Chapter 5: Matrix Inverses

Definition: For each $n \geq 2$, the **identity matrix** I_n is the $n \times n$ matrix with 1s on the main diagonal (upper left to lower right), and zeros elsewhere.

Key property of identity matrices If A is any $m \times n$ matrix, then

$$AI_n = A \text{ and } I_m A = A.$$

Example: Let's verify this fact for I_3.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Definition: An $n \times n$ matrix A is called **nonsingular**, or **invertible**, if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Such a B is called the **inverse** of A. If no such B exists, A is called **singular**, or **noninvertible**.

Idea: The inverse of a matrix mimics the reciprocal of a real number.

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$. Compute both AB and BA, and make a conclusion using the language of inverses.

$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, B is the inverse of A

Fact: If A, B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

Fact: The inverse of a matrix, if it exists, is unique. Therefore, we can write A^{-1} for the inverse of A.

Example: Does $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ have an inverse?

If it does, then $\exists\, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, s.t.

$A \cdot B = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$\Rightarrow C_{22} = 0 \times b_{12} + 0 \times b_{22} = 1$, contradiction!

Theorem: If both A and B are nonsingular $n \times n$ matrices, then the matrix AB is nonsingular and its inverse is $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

A, B are invertible (nonsingular),

$\exists\, A^{-1}, B^{-1}$, s.t. $A \cdot A^{-1} = B \cdot B^{-1} = I_n$.

$$AB \cdot (B^{-1}A^{-1}) \underset{\text{associativity}}{=} A \cdot (B \cdot B^{-1}) A^{-1}$$

$$= A \cdot I_n \cdot A^{-1}$$

$$= A \cdot A^{-1} = I_n$$

Follow up facts:

- If $A_1, A_2, \ldots, A_{k-1}, A_k$ are $n \times n$ invertible/nonsingular matrices, then $A_1 A_2 \cdots A_{k-1} A_k$ is invertible/nonsingular and $(A_1 A_2 \cdots A_{k-1} A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$
- If A is invertible/nonsingular, then A^{-1} is invertible/nonsingular and $(A^{-1})^{-1} = A$.
- If A is invertible/nonsingular, then A^T is invertible/nonsingular and $(A^{-1})^T = (A^T)^{-1}$.
- With the convention that $A^0 = I_n$ for an $n \times n$ invertible matrix A, the rules $A^p A^q = A^{p+q}$ and $(A^p)^q = A^{pq}$ hold for all integers p and q.

Example: If A is invertible, and $k \neq 0$, then $(kA)^{-1} = \underline{k^{-1}} A^{-1}$.

Theorem: (Inverse of 2×2 matrix) The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We'll learn how to compute inverses for bigger matrices soon.

Example: Find the inverse of each matrix, if possible.

$$A = \begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$(-2) \times 6 - (-3) \times 4 = 0$, A is not invertible.

$1 \times 3 - 2 \times 6 \neq 0$, B is invertible, and

$$B^{-1} = -\frac{1}{9} \begin{bmatrix} 3 & -6 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

Linear Systems and Inverses

If A is an $n \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ is a system of $\mathbf{n}$ equations in $\mathbf{n}$ unknowns. Suppose A is nonsingular. How can we use A^{-1} to solve the system $A\mathbf{x} = \mathbf{b}$?

Multiply both sides by A^{-1}, we get

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b} \Rightarrow (A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$\Rightarrow \vec{x} = A^{-1}\cdot\vec{b}.$$

Consequences:

- When A^{-1} exists, then $A\mathbf{x} = \mathbf{b}$ has a *unique* solution.
- If A is invertible/nonsingular, then the ONLY solution to the homogeneous system $Ax = 0$ is $\mathbf{x} = \mathbf{0}$.

Example: Use the inverse of A to solve the linear systems $A\mathbf{x} = \mathbf{b}$, $A\mathbf{x} = \mathbf{c}$, and $A\mathbf{x} = \mathbf{0}$, where A, $\mathbf{b}$, and $\mathbf{c}$ are given below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

1. $\vec{x} = A^{-1}\cdot\vec{b} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\cdot\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$

2. $\vec{x} = A^{-1}\cdot\vec{c} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ \frac{7}{2} \end{bmatrix}$

3. $\vec{x} = A^{-1}\cdot\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$