On the Efficient Solution to the Filters' Dual Variables

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In this draft we discuss the solution to the dual variables α_1, and α_2 of the filter during the training as given in [1]. The linear system asscoiated in training the first filter is given as follows:

$$\Big(\Phi_1\Phi_1^T + (\lambda+\mu)\mathbf{I}\Big)\alpha_1 = \mathbf{y} - \Big(k\mathbf{I} + [k\lambda + \mu(k-1)](\Phi_1\Phi_1^T)^{-1}\Big)\Phi_1\Phi_2^T\alpha_2 \tag{1}$$

Similarly, the second filter's dual variables are given as follows:

$$\Big(\Phi_2\Phi_2^T + (\lambda+\mu)\mathbf{I}\Big)\alpha_2 = \mathbf{y} - \Big(k\mathbf{I} + [k\lambda + \mu(k-1)](\Phi_2\Phi_2^T)^{-1}\Big)\Phi_2\Phi_1^T\alpha_1 \tag{2}$$

Note that in [1], $\tilde{\mathbf{b}} = \Phi_1\Phi_2^T\alpha_2$. Here we show the detailed solution to 1 and infer the solution to the other directly by mirroring the variables. To solve equation 1, we first introduce a variable $c = k\lambda + \mu(k-1)$ for ease; therefore, the system to be solved is as follows:

$$\Big(\Phi_1\Phi_1^T + (\lambda+\mu)\mathbf{I}\Big)\alpha_1 = \mathbf{y} - \Big(k\mathbf{I} + c(\Phi_1\Phi_1^T)^{-1}\Big)\Phi_1\Phi_2^T\alpha_2 \tag{3}$$

The solution steps is given below where Φ is a circulant matrix.

$$\Big(\mathbf{F}(diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^*)\mathbf{F}^H + (\lambda+\mu)\mathbf{F}\mathbf{F}^H\Big)\alpha_1 = \mathbf{y} - \Big(k\mathbf{F}\mathbf{F}^H + c\mathbf{F}(diag^{-1}(\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*)\mathbf{F}^H\Big)\mathbf{F}diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\mathbf{F}^H\alpha_2$$

$$\mathbf{F}\Big((diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu)\Big)\mathbf{F}^H\alpha_1 = \mathbf{y} - \mathbf{F}diag\Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big)diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\mathbf{F}^H\alpha_2$$

$$\mathbf{F}\Big((diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu)\Big)\hat{\alpha_1}^* = \mathbf{y} - \mathbf{F}diag\Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big)diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\hat{\alpha_2}^*$$

$$diag\Big(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu\Big)\hat{\alpha_1}^* = \hat{\mathbf{y}}^* - diag\Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big)diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\hat{\alpha_2}^*$$

$$\hat{\alpha}_1^* = \frac{\hat{\mathbf{y}}^* - \Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big) \odot (\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^* \odot \hat{\alpha}_2^*)}{\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu}$$

$$\hat{\alpha}_1^* = \frac{\hat{\mathbf{y}}^* - \Big(k + \frac{k\lambda + \mu(k-1)}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big) \odot (\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^* \odot \hat{\alpha}_2^*)}{\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu}$$

$$\hat{\alpha}_1 = \frac{\hat{\mathbf{y}} - \Big(k + \frac{k\lambda + \mu(k-1)}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big) \odot (\hat{\mathbf{a}}_1^* \odot \hat{\mathbf{a}}_2 \odot \hat{\alpha}_2)}{\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu}$$

Note that $\hat{\mathbf{x}}$ is the FFT of $\mathbf{x}$, and $\mathbf{F}$ is the normalized DFT matrix and all operations are element wise.

By following the same derivation but for equation 2, the solution is given as follows:

$$\hat{\alpha}_2 = \frac{\hat{\mathbf{y}} - \Big(k + \frac{k\lambda + \mu(k-1)}{\mathbf{a}_2 \odot \hat{\mathbf{a}}_2^*}\Big) \odot (\hat{\mathbf{a}}_2^* \odot \hat{\mathbf{a}}_1 \odot \hat{\alpha}_1)}{\hat{\mathbf{a}}_2 \odot \hat{\mathbf{a}}_2^* + \lambda + \mu}$$

Note that the code implemented and available online is for correlation operation not convolution. This will result into having a symmetric conjugation over all variables. Equivalently, having all circulate matrices to be the transpose(Hermitian) of the current derivation.

[1] Bibi, A., Ghanem, B. "Multi-template scale-adaptive kernelized correlation filters". In: Proceedings of the IEEE International Conference on Computer Vision Workshops.(2015) 5057