Chapter 5: Matrix Inverses

Definition: For each $n \geq 2$, the **identity matrix** I_n is the $n \times n$ matrix with 1s on the main diagonal (upper left to lower right), and zeros elsewhere.

Key property of identity matrices If A is any $m \times n$ matrix, then

$$AI_n = A \text{ and } I_m A = A.$$

Example: Let's verify this fact for I_3.

Definition: An $n \times n$ matrix A is called **nonsingular**, or **invertible**, if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Such a B is called the **inverse** of A. If no such B exists, A is called **singular**, or **noninvertible**.

Idea: The inverse of a matrix mimics the reciprocal of a real number.

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$. Compute both AB and BA, and make a conclusion using the language of inverses.

Fact: If A, B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

Fact: The inverse of a matrix, if it exists, is unique. Therefore, we can write A^{-1} for the inverse of A.

Example: Does $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ have an inverse?

Theorem: If both A and B are nonsingular $n \times n$ matrices, then the matrix AB is nonsingular and its inverse is $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

Follow up facts:

- If $A_1, A_2, \ldots, A_{k-1}, A_k$ are $n \times n$ invertible/nonsingular matrices, then $A_1 A_2 \cdots A_{k-1} A_k$ is invertible/nonsingular and $(A_1 A_2 \cdots A_{k-1} A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$
- If A is invertible/nonsingular, then A^{-1} is invertible/nonsingular and $(A^{-1})^{-1} = A$.
- If A is invertible/nonsingular, then A^T is invertible/nonsingular and $(A^{-1})^T = (A^T)^{-1}$.
- With the convention that $A^0 = I_n$ for an $n \times n$ invertible matrix A, the rules $A^p A^q = A^{p+q}$ and $(A^p)^q = A^{pq}$ hold for all integers p and q.

Example: If A is invertible, and $k \neq 0$, then $(kA)^{-1} = \underline{\quad} A^{-1}$.

Theorem: (Inverse of 2×2 matrix) The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We'll learn how to compute inverses for bigger matrices soon.

Example: Find the inverse of each matrix, if possible.

$$A = \begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix}$$

Linear Systems and Inverses

If A is an $n \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ is a system of **n** equations in **n** unknowns. Suppose A is nonsingular. How can we use A^{-1} to solve the system $A\mathbf{x} = \mathbf{b}$?

Consequences:

- When A^{-1} exists, then $A\mathbf{x} = \mathbf{b}$ has a *unique* solution.
- If A is invertible/nonsingular, then the ONLY solution to the homogeneous system $Ax = 0$ is $\mathbf{x} = \mathbf{0}$.

Example: Use the inverse of A to solve the linear systems $A\mathbf{x} = \mathbf{b}$, $A\mathbf{x} = \mathbf{c}$, and $A\mathbf{x} = \mathbf{0}$, where A, $\mathbf{b}$, and $\mathbf{c}$ are given below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Chapter 6: Elementary Matrices

Definition: An $n \times n$ **elementary** matrix is a matrix obtained from the identity matrix by performing a single elementary row operation.

Example: Fill in the row operation that is performed on I_3 to get each elementary matrix, then perform the given matrix multiplication on an arbitrary 3×3 matrix. What do you observe?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\boxed{}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\boxed{}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\boxed{}} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

Theorem: If an elementary row operation is performed on the $m \times n$ matrix A, then the result is the product EA, where E is the elementary matrix obtained by performing the same row operation on the $m \times m$ identity matrix I_m.

Fact: Every elementary matrix is invertible, and its inverse is an elementary matrix.

Example: Find the inverse of $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem: Suppose that $E_1, E_2, \ldots, E_k$ are the elementary matrices that row reduce A to its reduced row echelon form, i.e. $E_k E_{k-1} \cdots E_2 E_1 A = RREF(A)$. Then:

1. A is invertible if and only if the reduced row echelon form of A is I_n.

2. When A is invertible, its inverse is $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$.

Finding A^{-1}

Let A be a nonsingular $n \times n$ matrix. By the above theorem, $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$. This gives us a method for finding A^{-1}: Perform elementary row operations E_1, E_2, ..., E_k on A until we get I_n, then $E_k E_{k-1} \ldots E_2 E_1$ equals A^{-1}.

A convenient way to track this computation is to write down the partitioned matrix $[A \mid I_n]$. Then applying the elementary row operations we transform $[A \mid I_n]$ into $[I_n \mid A^{-1}]$.

Algorithm for computing A^{-1}

1. Set up the augmented matrix $[A \mid I_n]$.

2. Apply elementary row operations to $[A \mid I_n]$ to reduce it to $[I_n \mid A^{-1}]$.

If A is an $n \times n$ matrix, either A can be reduced to I_n by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

Example: Find A^{-1} where $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Box Of Facts/Invertibility Criteria
Let A be an $n \times n$ matrix. The following are equivalent:

1. A is nonsingular/invertible.
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. $\text{rank}(A) = n$.
6. A can be written as a product of elementary matrices.