Unit 1: Systems of Linear Equations and Matrices

Chapter 1: Systems of Linear Equations

These notes follow and use material from your textbook *Discover Linear Algebra* by Jeremy Sylvestre.

Example: In a Wisconsin forest, there are robins and badgers. Together they have 18 heads and 56 legs. How many robins and badgers are in the forest?

$$\begin{cases} x+y=18 \\ 2x+4y=56 \end{cases} \Rightarrow \begin{cases} x=8 \\ y=10 \end{cases}$$

Linear Equations

An equation of the form $a_1x_1+a_2x_2+\cdots+a_nx_n=b$ is called a **linear equation**. The real (or complex) quantities $a_1,\ldots,a_n,b$ are **constants**, while the variables $x_1,\ldots,x_n$ represent **unknowns**.

Example: Identify the constants and unknowns in each linear equation in the first example.

Example: What are some equations that are not linear equations?

More generally, a **system of m linear equations in n unknowns** $x_1,\ldots,x_n$, or a **linear system**, is a set of m linear equations each in n unknowns. A linear system has the form

$$\begin{array}{llllll} a_{11}x_1+ & a_{12}x_2+ & \cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1+ & a_{22}x_2+ & \cdots+ & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1}x_1+ & a_{m2}x_2+ & \cdots+ & a_{mn}x_n & = & b_m. \end{array} \qquad (*)$$

A **solution** to the linear system $(*)$ is a sequence of n numbers $s_1,\ldots,s_n$ so that *each* equation in $(*)$ satisfied when $x_1=s_1$, $x_2=s_2$, …, $x_n=s_n$ are substituted.

Let $(*)$ be a linear system as above.

- If $(*)$ has *no* solution, it is called **inconsistent**.
- If $(*)$ has a solution, it is called **consistent**. (Note that it may have *infinitely many* solutions!)
- If $(*)$ has $b_1 = b_2 = \cdots = b_n = 0$, it is called a **homogeneous system**.
  - Note that $x_1 = x_2 = \cdots = x_n = \mathbf{0}$ is always a solution to a homogeneous system. It is called the **trivial** solution.
  - A solution to a homogeneous system where *not* each x_i is 0 is called a **nontrivial** solution.

Consider another system of r linear equations in n unknowns:

$$\begin{array}{ccccccc} c_{11}x_1+ & c_{12}x_2+ & \cdots+ & c_{1n}x_n & = & d_1 & \\ c_{21}x_1+ & c_{22}x_2+ & \cdots+ & c_{2n}x_n & = & d_2 & \\ \vdots & \vdots & & \vdots & & \vdots & (**) \\ c_{r1}x_1+ & c_{r2}x_2+ & \cdots+ & c_{rn}x_n & = & d_r. & \end{array}$$

The systems $(*)$ and $(**)$ are **equivalent** if they have exactly the same solutions.

Example: In another Wisconsin forest, there are also robins and badgers. Together they have 18 heads, 36 eyes, and 56 legs. How many robins and badgers are in the forest? Can we use any new terminology to describe this system?

$$\begin{cases} x+y=18 \\ 2x+2y=36 \\ 2x+4y=56 \end{cases} \Rightarrow \begin{cases} x=8 \\ y=10. \end{cases}$$

Example: In a third Wisconsin forest, there are deer and badgers. Together they have 18 heads and 70 legs. How many deer and badgers are in the forest? Can we use any new terminology to describe this system?

$$\begin{cases} x+y=18 \\ 4x+4y=70 \end{cases}$$, no sol'n , inconsistent.

Definition/Theorem
The following operations, called **elementary operations**, can be performed on systems of linear equations to produce equivalent systems.

1. **Interchange** the ith and jth equations.
2. **Multiply** an equation by a **nonzero constant**.
3. **Replace** the ith equation by c times the jth equation *plus* the ith equation.

We can track these operations more easily using an array/matrix to record the coefficients. The **coefficient matrix** of the linear system is the $m \times n$ matrix A:

$$\begin{array}{llllll} a_{11}x_1+ & a_{12}x_2+ & \cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1+ & a_{22}x_2+ & \cdots+ & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1}x_1+ & a_{m2}x_2+ & \cdots+ & a_{mn}x_n & = & b_m \end{array} \longrightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$$

We can also build a matrix that includes the constants on the right-hand side of the linear system. First, note that

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

is an $m \times 1$ matrix (called the **constant matrix**) representing the constant terms of the linear equations (this is also called a **column vector**). We can adjoin $\mathbf{b}$ to matrix A to create the **augmented matrix** representing our linear system:

$$[A \,\vdots\, \mathbf{b}] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example: Consider the linear system

$$\begin{array}{rcrcrcr} -2x & + & y & - & 3z & = & 1 \\ & & y & & 4z & = & -8 \\ x & + & y & + & z & = & 7 \end{array}$$

The system is reduced to a simpler system using elementary operations. Find the augmented matrix for the linear system, then record the resulting matrices.

$$\begin{array}{rcrcrcr} -2x & + & y & - & 3z & = & 1 \\ & & y & & 4z & = & -8 \\ x & + & y & + & z & = & 7 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & & 4z & = & -8 \\ -2x & + & y & - & 3z & = & 1 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & & 4z & = & -8 \\ & & 3y & - & z & = & 15 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & + & 4z & = & -8 \\ & & 0 & & -13z & = & 39 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & + & 4z & = & -8 \\ & & & & z & = & -3 \end{array}$$

Definition: An **elementary row operation** on a matrix A is any one of the following operations:

(a) Type I: **Interchange** any two rows.

(b) Type II: **Multiply** a row by a **nonzero number**.

(c) Type III: **Add** a multiple of one row to another.

Chapter 2: Solving Systems Using Matrices

Example: The following are augmented matrices representing systems of linear equations in three unknowns (x, y, z). Rewrite the equations, then solve the systems.

(a) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \iff \begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_2 + 2x_3 = 2 \\ x_3 = -3. \end{cases} \Rightarrow \begin{cases} x_1 = -9 \\ x_2 = 8 \\ x_3 = -3 \end{cases}$

(b) $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \iff \begin{cases} x_1 = -5 \\ x_2 = 2 \\ x_3 = 4 \end{cases} \Rightarrow \begin{cases} x_1 = -5 \\ x_2 = 2 \\ x_3 = 4 \end{cases}$

Definition: An $m \times n$ matrix A is in **row echelon form** (REF) if it satisfies the following properties:

(a) All zero rows, if there are any, appear at the **bottom** of the matrix.

(b) The first nonzero entry from the left of a nonzero row is a 1 . This entry is called the leading one of its row.

(c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

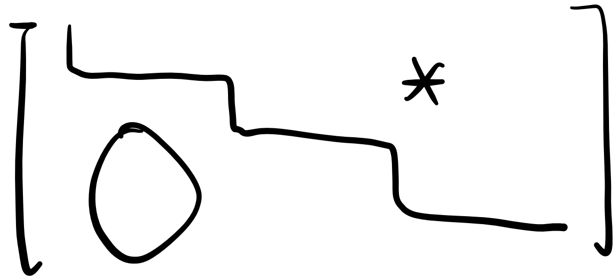
We say A is in **reduced row echelon form** (RREF) if A is in REF and also satisfies

(d) If a column contains a leading one, then all the other entries in that column are zero.

A matrix in RREF appears as a *staircase* (or *echelon*) pattern of leading ones descending from the *upper left corner* of the matrix.

Note: There is a similar definition for (reduced) column echelon form.

Let's draw a schematic of a matrix in (R)REF.



Example: Which of the following matrices are in REF? Which are in RREF?

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

$$C = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

none of them

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

RREF.

$$E = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

none of them.

Definition: For a matrix in REF, the first 1 that appears in a row is called a **pivot**. The columns containing pivots are called **pivot columns**.

Definition: An $m \times n$ matrix A is **row equivalent** to an $m \times n$ matrix B if B can be produced by applying a finite sequence of elementary row operations to A.

Theorem: Every $m \times n$ matrix A is **row equivalent** to a **unique** matrix in RREF.

Gaussian Algorithm: How to find a row-echelon matrix from a given matrix
Heuristic: Move from top to bottom and outside in until you get a staircase-shape with ones as leading entries.

1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.
2. Otherwise, find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.
3. Now multiply the new top row by $1/a$ to create a leading 1.
4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row, and all further row operations are carried out on the remaining rows.

5. Repeat steps 1–4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

Example: Find a row echelon form of the given matrix A.

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{r_1 \to \frac{1}{2} r_1} \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 3 & 3 & 6 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2 \to r_2 - 3r_1} \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & -\frac{3}{2} & 6 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2 \to -\frac{2}{3} r_2} \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Theorem: Consider two linear systems each of m equations in n unknowns. If the augmented matrices $[A \mid \mathbf{b}]$ and $[C \mid \mathbf{d}]$ are *row equivalent*, then the linear systems are *equivalent*, i.e. the systems have the *same solutions*.

Example: Solve the linear system

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 9 \\ 2x & - & y & + & z & = & 8 \\ 3x & & & - & z & = & 3 \end{array}$$

by transforming the associated matrix $[A \mid \mathbf{b}]$ to row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow[r_3 \to r_3 - 3r_1]{r_2 \to r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\xrightarrow[r_3 \to r_3 + 6r_2]{r_2 \to -\frac{1}{5}r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \xrightarrow{r_3 \to -\frac{1}{4}r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example: Solve the previous linear system by transforming $[A \mid \mathbf{b}]$ to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[r_2 \to r_2 - r_3]{r_1 \to r_1 - 2r_2 - r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

In this example, the RREF of A is the **identity matrix**. When this happens, the system corresponding to $[A \mid \mathbf{b}]$ has *a unique solution*.

Definition: If the augmented matrix of a linear system is a **row echelon matrix**, we say the linear system is in **echelon form**. The variables corresponding to columns with leading entries are called **leading variables** or **basic variables**, while the other variables are called **free variables**.

If the augmented matrix of a linear system is a REF matrix, a variable whose corresponding column does not have a leading 1 is called a **free** variable.

Example: Let

$$[A \,\vdots\, \mathbf{b}] = \left[\begin{array}{ccccc:c} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & -1 & 7 \\ 0 & 0 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{array}\right]$$

represent a linear system. Solve the system.

Example: Let

$$[A \,\vdots\, \mathbf{b}] = \left[\begin{array}{ccccc:c} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & -1 & 7 \\ 0 & 0 & 1 & 2 & 4 & 7 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{array}\right] \xrightarrow[\substack{r_2 \to r_2 - 3r_4 \\ r_1 \to r_1 - 4r_4}]{r_3 \to r_3 - 2r_4} \left[\begin{array}{ccccc:c} 1 & 2 & 3 & 0 & -3 & -30 \\ 0 & 1 & 2 & 0 & -7 & -20 \\ 0 & 0 & 1 & 0 & 0 & -11 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{array}\right]$$

represent a linear system. Solve the system.

$$\xrightarrow[r_1 \to r_1 - 3r_3]{r_2 \to r_2 - 2r_3} \left[\begin{array}{ccccc:c} 1 & 2 & 0 & 0 & -3 & 3 \\ 0 & 1 & 0 & 0 & -7 & 2 \\ 0 & 0 & 1 & 0 & 0 & -11 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{array}\right]$$

$$\xrightarrow{r_1 \to r_1 - 2r_2} \left[\begin{array}{ccccc:c} 1 & 0 & 0 & 0 & 11 & -1 \\ 0 & 1 & 0 & 0 & -7 & 2 \\ 0 & 0 & 1 & 0 & 0 & -11 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{array}\right]$$

x_5 is free variable.

let $x_5 = s$, then

$$\begin{cases} x_1 = -1 - 11s \\ x_2 = 2 + 7s \\ x_3 = -11 \\ x_4 = 9 - 2s. \end{cases}$$

Example: Let

$$[C \,\vdots\, \mathbf{d}] = \left[\begin{array}{cccc:c} 1 & 2 & -3 & 4 & 5 \\ 0 & 1 & 2 & 1 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

represent a linear system. Solve the system.

No sol'n.

> **Gaussian Elimination: Solving a Linear System**
>
> 1. Convert your linear system to an augmented matrix $[A \,\vdots\, \mathbf{b}]$.
> 2. Use elementary row operations to obtain a matrix $[C \,\vdots\, \mathbf{d}]$ in (reduced) row echelon form. Note that the two systems are equivalent.
> 3. If a row $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & : & 1 \end{bmatrix}$ occurs, the system is inconsistent.
> 4. Otherwise, assign the free variables as parameters, and use the equations in the row echelon form to solve for the leading variables in terms of the parameters.

<u>Rank</u>

Row echelon matrices are not unique for a given starting matrix. For example, $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are row equivalent row echelon matrices, but they are not equal.

However, the number of leading 1s will be the same in each row-equivalent row-echelon matrices.

Definition: The **rank** of a matrix A is the number of leading 1s (i.e. the number of pivots) in any row-echelon form that is row-equivalent to A.

Example: Find the rank of $M = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$$\xrightarrow[r_2 \to r_2 - 3r_1]{r_1 \to \frac{1}{2} r_1} \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & -3/2 & 6 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_2 \to -\frac{2}{3} r_2} \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Rank $M = 3$.

If rank $A = r$, and A has m rows and n columns, what do we know about r?

$r \le \min\{m, n\}$.

Theorem: Suppose a system of m equations in n variables is **consistent** (i.e. it has at least one solution), and the rank of the augmented matrix is r.

a. The set of solutions involves exactly $n - r$ parameters.

b. If $r < n$, the system has infinitely many solutions.

c. If $r = n$, the system has a unique solution.

For any system of linear equations, exactly three possibilities exist:

1. No solution. This occurs when a row $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & : & 1 \end{bmatrix}$ occurs in row-echelon form.

2. Unique solution. This occurs when every variable is a leading variable, i.e. every variable corresponds to a *pivot*.

3. Infinitely many solutions. This occurs when the system is *consistent* and there is at least one *free variable*.

Example: Discuss the rank and the number of solutions for each augmented matrix for a 3×3 system (RREF is also given):

a. $[A \,\vdots\, \mathbf{b}] = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & 4 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank $A = 2$, $2 < 3$, infinitely sol'ns.

b. $[A \,\vdots\, \mathbf{b}] = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rank $A = 2$, but last row is $[0\ 0\ 0\ 1)$, no soln.

Homogeneous Equations

Recall: A system of linear equations in variables $x_1, x_2, \ldots, x_n$ is called **homogeneous** if all the constant terms are 0, i.e. every equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0.$$

Note that $x_1 = x_2 = \cdots = x_n = \mathbf{0}$ is always a solution to a homogeneous system. It is called the **trivial** solution. Thus, a homogeneous system is always *consistent*.
A solution to a homogeneous system where *not* each x_i is 0 is called a **nontrivial** solution.
Example: Solve the homogeneous system

$$\begin{aligned} 2x_1 + 4x_3 + 6x_4 &= 0 \\ x_1 + 2x_2 + 7x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 0 & 4 & 6 \\ 1 & 2 & 0 & 7 \end{bmatrix} \xrightarrow{r_1 \to \frac{1}{2}r_1} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 7 \end{bmatrix}$$

$$\xrightarrow{r_2 \to r_2 - r_1} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -2 & 4 \end{bmatrix} \xrightarrow{r_2 \to \frac{1}{2}r_2} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

Rank $A = 2 < 4$, x_3 & x_4 are free variable,
let $x_3 = s$, $x_4 = t$, then $\begin{cases} x_1 = -2s - 3t \\ x_2 = s - 2t. \end{cases}$

Theorem: A homogeneous system of m linear equations in n unknowns *always* has a non-trivial solution if $n > m$, that is if the number of *unknowns* exceeds the number of *equations*.
Does a homogeneous system NEED to have more variables than equations in order to have infinitely many solutions?

No need. e.g. $\begin{cases} x + y = 10 \\ 2x + y = 20. \end{cases}$ $\Rightarrow$ inf. many sol'ns.