Simplicity in Composition

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Scala World 2017

Composition

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$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

$$x = x \oplus 1_A = 1_A \oplus x$$

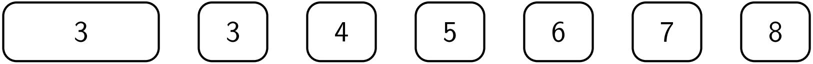
$$(\mathbb{Z}, +, 0)$$

$$(\texttt{[a]}, +\!\!+, \texttt{[]})$$

$$(A \to A, \circ, a \mapsto a)$$

```scala
trait Monoid[A] {
  def combine(x: A, y: A): A
  def empty: A
}
```

1	2	3	4	5	6	7	8


3
3
4
5
6
7
8

6	4	5	6	7	8

15	6	7	8

36

1 2 3 4 5 6 7 8

3	7	11	15

10

26

36

36

10 26

Associative composition allows for modular decomposition and reasoning.

Composing programs

`A`

Composing programs

```
F[A]
```

Composing programs

```
(F[A], F[A]) => F[A]
```

Composing programs

```
(F[A], F[B]) => F[?]
```

Composing programs

```
(F[A], F[B]) => F[(A, B)]
```

Composing programs

```scala
def zipOption[A, B]
  (oa: Option[A], ob: Option[B]): Option[(A, B)] =

  (oa, ob) match {
    case (Some(a), Some(b)) => Some((a, b))
    case _                  => None
  }
```

Composing programs

```
def zipList[A, B]
  (la: List[A], lb: List[B]): List[(A, B)] =

  la match {
    case Nil    => Nil
    case h :: t => lb.map((h, _)) ++ zipList(t, f)
  }
```

Composing programs

```scala
def zipFunction[A, B, X]
  (f: X => A, g: X => B): X => (A, B) =

  (x: X) => (f(x), g(x))
```

Composing programs

- $F(\_)$: type of program

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Composing programs

- $F(\_)$: type of program
- $\otimes : F(A) \times F(B) \rightarrow F((A, B))$
- $\eta : A \rightarrow F(A)$

$$fa \otimes (fb \oplus fc) \cong (fa \oplus fb) \oplus fc$$

$$fa \cong fa \otimes \eta_{Unit} \cong \eta_{Unit} \otimes fa$$

```scala
trait Monoidal[F[_]] {
  def zip[A, B](fa: F[A], fb: F[B]): F[(A, B)]
  def pure[A](a: A): F[A]

  /*
  def map[A, B](fa: F[A])(f: A => B): F[B]
  */
}
```

Composing programs

```
(F[A], F[B]) => F[(A, B)]
```

Composing programs

`F[A]`

Composing dependent programs

```
(F[A], A => F[B]) => F[B]
```

Composing dependent programs

```scala
def flatMapOption[A, B]
  (oa: Option[A], f: A => Option[B]): Option[B] =

  oa match {
    case Some(a) => f(a)
    case None    => None
  }
```

Composing dependent programs

```scala
def flatMapList[A, B]
  (la: List[A], f: A => List[B]): List[B] =

  la match {
    case Nil => Nil
    case h :: t => f(h) ++ flatMapList(t, f)
  }
```

Composing dependent programs

```scala
def flatMapFunction[A, B, X]
  (fa: X => A, f: A => (X => B)): X => B =

  (x: X) => f(fa(x))(x)
```

Composing dependent programs

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- $\gg=: (F(A) \times A \rightarrow F(B)) \rightarrow F(B)$
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$$(fa \gg= f) \gg= g \quad = \quad fa \gg= (f >=> g)$$

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- $\gg=: (F(A) \times A \to F(B)) \to F(B)$
- $\eta : A \to F(A)$

$$
\begin{aligned}
(fa \gg= f) \gg= g \quad &= \quad fa \gg= (f >=> g) \\
f(x) \quad &= \quad \eta(x) \gg= f
\end{aligned}
$$

Composing dependent programs

- $F(\_)$: type of program
- $\gg=: (F(A) \times A \to F(B)) \to F(B)$
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$$
\begin{aligned}
(fa \gg= f) \gg= g &= fa \gg= (f >=> g) \\
f(x) &= \eta(x) \gg= f \\
fa &= fa \gg= \eta
\end{aligned}
$$

```scala
trait Monad[F[_]] {
  def flatMap[A, B](fa: F[A], f: A => F[B]): F[B]
  def pure[A](a: A): F[A]
}
```

A nicer monad

$$
\begin{aligned}
(fa \ggg= f) \ggg= g &= fa \ggg= (f >=> g) \\
f(x) &= \eta(x) \ggg= f \\
fa &= fa \ggg= \eta
\end{aligned}
$$

A nicer monad

$$f : A \to F(B)$$
$$g : B \to F(C)$$
$$h : C \to F(D)$$

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$$f : A \to F(B)$$
$$g : B \to F(C)$$
$$h : C \to F(D)$$

$$(f >=> g) >=> h \quad = \quad f >=> (g >=> h)$$

A nicer monad

$$f : A \to F(B)$$
$$g : B \to F(C)$$
$$h : C \to F(D)$$

$$(f \mathrel{>=>} g) \mathrel{>=>} h \quad = \quad f \mathrel{>=>} (g \mathrel{>=>} h)$$
$$f \quad = \quad f \mathrel{>=>} \eta \quad = \quad \eta \mathrel{>=>} f$$

Category theory studies the algebra of composition.

Category theory

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

Category theory

- objects: A, B, C ...

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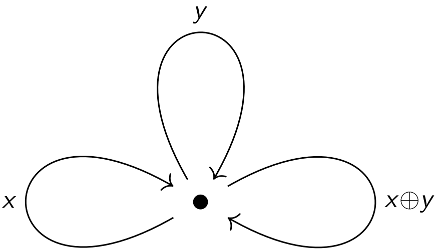
$$(f \circ g) \circ h = f \circ (g \circ h)$$

Category theory

- objects: A, B, C ...
- arrows: $f : A \to B$, $g : B \to C$ ...
- $1_A : A \to A$

$$(f \circ g) \circ h = f \circ (g \circ h)$$
$$f = f \circ 1_A = 1_A \circ f$$

Category theory: monoids



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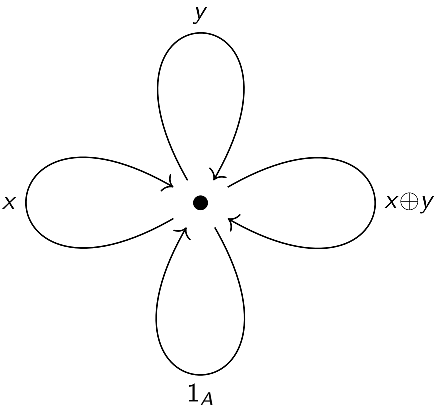

Category theory: monads

Category theory: monads

$$
\begin{tikzcd}
 & T(B) & \\
A \arrow[ur, "f"] \arrow[rr, "? \circ f"'] & & T(C)
\end{tikzcd}
$$

Category theory: monads

$$\begin{tikzcd}
& T(B) & \\
A \arrow[ur, "f"] \arrow[rr, "? \circ f"'] & & T(C)
\end{tikzcd}$$

$$\gg=: (F(A) \times A \to F(B)) \to F(B)$$

Category theory: monads

$$\begin{tikzcd} & T(B) & \\ A \arrow[ur, "f"] \arrow[rr, "? \circ f"'] & & T(C) \end{tikzcd}$$

$$\gg\!=: (A \to F(B)) \to F(A) \to F(B)$$

Category theory: monads

$$\begin{tikzcd} & T(B) \arrow[dr, "g^*"] & \\ A \arrow[ur, "f"] \arrow[rr, "g^* \circ f"'] & & T(C) \end{tikzcd}$$

$$\gg=: (A \to F(B)) \to F(A) \to F(B)$$

Category theory: monads

$$
\begin{tikzcd}
 & B \arrow[dr, dashed, "g'"] & \\
A \arrow[ur, dashed, "f'"] \arrow[rr, dashed, "g' \circ f'"'] & & C
\end{tikzcd}
$$

Category theory: monads

$$
\begin{tikzcd}
& B \arrow[dr, dashed, "g'"] \arrow[loop above, dashed, "1_B"] & \\
A \arrow[ur, dashed, "f'"] \arrow[rr, dashed, "g' \circ f'"'] \arrow[loop below, dashed, "1_A"] & & C \arrow[loop below, dashed, "1_C"]
\end{tikzcd}
$$

Category theory: monads

$$
\begin{tikzcd}
 & B \arrow[loop above, dashed, "B \to T(B)"] \arrow[dr, dashed, "g'"] & \\
A \arrow[ur, dashed, "f'"] \arrow[rr, dashed, "g' \circ f'"'] \arrow[loop below, dashed, "A \to T(A)"'] & & C \arrow[loop below, dashed, "C \to T(C)"']
\end{tikzcd}
$$

Can we compose the composers?

```scala
(A, A) => A
```

```
(Monoid[A], Monoid[B]) => Monoid[?]
```

```
(Monoid[A], Monoid[B]) => Monoid[(A, B)]
```

$$(x_1, y_1) \oplus (x_2, y_2)$$

$$(x_1 \oplus x_2, y_1 \oplus y_2)$$

```
(F[A], F[B]) => F[(A, B)]
```

```
(Monoidal[F], Monoidal[G]) => Monoidal[?]
```

```scala
type L[X] = (F[X], G[X])
(Monoidal[F], Monoidal[G]) => Monoidal[L]
```

```
type L[X] = (F[X], G[X])
(Monad[F], Monad[G]) => Monad[L]
```

```scala
type L[X] = F[G[X]]
(Monoidal[F], Monoidal[G]) => Monoidal[L]
```

```scala
type L[X] = F[G[X]]
(Monad[F], Monad[G]) => Monad[L]
```

```
type L[X] = F[G[X]]
(Monad[F], Monad[G]) => Monad[L] % TODO: Cros
```