Math 421

Wednesday, September 17

Announcements

- Homework 2 Due Friday
- Course assistant office hours had been moved to Tuesday this week – so none tomorrow.
- Grace's drop-in hours
  - Today, 12-1 VV 311
  - Tomorrow, 9-11 VV B205

Chapter 1: Introduction

Section 3: The Set $\mathbb{R}$ of Real Numbers

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$

$\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$

Real Numbers: $\mathbb{R}$

Axioms:

- Associative laws (**A1.** & **M1.**)
- Commutative laws (**A2.** & **M2.**)
- 0 1
  Existence of an identity element (**A3.** & **M3.**)
- $-x$ x^{-1}
  Existence of inverse elements (**A4.** & **M4.**)
- Distributive law (**DL**)
- $x + y \in \mathbb{R}$ $xy \in \mathbb{R}$
  Closure under addition and multiplication
- Order structure (**O1.**-**O5.**)

$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2$

$x^2 = 2$

Use contradiction.
(technically a proof by negation)

Example Proof: $\sqrt{2} \notin \mathbb{Q}$

$\sqrt{3}$

Thm: $\sqrt{2}$ is irrational. → not rational

Proof. Assume by way of contradiction that $\sqrt{2}$ is rational. Then there exist $p, q \in \mathbb{Z}$, $q \neq 0$, such that $\sqrt{2} = \frac{p}{q}$. By removing common factors from p and q, we may assume p & q are not both even. (do not have a common factor of 3,) Then

$$\sqrt{2} = \frac{p}{q} \Rightarrow (\sqrt{2})^2 = \left(\frac{p}{q}\right)^2 \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2$$

So p^2 is even $\Rightarrow$ p is even. So $p = 2k$ for some $k \in \mathbb{Z}$. So $2q^2 = p^2 = (2k)^2 = 4k^2 \Rightarrow q^2 = 2k^2$. Thus q^2 is even $\Rightarrow$ q is even. This is a condradiction. Therefore our initial assumption that $\sqrt{2}$ is rational is false.

$x \notin \mathbb{Q}$ $\mathbb{R} - \mathbb{Q}$

Discussion Questions

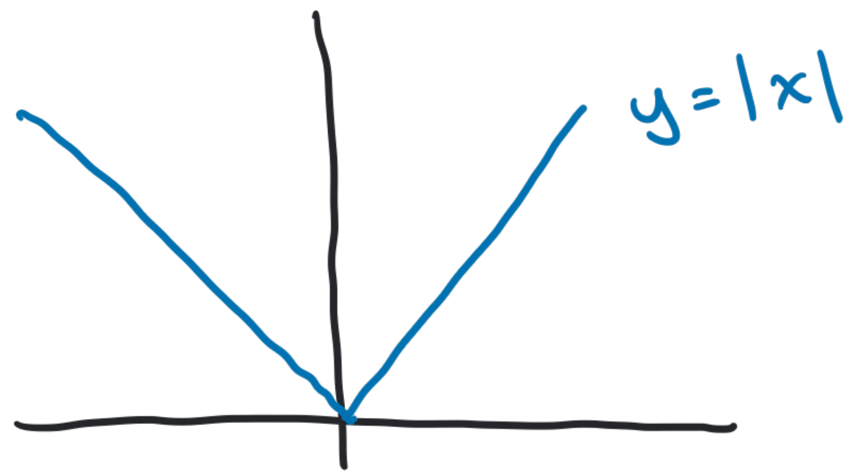
Answer the following questions, if the answer is yes or the statement is true, you do not need to provide a proof. If the answer is no or the statement is false, you should provide a counterexample.

- Are the rational numbers closed under multiplication? Closed under addition?
- Are the irrational numbers closed under multiplication? Closed under addition? $\sqrt{5}\cdot\sqrt{5} = 5$ $\sqrt{5} + (-\sqrt{5}) = 0$
- **True or False:** If x is rational and y is irrational, then $x + y$ is irrational.
- **True or False:** if x is rational and y is irrational, then xy is irrational. $x=0$

Absolute Value & Distance

Def: The **absolute value function** is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Notice that for all $x \in \mathbb{R}$

- $|-x| = |x|$
- $-|x| \leq x \leq |x|$

• Use a direct proof with cases.

Example Proof: Triangle Inequality

Thm. (The Triangle Inequality) If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.

Proof.

Case 1: $x \geq 0$, $y \geq 0$. Then $x+y \geq 0$, so

$$|x+y| = x+y = |x|+|y| \leq |x|+|y|$$

Case 2: $x<0$, $y<0$. Then $x+y<0$, so

$$|x+y| = -(x+y) = -x + -y = |x| + |y| \leq |x|+|y|$$

Example: Proof (cont.)

Case 3: $x \geq 0,\ y < 0$

3a: $x + y \geq 0$

$|x+y| = x+y = |x| + y < |x| + 0 < |x| + |y| \leq |x| + |y|.$

3b: $x + y < 0.$

$|x+y| = -(x+y) = -x + -y = -x + |y|$

$\leq 0 + |y| \leq |x| + |y|$

Case 4: $x < 0,\ y \geq 0$. By swapping the variables this follows from Case 3.