Chapter 2: Solving Systems Using Matrices

Example: The following are augmented matrices representing systems of linear equations in three unknowns (x, y, z). Rewrite the equations, then solve the systems.

(a) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Definition: An $m \times n$ matrix A is in **row echelon form** (REF) if it satisfies the following properties:

(a) All zero rows, if there are any, appear at the **bottom** of the matrix.

(b) The first nonzero entry from the left of a nonzero row is a 1 . This entry is called the leading one of its row.

(c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

We say A is in **reduced row echelon form** (RREF) if A is in REF and also satisfies

(d) If a column contains a leading one, then all the other entries in that column are zero.

A matrix in RREF appears as a *staircase* (or *echelon*) pattern of leading ones descending from the *upper left corner* of the matrix.

Note: There is a similar definition for (reduced) column echelon form.

Let's draw a schematic of a matrix in (R)REF.

Example: Which of the following matrices are in REF? Which are in RREF?

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: For a matrix in REF, the first 1 that appears in a row is called a **pivot**. The columns containing pivots are called **pivot columns**.

Definition: An $m \times n$ matrix A is **row equivalent** to an $m \times n$ matrix B if B can be produced by applying a finite sequence of elementary row operations to A.

Theorem: Every $m \times n$ matrix A is **row equivalent** to a **unique** matrix in RREF.

Gaussian Algorithm: How to find a row-echelon matrix from a given matrix
Heuristic: Move from top to bottom and outside in until you get a staircase-shape with ones as leading entries.

1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.
2. Otherwise, find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.
3. Now multiply the new top row by $1/a$ to create a leading 1.
4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row, and all further row operations are carried out on the remaining rows.

5. Repeat steps 1–4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

Example: Find a row echelon form of the given matrix A.

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \end{bmatrix}$$

Theorem: Consider two linear systems each of m equations in n unknowns. If the augmented matrices $[A \,\vdots\, \mathbf{b}]$ and $[C \,\vdots\, \mathbf{d}]$ are *row equivalent*, then the linear systems are *equivalent*, i.e. the systems have the *same solutions*.

Example: Solve the linear system

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 9 \\ 2x & - & y & + & z & = & 8 \\ 3x & & & - & z & = & 3 \end{array}$$

by transforming the associated matrix $[A \,\vdots\, \mathbf{b}]$ to row echelon form.

Example: Solve the previous linear system by transforming $[A \,\vdots\, \mathbf{b}]$ to reduced row echelon form.

In this example, the RREF of A is the **identity matrix**. When this happens, the system corresponding to $[A \,\vdots\, \mathbf{b}]$ has *a unique solution*.