Finding Discrete Subspaces of Hausdorff CSC Spaces

Andrew DeLapo

University of Connecticut
Storrs, CT

Reverse Mathematics: New Paradigms
August 6, 2025

Outline

Outline

Ginsburg-Sands Theorem

Benham, D., Dzhafarov, Solomon, and Villano analyzed the following theorem from the perspective of reverse mathematics by restricting to CSC spaces.

Theorem (Ginsburg-Sands, 1979)

Every infinite topological space has a subspace homeomorphic to one of the following topologies on ω:

1. *indiscrete*
2. *initial segment*
3. *final segment*
4. *cofinite*
5. *discrete*

Notice: if X is Hausdorff, the Ginsburg-Sands theorem implies X has a discrete subspace.

CSC Spaces

Definition (Dorais 2011)

A **countable second-countable space** (**CSC space**) is a triple $(X, \mathcal{U}, k)$ where X is a countable set, $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ is a countable basis for open sets in X, and k is a function $X \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

- for all $x \in X$, there is $i \in \mathbb{N}$ such that $x \in U_i$,
- for all $x \in X$ and $i, j \in \mathbb{N}$, if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

CSC spaces provide an excellent context for studying topological facts in computability theory and reverse mathematics (Dorais 2011, Shafer 2020, Benham, D., Dzhafarov, Solomon, and Villano 2024, Genovesi 2024, D. and Gonzalez 2025).

Reverse Mathematics

Definition (RCA_0)

Let GST_2 be the following principle: every Hausdorff CSC space $X = (\mathbb{N}, \mathcal{U}, k)$ has an infinite discrete subspace $A \subseteq X$.

Theorem (D.)

$\mathsf{RCA}_0 \vdash \mathsf{GST}_2$.

Classical Proof of GST_2

Proof.

Let $X = (\mathbb{N}, \mathcal{U}, k)$ be a Hausdorff CSC space. If X is discrete, then X is a discrete subspace of itself.

Assume X is not discrete. Then X has a limit point: a point $p \in X$ such that if $p \in U_i$, then there is $x \in U_i$ with $x \neq p$. Fix such a point p.

Let $x_0 \in X$, $x_0 \neq p$. Since X is Hausdorff, there are i_0, j_0 such that $x_0 \in U_{i_0}$, $p \in U_{j_0}$, and $U_{i_0} \cap U_{j_0} = \emptyset$.

Since p is a limit point, there is $x_1 \in U_{j_0}$, $x_1 \neq p$. Repeat the argument within U_{j_0}, taking necessary intersections. Obtain a sequence of points $x_0, x_1, x_2, \ldots \in X$, and see that $A = \{x_0, x_1, x_2, \ldots\}$ is an infinite discrete subspace. $\square$

Formalizing the proof in RCA_0 uses the following lemma:

Lemma (D.)

The following is provable in RCA_0. Let $(X, \mathcal{U}, k)$ be a Hausdorff CSC space with $\mathcal{U} = \langle U_n : n \in \mathbb{N} \rangle$. There exists a function $c : X \times X \times \mathbb{N} \to \mathbb{N}$ such that for all distinct $x, y \in X$,

1. *for all $s \in \mathbb{N}$, $c(x, y, s)$ is the least pair $\langle m, n \rangle$ such that $x \in U_m \setminus U_n$, $y \in U_n \setminus U_m$, and $(U_m \upharpoonright s) \cap (U_n \upharpoonright s) = \emptyset$, and*
2. *for all sufficiently large $s \in \mathbb{N}$, $c(x, y, s)$ is the least pair $\langle m, n \rangle$ such that $x \in U_m$, $y \in U_n$, and $U_m \cap U_n = \emptyset$.*

Think of c as a Δ^0_2 approximation to a function taking pairs of points to pairs of disjoint separating basic open sets.

Reverse Mathematics

Definition (Dorais 2011)

The following definitions are made within RCA_0. Let $(X, \mathcal{U}, k)$ be a CSC space.

- X is **effectively Hausdorff** if there is a function $h : X \times X \to \mathbb{N}$ such that for all distinct $x, y \in X$, $h(x, y) = \langle m, n \rangle$ where $x \in U_m$, $y \in U_n$, and $U_m \cap U_n = \emptyset$.
- A subspace $A \subseteq X$ is **effectively discrete** if there is a function $d : A \to \mathbb{N}$ such that for all $x \in A$, $A \cap U_{d(x)} = \{x\}$.

Theorem (Dorais 2011)

The following are equivalent over RCA_0*.*

1. ACA_0.
2. *Every Hausdorff CSC space is effectively Hausdorff.*
3. *Every discrete CSC space is effectively discrete.*

Reverse Mathematics

We have a slight improvement:

Theorem (D.)

The following are equivalent over RCA_0.

1. ACA_0.
2. *Every discrete, effectively Hausdorff CSC space is effectively discrete.*

For finding subspaces, we have:

Theorem (Benham, D., Dzhafarov, Solomon, and Villano 2024)

The following are equivalent over RCA_0.

1. ACA_0.
2. *Every discrete CSC space has an effectively discrete subspace.*

Reverse Mathematics

So far:

- RCA_0 proves GST_2 ("every Hausdorff CSC space has an effectively discrete subspace").
- ACA_0 is needed to prove "every discrete CSC space has an effectively discrete subspace."

What's missing?

Definition

Let EGST_2 be the following principle: every infinite discrete effectively Hausdorff CSC space has an infinite effectively discrete subspace.

Theorem (D.)

$\mathsf{RCA}_0 \nvdash \mathsf{EGST}_2$.

A Detour

We step outside of reverse mathematics for this slide. Let $(X, \mathcal{U}, k)$ be a CSC space.

Definition

The CSC space X is **computable** if $\mathcal{U} = (U_i)_{i \in \omega}$ is uniformly computable and k is computable.

- X is **effectively Hausdorff** if there is a computable function $h : X \times X \to \omega$ such that for all distinct $x, y \in X$, $h(x, y) = \langle m, n \rangle$ where $x \in U_m$, $y \in U_n$, and $U_m \cap U_n = \emptyset$.
- A subspace $A \subseteq X$ is **effectively discrete** if there is a computable function $d : A \to \omega$ such that for all $x \in A$, $A \cap U_{d(x)} = \{x\}$.

Proposition (D.)

If $(X, \mathcal{U}, k)$ is effectively Hausdorff and **not discrete**, then the classical proof is effective: X has an infinite effectively discrete subspace.

Theorem (D.)

There is an infinite, computable, discrete, effectively Hausdorff CSC space having no computable, infinite, effectively discrete subspace.

Proof sketch

Build a sequence $(V_n)_{n\in\omega}$ of basic open sets satisfying the following requirements via finite injury:

$$\begin{aligned} \mathcal{D}_x &: \text{there exists } n \in \omega \text{ such that } V_n = \{x\} \\ \mathcal{R}_{e,u} &: \text{if } \Phi_e \text{ is the characteristic function of an infinite set } Y \\ &\quad \text{and } \Phi_u \text{ is total on } Y\text{, then there is } x \in Y \text{ such that} \\ &\quad \Phi_u(x)\downarrow = \langle n_0, \dots, n_r \rangle \text{ and } \bigcap_{i\le r} V_{n_i} \neq \{x\} \end{aligned}$$

Then close $(V_n)_{n\in\omega}$ under finite intersection (via primitive recursion) and get $(\omega, \mathcal{U}, k)$ with the desired properties.

Observations

The classical proof of GST_2 broke up into two cases:

- If X is discrete, then X is a discrete subspace of itself.
- If X is not discrete, then given a limit point of X, we compute an infinite discrete subspace of X.

But for EGST_2, the "difficult" case is when X is already discrete!

Definition

Let SEGST_2 be the restriction of EGST_2 to stable CSC spaces, where every basic open set is either finite or cofinite.

The earlier construction in fact produces a stable CSC space.

Corollary

$\mathsf{RCA}_0 \nvdash \mathsf{SEGST}_2$.

Reverse Mathematics

Theorem (D., Dzhafarov, Genovesi, Shafer)

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{EGST}_2$, *and* $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{SEGST}_2$.

Proof.

Let $(\mathbb{N}, \mathcal{U}, k)$ be an effectively Hausdorff CSC space, witnessed by h. For $\mathsf{RT}^2_2 \to \mathsf{EGST}_2$, note that $\mathsf{RT}^2_2 \to \mathsf{COH}$, so we can assume the space is stable. For each i, let $(p_i, q_i) = h(2i, 2i+1)$. Define $c : [\mathbb{N}]^2 \to 2$ by

$$c(i,x) = \begin{cases} 0 & \text{if } x \in U_{p_i} \\ 1 & \text{if } x \notin U_{p_i} \end{cases}.$$

Let H be an infinite homogeneous set for c. If H has color 0, then for all $i \in H$, U_{p_i} is infinite, therefore cofinite, so each U_{q_i} is finite. Using $\mathsf{B}\Sigma^0_2$, we can find an effectively discrete subspace. The argument is symmetric if H has color 1. Finally, see that c is a stable coloring when the CSC space is stable. $\square$

Outline

GST_2 as an Instance-Solution Problem

Definition

GST_2 is the problem whose instances are infinite Hausdorff CSC spaces $X = (\omega, \mathcal{U}, k)$, and the solutions to such an X are exactly the infinite discrete subspaces $A \subseteq X$.

Theorem (D.)

GST_2 is computably true, but it does not uniformly admit computable solutions (there is no fixed Turing functional Φ computing an infinite discrete subspace of every CSC space).

The following lemma is helpful for this line of inquiry:

Lemma

For any $\sigma \in 2^{<\omega}$ and any $p \in \omega$, there is a Hausdorff CSC space $(\omega, \mathcal{U}, k)$ "extending σ" such that every U_i containing p is cofinite.

GST_2 as an Instance-Solution Problem

Proof.

Suppose Φ witnesses that GST_2 uniformly admits computable solutions. Search for $\sigma \in 2^{<\omega}$ and $p \in \omega$ such that $\Phi^\sigma(p)\downarrow= 1$.

By the lemma, there is an instance $X = (\omega, \mathcal{U}, k)$ of GST_2 such that $X \succeq \sigma$ and where every U_i containing p is cofinite. Note that $A = \Phi(X)$ is an infinite discrete subspace of X containing p. However, we claim that p cannot be a member of any infinite discrete subspace of X.

If A is an infinite discrete subspace containing p, then $A \cap U_i = \{p\}$ for some i. But U_i is cofinite, and A is infinite, so this is a contradiction. $\square$

Weihrauch Reduction

Definition

Let P and Q be problems. P is **Weihrauch reducible** to Q, written $\mathsf{P} \leq_W \mathsf{Q}$, if there are Turing functionals Φ and Ψ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\Phi"] \arrow[d, dashed, "\text{is solved by}"'] & Y \arrow[d, "\text{is solved by}"] \\
\hat{X} & \hat{Y} \arrow[l, "\Psi(X \oplus -)"]
\end{tikzcd}$$

GST_2 and other Problems

Definition

$\mathsf{IsDiscrete}$ is the problem whose instances are infinite Hausdorff CSC spaces $X = (\omega, \mathcal{U}, k)$, and the solution to such an X is 1 if X is discrete, and 0 otherwise.

Definition

$\mathsf{LimPoint}$ is the problem whose instances are infinite, non-discrete Hausdorff CSC spaces $X = (\omega, \mathcal{U}, k)$, and the solutions to X are exactly the limit points of X.

The proof of GST_2 within RCA_0 shows:

Proposition (D.)

$\mathsf{GST}_2 \leq_W \mathsf{LimPoint} * \mathsf{IsDiscrete}$.

GST_2 and other Problems

Theorem (D.)

$\mathsf{LimPoint} \not\leq_W \mathsf{GST}_2$.

Proof sketch.

Suppose Φ and Ψ witness $\mathsf{LimPoint} \leq_W \mathsf{GST}_2$. Search for finite sets X_0' and D_0' such that $\Psi(X_0' \oplus D_0') \downarrow= 0$, and write $D_0' = \{y_1, \ldots, y_n\}$. For any instance $X \succeq X_0$ of $\mathsf{LimPoint}$ which does not have 0 as a limit point, we have $\Psi(X \oplus D_0') \downarrow= 0$, so D_0' cannot be an initial segment of any infinite discrete subspace of $\Phi(X)$, for any such X. That means if $X \succeq X_0'$, $\Phi(X)$ must have a limit point among $y_1, \ldots, y_n$.

There is a functional Δ such that if X an instance of $\mathsf{LimPoint}$ and q is a limit point of $\Phi(X)$, then $\Delta(X, q)$ is a limit point p of X. Diagonalize against p being any of the y_i. $\square$

GST_2 and other Problems

Theorem (D.)

$\mathsf{LimPoint} \not\leq_W \mathsf{GST}_2$.

Corollary

$\mathsf{GST}_2 <_W \mathsf{LimPoint} * \mathsf{IsDiscrete}$.

In D. and Gonzalez 2025, we show that determining whether a Hausdorff CSC space has a limit point is Σ^0_3-complete. This allows us to formulate $\mathsf{LimPoint}$ and $\mathsf{IsDiscrete}$ in terms of known problems:

Theorem (D.)

- $\mathsf{LimPoint} \equiv_W \mathsf{C}'_{\mathbb{N}}$, *the problem of finding an element of a non-empty Π^0_2 set.*
- $\mathsf{IsDiscrete} \equiv_W \mathsf{sTC}''_1$, *the problem of finding an element of a Π^0_3 set with at most one element (or -1 if the set is empty).*

Outline

Computable Structure Theory for CSC Spaces

Definition

Let $(X, \mathcal{U}, k)$ be a CSC space.

- The **weak computable dimension** of a CSC space X is the number of computable copies of X up to computable homeomorphism.
- X is **weakly computably categorical** if it has weak computable dimension 1.

Proposition

The discrete topology $\mathbb{N}_{DIS} = (\omega, (\{i\})_{i \in \omega}, k)$ is weakly computably categorical.

However, we have seen that whether or not a discrete CSC space is effectively discrete depends on its presentation.

Computable Structure Theory for CSC Spaces

Definition (Dorais 2011)

Let $(X, \mathcal{U}, k)$ and $(Y, \mathcal{V}, \ell)$ be CSC spaces.

- A function $f : X \to Y$ is **effectively continuous** if f is computable and there is a computable function Φ such that for all x and i, if $f(x) \in V_i$, then $x \in U_{\Phi(x,i)} \subseteq f^{-1}(V_i)$.
- A function $f : X \to Y$ is an **effective homeomorphism** if f is a bijection and both f and f^{-1} are effectively continuous.

Definition

Let X be a CSC space.

- The **computable dimension** of X is the number of computable copies of X up to effective homeomorphism.
- X is **computably categorical** if X has computable dimension 1.

Computable Structure Theory for CSC Spaces

Definition

For a CSC space $(X, \mathcal{U}, k)$, a **discreteness function** for X is a function $d : X \to \omega$ such that $U_{d(x)} = \{x\}$ for all $x \in X$.

Lemma (D.)

Let X and Y be effectively homeomorphic CSC spaces, and let d_X be a discreteness function for X. Then Y has a discreteness function d_Y computable from d_X. In particular, if X is effectively discrete, then so is Y.

Theorem (D.)

For each $e \in \omega$, there is a computable discrete CSC space X_e such that X_e has a unique discreteness function d_e, and $d_e \equiv_T W_e$. Thus $\mathbb{N}_{DIS}$ has computable dimension ω.

References I

- Benham, Heidi et al. (2024). "The Ginsburg–Sands theorem and computability theory". In: *Advances in Mathematics* 444, p. 109618. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2024.109618.
- D., Andrew and David Gonzalez (2025). *Computability of Separation Axioms in Countable Second Countable Spaces*. arXiv: 2507.18564 [math.LO].
- Dorais, François G. (2011). *Reverse mathematics of compact countable second-countable spaces*. arXiv: 1110.6555 [math.LO].
- Genovesi, Giorgio G. (2024). *Reverse mathematics of regular countable second countable spaces*. arXiv: 2410.22227 [math.LO].
- Shafer, Paul (2020). "The strength of compactness for countable complete linear orders". In: *Computability* 9.1, pp. 25–36. ISSN: 2211-3568. DOI: 10.3233/com-190262.