# UNIT 1: SYSTEMS OF LINEAR EQUATIONS AND MATRICES

## Chapter 1: Systems of Linear Equations

---

These notes follow and use material from your textbook *Discover Linear Algebra* by Jeremy Sylvestre.

**Example:** In a Wisconsin forest, there are robins and badgers. Together they have 18 heads and 56 legs. How many robins and badgers are in the forest?

### Linear Equations

An equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is called a **linear equation**. The real (or complex) quantities $a_1, \ldots, a_n, b$ are **constants**, while the variables $x_1, \ldots, x_n$ represent **unknowns**.

**Example:** Identify the constants and unknowns in each linear equation in the first example.

**Example:** What are some equations that are not linear equations?

More generally, a **system of $m$ linear equations in $n$ unknowns** $x_1, \ldots, x_n$, or a **linear system**, is a set of $m$ linear equations each in $n$ unknowns. A linear system has the form

$$\begin{array}{llllll} a_{11}x_1+ & a_{12}x_2+ & \cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1+ & a_{22}x_2+ & \cdots+ & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1}x_1+ & a_{m2}x_2+ & \cdots+ & a_{mn}x_n & = & b_m. \end{array} \qquad (*)$$

A **solution** to the linear system $(*)$ is a sequence of $n$ numbers $s_1, \ldots, s_n$ so that *each* equation in $(*)$ satisfied when $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ are substituted.

Let $(*)$ be a linear system as above.

- If $(*)$ has *no* solution, it is called **inconsistent**.
- If $(*)$ has a solution, it is called **consistent**. (Note that it may have *infinitely many* solutions!)
- If $(*)$ has $b_1 = b_2 = \cdots = b_n = 0$, it is called a **homogeneous system**.
  - Note that $x_1 = x_2 = \cdots = x_n = \mathbf{0}$ is always a solution to a homogeneous system. It is called the **trivial** solution.
  - A solution to a homogeneous system where *not* each $x_i$ is 0 is called a **nontrivial** solution.

Consider another system of $r$ linear equations in $n$ unknowns:

$$\begin{array}{llllll} c_{11}x_1+ & c_{12}x_2+ & \cdots+ & c_{1n}x_n & = & d_1 \\ c_{21}x_1+ & c_{22}x_2+ & \cdots+ & c_{2n}x_n & = & d_2 \\ \vdots & \vdots & & \vdots & & \vdots \quad (**) \\ c_{r1}x_1+ & c_{r2}x_2+ & \cdots+ & c_{rn}x_n & = & d_r. \end{array}$$

The systems $(*)$ and $(**)$ are **equivalent** if they have exactly the same solutions.

**Example:** In another Wisconsin forest, there are also robins and badgers. Together they have 18 heads, 36 eyes, and 56 legs. How many robins and badgers are in the forest? Can we use any new terminology to describe this system?

**Example:** In a third Wisconsin forest, there are deer and badgers. Together they have 18 heads and 70 legs. How many deer and badgers are in the forest? Can we use any new terminology to describe this system?

**Definition/Theorem**
The following operations, called **elementary operations**, can be performed on systems of linear equations to produce equivalent systems.

1. **Interchange** the $i$th and $j$th equations.
2. **Multiply** an equation by a **nonzero constant**.
3. **Replace** the $i$th equation by $c$ times the $j$th equation *plus* the $i$th equation.

We can track these operations more easily using an array/matrix to record the coefficients. The **coefficient matrix** of the linear system is the $m \times n$ matrix $A$:

$$\begin{array}{lllll} a_{11}x_1 + & a_{12}x_2 + & \cdots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 + & a_{22}x_2 + & \cdots + & a_{2n}x_n & = b_2 \\ \vdots & \vdots & & \vdots & \quad \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots + & a_{mn}x_n & = b_m \end{array} \longrightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$$

We can also build a matrix that includes the constants on the right-hand side of the linear system. First, note that

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

is an $m \times 1$ matrix (called the **constant matrix**) representing the constant terms of the linear equations (this is also called a **column vector**). We can adjoin $\mathbf{b}$ to matrix $A$ to create the **augmented matrix** representing our linear system:

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

**Example:** Consider the linear system

$$\begin{array}{rcrcrcr} -2x & + & y & - & 3z & = & 1 \\ & & y & & 4z & = & -8 \\ x & + & y & + & z & = & 7 \end{array}$$

The system is reduced to a simpler system using elementary operations. Find the augmented matrix for the linear system, then record the resulting matrices.

$$\begin{array}{rcrcrcr} -2x & + & y & - & 3z & = & 1 \\ & & y & & 4z & = & -8 \\ x & + & y & + & z & = & 7 \end{array}$$

---

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & & 4z & = & -8 \\ -2x & + & y & - & 3z & = & 1 \end{array}$$

---

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & & 4z & = & -8 \\ & & 3y & - & z & = & 15 \end{array}$$

---

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & + & 4z & = & -8 \\ & & 0 & & -13z & = & 39 \end{array}$$

---

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 7 \\ & & y & + & 4z & = & -8 \\ & & & & z & = & -3 \end{array}$$

*Definition:* An **elementary row operation** on a matrix $A$ is any one of the following operations:

(a) Type I: **Interchange** any two rows.

(b) Type II: **Multiply** a row by a **nonzero number**.

(c) Type III: **Add** a multiple of one row to another.