Math 421

Monday, September 15

Announcements

- Homework 2 due Friday, September 19 at 11:59 pm
- Drop-in hours
  - Wednesday 12-1 pm in VV 311
  - Thursday 9-11 in VV B205
- MLC
  - Proof Table (M-Th 3:30-7 VV B227)
  - Course Assistant (~~Th 4-7~~ Table 4)

This week Tues 4-7

Example 2 – Discussion

Theorem: If $n \in \mathbb{N}$, then $n! \leq n^n$. P_n: "$n! \leq n^n$"

- What proof technique should we use? *Why*?

- Do you believe the statement? Verify for $n = 2, 3$. ✓

  $2 \leq 4$ ✓ $6 \leq 27$ ✓

- What is the base case?

  $n = 1$

  If $n! \leq n^n$ then $(n+1)! \leq (n+1)^{n+1}$.

- What is the inductive assumption? What is the conditional statement you are trying to prove.

Note: In "simple" cases we don't need to explicitly state P_n.

$n=1$ ✓ If k ✓ $\Rightarrow$ $k+1$ ✓ P_n: "$n! \leq n^n$"

Example 2 – Proof

Proof. We argue by induction.

Base Case: If $n=1$, then $1! = 1 = 1^1$. So true for $n=1$.

If $n! \leq n^n$ then $(n+1)! \leq (n+1)^{n+1}$.

Induction Step: Suppose $k! \leq k^k$ for some $k \in \mathbb{N}$. Then

rewrite to use induction hypothesis

LHS

$$(k+1)! = k!\,(k+1) \leq k^k (k+1) \leq (k+1)^k (k+1) = (k+1)^{k+1}$$

RHS

use properties

$\underbrace{1 \cdot 2 \cdot 3 \cdots k}_{} \cdot k+1 = k! \cdot (k+1)$

use induction hypothesis

Therefore $n! \leq n^n$ for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Activity – ~~Proof~~ Sketch of algebra.

OK: $1+3+5+\cdots+(2n-1)=n^2$

If $1+3+5+\cdots+(2k-1)=k^2$

then $1+3+5+\cdots+(2k-1)+(2(k+1)-1)=(k+1)^2$

Theorem: For all $n \in \mathbb{N}$,

$$\sum_{j=1}^{n}(2j-1)=n^2.$$

Let $k \in \mathbb{N}$

If $\sum_{j=1}^{k}(2j-1)=k^2$ then $\sum_{j=1}^{k+1}(2(j)-1)=(k+1)^2$

$$\sum_{j=1}^{k+1}(2j-1)=(2(k+1)-1)+\sum_{j=1}^{k}(2j-1)=(2(k+1)-1)+k^2$$

$$=2k+2-1+k^2=k^2+2k+1=(k+1)^2. \quad \square$$

$$\sum_{j=1}^{k}(2j-1)=k^2$$

Example 3 - Discussion

Theorem: For all $n \in \mathbb{N}$ and $n \geq 4$, we have $2^n < n!$.

Note: We can use any number as the "base case".

- What proof technique should we use? *Why*?
- Do you believe the statement? *Why does it start at 4*?
- What is the base case?
- What is the inductive assumption?

Example 3 – Proof

Proof. We argue by induction.

Base case: Let $n = 4$ $\quad 2^4 = 16 < 24 = 4!$

Induction step: Suppose $2^k < k!$ for some $k \in \mathbb{N}$, with $k \geq 4$. Then,

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot k! < (k+1)k! = (k+1)!$$

Therefore, $2^n < n!$ is true for all $n \geq 4$.

Definition – Power Set

Def: The power set, $\mathcal{P}(S)$, of a set S is a set of all subsets of S.

Examples

1. $S = \{a\}, \mathcal{P}(S) =$

2. $T = \{1,2\}, \mathcal{P}(T) =$

Group Activity

Theorem: Let $n \in \mathbb{N}$. The power set of a set S with n elements has 2^n elements.

1. Convince yourselves this statement is true.
2. This statement can be proven using a proof by induction.
   A. What is the base case?
   B. Assume P_2 is true, how would you prove P_3 is true?
   C. Can you generalize this to the case where you assume P_k is true and prove that P_{k+1} is true?