COLUMN GENERATION

OR Software Tools, IAP 2015

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Background

- Many real-world optimization problems involve complex and tedious constraints which are hard to formulate through conventional approach:
  - *Amazon Fresh* wants to route n vehicles to serve m customers' grocery everyday. Each customer has a specific time window for pick up, each vehicle has a maximum distance to travel due to fuel constraints.
  - *American Airlines* wants to route n aircraft to serve m flights. Aircraft can only be routed via connectable flights (flight 1's destination=flight 2's origin, flight 2's departure time is later than flight 1's arrival time).

In this lecture, you will learn…

- Column-wise modeling approach:
  - Transform *"Complex Constraints"* into *"Definition of Variables"*
- Column generation solution approach:
  - The above transformation is usually accomplished at the cost of huge increasing in number of decision variables
  - (Delayed) column generation helps to solve the program by generating only a small subset of the variables
- Implementing column generation and solving *American Airlines' aircraft routing problem* in Julia/JuMP.
- (Hopefully) introduce you a new way to formulate optimization problems in your own research projects.

Column Generation Essentials

- Consider the following standard form LP.

$$\begin{array}{ll} Min & c^T x \\ s.t. & Ax = b \\ & x \geq 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} Min & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ s.t. & A_1 x_1 + A_2 x_2 + \cdots + A_n x_n = b \\ & x_1, x_2, \cdots, x_n \geq 0 \end{array}$$

- Sufficient condition for basis B to be optimal
  - Reduced cost of any variable $x_k = c_k - c_B^T B^{-1} A_k = c_k - r^T A_k \geq 0$
  - r is the dual prices associated with basis B
- Start the solution process with a subset of variables, WLOG, only x_1 and x_2. Get the optimal dual solution r.

(Restricted Master Problem)

$$\begin{array}{lll} Min & c_1 x_1 + c_2 x_2 & \\ s.t. & A_1 x_1 + A_2 x_2 = b & (r) \\ & x_1, x_2 \geq 0 & \end{array}$$

- Solve the following sub-problem. If $z^* < 0$, then add optimal column A_{k^*} to the restricted master problem. Resolve it. If $z^* \geq 0$, the current solution from restricted master problem is optimal.

(Sub-problem)

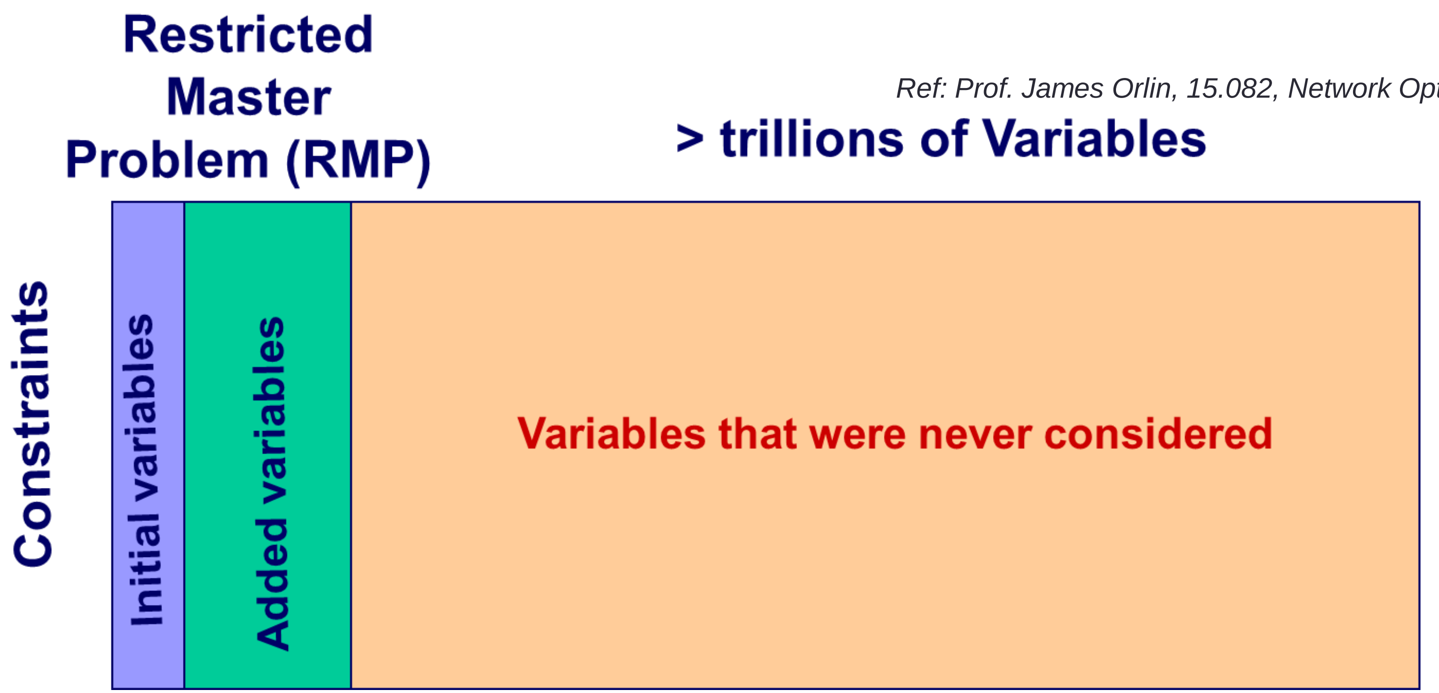
$$z^* = Min_{k \in remaining\ variables} \quad c_k - r^T A_k$$

Column Generation Essentials

- The beauty of column generation is the sub-problem in certain cases are straightforward to model and solve, *than it appears to be.*☺

(Sub-problem) $$z^* = Min_{k \in remaining\ variables} \quad c_k - r^T A_k \begin{cases} \text{shortest path problem} \\ \text{knapsack problem} \\ \ldots \end{cases}$$

- Column Generation in one figure:



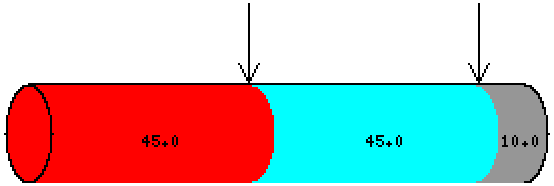
Ref: Prof. James Orlin, 15.082, Network Optimization

Why we care about problems with lots of variables?

- **Many problems are easy to model column-wisely, though at the cost of increasing number of variables.**

Warm-Up Example: Cutting Stock Problem

- Given paper rolls of fixed width (100 inches) and a set of orders for rolls of smaller widths (14, 31, 36, 45 inches), the objective of the Cutting Stock Problem is to determine how to cut the rolls into smaller widths to fulfill the orders in such a way as to minimize the number of paper rolls used.



Order Width	Quantity Ordered
14	211
31	395
36	610
45	97

Define n_1, n_2, n_3, n_4 as the number of cuts for each order width for a paper roll,

$$14n_1 + 31n_2 + 36n_3 + 45n_4 \le 100$$

n_1, n_2, n_3, n_4 are integers

However, each paper roll may have different cutting patterns, how to formulate the problem?

Answer: *Bring constraints into definition of variables.*

Ref: *http://www.neos-guide.org/content/cutting-stock-problem*

Warm-Up Example: Cutting Stock Problem

- Sets
  - I, set of order widths; J, set of feasible cutting patterns
- Parameters
  - a_{ij}, is the number of cuts of width i in cutting pattern j, $\forall i \in I, j \in J$
- Decision Variables
  - x_j, is the number of paper rolls using cutting pattern j, $\forall i \in I, j \in J$

$$
\begin{aligned}
MIN \quad & \sum_{j \in J} x_j \\
s.t. \quad & \sum_{j \in J} a_{ij} x_j \geq b_i, && \forall i \in I \\
& x_j \textit{ is integer}, && \forall j \in J
\end{aligned}
$$

|J| equals to number of different solutions for

$$14n_1 + 31n_2 + 36n_3 + 45n_4 \leq 100$$

n_1, n_2, n_3, n_4 are integers

Difficulty: |J| is large. Hard to enumerate $x_j, \forall j \in J$ and include them into the model…

Solution: Generate variables on the fly!

Warm-Up Example: Cutting Stock Problem

- Solve the LP relaxation using column generation:

STEP 1: • Initialize the restricted master problem with two patterns
  - width 14: 1 cut; width 31: 1 cut; width 36: 0 cut; width 45: 1 cut (14+31+45=90<100)
  - width 14: 0 cut; width 31: 0 cut; width 36: 2 cut; width 45: 0 cut (36*2=72<100)
  - Why those two patterns?

restricted master problem:

$$MIN \quad x_1 + x_2$$

$$s.t. \quad \begin{pmatrix}1\\1\\0\\1\end{pmatrix} x_1 + \begin{pmatrix}0\\0\\2\\0\end{pmatrix} x_2 \geq \begin{pmatrix}211\\395\\610\\97\end{pmatrix} \begin{matrix}r_1\\r_2\\r_3\\r_4\end{matrix}$$

$$x_1, x_2 \geq 0$$

STEP 2: • Solve the restricted master problem
  - Optimal dual solution: (r_1, r_2, r_3, r_4)

STEP 3: • Recall the sufficient optimal condition for LP
  - All variables' reduced costs ≥ 0
  - Reduced cost of a potential variable x_k

$$rc(x_k) = 1 - (r_1\ r_2\ r_3\ r_4)\begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\\a_{4k}\end{pmatrix}$$

sub-problem (knapsack problem):

$$z^* = MAX \quad \sum_{i=1}^{4} r_i a_{ik}$$

$$s.t. \quad 14a_{i1} + 31a_{i2} + 36a_{i3} + 45a_{i4} \leq 100$$

$$a_{i1}, a_{i2}, a_{i3}, a_{i4}\ are\ integers$$

Order Width	Quantity Ordered
14	211
31	395
36	610
45	97

STEP 4: • If $z^* \leq 1$, the optimal solution of restricted master problem is optimal for the original LP relaxation.

STEP 5: • If $z^* > 1$, add optimal solution $(a_{1k}, a_{2k}, a_{3k}, a_{4k})$ from sub-problem as a new column (variable) to restricted master problem. GOTO Step 2.

Warm-Up Example: Cutting Stock Problem

- Solve the LP relaxation using column generation:

STEP 1: • Initialize the restricted master problem w
  - width 14: 1 cut; width 31: 1 cut; width 36: 0
    (14+31+45=90<100)
  - width 14: 0 cut; width 31: 0 cut; width 36: 2
    (36*2=72<100)
  - Why those two patterns?

restricted master problem, after one column added:

$$MIN \quad x_1 + x_2 + x_k$$

$$s.t. \quad \begin{pmatrix}1\\1\\0\\1\end{pmatrix} x_1 + \begin{pmatrix}0\\0\\2\\0\end{pmatrix} x_2 + \begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\\a_{4k}\end{pmatrix} x_3 \geq \begin{pmatrix}211\\395\\610\\97\end{pmatrix}$$

$$x_1, x_2, x_k \geq 0$$

STEP 2: • Solve the restricted master problem
  - Optimal dual solution: (r_1, r_2, r_3, r_4)

sub-problem (knapsack problem):

$$z^* = MAX \quad \sum_{i=1}^{4} r_i a_{ik}$$

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STEP 3: • Recall the sufficient optimal condition for LP
  - All variables' reduced costs ≥ 0
  - Reduced cost of a potential variable x_k

$$rc(x_k) = 1 - (r_1 \; r_2 \; r_3 \; r_4) \begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\\a_{4k}\end{pmatrix}$$

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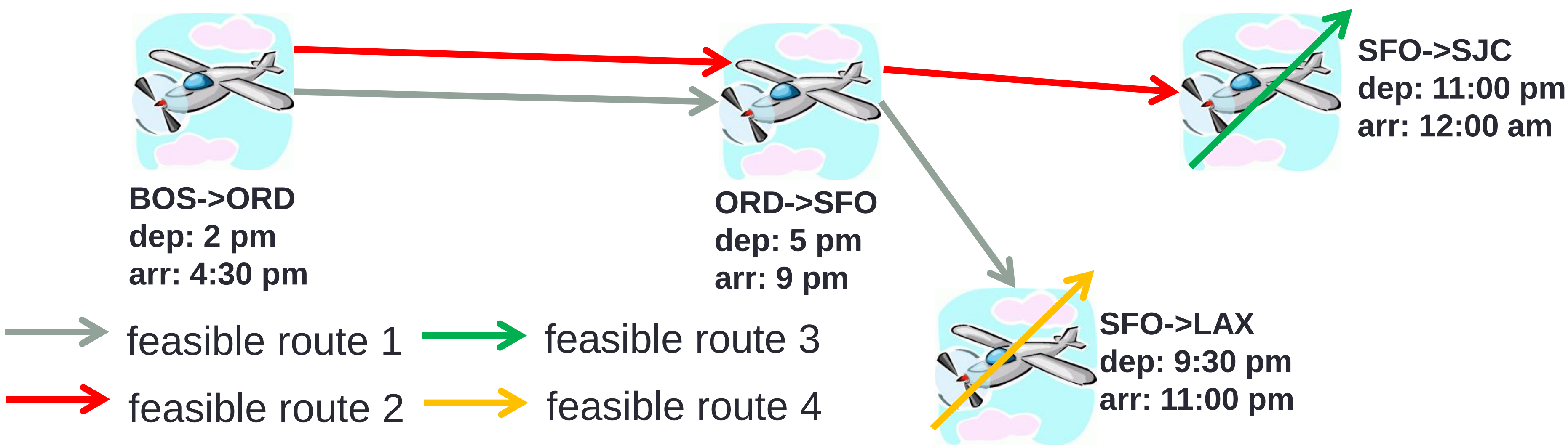
Warm-Up Example: Cutting Stock Problem

- Let’s now implement and solve this cutting stock problem in Julia/JuMP!

CG is much more powerful than just cutting paper rolls…

- *American Airlines* wants to route n aircraft to serve m flights. Aircraft can only be routed via connectable flights (flight 1's destination=flight 2's origin, flight 2's departure time is later than flight 1's arrival time).
- ***How to model the flight connection constraints?***
- **Solution:**
  - Forget about complex constraints first, bring them to the definition of variable!
    - Set: R, set of feasible route can be flew by an aircraft (connectable)
    - $x_r \in \{0,1\}, r \in R$, equals 1 if feasible route r is used; 0, otherwise.



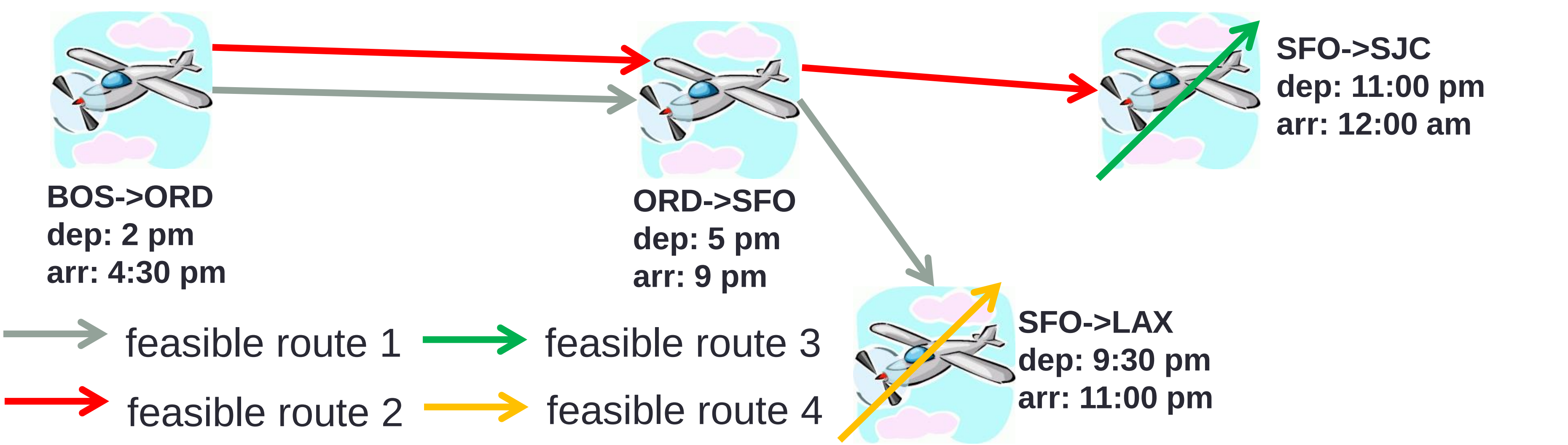
Aircraft Routing Problem

- In order to ensure each flight is covered by exactly one route

- $$A = \begin{array}{cl} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} & \begin{array}{l} \text{BOS->ORD} \\ \text{ORD->SFO} \\ \text{SFO->LAX} \\ \text{SFO->SJC} \end{array} \\ \begin{array}{cccc} \text{Route 1} & \text{Route 2} & \text{Route 3} & \text{Route 4} \end{array} & \end{array}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Updownarrow$$

$$Ax = 1$$

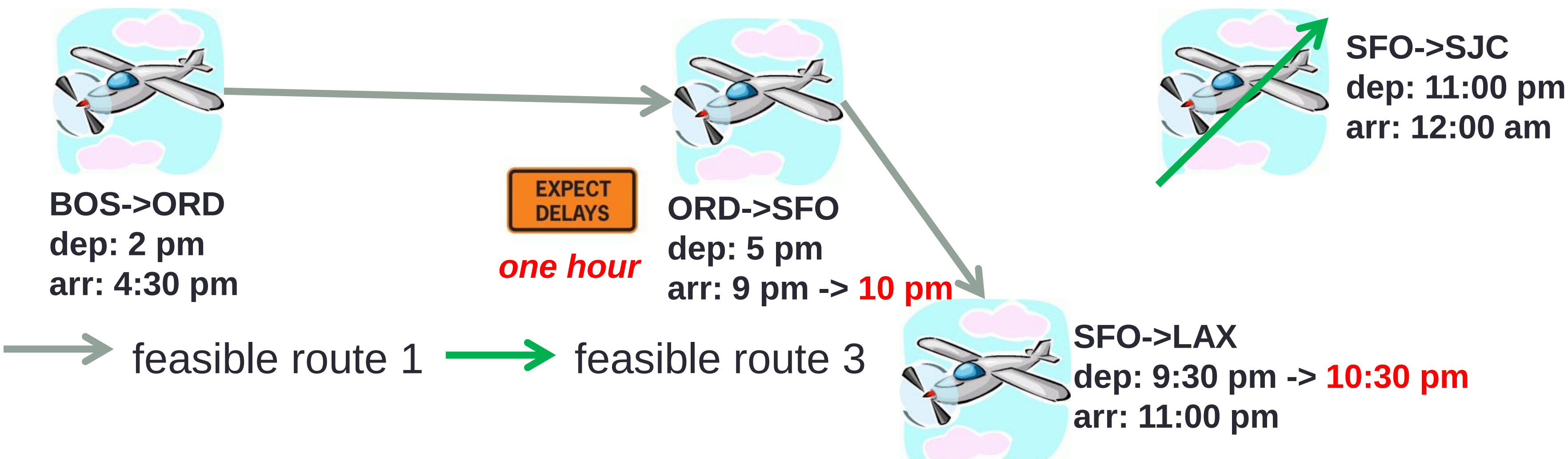


Aircraft Routing Problem

- In order to ensure each flight is covered by exactly one route
  - $A = \underbrace{\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}}_{\text{\# of feasible routes}=|R|}$ # of flights=m, $A_{ij} = 1$ if route j covers flight i
  - $Ax = 1 \quad \Longleftrightarrow \quad A_1 x_1 + A_2 x_2 + \cdots + A_{|R|} x_{|R|} = 1$
- In order to ensure at most n aircraft can be used
  - $x_1 + x_2 + \cdots + x_{|R|} \le n$
- Binary: $x_1, x_2, \cdots, x_{|R|} \in \{0,1\}$

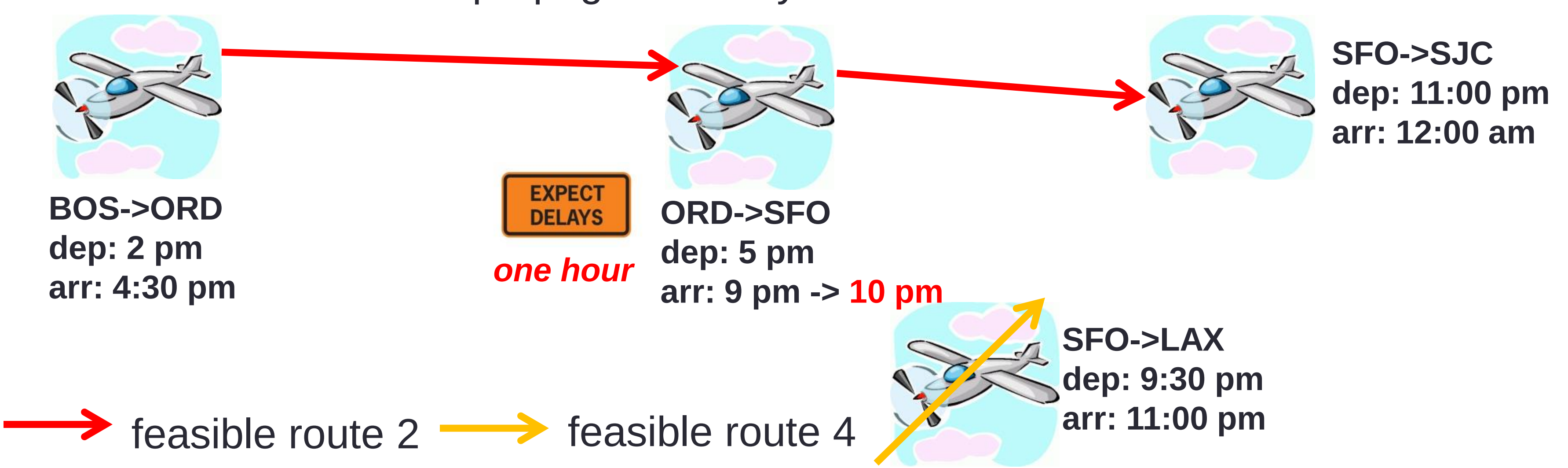
Aircraft Routing Problem

- Objective: Minimize *Propagated Delay*
  - Each flight has an arrival delay
  - Two connecting flight flew by same aircraft has a 30-minute minimum separation time
  - Flight ORD->SFO has 60 minutes arrival delay
  - Using route 1 and route 3
    - Route 1 has one hour propagated delay on flight SFO->LAX
    - Route 3 has 0 propagated delay



Aircraft Routing Problem

- Objective: Minimize *Propagated Delay*
  - Each flight has an arrival delay
  - Two connecting flight flew by same aircraft has a 30-minute minimum separation time
  - Flight ORD->SFO has 60 minutes arrival delay
  - Using route 2 and route 4
    - Route 2 has 0 propagated delay
    - Route 4 has 0 propagated delay



Aircraft Routing Problem

- Set
  - R, set of feasible route can be flew by an aircraft (connectable)
- Decision Variables
  - $x_r \in \{0,1\}, r \in R$, equals 1 if feasible route r is used; 0, otherwise.
- Parameter
  - m, number of flights; n, number of aircraft
  - $A \in \{0,1\}^{m \times |R|}, A_{ij} = 1$ if $route\ j$ covers $flight\ i$; 0, otherwise
  - $p_r, r \in R$, total propagated delay on route r
- Formulation:

$$
\begin{aligned}
Min \quad & \sum_{r \in R} p_r x_r \\
s.t. \quad & Ax = 1 \\
& \sum_{r \in R} x_r \leq n \\
& x_r \in \{0,1\}, \forall r \in R
\end{aligned}
$$

Huge number of variables

⇓

Column Generation!

Aircraft Routing Problem - CG

- Reduced cost of a potential route variable x_r
  - $p_r - s^T A_r - t$
  - $A_r[j] = 1$ if flight j is covered by route r; 0, otherwise.

$$p_r - s^T A_r - t = p_r - \sum_{f \in r} s_f - t,$$

$f \in r$ are the flights route r covers

$$\begin{aligned} Min \quad & \sum_{r \in R} p_r x_r \\ s.t. \quad & A_1 x_1 + \cdots A_{|R|} x_{|R|} = 1 \quad (s) \\ & \sum_{r \in R} x_r \leq n \quad (t) \\ & 0 \leq x_r \leq 1, \forall r \in R \end{aligned}$$

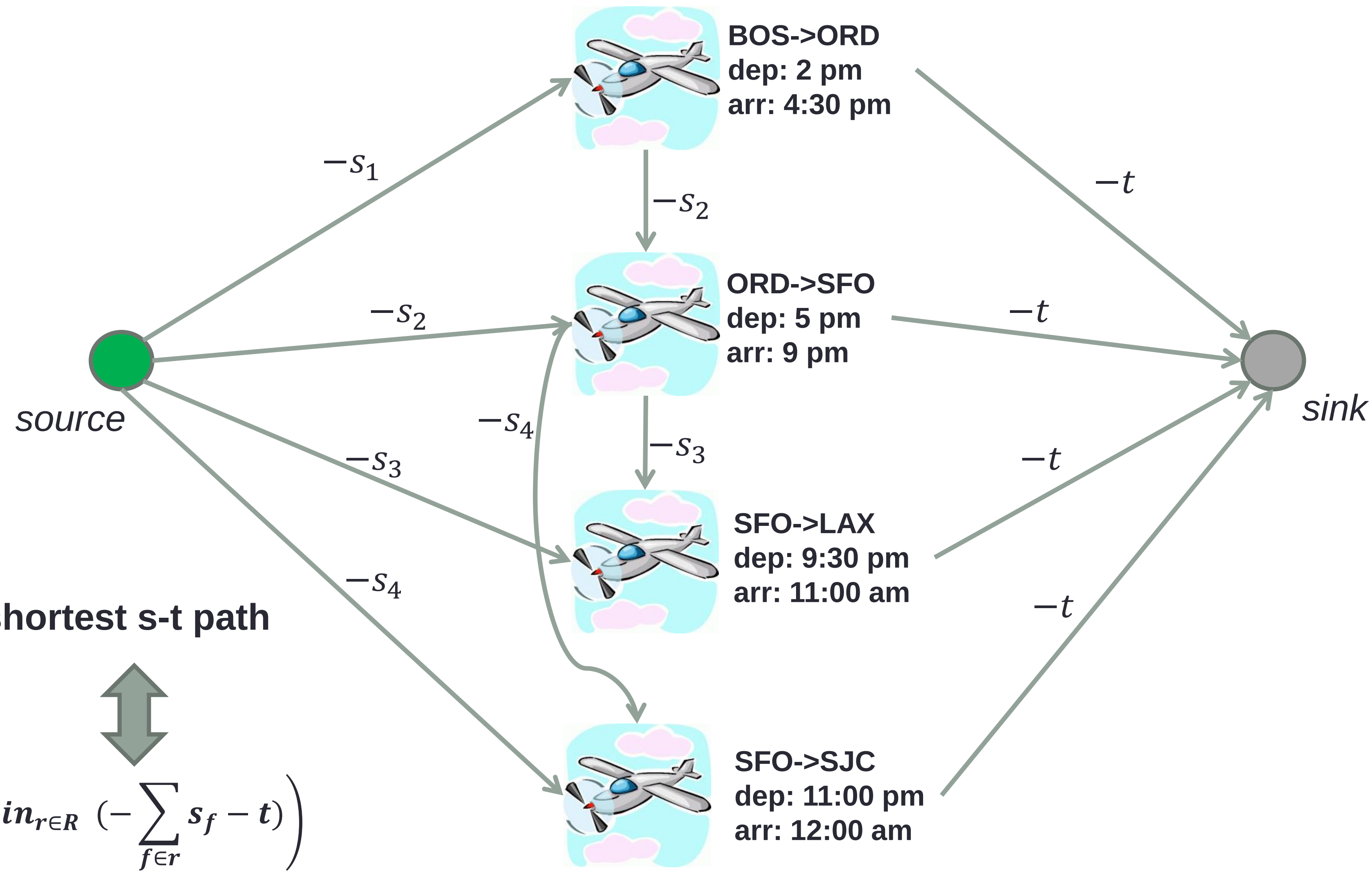
(sub-problem) $z^* = Min_{r \in R} \; (p_r - \sum_{f \in r} s_f - t) \; < 0\,??$

Heuristic

(sub-problem) $z^* = \boxed{\left(Min_{r \in R} \; (-\sum_{f \in r} s_f - t)\right)} + p_r \; < 0\,??$

A shortest path problem!

Aircraft Routing Problem - CG

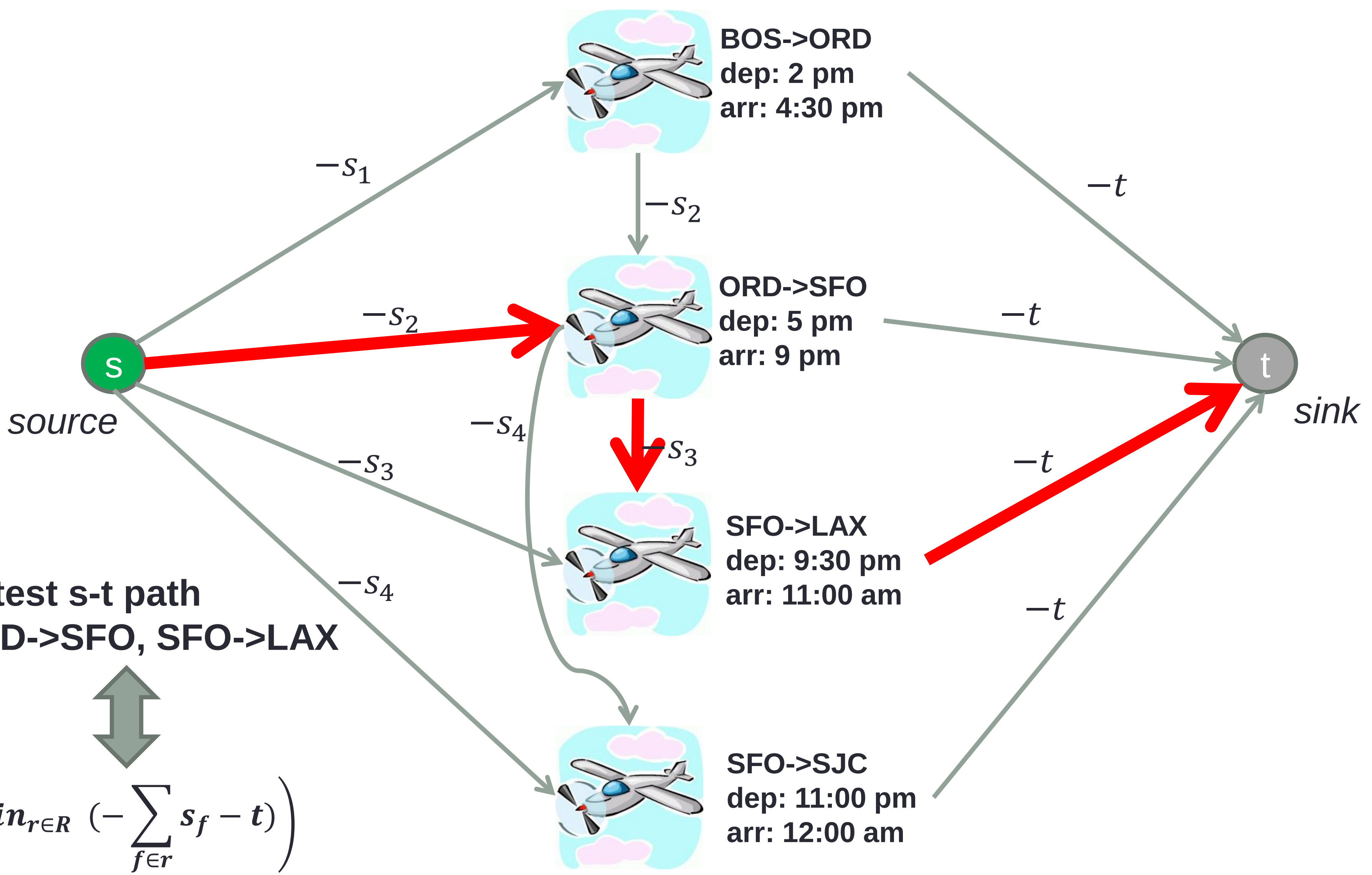


shortest s-t path

⇕

$$\left(\boldsymbol{Min}_{r \in R} \ \left(- \sum_{f \in r} s_f - t\right) \right)$$

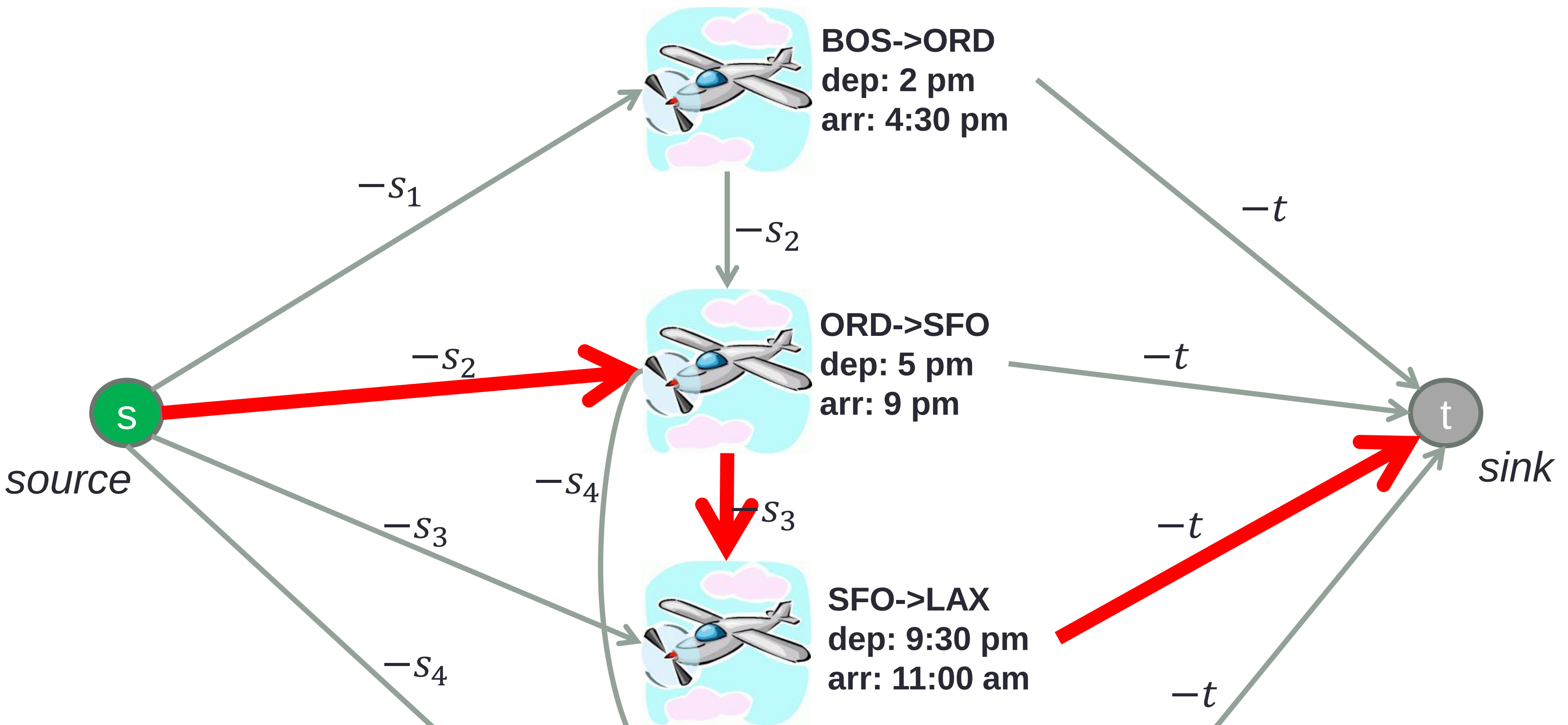
Aircraft Routing Problem - CG



shortest s-t path
= ORD->SFO, SFO->LAX

$$\left(\boldsymbol{Min}_{r \in R} \; \left(-\sum_{f \in r} \boldsymbol{s}_f - \boldsymbol{t}\right) \right)$$

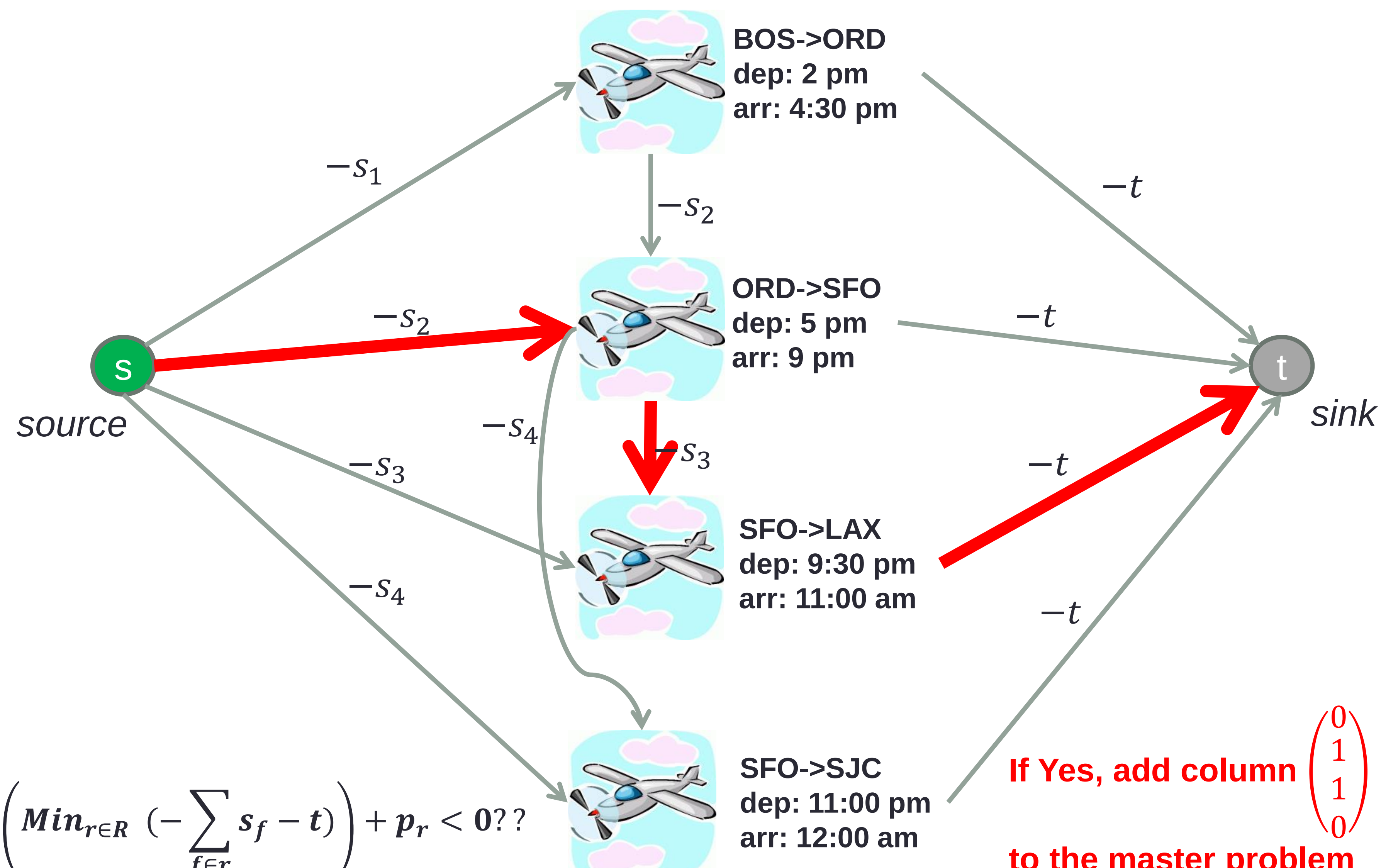
Aircraft Routing Problem - CG



Calculate p_r along route ORD->SFO, SFO->LAX

$$z^* = \left(Min_{r\in R}\ \left(-\sum_{f\in r} s_f - t\right)\right) + p_r < 0??$$

Aircraft Routing Problem - CG



$$z^* = \left(Min_{r \in R} \; \left(-\sum_{f \in r} s_f - t\right) \right) + p_r < 0??$$

If Yes, add column $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ to the master problem

Aircraft Routing Problem - CG

- Let's now implement and solve this aircraft routing problem in Julia/JuMP!

More on column-wise modeling

- What if American Airlines has some additional constraints?
  - An aircraft cannot fly more than 18 hours consecutively?
  - An aircraft must return to either ORD or SFO for night-time maintenance?
- Can you now try to model *Amazon Fresh's* vehicle routing problem using column-wise modeling?
  - *Amazon Fresh* wants to route n vehicles to serve m customers' grocery everyday. Each customer has a specific time window for pick up, each vehicle has a maximum distance to travel due to fuel constraints.

Summary

- We introduce column-wise modeling for optimization problems. For certain types of problems, this approaches greatly simplifies the modeling efforts however increases decision variable dramatically.
- Column generation solution approach comes to rescue by generating only a small subset of variables.

More on column generation

- Speeding up…
  - Generating multiple columns at one iteration
  - Stabilized dual solution
  - Column management
- Column generation on integer programs
  - Branch-and-Price
- SCIP Optimization Suite, http://scip.zib.de/
- Try column-wise modeling and using CG in your future research.
- Questions? chiwei@mit.edu