Computing elliptic curves over $\mathbb{Q}$ via Thue-Mahler equations, and related problems

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Thue-Mahler equations

A definition

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- Let $S = \{p_1, \ldots, p_v\}$ be a set of rational primes

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- Let $S = \{p_1, \ldots, p_v\}$ be a set of rational primes
- A *Thue-Mahler equation* is a Diophantine equations of the form

$$F(x, y) = c_0 x^n + c_1 x^{n-1} y + \cdots + c_{n-1} x y^{n-1} + c_n y^n = u,$$

where u is an *S-unit*

Our main objective

Given S and F, find all solutions (x, y) satisfying $F(x, y) = u$

An obvious question

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- **Hambrook (2011)**: Implementation of a Thue-Mahler solver

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Elliptic Curves

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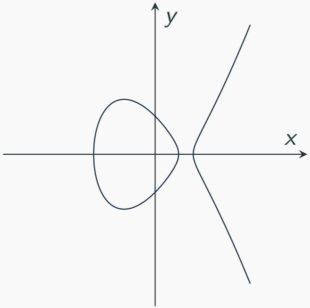
$$E : y^2 = x^3 + ax + b$$

- A curve is nonsingular $\iff \Delta_E = 4a^3 + 27b^2 \neq 0$.
- Together with the point "at infinity", the set of points on E form a group

$$E = \{(x, y) \ : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}.$$

Visualizing subgroups of E

$E : y^2 = x^3 - 2x + 1$ over $\mathbb{R}$



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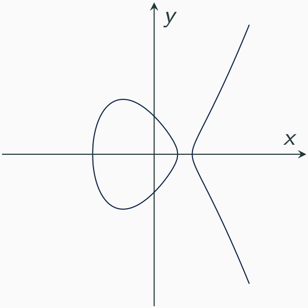
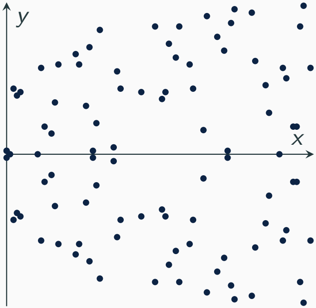


$E : y^2 = x^3 - 2x + 1$ over $\mathbb{F}_{89}$



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- $\Delta_E = p_1^{a_1} \cdots p_n^{a_n} \implies N = p_1^{b_1} \cdots p_n^{b_n}$

Computing Elliptic Curves

Shafarevich’s theorem

- Let S be a finite set of rational primes

Shafarevich's theorem

- Let S be a finite set of rational primes
- **Shafarevich (1963)**: There are at most finitely many elliptic curves over $\mathbb{Q}$ having good reduction outside S

Our motivation

Given S, find all such curves explicitly!

Some history

- Taniyama, Weil (1950s, 1960's): Are all $E/\mathbb{Q}$ of a given conductor N related to modular functions?

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    - $S = \{2\}$: Ogg (1966)
    - $S = \{2, 3\}$: Coghlan (1967)
    - $S = \{p\}$ for certain small primes p: Setzer (1975)
    - $S = \{11\}$: Agrawal, Coates, Hunt, and van der Poorten (1980)

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    - Antwerp IV (1972): $N \leq 200$
    - Tingley (1975): $N \leq 320$
    - Cremona (2019): $N \leq 500000$

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- Goal: compute all curves over $\mathbb{Q}$ having conductor $N \leq 10^6$

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- Ideal Goal: $N \leq 10^8$

Theorem (Bennett, G., Rechnitzer)

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = 2^{\alpha}3^{\beta}N_0$ where N_0 is coprime to 6.

Then there exists an integral binary cubic form F of discriminant

$$D_F = sign(\Delta_E)2^{\alpha_0}3^{\beta_0}N_1,$$

and relatively prime integers u and v with

$$F(u,v) = c_0u^3 + c_1u^2v + c_2uv^2 + c_3v^3 = 2^{\alpha_1}3^{\beta_1}\prod_{p|N_0} p^{\kappa_p}$$

such that E is isomorphic over $\mathbb{Q}$ to $E_{\mathcal{D}}$, where

$$E_{\mathcal{D}} : 3^{[\beta_0/3]}y^2 = x^3 - 27\mathcal{D}^2H_F(u,v)x + 27\mathcal{D}^3G_F(u,v).$$

Theorem (Bennett, G., Rechnitzer)

Here, $N_1 \mid N_0$,

$$(\alpha_0, \alpha_1) = \begin{cases} (2,0) \textit{ or } (2,3) & \textit{if } \alpha = 0 \\ (3, \geq 3) \textit{ or } (2, \geq 4) & \textit{if } \alpha = 1 \\ (2,1), (4,0) \textit{ or } (4,1) & \textit{if } \alpha = 2 \\ (2,1), (2,2), (3,2), (4,0) \textit{ or } (4,1) & \textit{if } \alpha = 3 \\ (2, \geq 0), (3, \geq 2), (4,0) \textit{ or } (4,1) & \textit{if } \alpha = 4 \\ (2,0) \textit{ or } (3,1) & \textit{if } \alpha = 5 \\ (2, \geq 0), (3, \geq 1), (4,0) \textit{ or } (4,1) & \textit{if } \alpha = 6 \\ (3,0) \textit{ or } (4,0) & \textit{if } \alpha = 7 \\ (3,1) & \textit{if } \alpha = 8, \end{cases}$$

Theorem (Bennett, G., Rechnitzer)

$$(\beta_0, \beta_1) = \begin{cases} (0,0) & \text{if } \beta = 0 \\ (0, \geq 1) \text{ or } (1, \geq 0) & \text{if } \beta = 1 \\ (3,0), (0, \geq 0), \text{ or } (1, \geq 0) & \text{if } \beta = 2 \\ (\beta, 0) \text{ or } (\beta, 1) & \text{if } \beta \geq 3, \end{cases}$$

$$\mathcal{D} = \prod_{p \mid \gcd(c_4(E), c_6(E))} p^{\min\{[\nu_p(c_4(E))/2], [\nu_p(c_6(E))/3]\}},$$

and $\kappa_p \in \mathbb{Z}_{>0}$ *with* $\kappa_p \in \{0, 1\}$ *whenever* $p^2 | N_1$.

Further,

$$\text{if } \beta_0 \geq 3, \text{ then } 3 | c_1 \text{ and } 3 | c_2$$

and

$$\text{if } \nu_p(N) = 1, \text{ for } p \geq 3, \text{ then } p \mid D_F F(u, v)$$

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3. Check "local" conditions and output the elliptic curves that arise

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Applying the Thue-Mahler solver

Representative forms: an example

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- Let $S = \{2, 3, 5, 7, 11, 13\}$
- There are 7893 corresponding forms which need to be solved
  - $F(x, y) = 19x^3 + 69x^2y + 76xy^2 + 126y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
  - $F(x, y) = 22x^3 + 22x^2y + 55xy^2 + 100y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
  - $F(x, y) = 46x^3 + 24x^2y + 85xy^2 + 7y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
  - $F(x, y) = 13x^3 + 3x^2y + 18xy^2 + 14y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
  - $F(x, y) = 17x^3 + 36x^2y + 39xy^2 + 26y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
  - $F(x, y) = 2x^3 + 18x^2y + 21xy^2 + 65y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$
  - $F(x, y) = x^3 + 12x^2y + 18xy^2 + 149y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5} \cdot 13^{z_6}$

$$\vdots$$

Some timing examples

- A nice example

$$x^3 + 3xy^2 + 44xy^2 + 66y^3 = 3^{z_1} \cdot 11^{z_2} \cdot 17^{z_3} \cdot 23^{z_4} \cdot 31^{z_5}$$

- A less nice example

$$3x^3 + 3xy^2 + 44xy^2 + 66y^3 = 3^{z_1} \cdot 11^{z_2} \cdot 17^{z_3} \cdot 23^{z_4} \cdot 31^{z_5}$$

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Total time: 247.580

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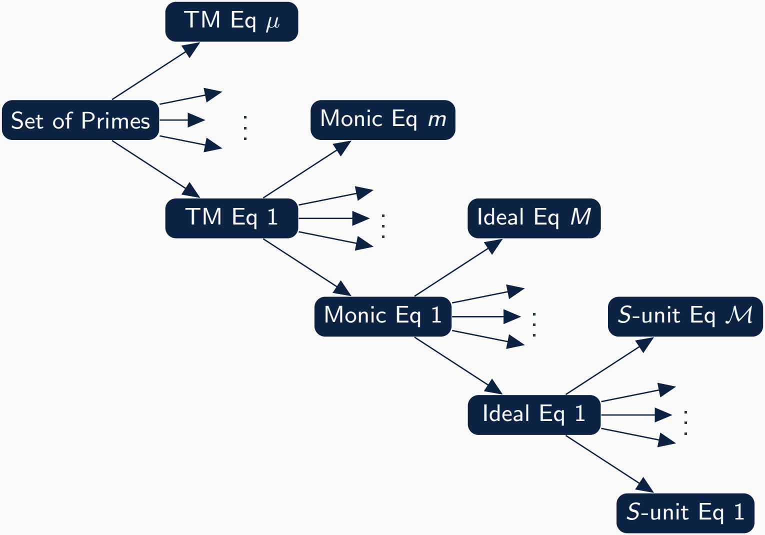
Total time: 2031.450

Algorithms for solving Thue-Mahler equations

Overview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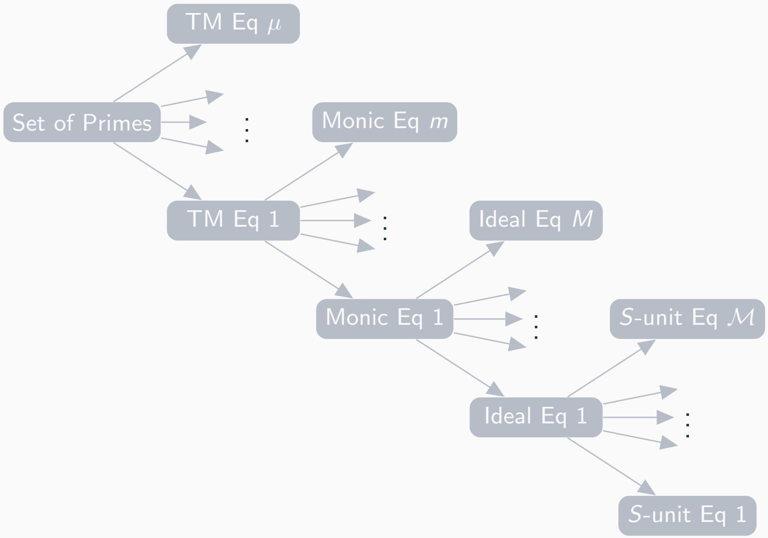
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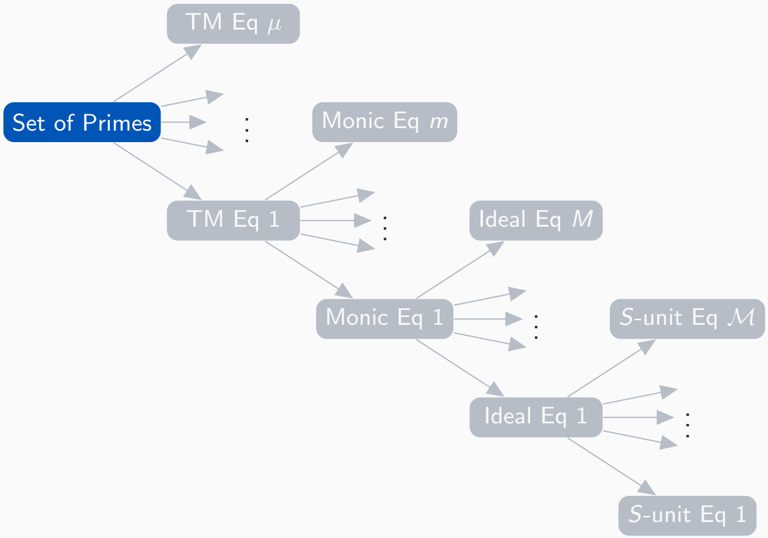

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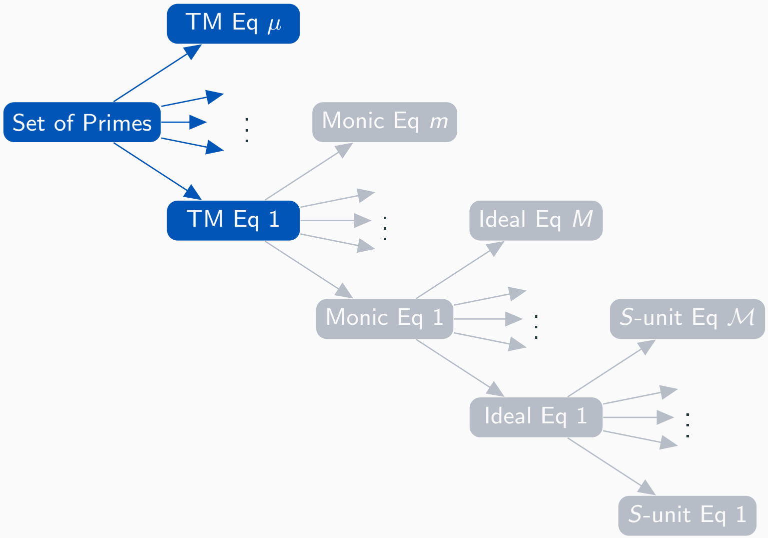
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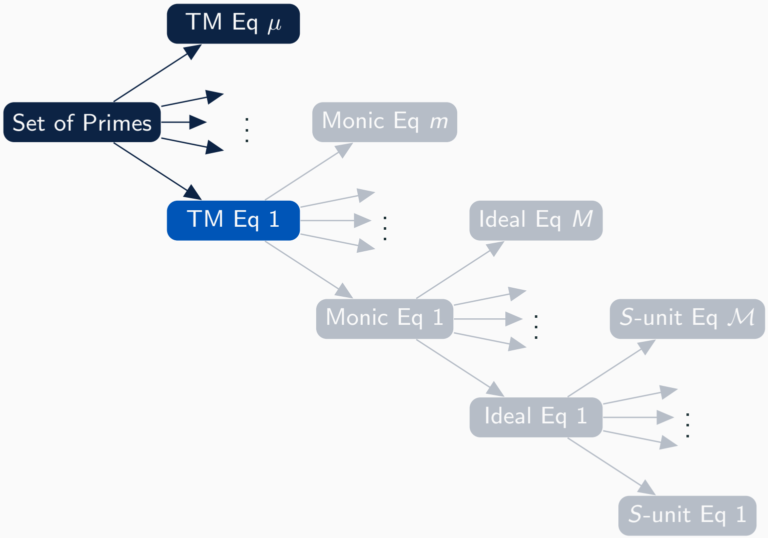

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First steps

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- Fix $a \in \mathbb{Z}$ and a set of distinct rational primes $S = \{p_1, \ldots, p_v\}$
- Reduce problem to solving a number of *Thue-Mahler equations*,

$$F(x, y) = c_0 x^3 + c_1 x^2 y + c_2 x y^2 + c_3 y^3 = a p_1^{z_1} \cdots p_v^{z_v}$$

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- Without loss of generality, we may assume

$$(x, y) = (y, c_0) = 1 \quad \text{and} \quad (a, p_1, \ldots, p_v) = 1$$

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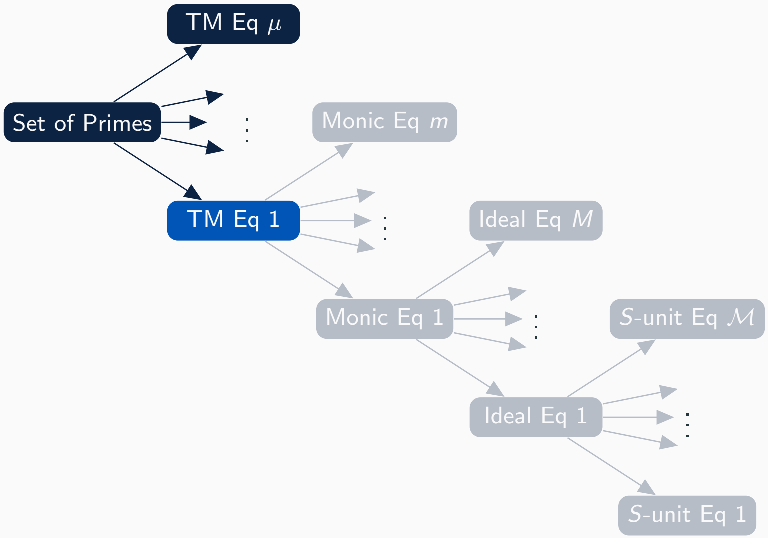
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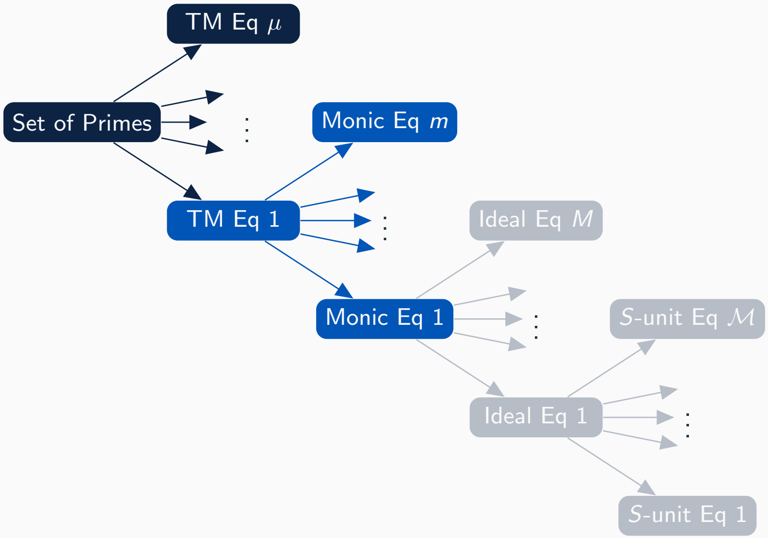

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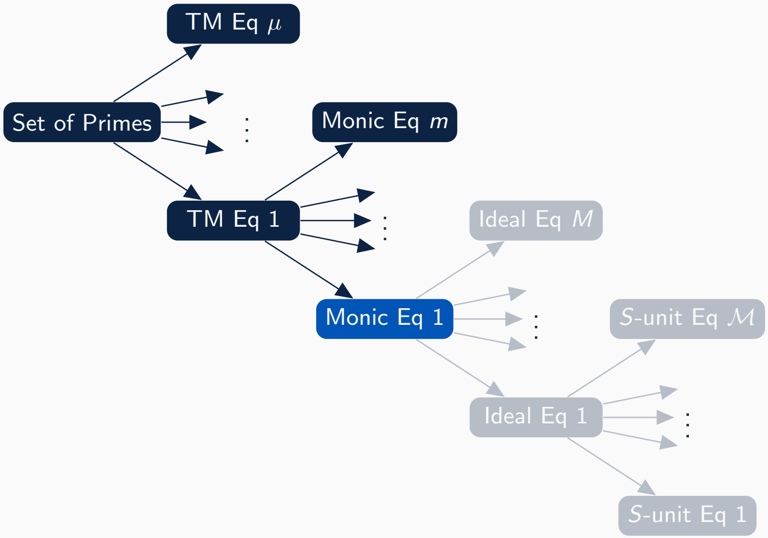
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- Solving $F(x, y) = a p_1^{z_1} \cdots p_v^{z_v}$ is equivalent to solving

$$(x - y\theta)\mathcal{O}_K = \mathfrak{a} \prod_{j=1}^{m_1} \mathfrak{p}_{1j}^{z_{1j}} \cdots \prod_{j=1}^{m_v} \mathfrak{p}_{vj}^{z_{vj}},$$

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- $\mathfrak{a}$ is an ideal with norm $|a|$
- z_{ij} are unknown non-negative integers to be solved for

The prime ideal removing lemma

- Applying the PIRL, we are left with a set of *ideal equations*

$$(x - y\theta)\mathcal{O}_K = \mathfrak{a}\mathfrak{p}_1^{u_1} \cdots \mathfrak{p}_\nu^{u_\nu}$$

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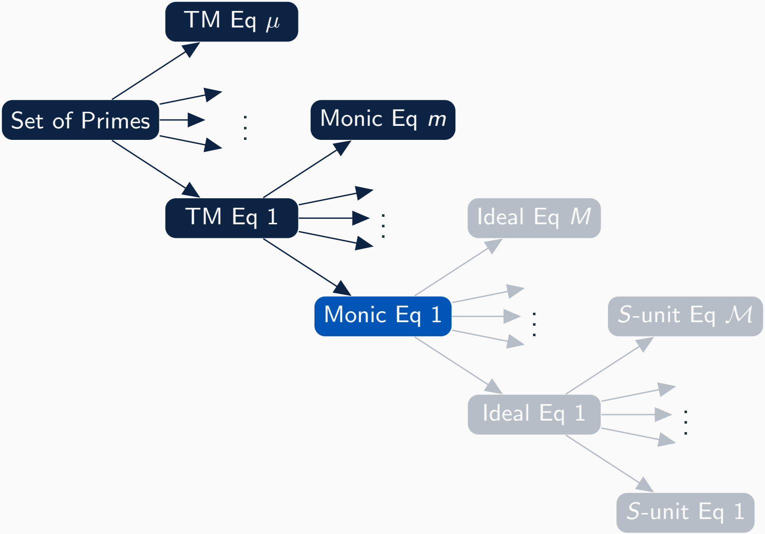
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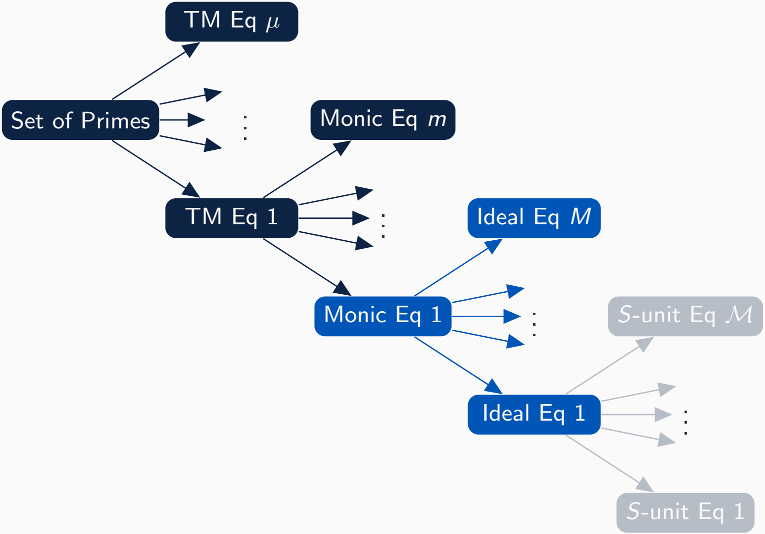
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Example: the number of ideal equations

- A nice example

$$x^3 + 20xy^2 + 24xy^2 + 15y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3} \cdot 37^{z_4} \cdot 53^{z_5}$$

- A less nice example

$$7x^3 + xy^2 + 29xy^2 - 25y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3}$$

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- Number of Ideal Equations: 32
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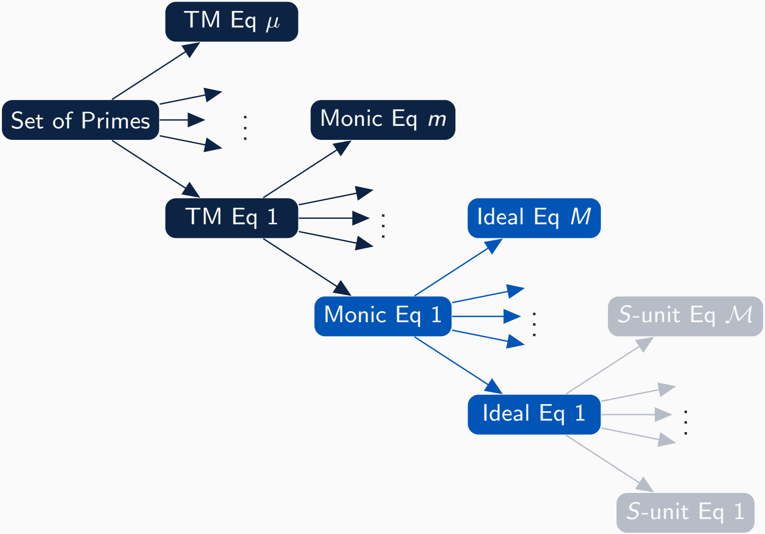
- Number of Ideal Equations: 26,136

Overview



TM Eq μ
Set of Primes
$\vdots$
Monic Eq m
TM Eq 1
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Ideal Eq M
Monic Eq 1
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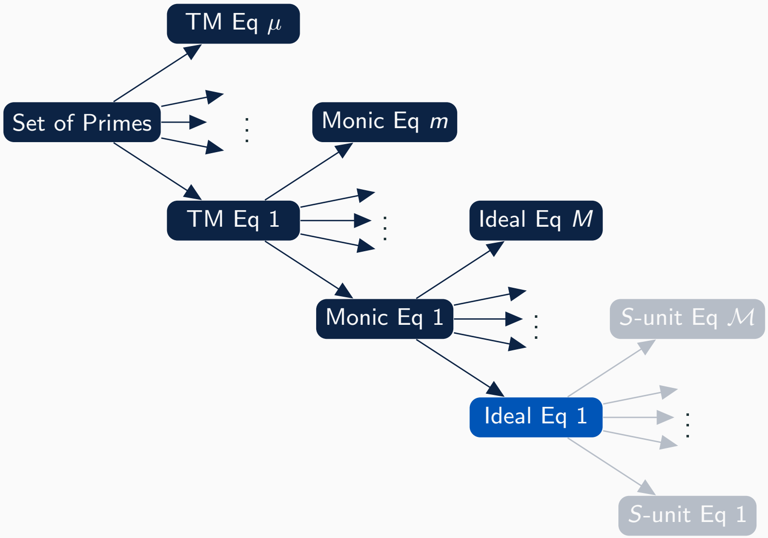
S-unit Eq $\mathcal{M}$
Ideal Eq 1
$\vdots$
S-unit Eq 1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Principalization tests

- Applying a number of principalization tests, we are left with a set of *S-unit equations*

$$x - y\theta = \alpha \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r} \gamma_1^{n_1} \cdots \gamma_\nu^{n_\nu}$$

Principalization tests

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  - $\alpha, \varepsilon_i, \gamma_i$ are computed directly from the ideal equation

Principalization tests

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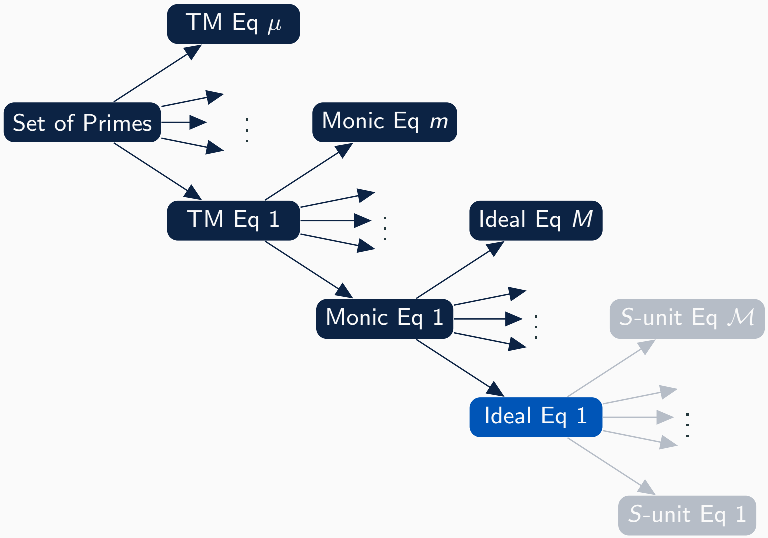
- Applying a number of principalization tests, we are left with a set of *S-unit equations*

$$x - y\theta = \alpha \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r} \gamma_1^{n_1} \cdots \gamma_\nu^{n_\nu}$$

  - $\alpha, \varepsilon_i, \gamma_i$ are computed directly from the ideal equation
  - a_i, n_i are unknown

Overview



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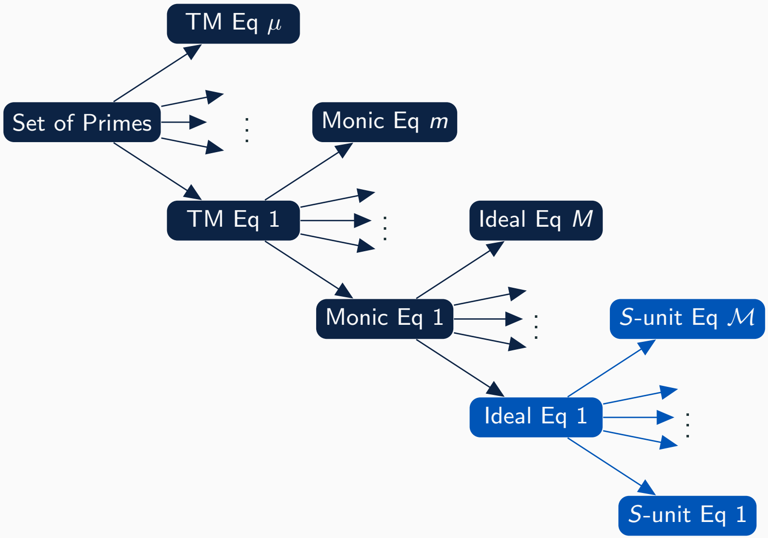
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Example: the number of S-unit equations

- A nice example

$$5x^3 + 6xy^2 - 17xy^2 - 8y^3 = 2^{z_1} \cdot 37^{z_2} \cdot 1657^{z_3}$$

Number of ideal equations: 48
Number of principalization tests: 48
Number of S-unit equations: 48
Total time: 1 second

Example: the number of S-unit equations

- A less nice example

$$7x^3 + 12xy^2 + 14y^3 = 7^{z_1} \cdot 11^{z_2} \cdot 37^{z_3}$$

Number of ideal equations: 16
Number of principalization tests: 11,664
Number of S-unit equations: 1296
Total time: 24 seconds

Example: the number of S-unit equations

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- A really bad example

$$2x^3 + 20x^2y - 14xy^2 + 37y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 11^{z_4} \cdot 13^{z_5} \cdot 17^{z_6}$$

  Number of ideal equations: 448
  Number of principalization tests: 7,560,000
  Number of S-unit equations: 139,264
  Total time: 4 hours

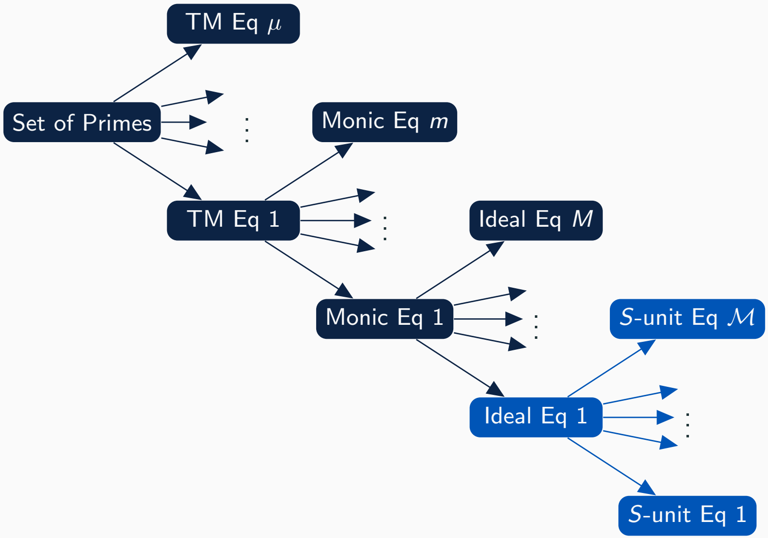
Example: the number of S-unit equations

- A really, really bad example

$$14x^3 + 20x^2y + 24xy^2 + 15y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3} \cdot 37^{z_4} \cdot 53^{z_5}$$

Number of ideal equations: 48
Number of principalization tests: 113,848,416
Number of S-unit equations: ????
Total time: ????

Overview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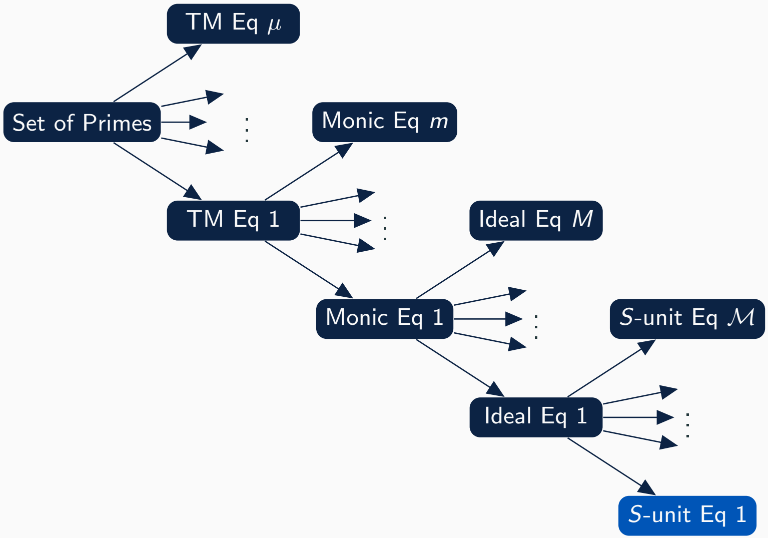
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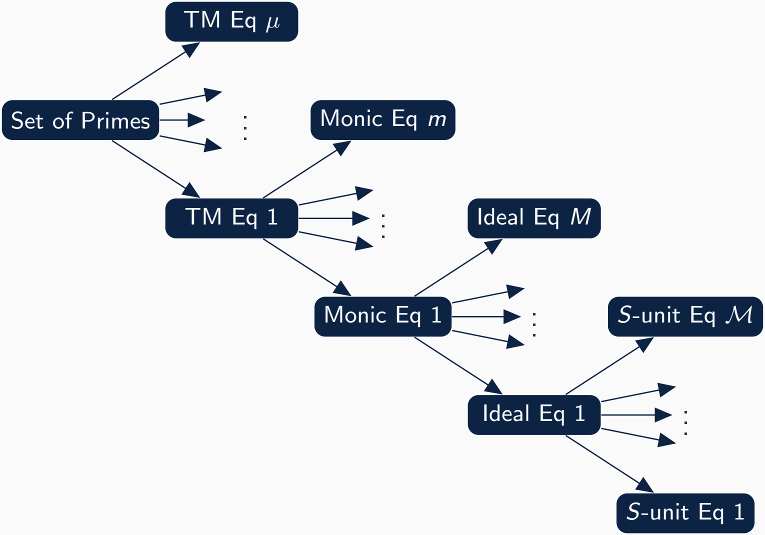
For each S-unit equation...

- Generate a very large upper bound on the solutions using the theory of linear forms in logarithms
- Reduce this bound via Diophantine approximation computations
- Search below this reduced bound

Refined algorithms for solving Thue-Mahler equations

Overview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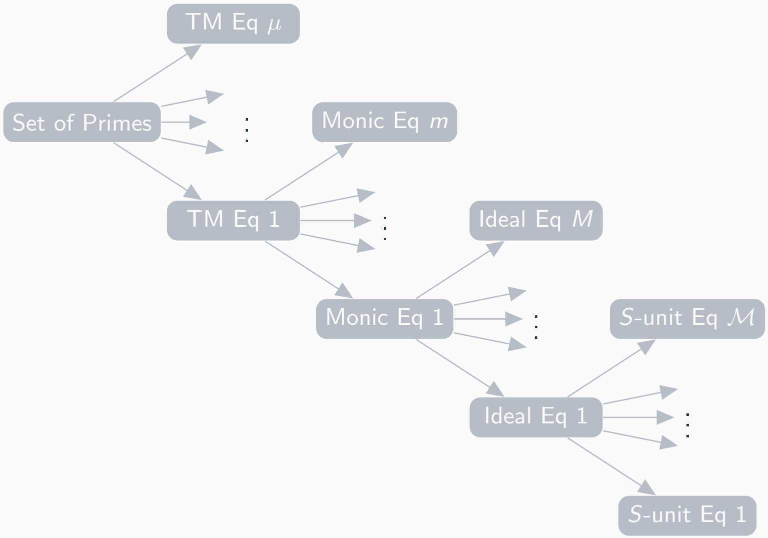
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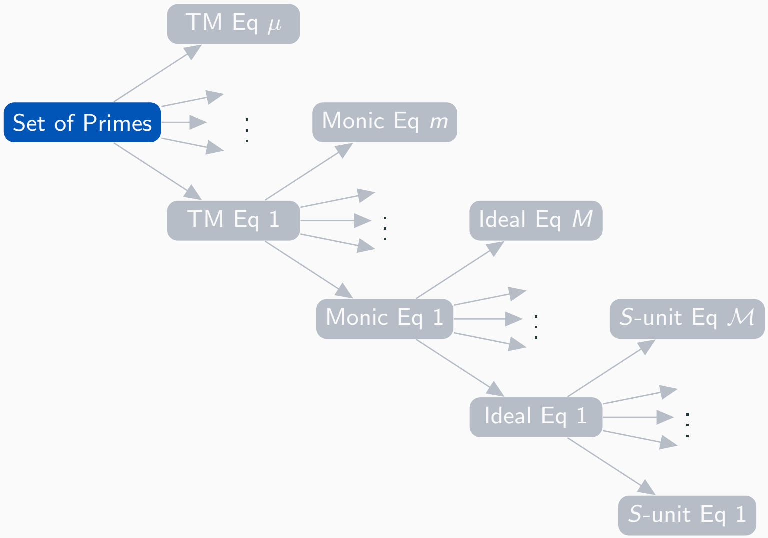
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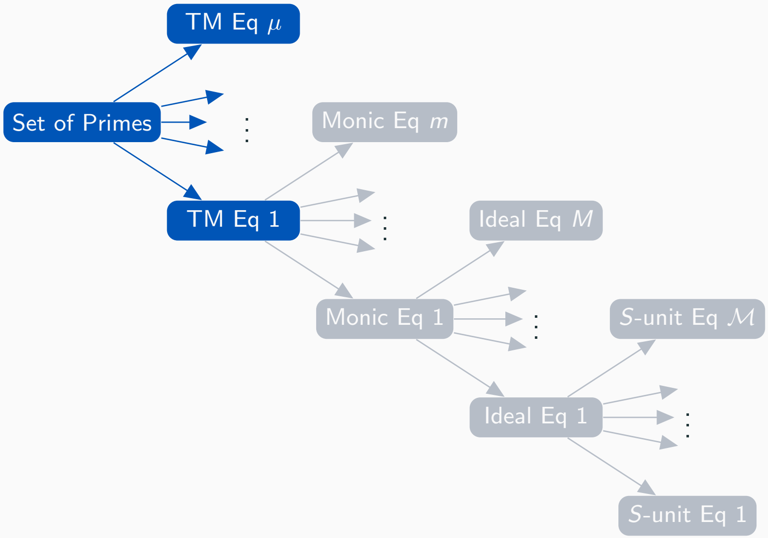
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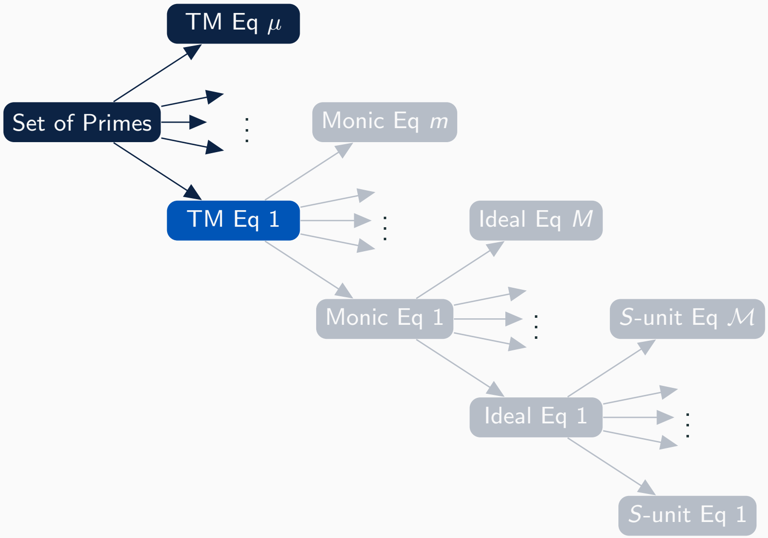
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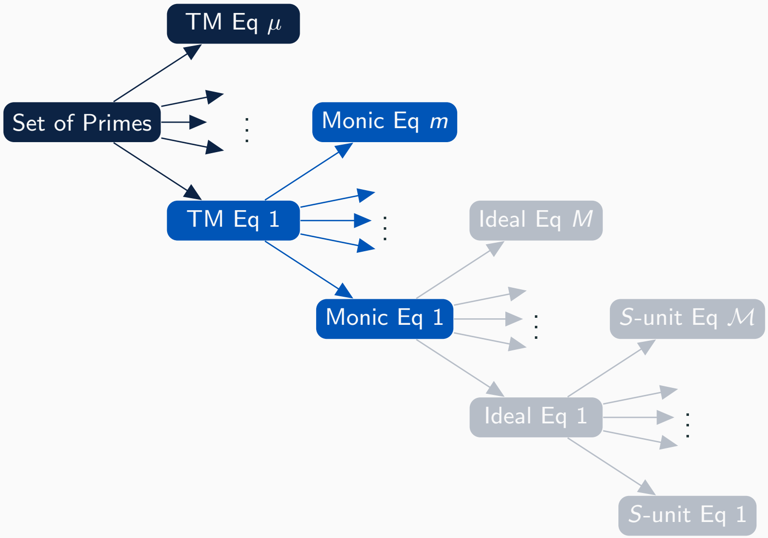

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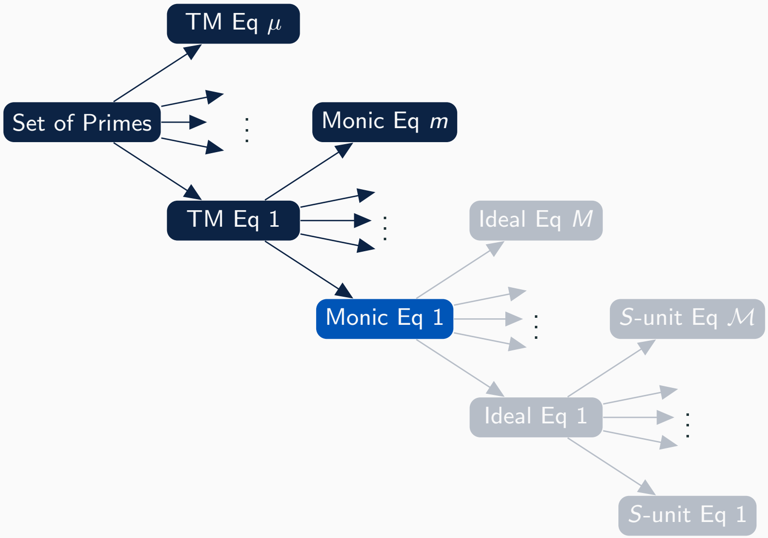
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Our refinements

- **Improved PIRL**: reduces the number of ideal equations

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- **Improved principalization test**: reduces the number of S-unit equations

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  - Reduced test run-time

Our refinements

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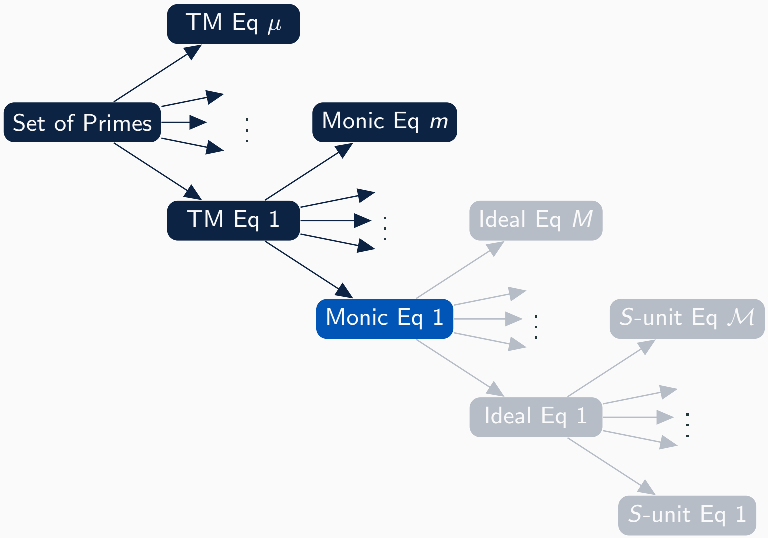
- **Improved PIRL**: reduces the number of ideal equations
- **Improved principalization test**: reduces the number of S-unit equations
  - Reduced test run-time
  - Yields only 1 S-unit equation for each ideal equation

Overview



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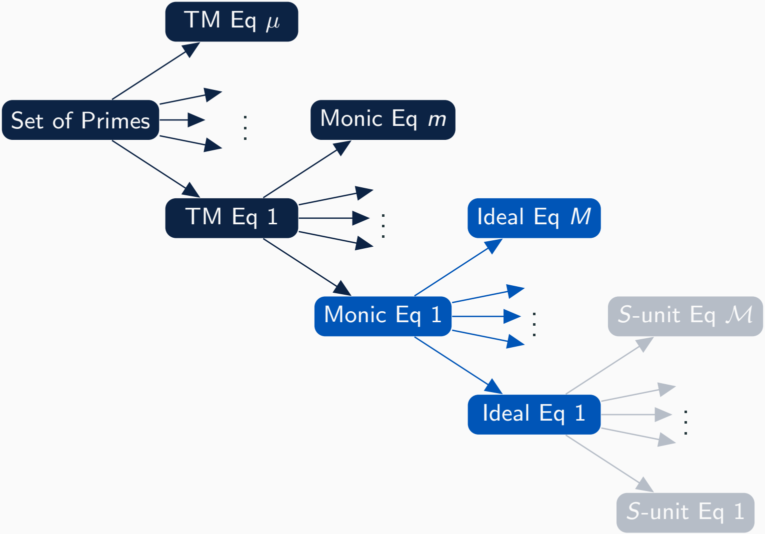
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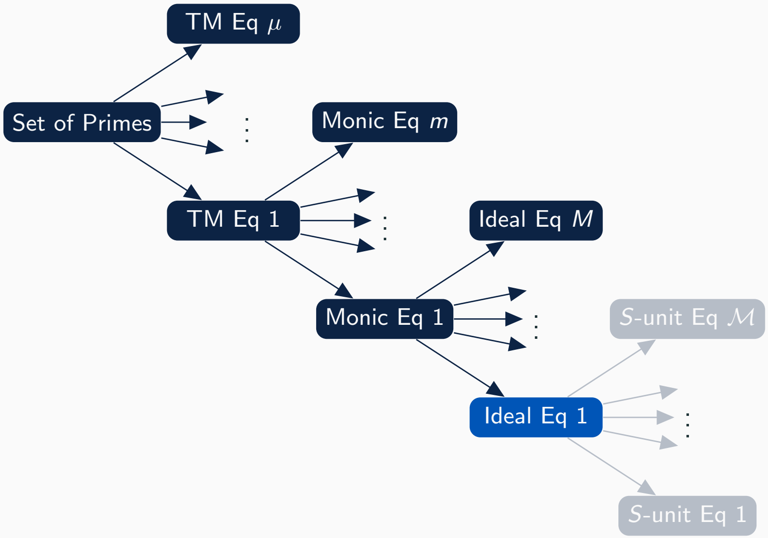
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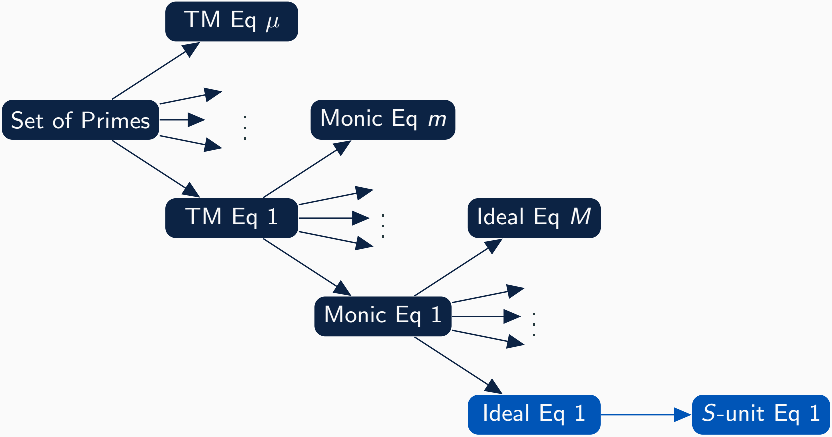
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Example: The number of S-unit equations

- A less nice example

$$7x^3 + 12xy^2 + 14y^3 = 7^{z_1} \cdot 11^{z_2} \cdot 37^{z_3}$$

Number of ideal equations: 16
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Example: The number of S-unit equations

- A less nice example

$$7x^3 + 12xy^2 + 14y^3 = 7^{z_1} \cdot 11^{z_2} \cdot 37^{z_3}$$

Number of ideal equations: 16

Now: 15

Number of principalization tests: 11,664

Now: 15

Number of S-unit equations: 1296

Now: 15

Total time: 24 seconds

Now: 0.12 seconds

Example: The number of S-unit equations

- A really bad example

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Number of ideal equations: 448

Now: 162

Number of principalization tests: 7,560,000

Now: 162

Number of S-unit equations: 139,264

Now: 122

Total time: 4 hours

Now: 0.85 seconds

Example: The number of S-unit equations

- This god-forsaken example

$$14x^3 + 20x^2y + 24xy^2 + 15y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3} \cdot 37^{z_4} \cdot 53^{z_5}$$

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Number of S-unit equations: ????
Total time: ????

Example: The number of S-unit equations

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- This god-forsaken example

$$14x^3 + 20x^2y + 24xy^2 + 15y^3 = 2^{z_1} \cdot 3^{z_2} \cdot 17^{z_3} \cdot 37^{z_4} \cdot 53^{z_5}$$

Number of ideal equations: 48

Now: 162

Number of principalization tests: 113,848,416

Now: 162

Number of S-unit equations: ????

Now: 132

Total time: ????

Now: 1.08 seconds

Current Ideas and Future Work

Our next steps

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- Refine and optimize the Thue-Mahler solver at each S-unit equation

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- Refine and optimize the Thue-Mahler solver at each S-unit equation
- Compute all elliptic curves of conductor $N < 10^6$

Our next steps

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- Refine and optimize the Thue-Mahler solver at each S-unit equation
- Compute all elliptic curves of conductor $N < 10^6$
- Generalize the Thue-Mahler solver over number fields

Thank You