Math 421

Friday, September 12

Announcements

- Homework 1 due at 11:59 pm
  - Don’t forget the collaboration statement
    - You may work with others, consult outside resources if needed, but all written work must be your own.
  - Follow the uploading instructions on Gradescope – including indicating on which pages of your document the problem solutions can be found.
  - Practice good proof writing techniques!

Sets – Definition

Definition: A ***set***, S, is an unordered collection of ***elements***.

Notation –

$x \in S$ – x is an element of S

$x \notin S$ – x is not an element of S

$T \subseteq S$ – every element in the set T is also in the set S

i.e. $x \in T$ implies $x \in S$.

Examples:

Some Important Sets

- $\mathbb{R}$ denotes the **real numbers**
- $\mathbb{Q}$ denotes the **rational numbers** – numbers of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$.
- $\mathbb{Z}$ denotes the **integers**
- $\mathbb{N}$ denotes the **natural numbers**
- $\emptyset$ denotes the **empty set** – the set with no elements

True or False?

If $x \in \emptyset$, then x is an odd number and an even number.

Set Builder Notation

Small, finite sets can be denoted using braces { }.

Example: $\{2, 3, 5, 7\}$

Larger finite, or infinite, sets can also be described by properties of their elements via the notation `{[variable]:[properties]}`.

Examples:

- $\{n : n \in \mathbb{N}$ and n is odd$\}$
- $\{x : x \in \mathbb{R}$ and $1 \leq x < 3\}$

Set Operations

Let A and B be two subsets of a set S.

- **Union** - $A \cup B = \{x \in S : x \in A \text{ or } x \in B\}$

- **Intersection** - $A \cap B = \{x \in S : x \in A \text{ and } x \in B\}$

More Set Operations

Let A and B be two subsets of a set S.

- **Complement** - $A^c = \{x \in S : x \notin A\}$

- **Set Difference** - $B - A = \{x \in S : x \in B \text{ and } x \notin A\}$

Example – Proving Subset Relation

Theorem: If A and B are subsets of a set S, then $A \cap B \subseteq A \cup B$.

Discussion.

Proof.

True or False

1. $\{1, 1, 2, 3\} = \{1, 2, 3\}$

2. $\{3, 2, 1\} = \{1,2,3\}$

Q: How do we determine if two sets are equal?

Chapter 1: Introduction

Section 1 : The set $\mathbb{N}$ of Natural Numbers

The Natural Numbers

The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ are the *unique* mathematical object satisfying five axioms (listed in the textbook), including:

Axiom N5: If $S \subseteq \mathbb{N}$ is a subset where

1. $1 \in S$,
2. $n + 1 \in S$ whenever $n \in S$,

then $S = \mathbb{N}$.

Informal justification:

$$1 \in S, \text{ so } 2 = 1 + 1 \in S, \text{ so } 3 = 2 + 1 \in S, \text{ so} \ldots$$

Mathematical Induction

Used when – you need to prove a statement for a discrete set of cases, i.e. prove P_n is true for all $n \in \mathbb{N}$.

Process – Let $P_1, P_2, P_3, \ldots$ be a list of statements that may or may not be true. Show

(I_1) Basis Step: P_1 is true

(I_2) Inductive Step: If P_n is true, then P_{n+1} is true.

Example 1

Theorem: If $n \in \mathbb{N}$, then

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- What proof technique should we use? *Why*?

- Do you believe the statement? Verify for $n = 2, 3$.

Inductive Proof Structure

P_n:

Base Case:

Inductive Step:

Activity – Proof

Theorem: For positive integers n,

$$\sum_{j=1}^{n}(2j-1) = 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

- What proof technique should we use? *Why*?

- Do you believe the statement? Verify for $n = 2, 3$.

Activity – Inductive Proof Structure

P_n:

Base Case:

Inductive Step: