Chapter 4: Matrices and matrix Operations

We've seen matrices used to help us solve systems of linear equations. We're now going to define them more formally and explore other ways to utilize them.
Definition: An $m \times n$ **matrix** A is a rectangular array of $m \cdot n$ real (or complex) numbers arranged in m **horizontal rows** and n **vertical columns**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & a_{ij} & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$$

The ***i*****th row** of A is $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$, and the ***j*****th column** of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj}. \end{bmatrix}$. The number a_{ij}, which is in the ith row and jth column of A, is the (i, j)**-entry** of A, and we often write $A = [a_{ij}]$. We say A is an "m **by** n" matrix.
Definition: An $m \times n$ matrix A is **square** if $m = n$.

Example: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -8 & -13 \end{bmatrix}$$

A is a **2qq** $\times$ **3qq** matrix. Complete the following:

- $a_{11} =$
- $a_{12} =$
- $a_{21} =$
- $a_{23} =$
- $a_{32} =$

Definition: Two matrices A and B are **equal** if they have the same **size** AND the corresponding entries are **equal**.
Example: If $A = \begin{bmatrix} 3 & y \\ 4 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 6+x & -2 \\ 4 & -7 \end{bmatrix}$, and $A = B$, find x and y.

Matrix Operations

Matrix Addition: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then their **sum** $A + B$ is the matrix $C = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$.

Scalar Multiplication: If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the **scalar multiple** of A by r, written rA, is the $m \times n$ matrix $C = [c_{ij}]$, where $c_{ij} = ra_{ij}$, that is, C is the matrix obtained by multiplying every entry of A by r.

Example: For the matrices A and B as below, calculate $A + B$ and calculate $-2A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -8 & -13 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$$

Definition: If $A_1, A_2, \ldots, A_k$ are $m \times n$ matrices and $c_1, c_2, \ldots, c_k$ are real numbers, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k$$

is called a **linear combination** of $A_1, A_2, \ldots, A_k$. The scalars $c_1, \ldots, c_k$ are called **coefficients**.

Example: The following is a linear combinations of matrices. Compute it.

$$4\begin{bmatrix} 0 & 2 \\ -3 & 3 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 4 & 2 \\ 6 & 2 \end{bmatrix}$$

Example: Is $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$?

<u>Matrix Addition/Scalar multiplication Properties</u> Let A, B, and C be $m \times n$ matrices, and let r, s be scalars/real numbers.

(a) $A + B = B + A$, i.e. matrix addition is **commutative**.

(b) $A + (B + C) = (A + B) + C$, i.e. matrix addition is **associative**.

(c) There is a unique $m \times n$ matrix O such that $A + O = A$ for any $m \times n$ matrix A. The matrix O is called the $m \times n$ **zero matrix**.

(d) For each $m \times n$ matrix A, there is a unique $m \times n$ matrix D such that $A + D = O$. The matrix D must, in fact, be the matrix $-A = (-1)A$. The matrix $-A$ is called the **negative** of A.

(e) $r(sA) = (rs)A$

(f) $(r + s)A = rA + sA$

(g) $r(A + B) = rA + rB$

Proof of (a) (if time):

Definition: If $A = [a_{ij}]$ is an $m \times n$ matrix, then the **transpose** of A, denoted $A^T = [{a_{ij}}^T]$, is the $n \times m$ matrix defined by ${a_{ij}}^T = a_{ji}$. In other words, the transpose of A is obtained by *interchanging* the rows and the columns of A.

Example: Compute the transpose for each of the given matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -8 & -13 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}.$$

Definition: If $A = [a_{ij}]$ is an $m \times n$ matrix, the elements $a_{11}, a_{22}, a_{33}, \ldots$ are called the **main diagonal** of A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

Thus forming the transpose of a matrix A can be viewed as flipping A over its main diagonal.

Matrix Transpose Properties If r is a scalar and A and B are matrices of the appropriate sizes, then

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$

Vectors

Definition: We say a matrix is a **row vector** if it is $1 \times n$, and is a **column vector** if it is $n \times 1$. Often we will call them just n-vectors or "vectors of length n" denoted with bold-face font, like $\mathbf{x}$ or $\mathbf{v}$.

$\mathbb{R}^n$ denotes the set of all n-vectors (with entries in $\mathbb{R}$).

Example: $\mathbf{v} = \begin{bmatrix} -3 \\ \sqrt{2} \\ 0 \end{bmatrix}$ is a vector in **Rtothethree**.

The **zero vector** in $\mathbb{R}^n$, denoted $\mathbf{0}$, is the n-vector consisting of all zeros.

An $m \times n$ matrix can be thought of a row of n column vectors of length m:
$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, where $\mathbf{a}_i$ is the **ith column** of A.

Matrix-Vector Product Consider the linear system of m equations in n unknowns,

$$\begin{array}{llll} a_{11}x_1+ & a_{12}x_2+ & \cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1+ & a_{21}x_2+ & \cdots+ & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1}x_1+ & a_{m2}x_2+ & \cdots+ & a_{mn}x_n & = & b_m. \end{array}$$

From this system, define the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We can rewrite this system as follows:

Example: Consider the linear system

$$\begin{array}{rcrcr} x & - & 3y & = & 5 \\ x & + & 2y & = & 0 \end{array}$$

Rewrite this system using a linear combination.

Definition: Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be an $m \times n$ matrix, written in terms of its columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any n-vector, the **matrix-vector product** $A\mathbf{x}$ is defined to be the m-vector given by:

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n.$$

Example: Compute the matrix-vector product $A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 4 & -5 \\ 3 & 1 & -2 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

Notice that we can now write a linear system of equations in terms of a single matrix equation: If A is the coefficient matrix of the linear system, and $\mathbf{b}$ is the constant vector, then the system takes the form of a single matrix equation: $A\mathbf{x} = \mathbf{b}$.

Example: Go back to the previous linear system example and rewrite it as a matrix equation.

Theorem:

1. Every system of linear equations has the form $A\mathbf{x} = \mathbf{b}$ where A is the coefficient matrix, $\mathbf{b}$ is the constant matrix/vector, and $\mathbf{x}$ is the matrix/vector of variables.
2. The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example: Is $\begin{cases} x - 3y &= 5 \\ x + 2y &= 0 \end{cases}$ a consistent system?

Example: If $\mathbf{x} = \mathbf{0}$, what is $A\mathbf{x}$?

If A is the zero matrix, what is $A\mathbf{x}$?

<u>**Matrix-Vector Product Properties**</u> Let A and B be $m \times n$ matrices, and let $\mathbf{x}$ and $\mathbf{y}$ be n-vectors in $\mathbb{R}^n$. Then:

a. $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

b. $A(r\mathbf{x}) = r(A\mathbf{x}) = (rA)\mathbf{x}$ for all scalars r.

c. $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$.

The Dot Product

We can also express Matrix-Vector multiplication using dot products.

Definition: An ordered sequence $(a_1, a_2, ..., a_n)$ of real numbers is called an **ordered n-tuple**.

Example: $(-2, \pi, 0, \frac{3}{4})$ is an ordered **four**-tuple.

Definition: If $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are two ordered n-tuples, their **dot product** is defined to be the number

$$a_1b_1 + a_2b_2 + \cdots a_nb_n$$

obtained by multiplying corresponding entries and adding the results.

Example: The dot product of $(-2, \pi, 0, \frac{3}{4})$ and $(3, 7, -4, 12)$ is

Theorem: Let A be an $m \times n$ matrix and let $\mathbf{x}$ be an n-vector. Then each entry of the vector $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$.

Example: Compute the product $A\mathbf{x}$ using the dot product, where

$$A = \begin{bmatrix} -1 & 4 & -5 \\ 3 & 1 & -2 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

Why do we have two definitions for matrix-vector product?

Matrix Multiplication

Definition: Let A be an $m \times n$ matrix, B be an $n \times k$ matrix, and write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$, where $\mathbf{b}_j$ is column j of B. The **product** matrix AB is the $m \times k$ matrix defined as follows:

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix}.$$

in other words, column j of AB is the matrix-vector product $A\mathbf{b}_j$ of A and the corresponding column $\mathbf{b}_j$ of B.

Example: Let

$$A = \begin{bmatrix} 1 & -3 & 5 \\ 3 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ -2 & -6 \\ -1 & 0 \end{bmatrix}.$$

Compute AB.

Dot Product Definition of Matrix Product As with matrix-vector products, we have another way to compute matrix products via the dot product:

Theorem: Let A be an $m \times n$ matrix, B be an $n \times k$ matrix. The (i, j) entry of AB is the dot product of row i of A with column j of B.
This process is shown in the schematic below:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2k} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nk} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} \end{bmatrix}$$

Example: Let

$$A = \begin{bmatrix} -3 & -4 & 1 \\ 2 & 4 & 0 \\ 1 & -4 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & -4 \\ -4 & -3 & -1 \\ 4 & 3 & 1 \end{bmatrix}.$$

What is the $(2,3)$ entry of AB?

Facts and Warnings about Matrix Multiplication: Let A be an $m \times p$ matrix and B be a $p \times n$ matrix, so that their product AB is defined and is a $m \times n$ matrix.

1. The product BA may not be defined!
2. If BA is also defined, the size of AB and BA may be the same or different.
3. In the case where AB and BA are the same size, AB and BA may be equal or they may be different.

Example: Let $A = \begin{bmatrix} 0 & -1 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 \\ -4 & -3 \end{bmatrix}$.
Calculate and compare AB and BA, if they are defined.

Properties of Matrix Multiplication Assume that A, B, C are matrices of appropriate sizes. Then:

(a) $A(BC) = (AB)C$, i.e. matrix multiplication is **associative**.

(b) $(A+B)C = AC + BC$ and $C(A+B) = CA + CB$

(c) $A(rB) = r(AB) = (rA)B$.

(d) $(AB)^T = B^T A^T$

Example: If A is 3×2, B is 2×4, what will be the size of A^T? B^T? $(AB)^T$? Is $A^T B^T$ possible?

Matrix Powers Using the definition of matrix multiplication, we can also define the powers of square matrices. Let A be a square matrix and p be a positive integer. Then A^p is A multiplied by itself p times.

Properties of Matrix Powers If A is square and p and q are positive integers, then

(a) $A^p A^q = A^{p+q}$

(b) $(A^p)^q = A^{pq}$

(c) $(A^p)^T = (A^T)^p$

Warning! It is not necessarily the case that $(AB)^p = A^p B^p$. Why?