On the Efficient Solution to the Filters' Dual Variables

Adel Bibi, and Bernard Ghanem

King Abdullah University of Science and Technology (KAUST), Saudi Arabia

In this draft we discuss the solution to the dual variables α_1, and α_2 of the filter during the training as given in [1]. The linear system associated in training the first filter is given as follows:

$$\Big(\Phi_1\Phi_1^T + (\lambda+\mu)\mathbf{I}\Big)\alpha_1 = \mathbf{y} - \Big(k\mathbf{I} + [k\lambda + \mu(k-1)](\Phi_1\Phi_1^T)^{-1}\Big)\Phi_1\Phi_2^T\alpha_2 \tag{1}$$

Similarly, the second filters' dual variables are given as follows:

$$\Big(\Phi_2\Phi_2^T + (\lambda+\mu)\mathbf{I}\Big)\alpha_2 = \mathbf{y} - \Big(k\mathbf{I} + [k\lambda + \mu(k-1)](\Phi_2\Phi_2^T)^{-1}\Big)\Phi_2\Phi_1^T\alpha_1 \tag{2}$$

Note that in [1], $\tilde{\mathbf{b}} = \Phi_1\Phi_2^T\alpha_2$. Here we show the detailed solution to 1 and infer the solution to the other directly by mirroring the variables. To solve equation 1, we first introduce a variable $c = k\lambda + \mu(k-1)$ for ease; therefore, the system to be solved is as follows:

$$\Big(\Phi_1\Phi_1^T + (\lambda+\mu)\mathbf{I}\Big)\alpha_1 = \mathbf{y} - \Big(k\mathbf{I} + c(\Phi_1\Phi_1^T)^{-1}\Big)\Phi_1\Phi_2^T\alpha_2 \tag{3}$$

The solution steps is given below where Φ is a circulant matrix.

$$\Big(\mathbf{F}(diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^*)\mathbf{F}^H + (\lambda+\mu)\mathbf{F}\mathbf{F}^H\Big)\alpha_1 = \mathbf{y} - \Big(k\mathbf{F}\mathbf{F}^H + c\mathbf{F}(diag^{-1}(\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*)\mathbf{F}^H\Big)\mathbf{F}diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\mathbf{F}^H\alpha_2$$

$$\mathbf{F}\Big((diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu)\Big)\mathbf{F}^H\alpha_1 = \mathbf{y} - \mathbf{F}diag\Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big)diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\mathbf{F}^H\alpha_2$$

$$\mathbf{F}\Big((diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu)\Big)\hat{\alpha_1}^* = \mathbf{y} - \mathbf{F}diag\Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big)diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\hat{\alpha_2}^*$$

$$diag\Big(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu\Big)\hat{\alpha_1}^* = \hat{\mathbf{y}}^* - diag\Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big)diag(\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^*)\hat{\alpha_2}^*$$

$$\hat{\alpha}_1^* = \frac{\hat{\mathbf{y}}^* - \Big(k + \frac{c}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big) \odot (\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^* \odot \hat{\alpha}_2^*)}{\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu}$$

$$\hat{\alpha}_1^* = \frac{\hat{\mathbf{y}}^* - \Big(k + \frac{k\lambda + \mu(k-1)}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big) \odot (\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_2^* \odot \hat{\alpha}_2^*)}{\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu}$$

$$\hat{\alpha}_1 = \frac{\hat{\mathbf{y}} - \Big(k + \frac{k\lambda + \mu(k-1)}{\mathbf{a}_1 \odot \hat{\mathbf{a}}_1^*}\Big) \odot (\hat{\mathbf{a}}_1^* \odot \hat{\mathbf{a}}_2 \odot \hat{\alpha}_2)}{\hat{\mathbf{a}}_1 \odot \hat{\mathbf{a}}_1^* + \lambda + \mu}$$

Note that $\hat{\mathbf{x}}$ is the FFT of $\mathbf{x}$, and $\mathbf{F}$ is the normalized DFT matrix and all operations are element wise.

By following the same derivation but for equation 2, the solution is given as follows:

$$\hat{\alpha}_2 = \frac{\hat{\mathbf{y}} - \Big(k + \frac{k\lambda + \mu(k-1)}{\mathbf{a}_2 \odot \hat{\mathbf{a}}_2^*}\Big) \odot (\hat{\mathbf{a}}_2^* \odot \hat{\mathbf{a}}_1 \odot \hat{\alpha}_1)}{\hat{\mathbf{a}}_2 \odot \hat{\mathbf{a}}_2^* + \lambda + \mu}$$

Note that the code implemented and available online is for correlation operation not convolution. This will result into having a symmetric conjugation over all variables. Equivalently, having all circulant matrices to be the transpose (hermitian) of the current derivation.

[1] Bibi, A., Ghanem, B. "Multi-template scale-adaptive kernelized correlation filters". In: Proceedings of the IEEE International Conference on Computer Vision Workshops.(2015) 5057