Homework 1

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MATH 421 – The Theory of Single Variable Calculus
Section 003
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Collaboration Statement

I worked on this homework independently. No outside collaborators or resources were used.

Problem 1

Determine whether the following statements are true or false and provide a short justification.

A. For every integer x, $x \leq 5$ and $x > 3$.

   Solution: This statement is **false**, by counterexample. For instance, $x = 7$ is an integer such that $x > 3$, but $x \not\leq 5$. Therefore, not every integer satisfies the conditions.

B. There exists an integer n such that $n \leq 5$ and $n > 3$.

   Solution: The statement claims that there exists an integer n such that $n \leq 5$ and $n > 3$. Consider $n = 4$. Clearly, $4 \in \mathbb{Z}$, and it satisfies both $4 \leq 5$ and $4 > 3$. Therefore, the statement is **true**.

C. There exists a unique integer x such that $x \leq 5$ and $x > 3$.

   Solution: The statement is **false**. Both $x = 4$ and $x = 5$ are integers with $x \leq 5$ and $x > 3$. Since more than one integer satisfies the condition, the solution is not unique.

D. There exists an integer x such that for all integers y, $xy = x$.

   Solution: This statement is **true**. We want to determine whether there is some integer x such that for all $y \in \mathbb{Z}$, the equality $xy = x$ holds.

   If $x = 0$, then for every integer y, we have $0 \cdot y = 0 = x$. Thus, $x = 0$ satisfies the condition.

   If $x \neq 0$, consider $y = 2$. Then $xy = 2x \neq x$. Therefore, no nonzero integer x works.

   Hence, the statement is true, and the only integer x with this property is $x = 0$.

E. For all integers x there exists an integer y such that $xy = x$.

Solution: The statement is **true**. Let $x \in \mathbb{Z}$ be arbitrary. Choose $y = 1$. Then $xy = x \cdot 1 = x$. Since x was arbitrary, we conclude that $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ with $xy = x$.

Problem 2

Create an (A) example of an ”if..., then...” statement (can be mathematical or not), then write the (B) converse and (C) contrapositive of your statement.

Solution:

A. **Example:** If there is no endogeneity, then there *might* be causality.

B. **Converse:** If there *might* be causality, then there is no endogeneity.

C. **Contrapositive:** If there is not causality, then there is endogeneity.

Problem 3

Use a truth table to show that the converse of an “if..., then...” statement is not equivalent to the original statement.

Solution: Let P and Q be statements. The original statement is $P \Rightarrow Q$, and the converse is $Q \Rightarrow P$. We construct the truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

From the table, we see that $P \Rightarrow Q$ and $Q \Rightarrow P$ do not always have the same truth value (for instance, in the second and third rows). Therefore, the converse of a conditional statement is **not** logically equivalent to the original statement.

Problem 4

Theorem 1. *If x and y are odd integers, then xy is an odd integer.*

Proof. Let x and y be odd integers. Then there exist integers $k, m \in \mathbb{Z}$ such that

$$x = 2k + 1 \quad \text{and} \quad y = 2m + 1.$$

Multiplying, we have

$$xy = (2k + 1)(2m + 1) = 4km + 2k + 2m + 1 = 2(2km + k + m) + 1.$$

Let $j = 2km + k + m$, which is an integer since $k, m \in \mathbb{Z}$. Thus,

$$xy = 2j + 1,$$

which is the definition of an odd integer. Therefore, xy is odd. □

Problem 5

Theorem 2. *If x and y are two integers with the same parity (that is, both even or both odd), then $x + y$ is even.*

Proof. We split the proof into two cases:

Case 1: x and y are both odd. Then there exist integers $k, m \in \mathbb{Z}$ such that $x = 2k+1$ and $y = 2m + 1$.

$$x + y = (2k + 1) + (2m + 1) = 2(k + m + 1).$$

Thus, $x + y$ is even.

Case 2: x and y are both even. Then there exist integers $k, m \in \mathbb{Z}$ such that $x = 2k$ and $y = 2m$.

$$x + y = 2k + 2m = 2(k + m).$$

Thus, $x + y$ is even.

Since both cases lead to $x + y$ being even, we conclude that if x and y have the same parity, then $x + y$ is even. □

Problem 6

Theorem 3. *For all real numbers x and y,*

$$(x+y)^2 = x^2 + y^2 \quad \textit{if and only if} \quad x = 0 \textit{ or } y = 0.$$

Proof. Let P be the statement $(x+y)^2 = x^2+y^2$ and let Q be the statement $x = 0$ or $y = 0$. We will prove the biconditional $P \Leftrightarrow Q$ by showing both directions: $P \Rightarrow Q$ and $Q \Rightarrow P$.

($P \Rightarrow Q$) Suppose $(x+y)^2 = x^2 + y^2$. Expanding the left-hand side gives

$$x^2 + 2xy + y^2 = x^2 + y^2.$$

Subtracting x^2+y^2 from both sides yields $2xy = 0$, hence $xy = 0$. Over $\mathbb{R}$ this implies $x = 0$ or $y = 0$. Thus, Q holds.

($Q \Rightarrow P$) Conversely, suppose $x = 0$ or $y = 0$.

- If $x = 0$, then $(x+y)^2 = (0+y)^2 = y^2 = x^2 + y^2$.
- If $y = 0$, then $(x+y)^2 = (x+0)^2 = x^2 = x^2 + y^2$.

In either case, P holds.

Since we have shown both $P \Rightarrow Q$ and $Q \Rightarrow P$, we conclude that

$$(x+y)^2 = x^2 + y^2 \quad \text{if and only if} \quad x = 0 \text{ or } y = 0.$$

$\square$