Chapter 4: Matrices and matrix Operations

We've seen matrices used to help us solve systems of linear equations. We're now going to define them more formally and explore other ways to utilize them.
Definition: An $m \times n$ **matrix** A is a rectangular array of $m \cdot n$ real (or complex) numbers arranged in m **horizontal rows** and n **vertical columns**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & a_{ij} & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$$

The ***i*th row** of A is $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$, and the ***j*th column** of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj}. \end{bmatrix}$. The number a_{ij}, which is in the ith row and jth column of A, is the (i, j)**-entry** of A, and we often write $A = [a_{ij}]$. We say A is an "m **by** n" matrix.
Definition: An $m \times n$ matrix A is **square** if $m = n$.

Example: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -8 & -13 \end{bmatrix}$$

A is a **2**[redacted] $\times$ **3**[redacted] matrix. Complete the following:

- $a_{11} =$ 1
- $a_{12} =$ 2
- $a_{21} =$ 5
- $a_{23} =$ -13
- $a_{32} =$ none

Definition: Two matrices A and B are **equal** if they have the same **size** AND the corresponding entries are **equal**.
Example: If $A = \begin{bmatrix} 3 & y \\ 4 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 6+x & -2 \\ 4 & -7 \end{bmatrix}$, and $A = B$, find x and y.

$$\begin{cases} x = -3 \\ y = -2 \end{cases}$$

Matrix Operations

Matrix Addition: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then their **sum** $A+B$ is the matrix $C = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$.

Scalar Multiplication: If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the **scalar multiple** of A by r, written rA, is the $m \times n$ matrix $C = [c_{ij}]$, where $c_{ij} = ra_{ij}$, that is, C is the matrix obtained by multiplying every entry of A by r.

Example: For the matrices A and B as below, calculate $A+B$ and calculate $-2A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -8 & -13 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 4 & 4 \\ 6 & -5 & -17 \end{bmatrix}$$

$$-2A = \begin{bmatrix} -2 & -4 & -6 \\ -10 & 16 & 26 \end{bmatrix}$$

Definition: If $A_1, A_2, \ldots, A_k$ are $m \times n$ matrices and $c_1, c_2, \ldots, c_k$ are real numbers, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k$$

is called a **linear combination** of $A_1, A_2, \ldots, A_k$. The scalars $c_1, \ldots, c_k$ are called **coefficients**.

Example: The following is a linear combinations of matrices. Compute it.

$$4\begin{bmatrix} 0 & 2 \\ -3 & 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -12 & 12 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -15 & 11 \end{bmatrix}$$

Example: Is $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$?

$$\boxed{\text{So } \begin{cases} c_1 = 8 \\ c_2 = 1 \end{cases}}$$

If so, there exist c_1, c_2, s.t.

$$c_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} 1 \cdot c_1 - 3c_2 = 5 \\ 1 \cdot c_1 + 2c_2 = 0 \end{cases}$$

Augmented matrix $[A \,\vdots\, b]$

$$= \begin{bmatrix} 1 & -3 & \vdots & 5 \end{bmatrix} \quad \underline{r_2 \to r_2 - r_1}$$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \end{array}\right]$

$\begin{bmatrix} 1 & -3 & 5 \\ 0 & -5 & -5 \end{bmatrix} \xrightarrow{r_2 \to -\frac{1}{5} r_2} \left[\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{r_1 \to r_1 + 3r_2} \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 1 \end{array}\right]$

Matrix Addition/Scalar multiplication Properties Let A, B, and C be $m \times n$ matrices, and let r, s be scalars/real numbers.

(a) $A + B = B + A$, i.e. matrix addition is **commutative**.

(b) $A + (B + C) = (A + B) + C$, i.e. matrix addition is **associative**.

(c) There is a unique $m \times n$ matrix O such that $A + O = A$ for any $m \times n$ matrix A. The matrix O is called the $m \times n$ **zero matrix**.

(d) For each $m \times n$ matrix A, there is a unique $m \times n$ matrix D such that $A + D = O$. The matrix D must, in fact, be the matrix $-A = (-1)A$. The matrix $-A$ is called the **negative** of A.

(e) $r(sA) = (rs)A$

(f) $(r + s)A = rA + sA$

(g) $r(A + B) = rA + rB$

Proof of (a) (if time):

$$\begin{aligned} A + B = [a_{ij}] + [b_{ij}] &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \\ &= [b_{ij}] + [a_{ij}] \\ &= B + A. \end{aligned}$$

Definition: If $A = [a_{ij}]$ is an $m \times n$ matrix, then the **transpose** of A, denoted $A^T = [{a_{ij}}^T]$, is the $n \times m$ matrix defined by ${a_{ij}}^T = a_{ji}$. In other words, the transpose of A is obtained by *interchanging* the rows and the columns of A.

Example: Compute the transpose for each of the given matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -8 & -13 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} 1 & 5 \\ 2 & -8 \\ 3 & -13 \end{bmatrix}, \quad B^T = \begin{bmatrix} 5 & 6 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 10 & 20 & 30 \end{bmatrix}$$

Definition: If $A = [a_{ij}]$ is an $m \times n$ matrix, the elements $a_{11}, a_{22}, a_{33}, \dots$ are called the **main diagonal** of A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

Thus forming the transpose of a matrix A can be viewed as flipping A over its main diagonal.

Matrix Transpose Properties If r is a scalar and A and B are matrices of the appropriate sizes, then

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$

Vectors

Definition: We say a matrix is a **row vector** if it is $1 \times n$, and is a **column vector** if it is $n \times 1$. Often we will call them just n-vectors or "vectors of length n" denoted with bold-face font, like $\mathbf{x}$ or $\mathbf{v}$.

$\mathbb{R}^n$ denotes the set of all n-vectors (with entries in $\mathbb{R}$).

Example: $\mathbf{v} = \begin{bmatrix} -3 \\ \sqrt{2} \\ 0 \end{bmatrix}$ is a vector in [redacted]. $\mathbb{R}^3$

The **zero vector** in $\mathbb{R}^n$, denoted $\mathbf{0}$, is the n-vector consisting of all zeros.

An $m \times n$ matrix can be thought of a row of n column vectors of length m:
$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, where $\mathbf{a}_i$ is the **ith column** of A.

Matrix-Vector Product Consider the linear system of m equations in n unknowns,

$$\begin{array}{llcll} a_{11}x_1+ & a_{12}x_2+ & \cdots+ & a_{1n}x_n & = b_1 \\ a_{21}x_1+ & a_{21}x_2+ & \cdots+ & a_{2n}x_n & = b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1+ & a_{m2}x_2+ & \cdots+ & a_{mn}x_n & = b_m. \end{array}$$

From this system, define the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We can rewrite this system as follows:

$$x_1 \cdot \underbrace{\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}}_{= \vec{a}_1} + x_2 \underbrace{\begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}}_{= \vec{a}_2} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{= \vec{a}_n} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{= \vec{b}}$$

$$\iff x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 + \cdots + x_n \cdot \vec{a}_n = \vec{b}$$

Example: Consider the linear system

$$\begin{array}{rcrcr} x & - & 3y & = & 5 \\ x & + & 2y & = & 0 \end{array}$$

Rewrite this system using a linear combination.

$$x \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Definition: Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be an $m \times n$ matrix, written in terms of its columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any n-vector, the **matrix-vector product** $A\mathbf{x}$ is defined to be the m-vector given by:

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n.$$

Example: Compute the matrix-vector product $A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 4 & -5 \\ 3 & 1 & -2 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

$$A\vec{x} = \begin{bmatrix} (-1)\times 2 + 4\times(-3) + (-5)\times 4 \\ 3\times 2 + 1\times(-3) + (-2)\times 4 \end{bmatrix} = \begin{bmatrix} -34 \\ -5 \end{bmatrix}$$

Notice that we can now write a linear system of equations in terms of a single matrix equation: If A is the coefficient matrix of the linear system, and $\mathbf{b}$ is the constant vector, then the system takes the form of a single matrix equation: $A\mathbf{x} = \mathbf{b}$.

Example: Go back to the previous linear system example and rewrite it as a matrix equation.

$$\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$