Theorem:

1. Every system of linear equations has the form $A\mathbf{x} = \mathbf{b}$ where A is the coefficient matrix, $\mathbf{b}$ is the constant matrix/vector, and $\mathbf{x}$ is the matrix/vector of variables.

2. The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example: Is $\begin{cases} x - 3y &= 5 \\ x + 2y &= 0 \end{cases}$ a consistent system?

$$2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

So $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is a linear comb. of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

By the thm above, the system is consistent.

Example: If $\mathbf{x} = \mathbf{0}$, what is $A\mathbf{x}$?

$$A\vec{x} = \vec{0}$$

If A is the zero matrix, what is $A\mathbf{x}$?

$$0 \cdot \vec{x} = \vec{0}$$

<u>Matrix-Vector Product Properties</u> Let A and B be $m \times n$ matrices, and let $\mathbf{x}$ and $\mathbf{y}$ be n-vectors in $\mathbb{R}^n$. Then:

a. $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

b. $A(r\mathbf{x}) = r(A\mathbf{x}) = (rA)\mathbf{x}$ for all scalars r.

c. $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$.

<u>The Dot Product</u>

We can also express Matrix-Vector multiplication using dot products.

Definition: An ordered sequence $(a_1, a_2, ..., a_n)$ of real numbers is called an **ordered n-tuple**.

Example: $(-2, \pi, 0, \frac{3}{4})$ is an ordered **four**-tuple.

Definition: If $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are two ordered n-tuples, their **dot product** is defined to be the number

$$a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

obtained by multiplying corresponding entries and adding the results.

Example: The dot product of $(-2, \pi, 0, \frac{3}{4})$ and $(3, 7, -4, 12)$ is

$$-2 \times 3 + \pi \times 7 + 0 \times (-4) + \frac{3}{4} \times 12 = 7\pi + 3$$

Theorem: Let A be an $m \times n$ matrix and let $\mathbf{x}$ be an n-vector. Then each entry of the vector $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$.

Example: Compute the product $A\mathbf{x}$ using the dot product, where

$$A = \begin{bmatrix} -1 & 4 & -5 \\ 3 & 1 & -2 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} (-1) \times 2 + 4 \times (-3) + (-5) \times 4 \\ 3 \times 2 + 1 \times (-3) + (-2) \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} -34 \\ 5 \end{bmatrix}$$

Why do we have two definitions for matrix-vector product?

Dot product is a special case of M.-V. product.

Matrix Multiplication

Definition: Let A be an $m\times n$ matrix, B be an $n\times k$ matrix, and write $B=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k\end{bmatrix}$, where $\mathbf{b}_j$ is column j of B. The **product** matrix AB is the $m\times k$ matrix defined as follows:

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k\end{bmatrix}.$$

in other words, column j of AB is the matrix-vector product $A\mathbf{b}_j$ of A and the corresponding column $\mathbf{b}_j$ of B.

Example: Let

$$A=\begin{bmatrix}1 & -3 & 5\\ 3 & 1 & 4\end{bmatrix} \text{ and } B=\begin{bmatrix}5 & 1\\ -2 & -6\\ -1 & 0\end{bmatrix}.$$

Compute AB.

$$A\cdot B = \begin{bmatrix}6 & 19\\ 9 & -3\end{bmatrix}$$

<u>Dot Product Definition of Matrix Product</u> As with matrix-vector products, we have another way to compute matrix products via the dot product:

Theorem: Let A be an $m\times n$ matrix, B be an $n\times k$ matrix. The (i,j) entry of AB is the dot product of row i of A with column j of B.
This process is shown in the schematic below:

$$\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1k}\\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2k}\\ \vdots & \vdots & & \vdots & & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nk}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & \cdots & c_{1k}\\ c_{21} & c_{22} & \cdots & c_{2k}\\ \vdots & \vdots & c_{ij} & \vdots\\ c_{m1} & c_{m2} & \cdots & c_{mk}\end{bmatrix}$$

Example: Let

$$A = \begin{bmatrix} -3 & -4 & 1 \\ 2 & 4 & 0 \\ 1 & -4 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & -4 \\ -4 & -3 & -1 \\ 4 & 3 & 1 \end{bmatrix}.$$

What is the $(2,3)$ entry of AB?

$C_{23} = (2^{nd}$ row of $A) \cdot (3^{rd}$ col. of $B)$

$= 2 \times (-4) + 4 \times (-1) + 0 \times 1 = -12$

> *Facts and Warnings about Matrix Multiplication*: Let A be an $m \times p$ matrix and B be a $p \times n$ matrix, so that their product AB is defined and is a $m \times n$ matrix.
>
> 1. The product BA may not be defined!
> 2. If BA is also defined, the size of AB and BA may be the same or different.
> 3. In the case where AB and BA are the same size, AB and BA may be equal or they may be different.

Example: Let $A = \begin{bmatrix} 0 & -1 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 \\ -4 & -3 \end{bmatrix}$.
Calculate and compare AB and BA, if they are defined.

$$AB = \begin{bmatrix} 4 & 3 \\ -23 & -18 \end{bmatrix}$$

$AB \neq BA$.

$$BA = \begin{bmatrix} -3 & 12 \\ 3 & -11 \end{bmatrix}$$

<u>**Properties of Matrix Multiplication**</u> Assume that A, B, C are matrices of appropriate sizes. Then:

(a) $A(BC) = (AB)C$, i.e. matrix multiplication is **associative**.

(b) $(A+B)C = AC + BC$ and $C(A+B) = CA + CB$

(c) $A(rB) = r(AB) = (rA)B$.

(d) $(AB)^T = B^T A^T$

Example: If A is 3×2, B is 2×4, what will be the size of A^T? B^T? $(AB)^T$? Is $A^T B^T$ possible?

A^T has size 2×3.

B^T - - - 4×2

$(AB)^T$ - - - 4×3

$A^T B^T$ is not possible, since the # of col. of $A^T \neq$ # of rows of B^T

<u>**Matrix Powers**</u> Using the definition of matrix multiplication, we can also define the powers of square matrices. Let A be a square matrix and p be a positive integer. Then A^p is A multiplied by itself p times.

<u>**Properties of Matrix Powers**</u> If A is square and p and q are positive integers, then

(a) $A^p A^q = A^{p+q}$

(b) $(A^p)^q = A^{pq}$

(c) $(A^p)^T = (A^T)^p$

Warning! It is not necessarily the case that $(AB)^p = A^p B^p$. Why?

Because AB may not equal to BA.