Math 421

September 10, 2025

Announcements

- Applications for the Fall 2025 Directed Reading Program are due 9/11 at 11:59 pm.
- Homework 1 due Friday 9/12 at 11:59 pm
- Reminder of ways to get help –
    - Drop-in hours
        - Today 12-1 pm in VV 311
        - Thursday 9-11 in VV B205
    - MLC
        - Proof Table (M-Th 3:30-7 VV B227)
        - Course Assistant (Th 4-7 Table 4)

If and Only If Theorem

Theorem: For all integers x, x is even if and only if $x + 1$ is odd.

Discussion:

Activity - Proof Sketch: In groups of 4 divide yourselves in half each half takes one direction, writes a proof, then trade & critique.

First Proof Techniques

- **Direct:**
  - **Basic Direct** – Use axioms and definitions to go from hypotheses to conclusion.
  - **Cases** – First break the proof up into cases that exhaust all possibilities. Then use a basic direct proof for each case.
- **Indirect:**
  - **Contrapositive** – Give a direct proof of the contrapositive of the given conditional statement.
  - **Contradiction** – Assume that the hypotheses are true and the conclusion is false, derive a contradiction.

Examples

Consider the following statements, what type of proof should we use?

Theorem A: If x is an integer, then x and x^2 have the same parity (i.e. both even or both odd).

Theorem B: If $3x + 2$ is odd, then x is odd.

Theorem C: If x and y are integers, then $x^2 - 4y \neq 2$.

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Proof.

Theorem C

Theorem: If x and y are integers, then $x^2 - 4y \neq 2$.

Proof.

Sets – Definition

Definition: A ***set***, S, is an unordered collection of ***elements***.

Notation –

$x \in S$ – x is an element of S

$x \notin S$ – x is not an element of S

$T \subseteq S$ – every element in the set T is also in the set S

i.e. $x \in T$ implies $x \in S$.

Examples:

Some Important Sets

- $\mathbb{R}$ denotes the **real numbers**
- $\mathbb{Q}$ denotes the **rational numbers** – numbers of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$.
- $\mathbb{Z}$ denotes the **integers**
- $\mathbb{N}$ denotes the **natural numbers**
- $\emptyset$ denotes the **empty set** – the set with no elements

True or False?

If $x \in \emptyset$, then x is an odd number and an even number.

Set Builder Notation

Small, finite sets can be denoted using braces { }.

Example: $\{2, 3, 5, 7\}$

Larger finite, or infinite, sets can be described by properties of their elements via the notation `{[variable]:[properties]}`.

Examples:

- $\{n : n \in \mathbb{N} \text{ and } n \text{ is odd}\}$
- $\{x : x \in \mathbb{R} \text{ and } 1 \leq x < 3\}$

Activity

Determine the elements (if any) of the following sets:

1. $\{n \in \mathbb{N} : n \text{ is prime and } n < 10\} =$

2. $\{n^2 : n \in \mathbb{N}\} =$

3. $\left\{\sin \frac{n\pi}{4} : n \in \mathbb{N}\right\} =$

4. $\{x^3 : x > 3\} =$

5. $\{n \in \mathbb{N} : 2 < n < 3\} =$

Set Operations

Let A and B be two subsets of a set S.

- **Union** - $A \cup B = \{x \in S : x \in A \text{ or } x \in B\}$

- **Intersection** - $A \cap B = \{x \in S : x \in A \text{ and } x \in B\}$

More Set Operations

Let A and B be two subsets of a set S.

- **Complement** - $A^c = \{x \in S : x \notin A\}$

- **Set Difference** - $B - A = \{x \in S : x \in B \text{ and } x \notin A\}$

Activity

Let $S = \mathbb{R}$, and $A = \{x \in \mathbb{R} : x \in \mathbb{N} \text{ and } x \text{ is odd}\}$, $B = \{x \in \mathbb{R} : 1 \leq x < 3\}$, and $C = \{x \in \mathbb{R} : x \in \mathbb{N} \text{ and } x \text{ is even}\}$. Find the following:

1. $A \cap B =$
2. $A \cup C =$
3. $B^c =$
4. $B - C =$

True or False

1. $\{1,1,2,3\} = \{1,2,3\}$

2. $\{3,2,1\} = \{1,2,3\}$

Q: How do we determine if two sets are equal?