# Math 421

September 8, 2025

## Announcements

- Homework 1 due Friday
  - LaTeX is optional, but encouraged for proofs!
- MLC opens today
  - Proof table
    - Mon-Thurs: 3pm-7:30pm VV B227
  - Course Assistant
    - Thurs: 4-7 in MLC
- Errors on Set Basics & 1.1 Reading Quizzes

# Conditional

If $P$ and $Q$ are statements, then the statement if $\boldsymbol{P}$***, then*** $\boldsymbol{Q}$ is

- True when either
  - $P$ and $Q$ are both true
  - $P$ is false
- False when $P$ is true and $Q$ is false

We sometimes write "$P \Rightarrow Q$" or "$P$ implies $Q$".

## True or False?

1. If $f$ is differentiable, then $f$ is continuous.

2. If $x$ is prime, then $x$ is odd.

# Biconditional/Equivalence

If $P$ and $Q$ are statements, then the statement ***$P$ if and only if $Q$*** is

- True when both $P$ and $Q$ are either both true or both false
- False otherwise

We sometimes write "$P \Leftrightarrow Q$" or "$P$ iff $Q$".

## True or False?

1. A function $f$ is differentiable if and only if it is continuous.

2. A number is divisible by 6 if and only if it is divisible by 2 and 3.

# Converse & Contrapositive

**Definition:** The converse of "if $P$, then $Q$" is "if $Q$, then $P$".

**Definition:** The contrapositive of "if $P$, then $Q$" is "if not $Q$, then not $P$".

**Example:** If $f$ is differentiable, then $f$ is continuous.

**Converse:**

**Contrapositive:**

**Q:** Is a statement equivalent to its converse? Its contrapositive?

# Activity

Use the following truth table to show that "$P \Rightarrow Q$" is logically equivalent to the contrapositive "not $Q \Rightarrow$ not $P$"

| $P$ | $Q$ | not $P$ | not $Q$ | $P \Rightarrow Q$ | not $Q \Rightarrow$ not $P$ |
|---|---|---|---|---|---|
| T | T | | | | |
| T | F | | | | |
| F | T | | | | |
| F | F | | | | |

# Pairs Activity

**Person A:** You **only** know the following information –

- The definition of the integers, i.e. that it is the set $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- The operations of addition, $+$, and multiplication, $\cdot$, and their basic properties, i.e. commutativity, associativity, distributive law, etc.

**Person B:** Convince Person A that the following theorem is true

**Theorem:** If $x$ is an even integer, then $x^2$ is an even integer.

# Definitions, Theorems & Proofs

Much of class will have the following structure

1. We will introduce a mathematical object using a **definition**.
   - **For yourself:** Think of an example and a non-example.
2. We will state a property of it using a **theorem**.
   - **For yourself:** What are the hypotheses and why are they needed? What is the conclusion?
3. We will explain why the property is true using a proof.
   - **For yourself:** What proof technique is being used? What is the big idea? What common strategies are being employed?

# Even & Odd – Definition & Examples

## Definition:

- An integer $x$ is ***even*** if there exists an integer $k$ such that $x = 2k$.
- An integer $x$ is ***odd*** if there exists an integer $k$ such that $x = 2k + 1$.

## Examples:

# Even & Odd – Theorem

**Theorem:** If $x$ is an even integer, then $x^2$ is an even integer.

Proof:

# Activity

**Theorem:** Suppose $x$ is an integer. If $x$ is odd, then $x^2 + 1$ is even.

*Discussion:*

Proof:

# If and Only If Theorem

**Theorem:** For all integers $x$, $x$ is even if and only if $x + 1$ is odd.

*Discussion:*

Proof:

# First Proof Techniques

- **Direct:**
  - **Basic Direct** – Use axioms and definitions to go from hypotheses to conclusion.
  - **Cases** – First break the proof up into cases that exhaust all possibilities. Then use a basic direct proof for each case.
- **Indirect:**
  - **Contrapositive** – Give a direct proof of the contrapositive of the given conditional statement.
  - **Contradiction** – Assume that the hypotheses are true and the conclusion is false, derive a contradiction.

# Example

**Theorem:** Suppose $x$ and $y$ are integers. If $x + y \geq 19$, then $x \geq 10$ or $y \geq 10$.

*Discussion:*

# Proof A: Cases

**Theorem:** Suppose $x$ and $y$ are integers. If $x + y \geq 19$, then $x \geq 10$ or $y \geq 10$.

Proof:

# Proof B: Contrapositive

**Theorem:** Suppose $x$ and $y$ are integers. If $x + y \geq 19$, then $x \geq 10$ or $y \geq 10$.

Proof:

# Proof C: Contradiction

**Theorem:** Suppose $x$ and $y$ are integers. If $x + y \geq 19$, then $x \geq 10$ or $y \geq 10$.

Proof:

# Common Language

- It is a **good idea** to start a proof by contrapositive with
  - “We prove the contrapositive: If [not Q], then [not P]. Assume [not Q]…”

- It is a **good idea** to start a proof by contradiction with
  - “We argue by contradiction. Suppose [P] and [not Q].”

  OR

  - “Suppose for a contradiction that [P] and [not Q].”

# Activity

**Theorem:** Suppose $a$ and $b$ are positive real numbers. If $ab \geq 9$, then $a \geq 3$ or $b \geq 3$.

Form groups of 6 and pair off for each of the three proof techniques. Then, use your chosen method to prove the statement. Once complete, discuss the details with the full group.

A – Cases

B – Contrapositive

C – Contradiction