263833900z^{13} \\
&+ 52973713260z^{12} - 696808074420z^{11} + 5748666613965z^{10} \\
&- 31768947077175z^{9} + 122537367297675z^{8} - 338203133741583z^{7} \\
&+ 676406267483166z^{6} - 981880065701370z^{5} + 1023885630009450z^{4} \\
&- 747159784060950z^{3} + 362085126121845z^{2} - 104655833796015z \\
&+ 13650760929915 \\
B^{\#}_{15,0}(z) = &-26158z^{15} + 9416880z^{14} - 672365232z^{13} \\
&+ 19666683036z^{12} - 305727527196z^{11} + 2882573827848z^{10} \\
&- 17804132466120z^{9} + 75540390606252z^{8} - 226621171818756z^{7} \\
&+ 488107139301936z^{6} - 757407629951280z^{5} + 839173226366475z^{4} \\
&- 647362203196995z^{3} + 330233350618710z^{2} - 100105580152710z \\
&+ 13650760929915 \\
K^{\#}_{15,0}(z) = &17898375136312z^{15} + 4351100634458147z^{14} - 259994842193246004z^{13} \\
&+ 6774021286078915836z^{12} - 96647473965326155740z^{11} \\
&+ 851160111253722402660z^{10} - 4967712033417551079585z^{9} \\
&+ 20080985161211047238715z^{8} - 57747913555709356686135z^{7} \\
&+ 119795821794959862592755z^{6} - 179717549112912333644550z^{5} \\
&+ 193102695263085916168575z^{4} - 144835773449177287631700z^{3} \\
&+ 71991953475180418982925z^{2} - 21304082910296402357625z \\
&+ 2840544388039520314350
\end{aligned}$$

setting

$$\begin{aligned}
F_{15,0}(x,y) = {} & 10608841267802658448327686619891890x^{14} \\
& - 68123327982300788729018622452576820x^{13}y \\
& + 194566233202521242858983803658624461x^{12}y^{2} \\
& - 325933705839696266676797826258367953x^{11}y^{3} \\
& + 355586047898432309109030891440420208x^{10}y^{4} \\
& - 265270226827715760641875285025265246x^{9}y^{5} \\
& + 138256478865942775048749452343172884x^{8}y^{6} \\
& - 50540942853850457834821171427766498x^{7}y^{7} \\
& + 12822620466797800991335912972826790x^{6}y^{8} \\
& - 2200439216462721116843319664341402x^{5}y^{9} \\
& + 244097842988546146854864994206186x^{4}y^{10} \\
& - 16231364736674936168446007289804x^{3}y^{11} \\
& + 567528671203688119104456558732x^{2}y^{12} \\
& - 8048892794225901061430511712xy^{13} \\
& + 22427241016498991359805734y^{14} \\
G_{15,0}(x,y) = {} & 10608841267802658448327686619891890x^{14} \\
& - 71659608404901674878461184659207450x^{13}y \\
& + 216095249055754543718842490407273191x^{12}y^{2} \\
& - 383874724987429350020909351317739910x^{11}y^{3} \\
& + 446380712028489137463269836702702830x^{10}y^{4} \\
& - 357111042505136534668378445916613290x^{9}y^{5} \\
& + 201089687590208498929678951602911649x^{8}y^{6} \\
& - 80163083272603204177118307682407900x^{7}y^{7} \\
& + 22445013271842962717917311303531660x^{6}y^{8} \\
& - 4318928869456449471643930302604020x^{5}y^{9} \\
& + 549333381838396230451655514623025x^{4}y^{10} \\
& - 43309263859855463266356355167060x^{3}y^{11} \\
& + 1898941985903806873256333221110x^{2}y^{12} \\
& - 37823099960348889581878989020xy^{13} \\
& + 211433204789117431160061665y^{14}
\end{aligned}$$

We can see that

$$\begin{aligned}
& F_{15}(x,y)A^{*}_{15}(x,y) - G_{15}(x,y)B^{*}_{15}(x,y) \\
& = -238083106529949930597670725667 50y^{15}x^{14} \\
& B^{*}_{14}(x,y)A^{*}_{15}(x,y) - A^{*}_{14}(x,y)B^{*}_{15}(x,y) = -586426690y^{29}
\end{aligned}$$

Since $B_{15}(0,1) = 26158$ and $K_{15}(z_1) > 2840544366735437404054$ formulas (A.0.2) and (A.0.1) imply:

$$|\varepsilon_1| < 117.11\Delta^{\frac{8}{45}}$$

this contradiction, proves non-vanishing of $\Sigma_{(15,0)}$.