Math 421

Friday, September 19

Announcements

- Homework 2 due today – make sure the submission has gone through and appears in gradescope!
- Information about makeup midterms have been posted – requires prior approval.

Proof Practice

Theorem. If x is a rational number and y is an irrational number, then $x + y$ is irrational.

Discussion.

- Determine proof method.
- Use definitions.
- Sketch out necessary algebra.

Proof.

Group Activity – Proof Practice

Theorem. If $x, y \in \mathbb{R}$, then $|xy| = |x||y|$.

Form groups of 4-6. Further divide in halves A & B. Each half proves 2 of the cases. Then regroup and compare.

Proof.

Case 1: Assume $x \geq 0$ and $y \geq 0$.

Case 2: Assume $x < 0$ and $y < 0$.

Case 3: Assume $x \geq 0$ and $y < 0$.

Case 4: Assume $x < 0$ and $y \geq 0$.

Distances

If $a, b \in \mathbb{R}$, then $|a - b|$ is the distance between a and b.

Notice that if $c \in \mathbb{R}$, then

$$|a - b| = \qquad\qquad\qquad \leq |a - c| + |c - b|$$

Proof Practice

Thm. Suppose $a, b \in \mathbb{R}$. Then $a = b$ if and only if for every positive real number $\varepsilon > 0$ we have $|a - b| < \varepsilon$.

Discussion.

- What is this theorem saying in your own words? Do you believe it?
- What are the two conditional statements we will have to prove?

Proof Practice

Proof.

Chapter 1: Introduction

Section 4: The Completeness Axiom

Maximum & Minimum

Def. Let S be a nonempty subset of $\mathbb{R}$. If S contains a *largest element*, s_0, then we call s_0 the **maximum of $\boldsymbol{S}$** and write $s_0 = \max S$.

Activity – Find the max/min (if any!)

1. $\mathbb{Z}$
2. $\mathbb{Q}$
3. $\mathbb{N}$
4. $\{1,2,3\}$
5. $(1,3)$
6. $(a, b]$

Upper & Lower Bounds

Def. Let S be a non-empty subset of $\mathbb{R}$. If a real number M satisfies $s \leq M$ for all $s \in S$, then M is an **upper bound** for S and the set S is **bounded above.**

Bounded

Def. Let S be a nonempty subset of $\mathbb{R}$. The set S is said to be **bounded** if there exist real numbers m and M such that $S \subseteq [m. M]$.

Activity – Is the set bounded?

1. $\mathbb{Z}$
2. $\mathbb{Q}$
3. $\mathbb{N}$
4. $\{1,2,3\}$
5. $(1,3)$
6. $(a, b]$

Supremum & Infimum

Def. Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above and S has a least upper bound, then we call it the **supremum of S** and denote it by $\sup S$.

Activity

Give an example of the following:

- A bounded set that has a minimum value but no maximum value.

- A set that has no least upper bound (or its least upper bound is infinite).

- A set that contains its least upper bound.

Discussion Questions

- When is it true that $\sup S \in S$ and $\inf S \in S$?
- Are $\sup S$ and $\inf S$ unique?
- Is the empty set bounded? Does it have a least upper bound? A greatest lower bound?
- What does it mean for a set to be **unbounded**?

Activity

Find the supremum/infimum of the following sets (if they exist).

1. $\{n \in \mathbb{N} : n \text{ is prime and } n < 10\}$
2. $\{n^2 : n \in \mathbb{N}\}$
3. $\left\{\sin \frac{n\pi}{4} : n \in \mathbb{N}\right\}$
4. $\{x^3 : x > 3\}$
5. $\{n \in \mathbb{N} : 2 < n < 3\}$

Equivalent Definition

Thm. Let $s \in \mathbb{R}$ be an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for every $\varepsilon > 0$ there exits some $a \in A$ with $s - \varepsilon < a$.

Discussion Questions:

- Describe what this theorem is saying in your own words.
- Convince yourselves that this theorem is true, it might be helpful to think in terms of a specific example and then see how it can generalize.