Math 421

Friday, September 12

Announcements

- Homework 1 due at 11:59 pm
  - Don’t forget the collaboration statement
    - You may work with others, consult outside resources if needed, but all written work must be your own.
  - Follow the uploading instructions on Gradescope – including indicating on which pages of your document the problem solutions can be found.
  - Practice good proof writing techniques!

Sets – Definition

Definition: A ***set***, S, is an unordered collection of ***elements***.

Notation –

$x \in S$ – x is an element of S

$x \notin S$ – x is not an element of S

$T \subseteq S$ – every element in the set T is also in the set S

i.e. $x \in T$ implies $x \in S$.

Examples:

$S = \{1, \{2,3\}, Red, B\}$

$1 \in S \qquad \{1\} \subseteq S$

$2 \notin S \qquad \{2,3\} \in S \qquad \{\{2,3\}\} \subseteq S. \qquad \{1, B\} \subseteq S$

$\{2,3\} \not\subseteq S$

$\{\} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Some Important Sets

- $\mathbb{R}$ denotes the **real numbers**
- $\mathbb{Q}$ denotes the **rational numbers** – numbers of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$.
- $\mathbb{Z}$ denotes the **integers**
- $\mathbb{N}$ denotes the **natural numbers**
- $\emptyset$ denotes the **empty set** – the set with no elements

  " $\{\}$

True or False?

T

If $\underbrace{x \in \emptyset}_{\text{always false.}}$, then x is an odd number and an even number.

Set Builder Notation

Small, finite sets can be denoted using braces $\{\ \}$.

Example: $\{2, 3, 5, 7\}$

Larger finite, or infinite, sets can also be described by properties of their elements via the notation
`{[variable]:[properties]}`.

Examples:

- $\{n : n \in \mathbb{N}$ and n is odd$\}$ $= \{1, 3, 5, \dots\}$

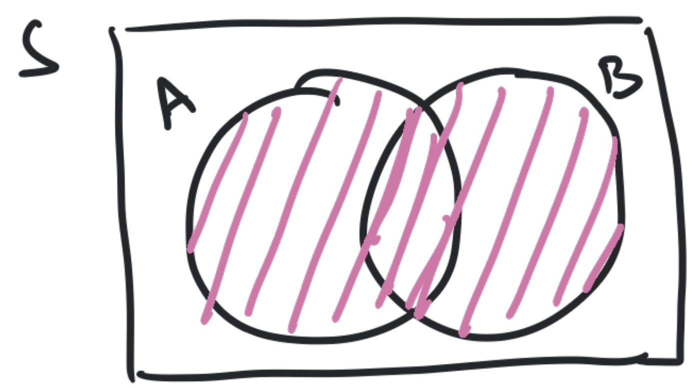
- $\{x : x \in \mathbb{R}$ and $1 \leq x < 3\}$ $[1, 3)$

$\boxed{\{[1,3)\}}$

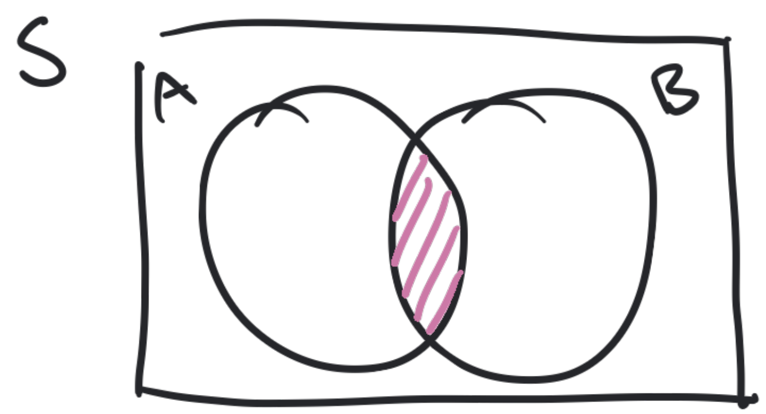
Set Operations

Let A and B be two subsets of a set S.

- **Union** - $A \cup B = \{x \in S : x \in A \text{ or } x \in B\}$



- **Intersection** - $A \cap B = \{x \in S : x \in A \text{ and } x \in B\}$



$S = \mathbb{Z}$

$A^c = \{\ldots, -2, -1, 0, 1, 3, 5, \ldots\}$

$A \subseteq \mathbb{N}$ $\quad S = \mathbb{N}$

$A = \{2, 4, 6, \ldots\}$

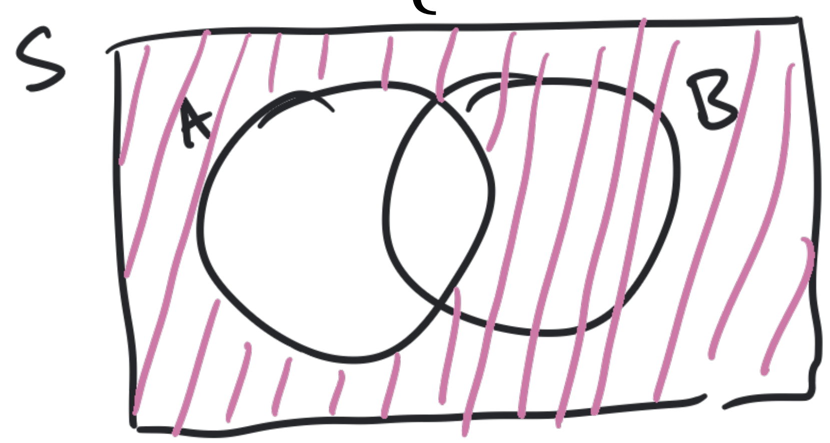
$A^c = \{1, 3, 5, \ldots\}$

More Set Operations

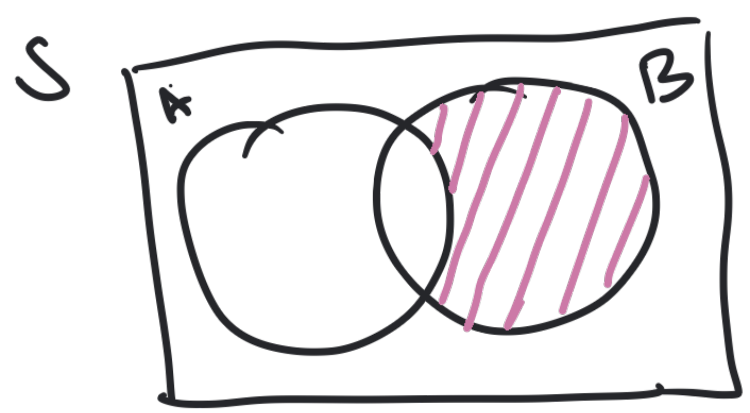
Let A and B be two subsets of a set S.

- **Complement** - $A^c = \{x \in S : x \notin A\}$ $= S - A$

$A^c \cup A = S$



- **Set Difference** - $B - A = \{x \in S : x \in B \text{ and } x \notin A\}$



Example – Proving Subset Relation

Theorem: If A and B are subsets of a set S, then $A \cap B \subseteq A \cup B$.

Discussion. $T \subseteq S$ means if $x \in T$ then $x \in S$.

Need to show: if $x \in A \cap B$ then $x \in A \cup B$.

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$.

Since $A \subseteq A \cup B$, $x \in A$ implies $x \in A \cup B$.

Therefore, $A \cap B \subseteq A \cup B$.

$\boxed{\text{Avoid: } \forall,\ \exists,\ \text{s.t.},\ \therefore,\ \because}$

True or False

1. $\{1, 1, 2, 3\} \overset{\subseteq}{\underset{\supseteq}{=}} \{1, 2, 3\}$ T

2. $\{3, 2, 1\} = \{1,2,3\}$ T

Q: How do we determine if two sets are equal? $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

Chapter 1: Introduction

Section 1 : The set $\mathbb{N}$ of Natural Numbers

The Natural Numbers

The natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ are the *unique* mathematical object satisfying five axioms (listed in the textbook), including:

Axiom N5: If $S \subseteq \mathbb{N}$ is a subset where

1. $1 \in S$,
2. $n + 1 \in S$ whenever $n \in S$,

then $S = \mathbb{N}$.

Informal justification:

$$1 \in S, \text{ so } 2 = 1 + 1 \in S, \text{ so } 3 = 2 + 1 \in S, \text{ so} \ldots$$

Mathematical Induction

Used when – you need to prove a statement for a discrete set of cases, i.e. prove P_n is true for all $n \in \mathbb{N}$.

Process – Let $P_1, P_2, P_3, \ldots$ be a list of statements that may or may not be true. Show

often $n=1$ but could be any natural

(I_1) Basis Step: P_1 is true

(I_2) Inductive Step: If P_n is true, then P_{n+1} is true.

Example 1

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

Theorem: If $n \in \mathbb{N}$, then

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- What proof technique should we use? *Why*?
- Do you believe the statement? Verify for $n = 2, 3$.

Inductive Proof Structure

P_n: "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$"

Base Case:

P_1 : $1 = \frac{1(1+1)}{2} = 1$ ✓

Inductive Step:

If "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$" then

"$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$"