



Elementary Mathematics I MTH 101

MIVA OPEN UNIVERSITY

HEADQUARTERS

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Study Session 1: Elementary Set Theory

Introduction

This is a branch of mathematics that deals with the study of sets and their properties, which will give you a basic understanding of operations done on sets and how to represent them with notations and symbols like set-builder notation and Venn diagrams.

Learning Outcome for Study Session 1

When you have studied this session, you should be able to:

- 1.1. Understand the concept of set operations and how to represent them using various notations.
- 1.2. Describing set operations such as union, intersection, and complement.
- 1.3. Applying set notations and symbols by being able to represent them with commas and braces and understanding the meaning of symbols such as subset, empty set, and proper subset.
- 1.4. Understanding set relationships and being able to draw and understand Venn diagrams.

1.1: Subsets

We say a set A is a subset of a set B if every element of A is an element of B (i.e., $x \in A \Rightarrow x \in B$). If A is a subset of B , we write $A \subseteq B$, and otherwise we write $A \not\subseteq B$. For example, $N \subseteq Z$, $Z \subseteq Q$, and $Q \subseteq R$. Also, $\{1, 3, 5\} \subseteq \{1, 3, 5\}$, and $\{2, 4\} \not\subseteq \{4, 5, 6\}$.

Proposition 1.1.1

Let A , B and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof

Take any $x \in A$. Since If $A \subseteq B$, the element $x \in B$. Since $B \subseteq C$, the element $x \in C$. Therefore, if $x \in A$ then $x \in C$. That is, $A \subseteq C$.

Proposition 1.1.2:

Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Proof: (\Rightarrow) Suppose $A = B$. Then every element of A is an element of B , and every element of B is an element of A . Thus, $A \subseteq B$ and $B \subseteq A$.

1.2: Proper Subsets:

The word “proper” occurs frequently in mathematics. Each time it has essentially the same meaning, roughly “and not equal to the whole thing”

A set A is a proper subset of a set B if $A \subseteq B$ and $A \neq B$. That is, A is a proper subset of B when every element of A belongs to B (so $A \subseteq B$) and there is an element in B which is not in A (so $A \neq B$). Three common ways to denote that A is a proper subset of B are $A \subset B$, $A \subsetneq B$ and $A \subsetneqq B$. The last two of these are clear. The first one is, unfortunately used by some authors to denote that A is a subset of B .

1.3: Union:

Let A and B be sets. The union of A and B is the set $A \cup B = \{x: (x \in A) \vee (x \in B)\}$.

1.4: Intersection:

Let A and B be sets. The intersection of A and B is the set $A \cap B = \{x: (x \in A) \wedge (x \in B)\}$.

Notice that the set union symbol looks vaguely like the symbol for the logical connective “or”, and the set intersection symbol looks vaguely like the symbol for the logical connective “and”. Indeed, union is defined using “or”, and intersection is defined using “and”. The definition of union and intersection allows us to use the laws of logic to prove statements about sets. As an example, we prove an associative law. The proof entails using set builder notation to show that the sets

on either side of the equals sign are each described by logical equivalent conditions.

Proposition 1.4.1:

Let A , B and C be sets. Then $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof:

$$\begin{aligned}
 (A \cup B) \cup C &= \{x: (x \in A \cup B) \vee (x \in C)\} \\
 &= \{x: ((x \in A) \vee (x \in B)) \vee (x \in C)\} \\
 &= \{x: (x \in A) \vee ((x \in B) \vee (x \in C))\} \\
 &= \{x: (x \in A) \vee (x \in B \cup C)\} \\
 &= A \cup (B \cup C)
 \end{aligned}$$

For each Law of Logic, there is a corresponding “Law of Set Theory”. For example, for sets A , B and C :

- (i) $(A \cap B) \cap C = A \cap (B \cap C)$
- (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- (iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

In each case, the proof can be carried out similarly to the above.

1.5: Complements:

The complement of A is the set $A^c = \{x: x \notin A\} = U \setminus A$. In analogy with a notation commonly used for the negation in logic (but not by us), the complement of A is also sometimes denoted by \overline{A} . For example, $\{a, b, c\} \setminus \{b, d\} = \{a, c\}$. The set $\{1, 2\}^c = \{3, 4, \dots\}$, and if $U = \{1, 2\}$, then $\{1, 2\}^c = \emptyset$. It is worth noticing that $\{b, d\} \setminus \{a, b, c\} = \{d\}$ which shows that, in general, $A \setminus B \neq B \setminus A$.

By definition, $A \setminus B = A \cap B^c$.

As in the situation for logical connectives, there is no precedence among set operations, except that complements are done first. The moral of the story is that one should always use brackets for clarity.

1.6: Venn Diagrams

Informally, a Venn diagram is a picture that shows all possible memberships between elements of the universe and a collection of sets.

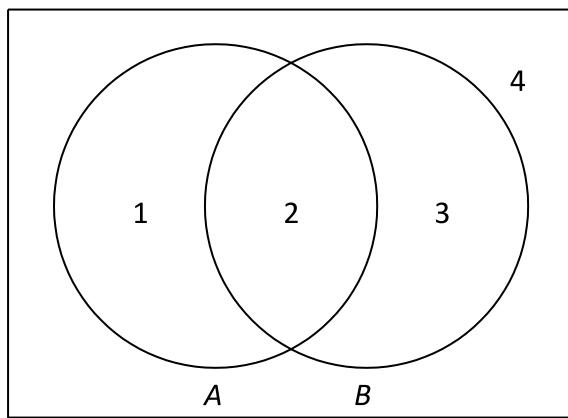
Let A and B be sets. For any element of the universe, there are four mutually-exclusive possibilities, where mutually exclusive means only one possibility holds at a time.

1. It belongs to A and not to B , that is, to $A \setminus B$;
2. It belongs to both A and B , that is, to $A \cap B$;
3. It belongs to B and not to A , that is, to $B \setminus A$; or

4. It belongs to neither A nor B , that is, to $(A \cup B)^c$.

These four possibilities correspond to the four regions as you can see in Figure 1.1 below, if we imagine each element of the universe being somehow located in the diagram depending on which of the possibilities holds.

Figure 1.1: Possibilities of a two-set Venn diagram



Each of the sets defined in the previous section can be associated with a collection of regions seen in Table 1.1.

Table 1.1: Region representation of Figure 1.1

Set	Represented by regions
A	1, 2
B	2, 3

$A \cup B$	1, 2, 3
$A \cap B$	2
U	1, 2, 3, 4
A^c	3, 4
B^c	1, 4
$A \setminus B$	1
$B \setminus A$	3
$A \oplus B$	1, 3

The regions that represent a set correspond exactly to its elements in the situation where $U = \{1, 2, 3, 4\}$, $A = \{1, 2\}$ and $B = \{2, 3\}$.

It is apparent from the table above that, for example, $A \setminus B \neq B \setminus A$, because the set on the left-hand side is represented by region 3. And the diagram can be used to get an example of a universe U and sets A and B such that $A \setminus B \neq B \setminus A$. Directly, from what was just done, if we take $U = \{1, 2, 3, 4\}$, $A = \{1, 2\}$ and $B = \{2, 3\}$, then $A \setminus B = \{1\}$ and $B \setminus A = \{3\}$.

We thus have an important principle: if two sets are represented by different collections of regions in a Venn diagram, then an example showing the sets are not equal can be obtained directly from the diagram. On the other hand, Venn diagrams can provide intuition about equality between sets. As a first example, let's investigate whether $A \cup B$ is equal to $(A \setminus B) \cup B$. Using table 1.2 from before, we have:

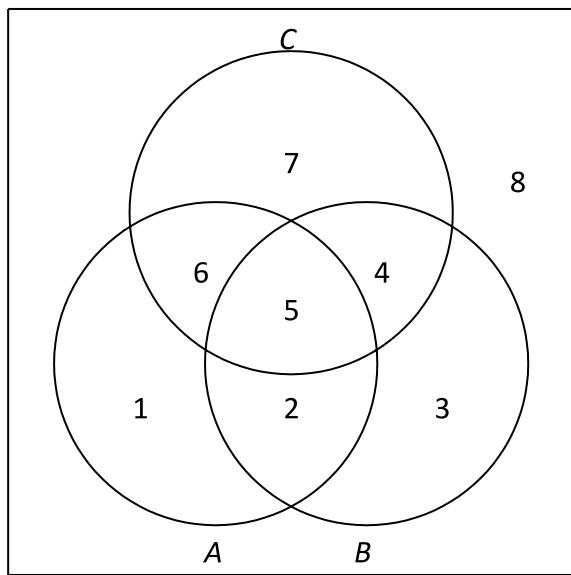
Table 1.2: Data showing $A \cup B$ is equal to $(A \setminus B) \cup B$.

Set	Represented by regions
A	1, 2
B	2, 3
$A \cup B$	1, 2, 3
$A \setminus B$	1
$(A \setminus B) \cup B$	1, 2, 3

Since both sets are represented by the same collection of regions, we expect that they are equal. There are several different ways to construct a proof. One of the ways is by adapting the definition of two sets to show they are described by logical equivalent conditions.

Let A , B and C be sets. For any element of the universe, there are eight mutually-exclusive possibilities: it belongs to none of them (one possibility), it belongs to exactly one of them (three possibilities), it belongs to exactly two of them (three possibilities), or it belongs to all of them (one possibility). These are represented by the eight regions in the Venn diagram below.

Figure 1.2: Possibilities of a three set Venn diagram



Let's use figure 1.2 to investigate whether $A \cup (B \cap C)$ equals $(A \cup B) \cap C$.

Table 1.3: Collection of regions represented by Figure 1.2

Set	Represented by regions
A	1, 2, 5, 6

B	2, 3, 4, 5
C	4, 5, 6, 7
$B \cap C$	4, 5
$A \cup (B \cap C)$	1, 2, 4, 5, 6
$A \cup B$	1, 2, 3, 4, 5, 6
$(A \cup B) \cap C$	4, 5, 6

As before, the regions correspond to the sets that would arise if we perform the set operations using $U = \{1, 2, \dots, 8\}$, $A = \{1, 2, 5, 6\}$, $B = \{2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7\}$. Hence, when the sets in question are represented by different regions, these sets provide a counterexample. Doing so for the example above

$$A \cup (B \cap C) = \{1, 2, 4, 5, 6\}$$

and

$$(A \cup B) \cap C = \{4, 5, 6\}.$$

Therefore, the two expressions determine different sets in general.

Summary

1. Showed how one set can be a subset of another.
2. Basic operations done on sets, i.e, union, intersection and difference of sets.
3. Venn diagram and how it shows possible memberships between two sets

Study Session 2: Real Numbers

Introduction

Real numbers are represented on the number line, where each point shows a unique real number in both directions allowing both positive and negative numbers. Real numbers can be classified into different sets such as natural numbers, integers, rational numbers, irrational numbers. Trichotomy's law is also defined stating a real number can either negative or positive.

Learning Outcomes for Study Session 2

When you have studied this session, you should be able to:

- 2.1 Understanding the number system, such as the classification of numbers such as natural numbers, whole numbers, integers, real numbers.
- 2.2 Understanding trichotomy's Law and the purpose of absolute numbers.

2.1: Set of Real Numbers

In this section, we shall introduce the basic properties of the set that is generally called the *x – axis*. This set is denoted in mathematical analysis by the letter R and is called the **set of real numbers**. Members of R are called real numbers. We shall denote the set of natural numbers by $N = \{1, 2, 3, \dots\}$. We shall also denote the set of all integers by $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and the set of rational numbers (or fractions) by $Q = \left\{ \frac{p}{q} : p, q \in Z, q \neq 0 \right\}$.

2.2: The Order relation

In this section, we defined an order relation ($<$) with the following order axioms:

Axiom 1: (Trichotomy Law). For all $x, y \in R$, exactly one of the following three relations must hold:

- (i) $x = y$ (ii) $x < y$ (iii) $y < x$.

It is conventional to agree that $x < y$ means $y > x$. Also, we write $x \leq y$ for $x < y$ or $x = y$. Similarly, $y \geq x$ means $y > x$ or $y = x$. Numbers $x \in R$ such that $y < 0$ are called negative. The trichotomy axioms assert that every real number other than zero is either positive or negative but never both, and zero is neither positive nor negative. Additionally, the trichotomy axioms assert that the order relation neglects no number.

We now introduce two more order relations.

Axiom 2: For all $x, y \in R$,

$$x < y \text{ if and only if } x + z < y + z.$$

Axiom 3: For all $x, y \in R$, if $x > 0$ and $y > 0$, then,

$$x + y > 0; \quad xy > 0.$$

with all these order axioms, we prove the following proposition.

Proposition 2.2.1: $x, y \in R$,

- (i) $x < y$ if and only if $x - y < 0$ and $y - x > 0$.
- (ii) If $x < y$ and $y < z$.

Proof:

$$x < y \text{ if and only if } x + (-y) < y + (-y), \text{ by (axiom 2)}$$

That is

$$x < y \text{ if and only if } x - y < 0.$$

Moreover,

$$x - y < 0 \text{ if and only if } x - y + (y - x) < 0 + (y - x),$$

i.e:

$$x - y < 0 \text{ if and only if } x - y + y - x < y - x \text{ if and only if } 0 < y - x.$$

From the above inequalities, we obtain that

$x < y$ if and only if $x - y < 0$ if and only if $y - x > 0$, thus establishing (i).

To prove (ii), we use **Axiom 3** as follows. Assume $x < y$ and $y < z$. This is same as:

$$y - x > 0 \text{ and } z - y > 0.$$

By **axiom 3**, the sum of these two positive numbers is positive. Hence,

$$(y - x) + (z - y) > 0, \text{ i.e., } z - x > 0, \text{ i.e.: } x < z.$$

With **proposition 2.1.1** proved, we can now use obvious statements such as these:

For all $x, y, z \in R$,

- (a) $x < y$ and $y \leq z \Rightarrow x < z$.
- (b) $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

We shall also need the notion of **absolute value** of a real number. If $a \in R$, we define the absolute value of a , denoted by $|a|$, by

$$|a| = \{a \text{ if } a > 0, 0 \text{ if } a = 0, -a \text{ if } a < 0\}.$$

For example, $|-3| = 3$; $|3| = 3$, and so on. Intuitively, the absolute value of a represents the distance from 0 to a . Some basic properties of the absolute value are given in the following theorem.

Theorem 2.2.1

- (i) $|a| \geq 0$ for all $a \in R$; $|a| = 0$ if and only if $a = 0$.
- (ii) $|a| \leq \epsilon$ if and only if $-\epsilon \leq a \leq \epsilon$.
- (iii) $|ab| = |a||b|$ for all $a, b \in R$;
- (iv) $|a + b| \leq |a| + |b|$ for all $a, b \in R$.

Proof:

(i) There are two cases, if $a \geq 0$, then $|a| = a \geq 0$. On the other hand, if $a < 0$, then $|a| = -a$. In both cases, $|a| \geq 0$.

(ii) Since $a = |a|$ or $a = -|a|$, it follows that $-|a| \leq a \leq |a|$.

Now, if $|a| < \epsilon$, then we have $-\epsilon \leq -|a| \leq a \leq |a| \leq \epsilon$.

And if $a < 0$, then $|a| = -a \leq \epsilon$. In both cases, $|a| \leq \epsilon$.

(iii) Consider four cases when a and b are both positive, both negative, or have opposite signs.

From the inequalities $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, we obtain that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This implies that $|a + b| \leq |a - b|$ for all $a, b \in R$.

Remark: Condition (iv) of **Theorem 2.2.1** is often called the triangle inequality. It roughly says that the length of any side of a triangle is less than or equal to the

sum of the lengths of the other two sides. A useful variant of the triangle inequality is

$$||a| - |b|| \leq |a - b| \text{ for all } a, b, \in R.$$

Exercise.

1. Prove that if $\epsilon > 0$ and $a, x, \in R$, then $|a - x| < \epsilon$ if and only if $x - \epsilon < a < x + \epsilon$.
2. Find all $x \in R$ that satisfy the following inequalities:
 - (i) $|4x - 3| \leq 11$
 - (ii) $|x - 2| > |x + 1|$
 - (iii) $|2 - 5x| \leq 7$
3. Prove that for all $a, b, \in R$,

$$|a - b| \leq |a| - |b|.$$

4. Prove that for all $a, b, c \in R$,

$$|a + b + c| \geq |a| - |b| - |c|.$$

5. Solve the inequality

$$\left| x - \frac{1}{2} \right| > 1.$$

Summary

1. Trichotomy's law shows that a real number can be either positive or negative.
2. Classification of real numbers.
3. Equation showing why the absolute value of a number is positive.

Study Session 3.: Mathematical Induction

Introduction

Mathematical induction is one of the techniques that can be used to prove a variety of mathematical statements that are formulated in terms of n , where n is a positive integer.

Learning Outcomes for Study Session 3

When you have studied this session, you should be able to:

- 3.1. Understand the principle of mathematical induction, which is a powerful proof technique used to establish the validity of statements for natural numbers.
- 3.2. Applying mathematical inductions to prove statements, such as, prove a base case, induction hypothesis for all natural numbers.
- 3.3. Being able to recognize the limitations of mathematical induction.

3.1: The principle of mathematical induction

Let $P(n)$ be a given statement involving the natural number n such that

- i. The statement is true for $n = 1$, i.e., $P(1)$ is true (or true for any fixed natural number) and
- (ii) If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of $P(k)$ implies the truth of $P(k + 1)$. Then $P(n)$ is true for all natural numbers n .

Example 3.1:

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

Solution: Let the given statement $P(n)$ be defined as $P(n): 1 + 3 + 5 + \dots + (2n - 1) = n^2$, for $n \in N$. Note that $P(1)$ is true, since:

$$P(1): 1 = 1^2$$

Assume that $P(k)$ is true for some $k \in N$, i.e:

$$P(k): 1 + 3 + 5 + \dots + (2k - 1) = k^2$$

Now, to prove that $P(k + 1)$ is true, we have:

$$1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$$

$$= k^2 + (2k + 1)$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in N$.

Example 3.2:

$$\sum_{n=1}^{n-1} t(t + 1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2.$$

Solution: Let the given statement $P(n)$, be given as:

$$P(n): \sum_{t=1}^{n-1} t(t + 1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2$$

We observe that:

$$P(2): \sum_{t=1}^{2-1} t(t + 1) = \sum_{t=1}^1 t(t + 1) = 1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3} = \frac{2 \cdot (2-1)(2+1)}{3}$$

Thus, $P(n)$ is true for $n = 2$

Assume that $P(n)$ is true for $n = k \in N$

i.e.,

$$P(k): \sum_{t=1}^{k-1} t(t + 1) = \frac{k(k-1)(k+1)}{3}$$

To prove that $P(k + 1)$ is true, we have

$$\sum_{t=1}^{(k+1)-1} t(t+1) = \sum_{t=1}^k t(t+1) = \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1) = k$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers $n \geq 2$

Example 3.3:

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for all natural numbers, } n \geq 2.$$

Solution: Let the given statement be $P(n)$, i.e:

$$P(n): \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for all natural numbers, } n \geq 2$$

we, observe that $P(2)$ is true, since:

$$\left(1 - \frac{1}{2^2}\right) = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4} = \frac{2+1}{2 \times 2}$$

assume that $P(n)$ is true for some $k \in N$, i.e.,

$$P(k): 1 - \frac{1}{2^2} \cdot 1 - \frac{1}{3^2} \cdots 1 - \frac{1}{k^2} = \frac{k+1}{2k}$$

now, to prove that $P(k+1)$ is true, we have

$$1 - \frac{1}{2^2} \cdot 1 - \frac{1}{3^2} \dots 1 - \frac{1}{k^2} \cdot 1 - \frac{1}{(k+1)^2} = \frac{k+1}{2k} \left(1 - \frac{1}{(k+1)^2} \right) = \frac{k^2 + 2k}{2k(k+1)} = \frac{(k+1)+1}{2(k+1)}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers, $n \geq 2$.

Example 3.4:

$2^{2n} - 1$ is divisible by 3 .

Solution: Let the statement $P(n)$ given as:

$P(n): 2^{2n} - 1$ is divisible by 3, for every natural number n .

We observe that $P(1)$ is true, since:

$$2^2 - 1 = 4 - 1 = 3. 1 \text{ is divisible by 3.}$$

Assume that $P(n)$ is true for some natural number k , i.e., $P(k): 2^{2k} - 1$ is divisible by 3, i.e., $2^{2k} - 1 = 3q$, where $q \in N$

Now, to prove that $P(k + 1)$ is true, we have:

$$\begin{aligned} P(k + 1): 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 = 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1 = 3 \cdot 2^{2k} + (2^{2k} - 1) \\ &= 3 \cdot 2^{2k} + 3q = 3(2^{2k} + q) = 3m, \text{ where } m \in N \end{aligned}$$

Thus $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(n)$ is true for all natural numbers n .

Example 3.5:

$2n + 1 < 2^n$, for all natural numbers $n \geq 3$.

Solution: Let $P(n)$ be the given statement, i.e., $P(n): (2n + 1) < 2^n$ for all natural numbers, $n \geq 3$. We observe that $P(3)$ is true, since:

$$2 \cdot 3 + 1 = 7 < 8 = 2^3$$

Assume that $P(n)$ is true for some natural number k , i.e., $2k + 1 < 2^k$

To prove $P(k + 1)$ is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have:

$$2(k + 1) + 1 = 2k + 3 = 2k + 1 + 2 < 2^k + 2 < 2^k \cdot 2 = 2^{k+1}.$$

Thus $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(n)$ is true for all natural numbers, $n \geq 3$.

Example 3.6:

The distributive law from algebra says that for all real numbers c, a_1 and a_2 , we have $c(a_1 + a_2) = ca_1 + ca_2$.

Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if c, a_1, a_2, \dots, a_n are any real numbers, then:

$$c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + \dots + ca_n$$

Solution: Let $P(n)$ be the given statement, i.e:

$P(n): c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + \dots + ca_n$ for all natural numbers $n \geq 2$, for $c, a_1, a_2, \dots, a_n \in R$.

We observe that that $P(2)$ is true since:

$$c(a_1 + a_2) = ca_1 + ca_2$$

(by distributive law)

Assume that $P(n)$ is true for some natural number k , where $k > 2$, i.e:

$$P(k): c(a_1 + a_2 + \dots + a_k) = ca_1 + ca_2 + \dots + ca_k$$

Now to prove $P(k + 1)$ is true, we have:

$$\begin{aligned} P(k + 1) : c(a_1 + a_2 + \dots + a_k + a_{k+1}) &= c((a_1 + a_2 + \dots + a_k) + a_{k+1}) = \\ c(a_1 + a_2 + \dots + a_k) + ca_{k+1} &= ca_1 + ca_2 + \dots + ca_k + ca_{k+1} \end{aligned}$$

(by distributive law)

Thus $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n \geq 2$.

Example 3.7:

Prove by induction that for all-natural number n :

$$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n - 1)\beta)$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right) \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Solution: Consider:

$$P(n): \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n - 1)\beta)$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right) \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}, \text{ for all natural number } n.$$

We observe that $P(1)$ is true, since:

$$P(1): \sin\alpha = \frac{\sin(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

assume that $P(n)$ is true for some natural numbers k , i.e:

$$P(k): \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (k - 1)\beta)$$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Now, to prove that $P(k + 1)$ is true, we have:

$$\begin{aligned}
 P(k + 1): & \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (k - 1)\beta) + \sin(\alpha + k\beta) \\
 & = \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha + k\beta) = \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \\
 & \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) + \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}} \\
 & = \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}} = \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{\sin\frac{\beta}{2}} = \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin(k+1)\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}
 \end{aligned}$$

Thus $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(n)$ is true for all-natural number n .

Example 3.8:

Prove by the Principle of Mathematical Induction that $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n + 1)! - 1$ for all natural numbers n .

Solution: Let $P(n)$ be the given statement, that is, $P(n): 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n + 1)! - 1$ for all natural numbers n

Note that $P(1)$ is true, since:

$$P(1): 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

assume that $P(n)$ is true for some natural number k , i.e:

$$P(k): 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! = (k + 1)! - 1$$

To prove $P(k + 1)$ is true, we have:

$$P(k + 1): 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k + 1) \times (k + 1)!$$

$$= (k + 1)! - 1 + (k + 1)! \times (k + 1)$$

$$= (k + 1 + 1)(k + 1)! - 1 = (k + 2)(k + 1)! - 1$$

$$= ((k + 2)! - 1$$

Thus $P(k + 1)$ is true, whenever $P(k)$ is true. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all-natural number n .

Example 3.9:

Show by the Principle of Mathematical Induction that the sum S_n of the n term of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 \dots$ is given by:

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Solution:

Here,

$$P(n): S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{when } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{when } n \text{ is odd} \end{cases}$$

also, note that any term T_n of the series is given by:

$$T_n = \begin{cases} n^2 & \text{if } n \text{ is odd} \\ 2n^2 & \text{if } n \text{ is even} \end{cases}$$

We observe that $P(1)$ is true since:

$$P(1): S_1 = 1^2 = 1 = \frac{1 \cdot 2}{2} = \frac{1^2 \cdot (1+1)}{2}$$

Assume that $P(k)$ is true for some natural number k , i.e.

Case 1: When k is odd, then $k + 1$ is even. We have:

$$\begin{aligned} P(k+1): S_{k+1} &= 1^2 + 2 \times 2^2 + \dots + k^2 + 2 \times (k+1)^2 = \frac{k^2(k+1)}{2} + 2 \times (k+1)^2 \\ &= \frac{(k+1)}{2} [k^2 + 4(k+1)] \left(\text{as } k \text{ is odd, } 1^2 + 2 \times 2^2 + \dots + k^2 = k^2 \frac{(k+1)}{2} \right) \\ &= \frac{k+1}{2} [k^2 + 4k + 4] = \frac{k+1}{2} (k+2)^2 = (k+1) \frac{[(k+1)+1]^2}{2} \end{aligned}$$

So $P(k+1)$ is true, whenever $P(k)$ is true in the case when k is odd.

Case 2: When k is even, then $k+1$ is odd. Now,

$$P(k+1): 1^2 + 2 \times 2^2 + \dots + 2 \cdot k^2 + (k+1)^2$$

$$\begin{aligned}
 &= \frac{k(k+1)^2}{2} + (k+1)^2 \quad (\text{as } k \text{ is even, } 1^2 + 2 \times 2^2 + \dots + 2k^2 = k \frac{(k+1)^2}{2}) \\
 &= \frac{(k+1)^2(k+2)}{2} = \frac{(k+1)^2((k+1)+1)}{2}
 \end{aligned}$$

Therefore, $P(k+1)$ is true, whenever $P(k)$ is true for the case when k is even. Thus $P(k+1)$ is true whenever $P(k)$ is true for any natural numbers k . Hence, $P(n)$ true for all natural numbers.

Summary

1. Mathematical induction is a proof technique used to show validity of statements for all natural numbers.
2. It involves three steps: the base step, the induction hypothesis, and the induction step.
3. The base step establishes the truth of the statement for the initial value.
4. The induction hypothesis assumes that the statement is true for an arbitrary value, usually denoted by "k".
5. It can be used to prove various statements, such as those for sequences, divisibility, inequalities, and properties of numbers.

Exercise

- 1 Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.
- 2 Give an example of a statement $P(n)$, which is true for all n . Justify your answer. Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction:
- 3 $4^n - 1$ is divisible by 3, for each natural number n .
- 4 $2^{3n} - 1$ is divisible by 7, for all natural numbers n .

- 5 $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n .
- 6 $3^{2n} - 1$ is divisible by 8, for all natural numbers n . 7. For any natural number n , $7^n - 2^n$ is divisible by 5.
- 7 For any natural number n , $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.
- 8 $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.
- 9 $n(n^2 + 5)$ is divisible by 6, for each natural number n .
- 10 $n^2 < 2^n$ for all natural numbers $n \geq 5$.
- 11 $2n < (n + 2)!$ for all natural number n .
- 12 $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.
- 13 $2 + 4 + 6 + \dots + 2n = n^2 + n$ for all natural numbers n .
- 14 $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ for all natural numbers n .
- 15 $1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$ for all natural numbers n .

Use the principle of mathematical induction in the following Exercises.

- 17 A sequence a_1, a_2, a_3, \dots is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

18 A sequence $b_0, b_1, b_2 \dots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k . Show that $b_n = 5 + 4n$ for all natural numbers n using mathematical induction.

19 A sequence $d_1, d_2, d_3 \dots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in N$.

20 Prove that for all $n \in N$

$$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n - 1)\beta)$$

$$= \frac{\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right) \sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

21 Prove that, $\cos\theta \cos 2\theta \cos 2^2\theta \dots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}$, for all $n \in N$.

Study Session 4:Sequences and Series

Introduction

Sequences are ordered lists of numbers, and series are the sums of these numbers. Sequences can be finite or infinite, and they follow patterns such as arithmetic or geometric progressions. Sigma notation (Σ) is used to represent the sum of terms in a series.

Arithmetic series have a common difference between terms, while geometric series have a common ratio. Finite series have a specific number of terms, while infinite series continue indefinitely. Infinite series can either converge to a finite sum or diverge.

Studying sequences and series provides a foundation for recognizing patterns, calculating sums, and analysing mathematical structures.

Learning Outcomes for Study Session 4

When you have studied this session, you should be able to:

- 4.1. Understand the concept of a sequence and how it differs from a set, by being an ordered list where each number is called a term.
- 4.2. Identifying arithmetic sequences and finding the n^{th} term of an arithmetic sequence.

- 4.3. Recognizing geometric sequences, finding the n^{th} term of a geometric sequence and determining the sum of a geometric series.
- 4.4. Understanding the concept of sigma notation and summation index.
- 4.5. Understanding the concept of the difference method and when to use it.
- 4.6. Recognizing the difference between finite and infinite series.

4.1: What is a Sequence?

Definition: A **sequence** is simply an ordered list $u_1, u_2, u_3, \dots, u_n$ of numbers (or terms). This is often abbreviated to $\{u_n\}$. For our purposes, each term u_n is usually used in one of two ways:

- (i) As a function of the preceding term(s), or
- (ii) As a function of its position in the sequence.

Example 4.1:

The sequence $\{u_n\}$ defined by

$$u_1 = 1 \text{ and } u_n = (u_1 u_2 \dots u_{n-1}) + 1 \text{ for } n \geq 2$$

is 1, 2, 3, 7, 43, 1807, ...,

where each term (after the first) is one greater than the product of all the previous terms of the sequence

You will appreciate, however, that, in such cases, should you wish to know (say) u_{100} , the hundredth term of this sequence, then you would first need to know each of the preceding 99 terms. It is preferable, then, to have sequences given in the second way, with each term defined as a function of n , its position in the sequence.

Example 4.2:

One of the most famous sequences of all is the Fibonacci sequence $\{F_n\}$ which is defined by

$$F_1 = 1, F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

This sequence begins

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

The general solution of sequences defined in this way is not within the scope of this course, but (for the interested reader) each term of the Fibonacci sequence is actually given by

$$F_n = \frac{1}{5} \left\{ \left(\frac{1+5}{2} \right)^n - \left(\frac{1-5}{2} \right)^n \right\}$$

4.2: Arithmetic and Geometric Sequences and Series.

The sequence defined by

$$u_1 = a \text{ and } u_n = u_{n-1} + d \text{ for } n \geq 2$$

begins

$$a, a + d, a + 2d, \dots$$

and you should recognize this an **arithmetic sequence** with first term a and common difference d

The nth term (i.e., the solution) is given by $u_n = a + (n - 1)d$

The arithmetic series with n terms

$$a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]$$

has sum

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

or

$$S_n = \frac{n}{2}(\text{first} + \text{last})$$

and the results should be well known to you.

Example 4.3:

An arithmetic series has the property that the sum of the first ten terms is half the sum of the **next** ten terms and the common difference.

Solution

Let the first term be a and the common difference be d . Then the sums of the first ten terms and the next ten terms are

$$\frac{10}{2}(a + (a + 9d)) \text{ and } \frac{10}{2}((a + 10d) + (a + 19d))$$

since the former is half the latter, we deduce that

$$2a + 9d = \frac{1}{2}(2a + 29d)$$

or $2a = 11d \quad (1)$

also, as the 100th term is 95, we know that

$$a + 99d = 95 \quad (2)$$

From (1)

$$99d = 18a \text{ and so in (2)}$$

$$a + 18q = 19a = 95$$

hence:

$$a = 5 \text{ and } d = \frac{10}{11}$$

Another important sequence defined by

$$u_1 = a \text{ and } u_n = ru_{n-1} \text{ for } n \geq 2$$

begins

$$a, ar, ar^2, \dots$$

This is the **geometric sequence** with first term a and common ratio r

The n th term is given by $u_n = ar^{n-1}$

The geometric series with n terms,

$$a + ar + ar^2 + \dots + ar^{n-1}$$

has sum

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad \frac{a(r^n-1)}{r-1} \quad \text{for } r \neq 1$$

Note that a **series** is the sum of a number of terms in a **sequence**. The terms 'arithmetic progression' (AP) and 'geometric progression' (GP) are not preferred here as the word 'progression' is used loosely in respect of both a sequence and a series. However, you should recognize the use of these terms and their abbreviations, since they are in common usage.

4.3: The Sigma Notation

When writing series, the shorthand Σ notation is used to represent the sum of a number of terms having a common form.

The series $f(1) + f(2) + \dots + f(n - 1) + f(n)$ would be written

$$\sum_{r=1}^n f(r)$$

The function f is the common form that each term of the series takes; r is called the **summation index**, being the variable quantity from term to term. The ' $r = 1$ ' below the sigma indicates the first value taken by r , and the ' n ' above the sigma denotes the final term taken by r .

Note that, while it is customary to take $r = 1$ for the first term, it is by no means essential. The above series would equally well be written as:

$$\sum_{r=1}^{n-1} f(r + 1)$$

in fact, later on you will encounter series which, for convenience, will start with $r = 0$

Also, the choice of the letter ' r ' is unimportant:

$$\sum_{r=1}^n f(r) \quad \sum_{k=1}^n f(k) \text{ and } \sum_{i=1}^n f(i)$$

All represent the series $f(1) + f(2) + \dots + f(n)$ (1)

You should know the following results relating to the Σ notation:

$$\begin{aligned} \sum_{r=1}^n \{f(r) + g(r)\} &= \{f(1) + g(1)\} + \{f(2) + g(2)\} + \dots + \{f(n) + g(n)\} \\ &= \{f(1) + f(2) + \dots + f(n)\} + \{g(1) + g(2) + \dots + g(n)\} \end{aligned}$$

$$= \sum_{r=1}^n f(r) + \sum_{r=1}^n g(r) \quad (2)$$

$$\sum_{r=1}^n af(r) = af(2) + \dots + af(n)$$

where a is some constant

$$= a\{f(1) + f(2) + \dots + f(n)\}$$

$$= a \sum_{r=1}^n f(r)$$

Note that $\sum_{r=1}^n nf(r) = n \sum_{r=1}^n f(r)$ also, since n is a fixed quantity, not a variable, in the summation.

$$(3) \quad a. \sum_{r=1}^n r = \frac{n(n+1)}{2}$$

$$b. \sum_{r=1}^n \frac{n}{6}(n+1)(2n+1)$$

$$(c) \sum_{r=1}^n \frac{n^2}{4}(n+1)^2 \text{ or } \left[\frac{n(n+1)}{2} \right]^2$$

$$d. \sum_{r=1}^n (1+1+\dots+1)_{\text{n times}} = n$$

Since a '1' is wanted for each of $r = 1, 2, \dots, n$

This result is entirely overlooked, and often incorrectly written as

$$\sum_{r=1}^n 1 = 1$$

Example 4.4:

Show that $\sum_{r=1}^n (6r^2 + 4r - 1) = n(n + 2)(2n + 1)$

Solution

$$\begin{aligned}
 \sum_{r=1}^n (6r^2 + 4r - 1) &= \sum_{r=1}^n 6r^2 + \sum_{r=1}^n 4r - \sum_{r=1}^n 1 && \text{by result (1)} \\
 &= 6 \sum_{r=1}^n r^2 + 4 \sum_{r=1}^n r - \sum_{r=1}^n 1 && \text{by result (2)} \\
 &= \frac{n}{6}(n + 1)(2n + 1) + 4 \cdot \frac{n}{2}(n + 1) - n && \text{by results (3)} \\
 &= n(n + 1)(2n + 1) + 2n(n + 1) - n \\
 &= n(2n^2 + 3n + 1 + 2n + 2 - 1) \\
 &= n(2n^2 + 5n + 2) \\
 &= n(n + 2)(2n + 1)
 \end{aligned}$$

Exercise 4.1

1. A geometric series has first term 4 and second term 7. Giving your answer to three significant figures, find the sum of the first twenty terms of the series. (AEB)
2. The first term of an arithmetic progression is -13 and the last term is 99. The sum of the series is 1419. Find the number of terms and common difference. Find also the sum of all the positive terms of the series. (AEB)
3. An arithmetic series has first term 7 and second term 11.
 - (a) Find the 17th term
 - (b) Show that the sum of the first 35 terms is equal to the sum of the next 15 terms. (AEB)
4. The first three terms of a geometric series are $2x$, $x - 8$ and $2x + 5$ respectively. Find the possible values of x . (AEB)
5. An arithmetic series has first term $\ln 2$ and common difference $\ln 4$. Show that the sum S_n of the first n terms is $n^2 \ln \ln 2$

Find the least value of n for which S_n is greater than fifty times nth term

6. A geometric series has a first term 4 and a common ratio r , where $0 < r < 1$. Given that the first, second, and fourth terms of this geometric series form three successive terms of an arithmetic series, show that $r^2 - 2r + 1 = 0$. Find the value of r . (AEB)

7. Given that $S(n) = n^2 + 4n$, write down an expression for $S(n - 1)$ and simplify $S(n) - S(n - 1)$

Proof by Induction

An inventor builds a climbing 'robot' which is designed to be able to climb any ladder that has equally spaced rungs, no matter how long it may be. (Being solar-powered, it can continue indefinitely if necessary.) The inventor tests it on a variety of ladders: on some of them, the robot succeeds; on others, it does not. Given any ladder, what conditions need to be satisfied for the robot to be able to climb the ladder?

You will see that there are, in fact, just two:

- (a) In the first instance, the robot needs to be able to get on the ladder (presumably at the first rung); and
- (b) If the robot is on any given rung (say rung k – as rungs are equally spaced, it does not matter where rung k is on the ladder), then it has to be able to get on to the next rung after this (rung $k + 1$)

If the robot's programming and construction enable it to satisfy both of these conditions, then it can climb as far up the ladder (and to infinity if necessary) as its inventor could wish for, since by condition a., the robot can get on the ladder (at rung 1). Then, by condition b., it can get on to rung 2. By b. again, it can get to rung 3. a. The robot can get on the ladder (at rung 1). Then, by condition b., it can get on to rung 2. By b. again, it can get to rung 3, hence to rung 4, hence to rung 5, etc. Thus, the robot can reach rung n for **any** positive integer n .

No, the authors have not gone senile! The above is actually an illustration of a very powerful technique called **proof by induction**. The method is a cunning means of proving the truth of some statement or formula that is found by experimental means, for instance, but that, without a general proof, is only known to be true for certain values of the variable concerned.

For instance, the result

$$\sum_{r=1}^n r = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Arises from the sum to n terms of an arithmetic series, and had already been proved.

What about the result $\sum_{r=1}^n \frac{n}{6}(n + 1)(2n + 1)$?

Where did that come from?

One way, is to first note that the formula for $\sum_{r=1}^n r$ is a quadratic in n , and so assume

that the formula for $\sum_{r=1}^n r^2$ is a cubic in n , say

$$an^3 + bn^2 + cn + d$$

Then use the values found in the cases when $n = 1, 2, 3$ and 4 to set up and solve a system of four simultaneous equations in the four unknowns a, b, c, d (which turn out to be $\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$ and 0 respectively)

However, this only proves that such an expression fits the bill in these four cases: namely $n = 1, 2, 3$ and 4 . You could try it out for $n = 5, 6, 7, \dots$ as far as you like. You could convince yourself that this expression for the sum of the squares of the first n positive integers just has to be true for every positive integer n ; but you would have proved it in only a few (or even many) particular cases.

Try the robot approach:

$$\text{When } n = 1, \sum_{r=1}^n r^2 = 1^2 = 1$$

While the formula with $n = 1$ gives

$$\frac{1}{6}(1)(1 + 1)(2 \times 1 + 1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$$

also.

Hence the formula is true when $n = 1$. (The robot is on the ladder at rung 1)

Next, **assume** that:

$$\sum_{r=1}^n r^2 = \frac{k}{6}(k + 1)(2k + 1)$$

(That is, the robot is on the ladder (somewhere) at rung k . Now what about rung $k + 1$?)

Then it follows that:

$$\begin{aligned}
 \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^k r^2 + (k + 1)^2 \\
 = \frac{k}{6}(k + 1)(2k + 1) + (k + 1)^2 &\quad (\text{The first was assumed above}) \\
 = \frac{k}{6}(k + 1)(2k + 1) + \frac{6}{6}(k + 1)^2 &\quad (\text{Common denominator}) \\
 = \frac{(k+1)}{6}\{k(2k + 1) + 6(k + 1)\} &\quad (\text{factorising}) \\
 &= \frac{(k+1)}{6}(2k^2 + 7k + 6) \\
 &= \frac{(k+1)}{6}(k + 2)(2k + 3)
 \end{aligned}$$

Now notice that this expression is

$$\frac{1}{6}(k + 1)([k + 1] + 1)(2[k + 1] + 1),$$

Which is precisely the formula expected, but with $n = k + 1$

So if the formula assumed when $n = k$ is true then, by adding on the $(k + 1)$ th term, it must also be true when $n = k + 1$. whenever k is. (The robot can get from rung k to rung $k + 1$). In itself, this step of the process is a big IF. But the formula is

true when $n = 1$, so this 'stepping up' bit proves it must be true when $n = 2$ also, since it is true for $n = 2$ (which it now is known to be) then it must be true for $n = 3$ as well; and then for $n = 4, n = 5, \dots$ and for all positive integers n .

Remember that the 'stepping up' part of the proof relies on an assumption (called **induction hypothesis**), and it is important, therefore, to use the word 'assume' (or 'suppose') otherwise there is a large amount of written explanation to do at the end of this proof in order to finalise matters conclusively. The proof is clinched by showing that the whole process starts in the first place.

Example 4.5

Use mathematical induction to prove that for all positive integers n ,

$$2 \cdot 2 + 3 \cdot 2^2 + \dots + (n + 1) \cdot 2^n = n \cdot 2^{n+1}$$

Solution

when $n = 1$, LHS = $2 \cdot 2 = 4$

and RHS = $1 \cdot 2^{1+1} = 1 \cdot 4 = 4$

and the statement is true when $n = 1$. (Starting step)

assume that the formula is true for $n = k$; that is

$$2 \cdot 2 + 3 \cdot 2^2 + \dots + (k + 1) \cdot 2^k = k \cdot 2^{k+1} \quad (\text{Induction step})$$

then, when $n = k + 1$,

$$\begin{aligned}
 & 2 \cdot 2 + 3 \cdot 2^2 + \dots + (k+1) \cdot 2^k + (k+2) \cdot 2^{k+1} \\
 = & \left\{ 2 \cdot 2 + 3 \cdot 2^2 + \dots + (k+1) \cdot 2^k \right\} + (k+2) \cdot 2^{k+1} && ((k+1) \text{ term added}) \\
 = & k \cdot 2^{k+1} + (k+2) \cdot 2^{k+1} && (\text{by the hypothesis}) \\
 = & 2^{k+1}(k+k+2). \\
 = & 2^{k+1}(2k+2). \\
 = & 2^{k+1} \cdot 2(k+1). \\
 = & (k+1) \cdot 2^{k+2}.
 \end{aligned}$$

which is the required formula with k replaced by $k+1$. Hence if the statement is true for $n = k$, then it is also true for $n = k+1$.

By induction $2 \cdot 2 + 3 \cdot 2^2 + \dots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$ is true for all positive integers n .

Note: because it is easy to decide what the final expression ('the formula for $n = k$ with k replaced by $k+1$ ') should be in advance, many students 'fiddle' it into existence, or simply write it straight down, without showing the necessary working to demonstrate that it does indeed arise as a result of adding on the $(k+1)$ th term to the assumed sum- k -terms, be careful to show your working!

The following example illustrate the minimal amount of working that needs to be put down to clinch an inductive proof

Example 4.6

Prove by induction that

$$\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2((n+1)(n+2))}$$

for all positive integers n

Solution

$$\text{For } n = 1, LHS = \sum_{r=1}^1 \frac{1}{r(r+1)(r+1)} = \frac{1}{1 \times 2 \times 3} = \frac{1}{6}$$

$$\text{while } RHS = \frac{1}{4} - \frac{1}{2 \times 2 \times 3} = \frac{1}{4} - \frac{1}{12} = \frac{1}{6} \quad \text{also.}$$

Now assume that :

$$\sum_{r=1}^k \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$$

then :

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^k \frac{1}{r(r+1)(r+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$\begin{aligned}
 &= \frac{1}{4} - \frac{1}{(k+1)(k+2)} \left\{ \frac{1}{2} - \frac{1}{k+3} \right\} \\
 &= \frac{1}{4} - \frac{1}{(k+1)(k+2)} \left\{ \frac{k+3-2}{2(k+3)} \right\} \\
 &= \frac{1}{4} - \frac{(k+1)}{2(k+1)(k+2)(k+3)} \\
 &= \frac{1}{4} - \frac{1}{2([k+1]+1)([k+1]+2)}
 \end{aligned}$$

as required. Proof follows by induction

Exercise 4.2

Use mathematical induction to prove the following results for all positive integers n

$$1. \quad \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n + 1)^2$$

$$2. \quad \sum_{r=1}^n r(r + 3) = \frac{1}{3}n(n + 1)(n + 5)$$

$$3. \quad \sum_{r=1}^n (2r - 1)(2r + 1) = \frac{1}{2} + \frac{1}{6}(2n - 1)(2n + 1)(2n + 3)$$

$$4. \quad \sum_{r=1}^n \frac{2}{(r+1)(r+2)(r+3)} = \frac{1}{6} - \frac{1}{(n+2)(n+3)}$$

4.3: The Difference Method

The process of proof by induction, while being a powerful mathematical tool, has the disadvantage that, in order to employ it, you really need to have the answers (or something you strongly suspect to be the answer) to begin with.

There are, however, direct methods of proof available in most cases. One of these is known as the method of differences, or the **difference method**.

The following example illustrates the method of determining differences. Although the working appears initially to be going nowhere, it will eventually lead to a direct proof that:

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

During the proof, it will be assumed that $\sum_{r=1}^n r = \frac{1}{2}n(n + 1)$ which has been established as the sum of an arithmetic series.

Consider the following identity $(n)^3 - (n - 1)^3 \equiv n^3 - (n^3 - 3n^2 + 3n - 1)$

$$\text{i.e } n^3 - (n - 1)^3 \equiv 3n^2 - 3n + 1 \quad (1)$$

from this, the similar identity:

$$(n - 1)^3 - (n - 2)^3 \equiv 3(n - 1)^2 - 3(n - 1) + 1$$

can be deduced by replacing n by $n - 1$.

Also, replacing n by $n - 2$ in equation (1) gives

$$(n - 1)^3 - (n - 3)^2 \equiv 3(n - 2)^2 - 3(n - 2) + 1, \text{ etc}$$

writing this sequence of results in a column form:

$$n^3 - (n - 1)^3 = 3n^2 - 3n + 1$$

$$(n - 1)^3 - (n - 2)^3 = 3(n - 1)^2 - 3(n - 1) + 1$$

$$(n - 2)^2 - (n - 3)^3 = 3(n - 2)^2 - 3(n - 2) + 1$$

$$(n - 3)^3 - (n - 4)^3 = 3(n - 3)^2 - 3(n - 3) + 1$$

⋮ ⋮ ⋮

$$3^3 - 2^3 = 3(3)^2 - 3(3) + 1 \quad 2^3 - 1^3 = 3(2)^2 - 3(2) + 1$$

$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$

now add up all the terms on the LHS: you get $n^3 - 0^3$, since all other terms appear once positively and once negatively, cancelling out. Adding up the RHS in three columns gives

$$3 \sum_{r=1}^n r^2 - 3 \sum_{r=1}^n r + n$$

$$\begin{aligned}
 \text{Thus, } n^3 &= 3 \sum_{r=1}^n r^2 - 3 \sum_{r=1}^n r + n \quad \left[\text{using } \sum_{r=1}^n r = \frac{1}{2}n(n+1) \right] \\
 \rightarrow \quad 3 \sum_{r=1}^n r^2 &= (n^3 - n) + 3 \times \frac{1}{2}n(n+1) \\
 &= n(n^2 - 1) + \frac{3}{2}n(n+1) \\
 &= \frac{2n}{2}(n-1)(n+1) + \frac{3}{n}n(n+1) \\
 &= \frac{n(n+1)}{2}\{2(n-1) + 3\} \\
 &= \frac{1}{2}n(n+1)(2n+1)
 \end{aligned}$$

Finally, dividing by 3 gives the required result.

Example 4.7

By considering $n^5 - (n-1)^5$ and similar expressions, find the formula for $\sum_{r=1}^n r^4$ in terms of n , assuming the results for $\sum_{r=1}^n r$, $\sum_{r=1}^n r^2$ and $\sum_{r=1}^n r^3$.

Solution

$$n^5 - (n-1)^5 = 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

$$(n-1)^5 - (n-2)^5 = 5(n-1)^4 - 10(n-1)^3 + 10(n-1)^2 - 5(n-1) + 1$$

$$(n - 5)^5 - (n - 3)^5 = 5(n - 2)^4 - 10(n - 2)^3 + 10(n - 2)^2 - 5(n - 2) + 1$$

⋮ ⋮ ⋮

$$3^5 - 2^5 = 5(3)^4 - 10(3)^3 + 10(3)^2 - 5(3) + 1$$

$$2^5 - 1 = 5(2)^4 - 10(2)^3 + 10(2)^2 - 5(2) + 1$$

$$1^5 - 0^5 = 5(1)^4 - 10(1)^3 + 10(1)^2 - 5(1) + 1$$

adding:

$$n^5 - 0^5 = 5 \sum_{r=1}^n r^4 - 10 \sum_{r=1}^n r^3 + 10 \sum_{r=1}^n r^2 - 5 \sum_{r=1}^n r + n$$

$$5 \sum_{r=1}^n r^4 = (n^5 - n) + 10 \frac{n^2}{4} (n + 1)^2 - 10 \frac{n}{6} (n + 1)(2n + 1) + 5 \frac{n}{2} (n + 1)$$

$$= \frac{6}{6} n(n - 1)(n + 1)(n^2 + 1) + \frac{15}{6} n^2 (n + 1)^2 - 10 \frac{n}{6} (n + 1)(2n + 1) + \frac{15}{6} n(n + 1)$$

$$= \frac{1}{6} n(n + 1) \{ 6(n - 1)(n^2 + 1) + 15n(n + 1) - 10(2n + 1) + 15 \}$$

$$= \frac{1}{6} n(n + 1) \{ 6n^3 - 6n^2 + 6n - 6 + 15n^2 + 15n - 20n - 10 + 15 \}$$

$$= \frac{1}{6} n(n + 1)(6n^3 + 9n^2 + n - 1)$$

$$= \frac{1}{6} n(n + 1)(2n + 1)(3n^2 + 3n - 1)$$

So that:

$$\sum_{r=1}^n r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$$

admittedly, not a very appealing result! You will note, however, that the method of differences consists of adding two sides of a set of identities of a common form, where one side is expressed as a difference of successive terms. In this way, almost all terms cancel on this side of the summation.

The difference method can be employed in other circumstances also.

Example 4.8

Prove that :

$$\sum_{r=1}^n \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

Solution

Some ingenuity is indeed here to turn the one term in the summation on the LHS into two terms. Note that

$$\frac{r}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$$

so that

$$\begin{aligned}
 \sum_{r=1}^n \frac{r}{(r+1)!} &= \sum_{r=1}^n \left(\frac{1}{r!} - \frac{1}{(r+1)!} \right) \\
 &= \left(\frac{1}{1!} - \frac{1}{2!} \right) + \left(\frac{1}{2!} - \frac{1}{3!} \right) + \left(\frac{1}{3!} - \frac{1}{4!} \right) + \dots + \left(\frac{1}{(n-1)!} - \frac{1}{n!} \right) + \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) \\
 &= \frac{1}{1!} - \frac{1}{(n+1)!} \quad [\text{since all other terms cancel}] \\
 &= 1 - \frac{1}{(n+1)!}
 \end{aligned}$$

The equation referred to as the fundamental theorem of summation can summarise the difference method:

$$\sum_{r=1}^n \{f(r) - f(r - 1)\} = f(n) - f(0)$$

where f is any function suitably defined on the non-negative integers.

As an example, take $f(r) = \sin \sin \left(ar + \frac{a}{2}\right)$ for some constant a (not equal to a multiple of 2π). Then

$$\begin{aligned}
 &\sin \sin \left(ar + \frac{a}{2}\right) - \sin \sin \left(a[r - 1] + \frac{a}{2}\right) \\
 &= \sin \sin \left(ar + \frac{a}{2}\right) - \sin \sin \left(ar - \frac{a}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \cos \cos \left(\frac{ar + \frac{a}{2} + ar - \frac{a}{2}}{2} \right) \sin \sin \left(\frac{ar + \frac{a}{2} - ar - \frac{a}{2}}{2} \right) \\
 &= 2 \cos \cos (ar) \sin \frac{a}{2}
 \end{aligned}$$

and

$$\sum_{r=1}^n 2 \cos \cos (ar) \sin \frac{a}{2} = f(n) - f(0) = \sin \sin \left(an + \frac{a}{2} \right) - \sin \left(\frac{a}{2} \right)$$

by the difference method; leading to the result

$$\sum_{r=1}^n \cos \cos (ar) = \frac{1}{2} \left\{ \frac{\sin \sin \left(an + \frac{a}{2} \right)}{\sin \frac{a}{2}} - 1 \right\}$$

or, since

$$\sin \sin \left(an + \frac{a}{2} \right) - \sin \frac{a}{2} = 2 \cos \cos \left[\frac{a(n+1)}{2} \right] \sin \frac{an}{2},$$

$$\sum_{r=1}^n \cos \cos (ar) = \frac{\cos \cos \left[\frac{a(n+1)}{2} \right] \sin \left(\frac{an}{2} \right)}{\sin \frac{a}{2}}$$

If you find this approach a little confusing, it only enables you to save a line or two of work. In tricky cases, resorting to writing out the series in question and seeing the terms which cancel will prove much simpler to handle correctly

Example 4.9

Express $\frac{1}{x(x+1)}$ in terms of partial fractions.

Hence show that $\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all positive integers n.

Solution

$$\text{Assume } \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

By the cover-up method (or multiplying through by $x(x + 1)$ and comparing coefficients or substituting values), A = 1 and B = - 1.

Then

$$\begin{aligned} \sum_{r=1}^n \frac{1}{r(r+1)} &= \sum_{r=1}^n \left(\frac{1}{r} - \frac{1}{r+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{1} - \frac{1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1} \\ &= \frac{n}{n+1} \end{aligned}$$

as required.

Alternatively, note that is $\sum_{r=1}^n \{f(r) - f(r - 1)\}$ with $f(r) = \frac{1}{r+1}$. The sum is then

$$f(n) - f(0) = -\frac{1}{n+1} - \left(-\frac{1}{0+1}\right) = 1 - \frac{1}{n+1}, \text{ etc. as before.}$$

Exercise 4.3

1. Show that:

$$3r(r + 1) = r(r + 1)(r + 2) - r(r - 1)(r + 1) \text{ and deduce that } \sum_{r=1}^n r(r + 1) \\ = \frac{1}{3}n(n + 1)(n + 2)$$

2. Express $\frac{1}{r(r+2)}$ in partial fractions, and hence show that:

$$\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$

3. Simplify the expression $(r + 1)! - r!$ and hence prove that:

$$\sum_{r=1}^n r \times r! = (n + 1)! - 1$$

4. Simplify the expression $(r + r - 1)(r - r - 1)$. Hence prove that

$$\sum_{r=1}^n \frac{1}{r+r-1} = n \text{ for all positive integers } n.$$

5. Given that r is a positive integer and $f(r) = \frac{1}{r^2}$, find a single expression for

$$f(r) - f(r + 1). \text{ Hence prove that } \sum_{r=1}^n \left(\frac{2r+1}{r^2(r+1)^2} \right) = \frac{(3n+1)(5n+1)}{n^2(4n+1)^2}$$

6. Prove the identity :

$$\frac{2r+3}{r(r+1)} - \frac{2r+5}{(r+1)(r+2)} = \frac{2(r+3)}{r(r+1)(r+2)}$$

Hence, or otherwise, find the sum of the series

$$S_n = \frac{8}{1 \times 2 \times 3} + \frac{10}{2 \times 3 \times 4} + \dots + \frac{2(n+3)}{n(n+1)(n+2)}$$

7. Use the method of difference to find:

$$\sum_{r=1}^n \frac{1}{(n+r-1)(n+r)} \text{ in terms of } n.$$

8. Prove that $\sum_{r=1}^n \frac{r^2+r+1}{r^2+r} = n + 1 - \frac{1}{n+1}$

9. Prove the identity $\cos \cos (A - B) - \cos \cos (A + B) = 2\sin A \sin B$. Hence prove that:

$$\sin \theta \{ \sin \theta + \sin 3\theta + \sin 5\theta + \dots \sin (2n-1)\theta \} = \sin^2 n\theta$$

10. Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions. Hence prove the result:

$$\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

11. By considering $\frac{1}{1+a^{n-1}} - \frac{1}{1+a^n}$, show that:

$$\sum_{r=1}^n \frac{a^{n-1}}{(1+a^{n-1})(1+a^n)} = \frac{a^n - 1}{2(a-1)(a^n + 1)}$$

Where a is positive and $a \neq 1$. Deduce that:

$$\sum_{r=1}^n \frac{n^n}{(1+2^{n-1})(1+2^n)} < 1 \text{ (Cambridge)}$$

4.4: Infinite Series

Thus far in this session, you have studied only finite series. You have, however. Already encountered at least one example of a series which can be 'summed-to-infinity' without simply obtaining an 'infinitely large number', namely the geometric series

$$a + ar + ar^2 + \dots ar^{n-1} + \dots$$

provided that $-1 < r < 1$

The sum to n terms of a geometric series can be expressed as:

$$S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$$

which can be split up as

$$\frac{a}{1-r} - \frac{ar^n}{1-r}$$

For $|r| < 1$, $r^n \rightarrow 0$ as $n \rightarrow \infty$ (said ' r^n tends to zero as n tends to infinity'). In which case the second term $\rightarrow 0$ also. This is written

$$\left(\frac{ar^n}{1-r} \right) = 0$$

and said, the limit as n tends to infinity of $\left(\frac{ar^n}{1-r} \right)$ is zero'. The infinite geometric series converges in this case to the number:

$$S_n = \frac{a}{1-r},$$

or $S_\infty = \frac{a}{1-r}$ for short

when $|r| > 1$, $\frac{ar^n}{1-r} \rightarrow 0$, and the geometric series 'diverges', having no limit.

The sum-to-infinity of any convergent series is defined in the following way: given S_n , the sum to n terms of a series,

$$S_\infty = (S_n)$$

Example 4.10

From an earlier example, $\sum_{r=1}^n \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!} = S_n$

Then

$$S_\infty = \left(1 - \frac{1}{(n+1)!}\right) = 1$$

since $\frac{1}{(n+1)!} \rightarrow 0$ as $n \rightarrow \infty$;

i.e., the sum of the infinite series

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots \text{ is } 1$$

Example 4.11

Given that $S_n = \frac{2n^2+3n+1}{23+5n-7}$, find (S_n)

Solution

By dividing the numerator and denominator of S_n by n^2 , S_n can be written in the form

$$S_n = \frac{\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)}{\left(3 + \frac{5}{n} - \frac{7}{n^2}\right)}$$

now, as $n \rightarrow \infty$, $\frac{1}{n} \rightarrow 0$ and $\frac{1}{n^2} \rightarrow 0$ so that $S_n \rightarrow \frac{2}{3}$.

Thus $S_\infty \approx \frac{2}{3}$.

Another way of deducing this is to observe that, for n 'large', the numerator is $\approx 2n^2$; the '3n' and the '1' paling into insignificance in comparison; while the denominator is $\approx 3n^2$ for similar reasons. Then $S_n \approx \frac{2n^2}{3n^2} = \frac{2}{3}$

Exercise 4.4

1. The first term of a geometric series is 8 and sum to infinity is 400. Find the common ratio.
2. The first, second and third terms of a geometric series are p , p^2 and q respectively, where $p < 0$. The first, second and third terms of an arithmetic series are p , q and p^2 respectively.
 - a. Show that $p = -\frac{1}{2}$ and find the value of q
 - b. Find the sum-to-infinity of the geometric series.
3. Given S_n , deduce the value of S_∞ in each of the following
 - a. $S_n = \frac{n}{2n+1}$
 - b. $S_n = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$
 - c. $S_n = \frac{(3n+1)(5n+1)}{(4n+1)^2}$
 - d. $S_n = \frac{(2n+3)(n+1)}{(n)(n+1)(n+4)}$
 - e. $S_n = \frac{n(2n+5)}{(n+1)(n+2)}$
 - f. $S_n = \frac{5n+11}{2(n+1)(n+2)}$
 - g. $S_n = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$
4. Given $S(n) = n \times 2^{n+1}$, find $\left(\frac{S(n+1)}{S(n)}\right)$

5. Assuming the results $\sum_{r=1}^n r = \frac{1}{2}n(n + 1)$ and $\sum_{r=1}^n r^2 = \frac{1}{6}n(n + 1)(2n + 1)$,

find an expression for $\sum_{r=1}^n r(r + 1)$ in terms of n. hence determine

$$\frac{\sum_{r=1}^n r(r+1)}{\left(\sum_{r=1}^n r\right)^{\frac{3}{2}}}$$

6. A geometric series is given by

$$e^{3x} + 3e^x + 9e^{-x} + \dots$$

- a. Find the value of the sum-to-infinity in the case when $x = \ln 2$
- b. Determine the ranges of values of x for which a sum-to-infinity exists.

Summary

1. Understand the concept of a sequence and how it differs from a set, by being an ordered list where each number is called a term.
2. Identifying arithmetic sequences and finding the nth term of an arithmetic sequence.
3. Recognizing geometric sequences, finding the nth term of a geometric sequence and determining the sum of a geometric series.
4. Understanding the concept of sigma notation and summation index.
5. Understanding the concept of the difference method and when to use it.
6. Recognizing the difference between finite and infinite series.

Study Session 5: Theory of Quadratic Equations

Introduction

A quadratic is an expression of the form $ax^2 + bx + c$, where a , b and c are given numbers and $a \neq 0$. The **standard form** of a **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0,$$

where a , b and c are given numbers and $a \neq 0$.

We seek to find the value(s) of x which make the statement true, or to show that there are no such values. Thus, for example, the values $x = 3$ and $x = 2$ satisfy the equation, $x^2 - 5x + 6 = 0$. This is easily checked by substitution.

These values are called the **solutions** of the equation. Linear equations that are written in the standard form, $ax + b = 0$, $a \neq 0$, have one solution. Quadratic equations may have no solution, or, as in the above example, two solutions.

There are two special types of quadratic equations that are best dealt with separately.

Learning Outcomes for Study Session 5

When you have studied this session, you should be able to:

- 5.1. Define a quadratic equation as an expression of the form $ax^2 + bx + c = 0$, where a, b, and c are given numbers and $a \neq 0$.
- 5.2. Recognize the standard form of a quadratic equation and understand its purpose in finding the solutions.
- 5.3. Identify the solutions of a quadratic equation as the values of x that satisfy the equation.
- 5.4. Differentiate between linear equations and quadratic equations, understanding that quadratic equations can have no solution or two solutions.
- 5.5. Solve quadratic equations with no term in x by moving the constant to the other side and solving for x.
- 5.6. Solve quadratic equations with a term in x by factoring or applying the zero-product property to identify the solutions.
- 5.7. Apply the factoring method to solve non-monic quadratic equations, considering cases where the coefficient of x^2 is not equal to one.
- 5.8. Solve quadratic equations with three terms using the factorization method, finding two numbers whose product and sum match the coefficients.

5.1: Quadratic Equations with no term in x

When there is no term in x we can move the constant to the other side.

Example 5.1:

Solve $x^2 - 9 = 0$.

Solution:

$$x^2 - 9 = 0$$

$$x^2 = 9$$

5.2: Quadratic Equations with term in x

$$x = 3 \text{ or } x = -3$$

Example 5.2:

Solve $x^2 - 9x = 0$

In this case, we can write

$$x(x - 9) = 0$$

since the product of the two factors is 0, one or both of the factors is zero,

$$x(x - 9) = 0.$$

so, $x = 0$ or $x - 9 = 0$

hence, the two solutions are $x = 0$ or $x = 9$.

These two methods work just as well when the coefficient of x^2 is not one.

The two previous examples were relatively easy since in the first case it was easy to isolate the unknown while in the second, a common factor enabled the left-hand side to be easily factored.

5.3: Solving Quadratic Equations with Three Terms

We will now deal with the equation $ax^2 + bx + c = 0$ in which neither a nor b nor c are zero.

There are three basic methods of solving such quadratic equations:

- i. By factoring (ii) By completing the square and (iii) By the quadratic formula
- i. Factorization Method:

Example 5.3:

Solve $x^2 - 7x + 12 = 0$ by factorization method

Solution: We factor the left-hand side by finding two numbers whose product is 12 and whose sum is -7. Clearly, -4, -3 are the desired numbers. We can then factor as:

$$x^2 - 7x + 12 = 0$$

$$(x - 4)(x - 3) = 0$$

Since the product of the two factors is zero, one of the factors is zero.

Thus,

$$x - 4 = 0 \text{ or } x - 3 = 0$$

$$\text{so, } x = 4 \text{ or } x = 3$$

The same method can also be applied to non-monic quadratic equations. A non-monic quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. This is the general case.

Thus, $2x^2 + 5x + 3 = 0$ is an example of a non-monic quadratic equation.

Example 5.4:

Solve the equation $2x^2 + 5x + 3 = 0$.

Solution:

Using the factoring method from the module factorization, we multiply 2 and 3 to give 6 and find two numbers that multiply to give 6 and add to give 5. The desired numbers are 2 and 3. We use these numbers to split the middle term and factor in pairs.

$$2x^2 + 5x + 3 = 0$$

$$2x^2 + 2x + 3x + 3 = 0$$

$$2x(x + 1) + 3(x + 1) = 0$$

$$(x + 1)(2x + 3) = 0$$

we can now equate each factor to zero and obtain

$$x + 1 = 0 \text{ or } 2x + 3 = 0$$

$$x = -1 \text{ or } x = -\frac{3}{2}.$$

Exercise:

Solving the equations:

(a) $4x^2 - 20 = 0$ b. $x^2 - x - 12 = 0$ c. $3x^2 + 2x - 8 = 0$

5.4: Completing the Square:

The quadratic equations encountered so far had one or two solutions that were rational. There are many quadratics that have irrational solutions, or in some cases no real solutions at all. For example, it is not easy at all to see how to factor the quadratic $x^2 - 5x - 3 = 0$. Indeed it has no rational solutions. We will see shortly that the solutions are $x = \frac{5+\sqrt{37}}{2}$ and $x = \frac{5-\sqrt{37}}{2}$.

To deal with more general quadratics, we employ a technique known as **completing the square**. Historically, this was the most commonly used method of solution.

Example 5.5:

Solve $x^2 + 2x - 6 = 0$.

Solution:

It is easiest to move the constant term onto the other side first and then complete the square

$$x^2 + 2x - 6 = 0$$

$$x^2 + 2x = 6$$

$$x^2 + 2x + 1 = 7$$

$$(x + 1)^2 = 7$$

we can now take the positive and negative square roots to obtain

$$x + 1 = \sqrt{7} \text{ or } x + 1 = -\sqrt{7}$$

So, $x = -1 + \sqrt{7}$ or $x = -1 - \sqrt{7}$.

Notice that the solutions are irrational, and so this equation could not be easily solved using the factoring method.

Example 5.6:

Solve $x^2 - 6x - 2 = 0$

Solution:

$$x^2 - 6x - 2 = 0$$

$$x^2 - 6x + 9 - 9 - 2 = 0 \text{ (Complete the square)}$$

$$(x - 3)^2 = 11$$

$$x - 3 = \sqrt{11} \text{ or } x - 3 = -\sqrt{11}$$

Hence, $x = 3 + \sqrt{11}$ or $x = 3 - \sqrt{11}$.

Exercise.

Solve $x^2 - 5x - 3 = 0$ by completing the square and also show that $x^2 - 5x + 7 = 0$ has no solutions.

Example 5.7:

Solve $3x^2 - 5x + 1 = 0$.

Solution:

Divide the equation by 3 and shift the constant term to the other side.

$$3x^2 - 5x + 1 = 0$$

$$x^2 - \frac{5}{3}x = -\frac{1}{3}$$

$$x^2 - \frac{5}{3}x + \frac{25}{36} = \frac{25}{36} - \frac{1}{3}$$

$$\left(x - \frac{5}{3}\right)^2 = \frac{13}{36}$$

$$\text{so, } x = \frac{5+\sqrt{13}}{6} \text{ or } x = \frac{5-\sqrt{13}}{6}$$

5.5 :The Quadratic Formula.

The method of completing the square always works. By applying it to the general quadratic equation $ax^2 + bx + c = 0$, we obtain the well-known **quadratic formula**.

Applying the method of completing the square to the aforementioned general form of a quadratic equation yields

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The last formula is called the quadratic formula, sometimes written as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If the quantity $b^2 - 4ac = 0$ then there will only be one solution, $x = -\frac{b}{2a}$. In this case, the quadratic will be a perfect square. The quantity $b^2 - 4ac$ plays an important role in the **theory of quadratic equations** and is

called the discriminant. Thus, in summary, when solving $ax^2 + bx + c = 0$, first calculate the discriminant $b^2 - 4ac$. Then

- a. If $b^2 - 4ac$ is negative, then there is no solution
- b. If $b^2 - 4ac$ is positive, then the solutions are $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$
- c. If $b^2 - 4ac$ is zero, then there is only one solution $x = -\frac{b}{2a}$.

Note: if $b^2 - 4ac$ is zero, then the quadratic is a perfect square.

Example 5.8: Solve $x^2 - 10x - 3 = 0$ by using the quadratic formula

Solution:

Here, $a = 1$, $b = -10$ and $c = -3$,

Now, $b^2 - 4ac = 100 + 12 = 112$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{10 \pm \sqrt{112}}{2}$$

$$= \frac{10 \pm 4\sqrt{7}}{2}$$

$$x = 5 + 2\sqrt{7} \text{ or } x = 5 - 2\sqrt{7}$$

Summary

1. A quadratic equation is an expression of the form $ax^2 + bx + c = 0$, where a , b , and c are given numbers and $a \neq 0$.
2. The standard form of a quadratic equation is $ax^2 + bx + c = 0$, which helps in finding the solutions.
3. The solutions of a quadratic equation are the values of x that satisfy the equation.
4. Quadratic equations can have no solution or two solutions, unlike linear equations which have one solution.
5. Quadratic equations with no term in x can be solved by moving the constant to the other side.
6. Quadratic equations with a term in x can be solved by factoring or using the zero-product property.
7. Non-monic quadratic equations (coefficients of x^2 not equal to one) can also be solved using the factoring method.

Study Session 6: Binomial Theorem.

Introduction

An algebraic expression containing two terms is called a binomial expression, **Bi** means **two** and **nom** means **term**. Thus, the general type of a binomial is $a + b$, $x - 2$, $3x + 4$, etc. The expression of a binomial raised to a small positive power can be solved by ordinary multiplication, but for large power the actual multiplication is laborious and for fractional power actual multiplication is not possible. By means of a binomial theorem, this work is reduced to a shorter form. This theorem was first established by Sir Isaac Newton.

Learning Outcomes for Study Session 6

When you have studied this session, you should be able to:

- 6.1. Understand the concept of a binomial expression and its general form.
- 6.2. Recognize the binomial theorem and its significance in simplifying the multiplication of binomial expressions raised to large powers.
- 6.3. Define the factorial of a positive integer and its properties.
- 6.4. Comprehend the concept of combinations and calculate the number of combinations using the binomial coefficient.
- 6.5. Apply the binomial theorem to expand expressions of the form $a+bn$, where n is a positive integer power.

- 6.6. Identify the middle term in the binomial expansion for even and odd values of n.
- 6.7. Determine the general term in the binomial expansion and its use in finding specific terms or coefficients.
- 6.8. Utilise the binomial series to approximate expressions when x is numerically less than one.
- 6.9. Apply the binomial series to derive approximations for given expressions under specific conditions.

6.1: Factorial of a Positive Integer

If n is a positive integer, the factorial of n denoted by $n!$ is defined as the product of n +ve integers from n to 1. That is

$$n! = n(n - 1)(n - 2) \dots 3 \cdot 2 \cdot 1$$

for example,

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

and

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

one important relationship concerning factorials is that

$$(n + 1)! = (n + 1)n! \quad (1)$$

for instance,

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 5(4 \cdot 3 \cdot 2 \cdot 1)$$

$$5! = 5 \cdot 4!$$

obviously, $1! = 1$ and this permits to define from equation (1)

$$n! = \frac{(n+1)!}{(n+1)}$$

substituting 0 for n , we obtain

$$0! = \frac{(0+1)!}{(0+1)} = \frac{1!}{1} = \frac{1}{1}$$

$$0! = 1$$

6.2: Combination

Each of the groups or selections which can be made out of a given number of things by taking some or all of them at a time is called combination. In combination the order in which things occur is not considered e.g., combination of a, b, c taken two at a time are ab, bc, ca .

The number of the combination of n different objects taken r at time is denoted by $\binom{n}{r}$ and defined as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For example,

$$\begin{aligned}\binom{6}{4} &= \frac{6!}{4!(6-4)!} \\ &= \frac{6 \times 5 \times 4}{4! \times 2!} = \frac{6 \times 5 \times 4!}{4! \times 2!} = 15\end{aligned}$$

Example 6.1:

Expand $\binom{7}{3}$

Solution:

$$\begin{aligned} \binom{7}{3} &= \frac{7!}{3!(7-3)!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 1 \cdot 4!} = 35. \end{aligned}$$

Some important results:

$$\begin{aligned} (\text{i}) \quad \binom{n}{0} &= \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1 \\ (\text{ii}) \quad \binom{n}{n} &= \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 0!} = \frac{n!}{n! \times 1} = 1 \\ (\text{iii}) \quad \binom{n}{r} &= \binom{n}{n-r} \end{aligned}$$

Note: that the number $\binom{n}{r}$ is called the binomial coefficient.

6.3: The Binomial Theorem

The rule or formula for expansion of $(a + b)^n$, where n is any positive integer power, is called binomial theorem.

For any positive integer n

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}b^n$$

or briefly

$$(a + b)^n = \sum_{r=0}^n \left(\frac{n}{r}\right) a^{n-r} b^r$$

Remarks: The coefficients of the successive terms are $\left(\frac{n}{0}\right)$, $\left(\frac{n}{1}\right)$, $\left(\frac{n}{2}\right)$, ..., $\left(\frac{n}{r}\right)$, ..., $\left(\frac{n}{n}\right)$ are called Binomial coefficients.

Note: Sum of binomial coefficients is 2^n .

$$(a + b)^n = a^n + \frac{n}{1!} a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^{2+\dots+\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}} a^{n-r} b^r + \dots + b^n$$

Note: Since

$$\left(\frac{n}{r}\right) = \frac{n!}{r!(n-r)!}$$

so,

$$\left(\frac{n}{0}\right) = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$$

$$\left(\frac{n}{1}\right) = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{1!(n-1)!} = \frac{n(n-1)}{2!}$$

$$\left(\frac{n}{2}\right) = \frac{n!}{2!(n-2)!} = \frac{n(n-2)!}{2!(n-2)!} = \frac{n(n-1)}{2!}$$

$$\left(\frac{n}{3}\right) = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

$$\left(\frac{n}{r}\right) = \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)!}{r!(n-r)!}$$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 1} = 1.$$

The following points can be observed in the expansion of $(a + b)^n$

- (i) There are $(n + 1)$ terms in the expansion
- (ii) The 1st term is a^n and $(n + 1)$ th term or the last term is b^n
- (iii) The exponent of a decreases from n to zero
- (iv) The exponent of b increases from 0 to n .
- (v) The sum of the exponents of a and b in any term is equal to the index n .
- (vi) The co-efficient of the term equidistant from the beginning and end of the expansion are equal as

$$\binom{n}{r} = \binom{n}{n-r}.$$

6.4: General Term.

The term $\binom{n}{r}a^{n-r}b^r$ in the expansion of binomial theorem is called the **General Term** or $(r + 1)$ th term. It is denoted by T_{r+1} . Hence,

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r$$

Note that the General Term is used to find out the specified term or the required co-efficient of the term in the binomial expansion.

Example 6.2:

$$\begin{aligned}
 \text{Expand } (x + y)^4 &= x^4 + \left(\frac{4}{1}\right)x^{4-1}y + \left(\frac{4}{2}\right)x^{4-2}y^2 + \left(\frac{4}{3}\right)x^{4-3}y^3 + y^4 \\
 &= x^4 + 4x^3y + \frac{4 \times 3}{2 \times 1}x^2y^2 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1}xy^3 + y^4 \\
 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
 \end{aligned}$$

Example 6.3:

Expand by binomial theorem $\left(a - \frac{1}{a}\right)^6$

Solution:

$$\begin{aligned}
 \left(a - \frac{1}{a}\right)^6 &= a^6 + \left(\frac{6}{1}\right)a^{6-1}\left(-\frac{1}{a}\right)^1 + \left(\frac{6}{2}\right)a^{6-2}\left(-\frac{1}{a}\right)^2 + \left(\frac{6}{3}\right)a^{6-3}\left(-\frac{1}{a}\right)^3 + \left(\frac{6}{4}\right)a^{6-4}\left(-\frac{1}{a}\right)^4 \\
 &\quad + \left(\frac{6}{5}\right)a^{6-5}\left(-\frac{1}{a}\right)^5 + \left(\frac{6}{6}\right)a^{6-6}\left(-\frac{1}{a}\right)^6 \\
 &= a^6 + 6a^5\left(-\frac{1}{2}\right) + \frac{6 \times 5}{2 \times 1}a^4\left(-\frac{1}{a^2}\right) + \frac{6 \times 5 \times 4}{3 \times 2 \times 1}a^3\left(-\frac{1}{a^3}\right) + \frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1}a^2\left(-\frac{1}{a^4}\right) \\
 &= \frac{6 \times 5 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2 \times 1}a^1\left(-\frac{1}{a^5}\right) + \left(-\frac{1}{a^6}\right) \\
 &= a^6 - 6a^4 + 15a^2 - 20 + \frac{15}{a^2} - \frac{6}{a^5} + \frac{1}{a^6}
 \end{aligned}$$

Example 6.4:

$$\text{Expand } \left(\frac{x^2}{2} - \frac{2}{x} \right)^4$$

Solution:

$$\begin{aligned}
\left(\frac{x^2}{2} - \frac{2}{x} \right)^4 &= \left(\frac{x^2}{2} \right)^4 + \left(\frac{4}{1} \right) \left(\frac{x^2}{2} \right)^{4-1} \left(-\frac{2}{x} \right)^1 + \left(\frac{4}{2} \right) \left(\frac{x^2}{2} \right)^{4-2} \left(-\frac{2}{x} \right)^2 + \left(\frac{4}{3} \right) \left(\frac{x^2}{2} \right)^{4-3} \left(-\frac{2}{x} \right)^3 \\
&\quad + \left(\frac{4}{4} \right) \left(\frac{x^2}{2} \right)^{4-4} \left(-\frac{2}{x} \right)^4 \\
&= \frac{x^8}{16} + 4 \left(\frac{x^2}{2} \right)^3 \left(-\frac{2}{x} \right) + \frac{4 \times 3}{2 \times 1} \left(\frac{x^2}{2} \right)^2 \left(-\frac{4}{x^2} \right) + \frac{4 \times 3 \times 2}{3 \times 2 \times 1} \left(\frac{x^2}{2} \right)^1 \left(-\frac{4}{x^3} \right) + \frac{16}{x^4} \\
&= \frac{x^8}{16} - 4 \cdot \frac{x^8}{8} \frac{2}{x} + 6 \cdot \frac{x^4}{4} \frac{4}{x^2} - 4 \cdot \frac{x^2}{2} \frac{8}{x^3} + \frac{16}{x^4} \\
&= \frac{x^8}{16} - x^2 + 6x^2 - \frac{16}{x} + \frac{16}{x^4}
\end{aligned}$$

Example 6.5:

Expand $(1.04)^5$ by the binomial formula and find its value to two decimal places.

Solution:

$$(1.04)^5 = (1 + 0.04)^5$$

$$(1 + 0.04)^5 = (1)^5 + \left(\frac{5}{1} \right) (1)^{5-1} (0.04) + \left(\frac{5}{2} \right) (1)^{5-2} (0.04)^2 + \left(\frac{5}{3} \right) (1)^{5-3} (0.04)^3 +$$

$$\begin{aligned}
 & \left(\frac{5}{4}\right)(1)^{5-4}(0.04)^4 + (0.04)^5 \\
 &= 1 + 0.2 + 0.016 + 0.00064 + 0.000128 + 0.0000001024 \\
 &= 1.22
 \end{aligned}$$

Example 6.6:

Find the eighth term in the expansion of $\left(2x^2 - \frac{1}{x^2}\right)^{12}$

The general term is, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here, $T_8 = ?$, $a = 2x^2$, $b = -\frac{1}{x^2}$, $n = 12$, $r = 7$.

Therefore,

$$\begin{aligned}
 T_{7+1} &= \binom{12}{7} \left(2x^2\right)^{12-7} \left(-\frac{1}{x^2}\right)^7 \\
 T_8 &= \frac{12.11.10.9.8.7.6}{7.6.5.4.3.2.1} \left(2x^2\right)^5 \frac{(-1)^7}{x^{14}} \\
 &= 793 \times 32x^{10} \frac{(-1)}{x^{14}} \\
 &= -\frac{25344}{x^4}
 \end{aligned}$$

$$\therefore T_8 = -\frac{25344}{x^4}$$

6.5: Middle Term in the Expansion $(a + b)^n$

In the expansion of $(a + b)^n$, there are $(n + 1)$ terms.

Case I: If n is even then $(n + 1)$ will be odd, so $\left(\frac{n}{2} + 1\right)$ th term will be the only one middle term in the expansion. For example, if $n = 8$ (even), number of terms will be 9 (odd), therefore, $\left(\frac{8}{2} + 1\right) = 5^{\text{th}}$ will be the middle term.

Case II: If n is odd then $(n + 1)$ will be even, in this case there will not be a single middle term, but $\left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+1}{2} + 1\right)$ th term will be the two middle terms in the expansion. For example, for $n = 9$ (odd), number of terms is 10 i.e., $\left(\frac{9+1}{2}\right)$ th and $\left(\frac{9+1}{2} + 1\right)$ th i.e. 5th and 6th terms are taken as middle terms and these middle terms are found by using the general term.

Example 6.7:

Find the middle term of $\left(1 - \frac{x^2}{2}\right)^{14}$

Solution: we have $n = 14$, then number of terms is 15.

$\therefore \left(\frac{14}{2} + 1\right)$ i.e., 8th will be the middle term.

$$a = 1, b = -\frac{x^2}{2}, n = 14, r = 7, T_8 = ?$$

$$\begin{aligned}
 T_{r+1} &= \left(\frac{n}{r}\right) a^{n-r} b^r \\
 T_{r+1} &= \left(\frac{14}{7}\right) (1)^{14-7} \left(-\frac{x^2}{2}\right)^7 = \frac{14!}{7!7!} (-1)^7 \frac{x^{14}}{2^7} \\
 &= \frac{14.13.12.11.10.9.8.7!}{7.6.5.4.3.2.1 7!} \frac{(-1)}{128} x^{14} \\
 &= - (2)(13)(11)(2)(3) \frac{1}{128} x^{14} \\
 \therefore T_8 &= - \frac{429}{16} x^{14}
 \end{aligned}$$

Example 6.8:

Find the coefficient of x^{19} in $(2x^3 - 3x)^9$.

Solution: Here, $a = 2x^5, b = -3x, n = 9$

first, we find r .

Since,

$$\begin{aligned}
 T_r + 1 &= \left(\frac{n}{r}\right) a^{n-r} b^r \\
 &= \left(\frac{9}{r}\right) (2x^3)^{9-r} (-3x)^r \\
 &= \left(\frac{9}{r}\right) 2^{9-r} (-3)^r x^{27-3r} x^r \\
 &= \left(\frac{9}{r}\right) 2^{9-r} (-3)^r x^{27-2r}
 \end{aligned} \tag{1}$$

But we require x^{19} , so put

$$19 = 27 - 2r$$

$$2r = 8$$

$$r = 4$$

Plugging the value of r in equation (1) gives

$$\begin{aligned} T_{4+1} &= \left(\frac{9}{4}\right) 2^{9-4} (-3)^4 x^{19} \\ &= \frac{9.8.7.6.5}{4.3.2.1} 2^5 \cdot 3^4 \cdot x^{19} \\ &= 630 \times 32 \times 81 \times x^{19} \\ \therefore T_5 &= 1632960 x^{19} \end{aligned}$$

Example 6.9:

Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$

Solution:

We have $a = 2x^2$, $b = \frac{1}{x}$, $n = 9$

$$T_r + 1 = \binom{n}{r} a^{n-r} b^r$$

$$\begin{aligned}
 &= \binom{9}{r} (2x^2)^{9-r} \left(\frac{1}{x}\right)^r \\
 &= \binom{9}{r} 2^{9-r} \cdot x^{18-2r} \cdot x^r
 \end{aligned} \tag{2}$$

Since T_{r+1} is the term independent of x i.e., x^0 thus the power of x must be zero.

i.e., $18 - 3r = 0$ thus $r = 6$.

Substitute $r = 6$ into equation (2)

$$\begin{aligned}
 T_{r+1} &= \binom{9}{6} 2^{9-6} x^0 = \frac{9!}{6!3!} \cdot 1 \\
 &= \frac{9.8.7.6!}{6!3.2.1} \cdot 8 \cdot 1 = 672
 \end{aligned}$$

Exercise:

1. Expand the following by the binomial formula

(a) $\left(x + \frac{1}{x}\right)^4$ b. $\left(\frac{2x}{3} + \frac{3}{2x}\right)^5$ c. $\left(\frac{x}{2} - \frac{2}{y}\right)^4$ d. $(2x - y)^5$ e. $\left(-x + y^{-1}\right)^4$

2. Compute to two decimal places of decimal by use of the binomial formula

(a) $(1.02)^4$ b. $(0.98)^6$ c. $(2.03)^5$

3. Find the value of

(a) $(x + y)^5 + (x - y)^5$ b. $(x + \sqrt{2})^4 - (x - \sqrt{2})^4$

4. Find

- a. The 5th term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{10}$
 - b. The 6th in the expansion of $\left(x^2 + \frac{y}{2}\right)^{15}$
 - c. The 8th in the expansion of $\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right)^{12}$
5. Find the middle term of the following expansions:
- a. $\left(3x^2 + \frac{1}{2x}\right)^{10}$ b. $\left(\frac{a}{2} - \frac{b}{3}\right)^{11}$ c. $\left(2x + \frac{1}{x}\right)^7$
6. Find the coefficient of
- a. x^5 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{10}$
 - b. x^{20} in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{10}$
 - c. x^5 in the expansion of $\left(2x^2 - \frac{1}{3x}\right)^{10}$
7. Find the constant term in the expansion of
- a. $\left(x^2 - \frac{1}{x}\right)^9$ b. $\left(\sqrt{x} + \frac{1}{3x^2}\right)^{10}$

6.6: Binomial Series.

Since by the Binomial formula for positive integers n , we have

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots + b^n \quad (1)$$

Put $a = 1$ and $b = x$, then the above form in (1) becomes:

$$(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \dots + x^n$$

if n is a negative or a fractional number (-ve or +ve), then

$$(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \dots \infty \quad (2)$$

The series on the right-hand side of (2) is called the **Binomial Series**. This series is valid only when x is numerically less than one. That is, $|x| < 1$ otherwise the expression will not be valid.

Note: The first term in the expression must be one. For example, when n is not a positive integer (negative or fraction) to expand $(a + x)^n$, we shall have to write it as

$$(a + x)^n = a^n \left(1 + \frac{x}{a}\right)^n$$

and then apply the binomial series, where $\left|\frac{x}{a}\right|$ must be less than one.

6.7: Application of the Binomial Series; Approximations

The binomial series can be used to find expressions approximately equal to the given expressions under given conditions.

Example 6.10:

If x is very small, so that its square and higher powers can be neglected then prove that

$$\frac{1+x}{1-x} = 1 + 2x$$

Solution:

$\frac{1+x}{1-x}$ can be written as $(1 + x)(1 - x)^{-1}$

$$\begin{aligned}\frac{1+x}{1-x} &= (1 + x)(1 + x + x^2 + \dots \text{higher powers of } x) \\ &= 1 + x + x + \text{neglecting higher powers of } x \\ &= 1 + 2x\end{aligned}$$

Example 6.11:

Find to 4 decimal places the value of $(1.02)^8$

Solution:

$$\begin{aligned}(1 + 0.02)^8 &= 1 + \frac{8}{1}(0.02) + \frac{8.7}{2.1}(0.02)^2 + \frac{8.7.6}{3.2.1}(0.02)^3 + \dots \\ &= 1 + 0.16 + 0.0112 + 0.000448 + \dots \\ &= 1.1716\end{aligned}$$

Example 6.12:

Write and simplify the first four terms in the expansion of $(1 - 2x)^{-1}$.

Solution:

$$(1 - 2x)^{-1} = [1 + (-2x)]^{-1}$$

using $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

$$= 1 + (-1)(-2x) + \frac{(-1)(-1-1)}{2!}(-2x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-2x)^3 + \dots$$

$$= 1 + 2x + \frac{(-1)(-2)}{2.1}4x^2 + \frac{(-1)(-2)(-3)}{3.2.1}(-8x^3) + \dots$$

$$= 1 + 2x + 4x^2 + 8x^3 + \dots$$

Example 6.13:

Write the first three terms in the expansion of $(2 + x)^{-3}$

Solution:

$$\begin{aligned} (2 + x)^{-3} &= (2)^{-3}\left(1 + \frac{x}{2}\right)^{-3} \\ &= (2)^{-3}\left[1 + (-3)\left(\frac{x}{2}\right) + \frac{(-3)(-3-1)}{2!}\left(\frac{x}{2}\right)^2 + \dots\right] \\ &= \frac{1}{8}\left[1 - \frac{3}{2}x + 3x^2 + \dots\right] \end{aligned}$$

6.8: Root Extraction

The second application of the binomial series is that of finding the root of any quantity.

Example 6.14:

Find the square root of 24 correct to 5 decimal places.

Solution:

$$\begin{aligned}
 \sqrt{24} &= (25 - 1)^{\frac{1}{2}} \\
 &= (25)^{\frac{1}{2}} \left(1 - \frac{1}{25}\right)^{\frac{1}{2}} \\
 &= 5 \left(1 - \frac{1}{5^2}\right)^{\frac{1}{2}} \\
 &= 5 \left[1 + \frac{1}{2} \left(-\frac{1}{5^2}\right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \left(-\frac{1}{5^2}\right)^2 + \dots\right] \\
 &= 5 \left[1 - \frac{1}{2.5^2} - \frac{1}{2^3.5^4} - \frac{1}{2^4.5^6} - \dots\right] \\
 &= 5[1 - (0.02 + 0.002 + 0.00004\dots)] \\
 &= 4.89898
 \end{aligned}$$

Example 6.15:

Evaluate $\sqrt[3]{29}$ to the nearest hundredth.

Solution:

$$\sqrt[3]{29} = (27 + 2)^{\frac{1}{3}} = \left[27 \left(1 + \frac{2}{27}\right)\right]^{\frac{1}{3}} = 3 \left[\left(1 + \frac{2}{27}\right)\right]^{\frac{1}{3}} + \dots$$

$$\begin{aligned}
 \sqrt[3]{29} &= 3 \left[1 + \frac{1}{3} \left(\frac{2}{27} \right) + \frac{\frac{1}{3} \left(\frac{1}{3} - 1 \right)}{2.1} \left(\frac{2}{27} \right)^2 + \dots \right] \\
 &= 3 \left[1 + \frac{2}{81} + \frac{1}{2} \left(\frac{1}{3} \right) \left(-\frac{2}{3} \right) \left(\frac{2}{27} \right)^2 + \dots \right] \\
 &= 3[1 + 0.247 - 0.0006 + \dots] \\
 &= 3(1.0212) = 3.07
 \end{aligned}$$

Exercise:

1. Expand up to four terms

(a) $(1 - 3x)^{\frac{1}{3}}$ b. $(1 - 2x)^{-\frac{3}{4}}$ c. $(1 + x)^{-3}$ d. $\frac{1}{\sqrt{1+x}}$

2. Using the binomial expansion, calculate to the nearest hundredth

(a) $\sqrt[4]{65}$ b. $\sqrt{17}$ c. $(1.01)^{-7}$ d. $\sqrt{28}$

3. Find the coefficient of x^5 in the expansion of

(a) $\frac{(1+x)^2}{(1-x)^2}$ b. $\frac{(1+x)^2}{(1-x)^3}$

Summary

1. Binomial Expression: An algebraic expression containing two terms is called a binomial expression. It consists of two terms, represented as $a+b$, $x-2$, $3x+4$, etc. Solving binomial expressions raised to small positive powers can be done through

ordinary multiplication, but for large powers or fractional powers, the binomial theorem is used to simplify the calculations.

2. Factorial of a Positive Integer: The factorial of a positive integer n , denoted as $n!$, is the product of all positive integers from n to 1. It follows the formula $n! = n(n-1)(n-2)\dots 3.2.1$. The factorial of 0 is defined as 1. Factorials have important relationships, such as $(n+1)! = (n+1)n!$.

3. Combination: Combination refers to the groups or selections that can be made from a given number of objects, taking some or all of them at a time. The order of the objects is not considered in combinations. The number of combinations of n different objects taken r at a time is denoted as ${}^n r$ and calculated using the formula ${}^n r = n! / (r!(n-r)!)$.

4. The Binomial Theorem: The binomial theorem is a rule or formula for expanding expressions of the form $a+bn$, where n is any positive integer power. It states that the expansion consists of $(n+1)$ terms and follows the pattern of coefficients multiplied by powers of a and b . The coefficients are known as binomial coefficients and can be calculated using the formula ${}^n r = n! / (r!(n-r)!)$.

5. General Term: The general term or $(r+1)$ th term in the binomial expansion is denoted as T_{r+1} and can be calculated using the formula $T_{r+1} = {}^n r * a^r * b^{n-r}$, where n is the power, r is the term number, and ${}^n r$ represents the binomial coefficient.

6. Middle Term: In the expansion of a binomial expression, the middle term(s) depends on whether the power n is even or odd. If n is even, there is a single middle term, while if n is odd, there are two middle terms.

7. Binomial Series: The binomial series is a series obtained by applying the binomial theorem when n is a negative or fractional number. It represents an approximation of expressions with powers of x , where x is numerically less than one.

Study Session 7: Complex Numbers

Introduction

The history of complex numbers goes back to the ancient Greeks who decided (but were perplexed) that no number existed that satisfies

$$x^2 = -1$$

For example, Diophantus (about 275AD) attempted to solve what seems a reasonable problem, namely

'Find the sides of a right-angled triangle of perimeter 12 units and area 7 squared units.'

Letting $AB = x$, $AC = h$ as shown,

then area = $\frac{1}{2}xh$

and perimeter = $x + h + x^2 + h^2$

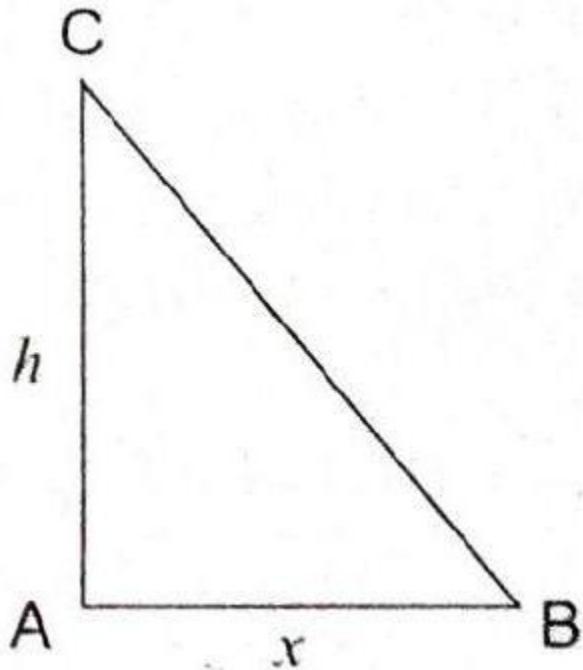


Figure 7.1: Right Angle Triangle

Show that the two equations above reduce to

$$6x^2 - 43x + 84 = 0$$

when perimeter = 12 and area = 7. Does this have real solutions?

A similar problem was posed by Cardan in 1545. He tried to solve the problem of finding two numbers, a and b , whose sum is 10 and whose product is 40;

i.e.

$$a + b = 10 \quad (1)$$

$$ab = 40 \quad (2)$$

eliminating b gives

$$a(10 - a) = 40$$

or

$$a^2 - 10a + 40 = 0.$$

Solving this quadratic gives

$$a = \frac{1}{2}(10 \pm -60) = 5 \pm -15$$

This shows that there are no real solutions, but if it is agreed to continue using the numbers

$$a = 5 + \sqrt{-15}, b = 5 - \sqrt{-15}$$

then equations (1) and (2) are satisfied.

Show that equations (1) and (2) are satisfied by these values of x and y .

So, these are solutions of the original problem but they are not real numbers. Surprisingly, it was not until the nineteenth century that such solutions were fully understood.

The square root of -1 is denoted by i , so that

$$i = \sqrt{-1}$$

and

$$a = 5 + \sqrt{15}i, b = 5 - \sqrt{15}i$$

are examples of **complex numbers**.

Learning Outcomes for Study Session 7

At the end of this learning session, students should be able to:

- 7.1. Solve quadratic equations involving complex numbers and determine their solutions.
- 7.2. Perform operations with complex numbers, including addition, subtraction, multiplication, and division.
- 7.3. Represent complex numbers graphically on an Argand diagram.
- 7.4. Express complex numbers in polar coordinates and convert between polar and rectangular forms.
- 7.5. Use the modulus and argument of a complex number to describe its length and direction in the complex plane.

Example 7.1:

Solve $x^2 - 2x + 2 = 0$

Solution

Using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \Rightarrow x = \frac{2 \pm \sqrt{-4}}{2}$$

But

$$\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1} = 2i$$

(Using the definition of i).

$$\text{Therefore, } x = \frac{2 \pm 2i}{2}$$

$$\Rightarrow x = 1 \pm i$$

Therefore, the two solutions are

$$x = 1 + i \text{ and } x = 1 - i$$

If the quadratic equation is expressed as $ax^2 + bx + c = 0$, then the expression that determines the type of solution is $b^2 - 4ac$, called the discriminant.

In a quadratic equation $ax^2 + bx + c = 0$, if:

$b^2 - 4ac > 0$ then solutions are real and different $b^2 - 4ac = 0$

then solutions are real and equal $b^2 - 4ac < 0$ then solutions are complex

7.1: Algebra of Complex Numbers

A number such as $3 + 4i$ is called a complex number. It is the sum of two terms (each of which may be zero).

The real term (not containing i) is called the real part and the coefficient of i is the imaginary part. Therefore, the real part of $3 + 4i$ is 3 and the imaginary part is 4.

A number is real when the coefficient of i is zero and is imaginary when the real part is zero.

e.g., $3 + 0i = 3$ is real and $0 + 4i = 4i$ is imaginary.

Having introduced a complex number, the ways in which they can be combined, i.e. addition, multiplication, division etc., need to be defined. This is termed the algebra of complex numbers. You will see that, in general, you proceed as in real numbers, but using

$$i^2 = -1$$

where appropriate.

But first equality of complex numbers must be defined.

If two complex numbers, say

$$a + bi, c + di$$

are equal, then both their real and imaginary parts are equal;

$$a + bi = c + di \Rightarrow a = c \text{ and } b = d$$

7.2: Addition and subtraction

Addition of complex numbers is defined by separately adding real and imaginary parts; so if

$$z = a + bi, w = c + di$$

then

$$z + w = (a + c) + (b + d)i.$$

Similarly, for subtraction.

Example 7.2:

Express each of the following in the form $x + yi$.

- a. $(3 + 5i) + (2 - 3i)$
- b. $(3 + 5i) + 6$
- c. $(3 + 5i) + 6$
- d. $7i - (4 + 5i)$

Solution

- a. $(3 + 5i) + (2 - 3i) = 3 + 2 + (5 - 3)i = 5 + 2i$
- b. $(3 + 5i) + 6 = 9 + 5i$
- c. $7i - (4 + 5i) = 7i - 4 - 5i = -4 + 2i$

7.3: Multiplication

Multiplication is straightforward provided you remember that $i^2 = -1$.

Example 7.3:

Simplify in the form $x + yi$:

- a. $3(2 + 4i)$
- b. $(5 + 3i)i$
- c. $(2 - 7i)(3 + 4i)$

Solution

- a. $3(2 + 4i) = 3(2) + 3(4i) = 6 + 12i$
- b. $(5 + 3i)i = (5)i + (3i)i = 5i + 3(i^2) = 5i + (-1)3 = -3 + 5i$
- c. $(2 - 7i)(3 + 4i) = (2)(3) - (7i)(3) + (2)(4i) - (7i)(4i)$
 $= 6 - 21i + 8i - (-28) = 6 - 21i + 8i + 28 = 34 - 13i$

In general, if

$$z = a + bi, w = c + di,$$

then

$$zw = (a + bi)(c + di) = ac - bd + (ad + bc)i$$

Simplify the following expressions:

- a. $(2 + 6i) + (9 - 2i)$
- b. $(8 - 3i) - (1 + 5i)$
- c. $3(7 - 3i) + i(2 + 2i)$
- d. $(3 + 5i)(1 - 4i)$
- e. $(5 + 12i)(6 + 7i)$
- f. $(2 + i)^2$
- g. i^3
- h. i^4
- i. $(1 - i)^3$
- j. $(1 + i)^2 + (1 - i)^2$
- k. $(2 + i)^4 + (2 - i)^4$
- l. $(a + ib)(a - ib)$

7.4: Division

The complex conjugate of a complex number is obtained by changing the sign of the imaginary part. So, if $z = a + bi$, its complex conjugate, \bar{z} , is defined by

$$\bar{z} = a - bi.$$

Note: an alternative notation often used for the complex conjugate is z^* .

Any complex number $a + bi$ has a complex conjugate $a - bi$ and from Activity 5 it can be seen that $(a + bi)(a - bi)$ is a real number. This fact is used in simplifying expressions where the denominator of a quotient is complex.

Example 7.4:

Simplify the following expressions:

- a. $\frac{1}{i}$
- b. $\frac{3}{1+i}$
- c. $\frac{4+7i}{2+5i}$

Solution

To simplify these expressions, you multiply the numerator and denominator of the quotient by the complex conjugate of the denominator.

- a. The complex conjugate of i is $-i$, therefore

$$\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{(1)(-i)}{(i)(-i)} = \frac{-i}{-(-1)} = -i$$

b. The complex conjugate of $1 + i$ is $1 - i$, therefore

$$\frac{3}{1+i} = \frac{3}{1+i} \times \frac{1-i}{1-i} = \frac{3(1-i)}{(1+i)(1-i)} = \frac{3-3i}{2} = \frac{3}{2} - \frac{3}{2}i$$

c. The complex conjugate of $2 + 5i$ is $2 - 5i$ therefore

$$\frac{4+7i}{2+5i} = \frac{4+7i}{2+5i} \times \frac{2-5i}{2-5i} = \frac{43-6i}{29} = \frac{43}{29} - \frac{6}{29}i$$

Exercise

Simplify to the form $a + ib$

- a. $\frac{4}{i}$
- b. $\frac{1-i}{1+i}$
- c. $\frac{4+5i}{6-5i}$
- d. $\frac{4i}{(1+2i)^2}$

Just as you can have equations with real numbers, you can have

equations with complex numbers, as illustrated in the example below.

Example 7.5:

Solve each of the following equations for the complex number

z .

a. $4 + 5i = z - (1 - i)$

b. $(1 + 2i)z = 2 + 5i$

Solution

a. Writing $z = x + iy$,

$$4 + 5i = (x + yi) - (1 - i) \quad 4 + 5i = x - 1 + (y + 1)i$$

Comparing real parts

$$\Rightarrow 4 = x - 1, x = 5$$

Comparing imaginary parts $\Rightarrow 5 = y + 1, y = 4$

So $z = 5 + 4i$. In fact there is no need to introduce the real and imaginary parts of z , since

$$4 + 5i = z - (1 - i)$$

$$\Rightarrow z = 4 + 5i + (1 - i)$$

$$\Rightarrow z = 5 + 4i$$

b. $(1 + 2i)z = 2 + 5i$

$$z = \frac{2+5i}{1+2i}$$

$$z = \frac{2+5i}{1+2i} \times \frac{1-2i}{1-2i} \quad z = \frac{12+i}{5} = \frac{12}{5} + \frac{1}{5}i$$

Exercise

a. Solve the following equations for real x and y

- (i) $3 + 5i + x - yi = 6 - 2i$
(ii) $x + yi = (1 - i)(2 + 8i)$.

b. Determine the complex number z which satisfies $z(3 + 3i) = 2 - i$.

Exercise

1 Solve the equations:

- a. $x^2 + 9 = 0$
b. $9x^2 + 25 = 0$
c. $x^2 + 2x + 2 = 0$
d. $x^2 + x + 1 = 0$
e. $2x^2 + 3x + 2 = 0$

2 Find the quadratic equation which has roots $2 \pm \sqrt{3}i$

3 Write the following complex numbers in the form $x + yi$.

- a. $(3 + 2i) + (2 + 4i)$
b. $(4 + 3i) - (2 + 5i)$
c. $(4 + 3i) + (4 - 3i)$
d. $(2 + 7i) - (2 - 7i)$

e. $(3 + 2i)(4 - 3i)$

f. $(3 + 2i)^2$

g. $(1 + i)(1 - i)(2 + i)$

4 Find the value of the real number y such that $(3 + 2i)(1 + iy)$ is a. real b. imaginary.

5 Simplify:

a. i^3

b. i^4

c. $\frac{1}{i}$

d. $\frac{1}{i^2}$

e. $\frac{1}{i^3}$

6 If $z = 1 + 2i$, find

a. z^2

b. $\frac{1}{z}$

c. $\frac{1}{z^2}$

7 Write in the form $x + yi$:

a. $\frac{2+3i}{1+i}$

b. $\frac{-4+3i}{-2-i}$

c. $\frac{4i}{2-i}$

d. $\frac{1}{2+3i}$

e. $\frac{3-2i}{i}$

f. $\frac{p+qi}{r+si}$

8 Simplify

a. $\frac{(2+i)(3-2i)}{1+i}$

b. $\frac{(1-i)^3}{(2+i)^2}$

c. $\frac{1}{3+i} - \frac{1}{3-i}$

9 Solve for z when

a. $z(2 + i) = 3 - 2i$

b. $(z + i)(1 - i) = 2 + 3i$

c. $\frac{1}{z} + \frac{1}{2-i} = \frac{3}{1+i}$

10 Find the values of the real numbers x and y in each of the following:

a. $\frac{x}{1+i} + \frac{y}{1-2i} = 1$

b. $\frac{x}{2-i} + \frac{yi}{i+3} = \frac{2}{1+i}$ 11. Given that p and q are real and that $1 + 2i$ is a root of the equation
complex numbers u, v and w are related by

$$z^2 + (p + 5i)z + q(2 - i) = 0$$

Determine:

a. the values of p and q ;

b. the other root of the equation.

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$$

Given that $v = 3 + 4i$, $w = 4 - 3i$, find u in the form $x + iy$.

7.5: The Argand Diagram

Any complex number $z = a + bi$ can be represented by an ordered pair (a, b) and hence plotted on xy -axes with the real part measured along the x -axis and the imaginary part along the y -axis. This graphical representation of the complex number field is called an Argand diagram, named after the Swiss mathematician Jean Argand (1768-1822).

Example 7.6:

Represent the following complex numbers on an Argand diagram:

- a. $z = 3 + 2i$ b. $z = 4 - 5i$ c. $z = -2 - i$

Solution

The Argand diagram is shown opposite.

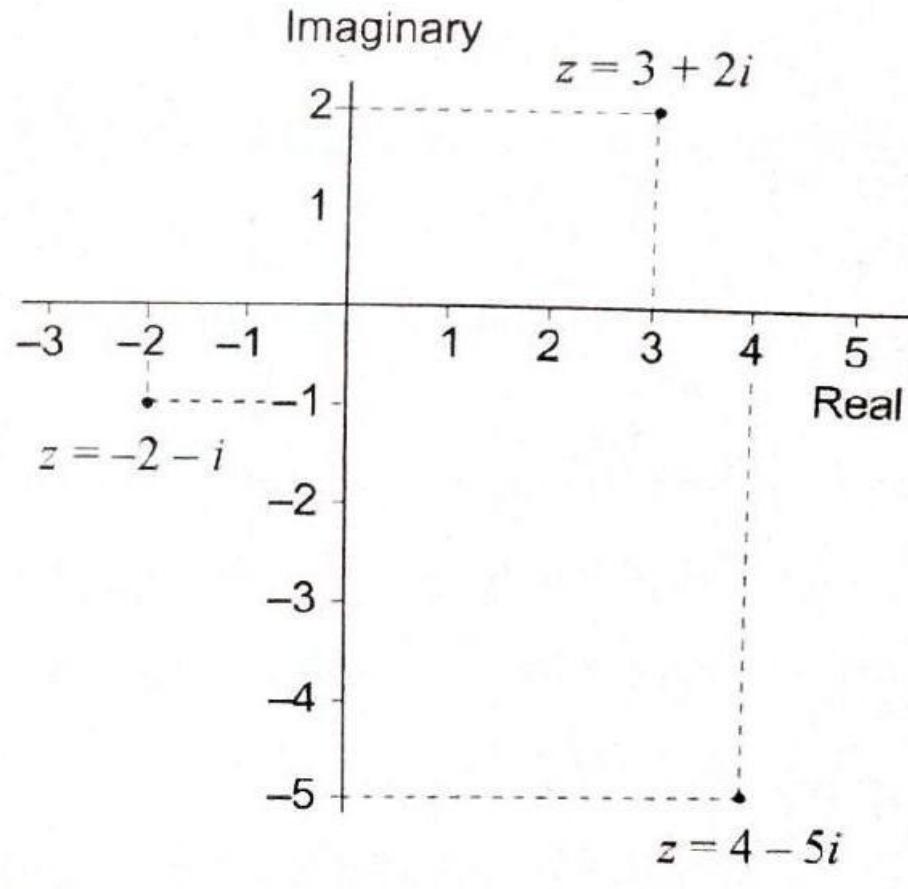


Figure 7.2: Argand Diagram

Let $z_1 = 5 + 2i$, $z_2 = 1 + 3i$, $z_3 = 2 - 3i$, $z_4 = -4 - 7i$.

- Plot the complex numbers z_1, z_2, z_3, z_4 on an Argand diagram and label them.
- Plot the complex numbers $z_1 + z_2$ and $z_1 - z_2$ on the same Argand diagram.
Geometrically, how do the positions of the numbers $z_1 + z_2$ and $z_1 - z_2$ relate to z_1 and z_2 ?

7.6: Polar Coordinates

Consider the complex number $z = 3 + 4i$ as represented on an Argand diagram. The position of A can be expressed as coordinates $(3, 4)$, the cartesian form, or in terms of the length and direction of OA .

Using Pythagoras' theorem, the length of $OA = \sqrt{3^2 + 4^2} = 5$.

This is written as $|z| = r = 5$, $|z|$ is read as the modulus or absolute value of z .

The angle that OA makes with the positive real axis is

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ \text{ (or } 0.927 \text{ radians)}.$$

This is written as $\arg(z) = 53.13^\circ$. You say $\arg(z)$ is the argument or phase of z .

The parameters $|z|$ and $\arg(z)$ are in fact the equivalent of polar coordinates r, θ as shown opposite. There is a simple connection between the polar coordinate form and the cartesian or rectangular form (a, b) :

$$a = r\cos\theta, b = r\sin\theta.$$

Therefore,

$$z = a + bi = r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta)$$

where $|z| = r$, and $\arg(z) = \theta$.

It is more usual to express the angle θ in radians. Note also that it is convention to write the i before $\sin\theta$, i.e., $i\sin\theta$ is preferable to $\sin\theta i$.

In the diagram opposite, the point A could be labelled $(2\sqrt{3}, 2)$ or as $2\sqrt{3} + 2i$.

The angle that OA makes with the positive x -axis is given by

$$\theta = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Therefore, $\theta = \frac{\pi}{6}$ or $2\pi + \frac{\pi}{6}$ or $4\pi + \frac{\pi}{6}$ or ... etc. There are an infinite number of possible angles. The one you should normally use is in the interval $-\pi < \theta \leq \pi$, and this is called the principal argument.

Figure 7.3: Polar Form

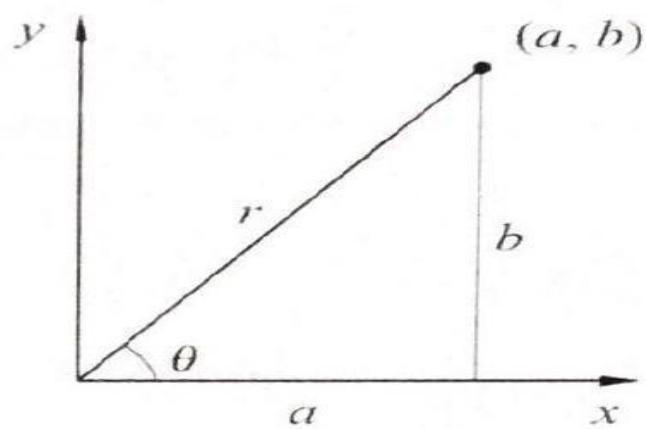
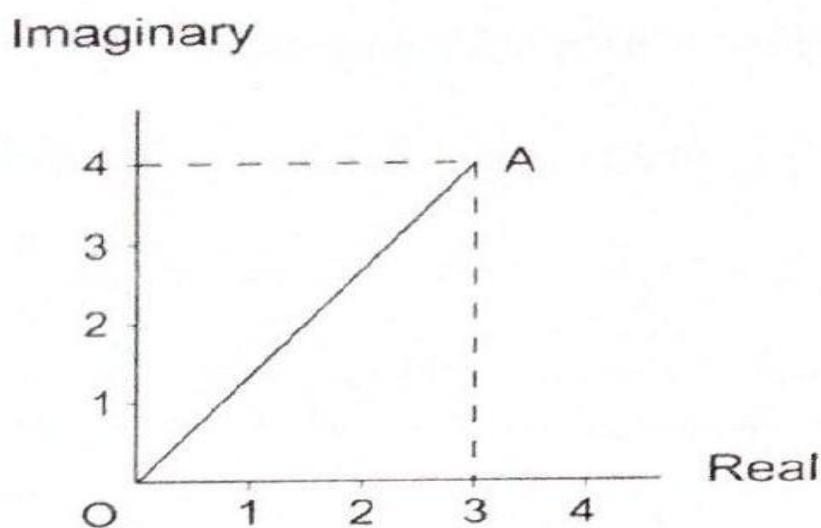
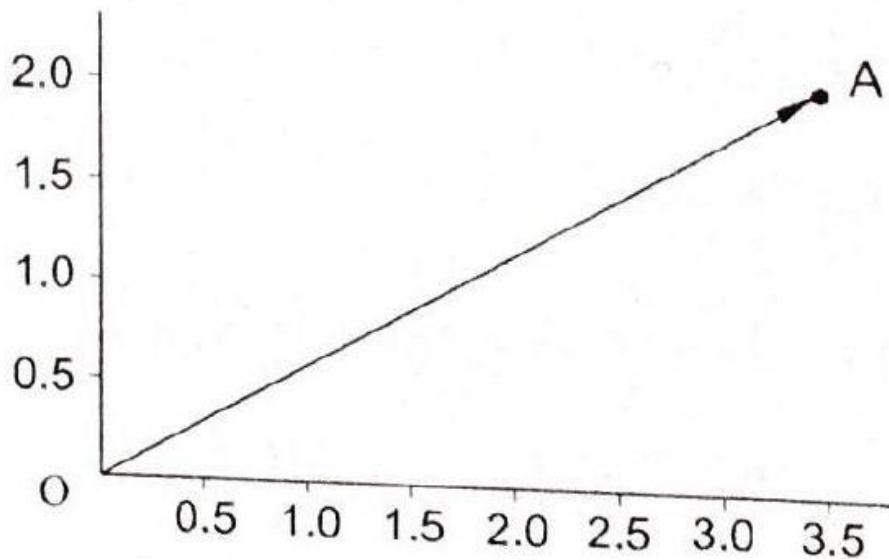


Figure 7.4: Polar Coordinate



Using polar coordinates, the point A could be labelled with its polar coordinates $[r, \theta]$ as $\left[4, \frac{\pi}{6}\right]$. Note the use of squared brackets when using polar coordinates. This is to avoid confusion with Cartesian coordinates.

$$\text{Thus } 2\sqrt{3} + 2i = 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

Important note: if you are expressing $a + ib$ in its polar form, where a and b are both positive, then the formula $\theta = \tan^{-1} \frac{b}{a}$ is quite sufficient. But in other cases, you need to think about the position of $a + ib$ in the Argand diagram.

Example 7.7:

Write $z = -1 - i$ in polar form.

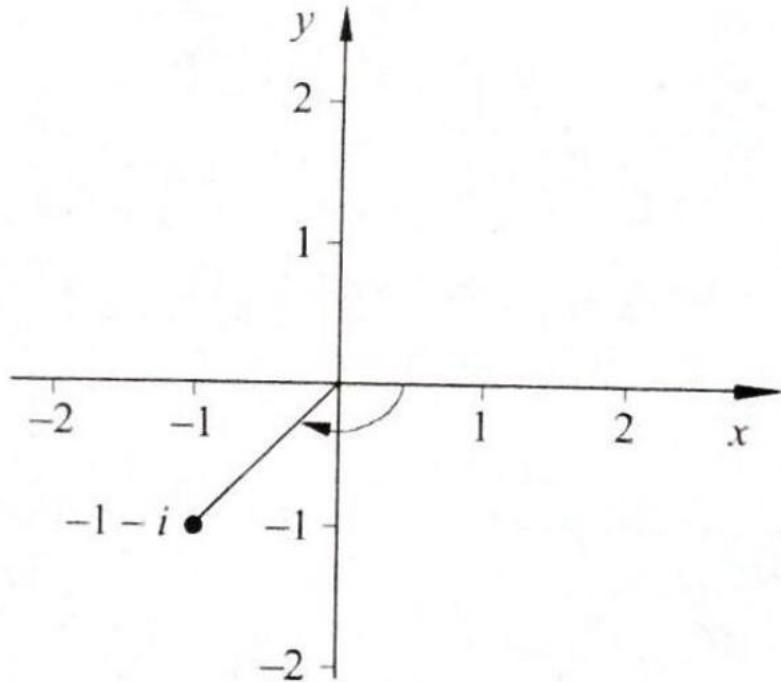
Solution

Now $z = a + ib$ where $a = -1$ and $b = -1$ and in polar form the modulus of

$z = |z| = r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and the argument is $\frac{5\pi}{4}$: its principal value is $-\frac{3\pi}{4}$.

Hence $z = [\sqrt{2}, -\frac{3\pi}{4}]$ in polar coordinates. (The formula $\tan^{-1} \frac{b}{a}$ would have given you $\frac{\pi}{4}$.)

Figure 7.5: Polar form diagram



7.7: DeMoivre's Theorem

An important theorem in complex numbers is named after the French mathematician, Abraham De Moivre (1667 – 1754).

Although born in France, he came to England where he made the acquaintance of Newton and Halley and became a private teacher of Mathematics. He never obtained the university position he sought but he did produce a considerable amount of research, including his work on complex numbers. The derivation of de Moivre's theorem now follows.

Consider the complex number $z = \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$.

$$\begin{aligned} \text{Then } z^2 &= \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \times \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \\ &= \cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3} + 2i \cos \frac{\pi}{3} \sin \frac{\pi}{3} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \end{aligned}$$

or with the modulus/argument notation

$$z = \left[1, \frac{\pi}{3} \right]$$

and

$$z^2 = \left[1, \frac{\pi}{3} \right] \times \left[1, \frac{\pi}{3} \right] = \left[1, \frac{2\pi}{3} \right].$$

Remember that any complex number $z = x + yi$ can be written in the form of an ordered pair $[r, \theta]$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

If the modulus of the number is 1, then $z = \cos\theta + i\sin\theta$

and

$$z^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos 2\theta + i\sin 2\theta$$

i.e.

$$z^2 = [1, \theta]^2 = [1, 2\theta]$$

Exercise

- (1) a. Use the principle that, with the usual notation,

$$[r_1, \theta_1] \times [r_2, \theta_2] = [r_1 r_2, \theta_1 + \theta_2]$$

to investigate $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n$ when $n = 0, 1, 2, 3, \dots, 12$. b. In the same way as in

a., investigate

$$\left(3\cos\frac{\pi}{6} + 3i\sin\frac{\pi}{6}\right)^n$$

for $n = 0, 1, 2, \dots, 6$.

You should find from the last activity that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

In $[r, \theta]$ form this is $[r, \theta]^n = [r^n, n\theta]$ and de Moivre's theorem states that this is true for any rational number n .

A more rigorous way of deriving de Moivre's theorem follows.

- (2) Show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for $n = 3$ and $n = 4$.
- (3) Show that

$$(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) = \cos((k+1)\theta) + i\sin((k+1)\theta).$$

Hence, show that if

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

$$\text{then } (\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta).$$

The principle of mathematical induction will be used to prove that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all positive integers.

Let $S(k)$ be the statement

$$'(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta'.$$

As $S(1)$ is true and you have shown in Activity 14 that $S(k)$ implies $S(k+1)$ then $S(2)$ is also true. But then (again by Activity 14) $S(3)$ is true. But then ... Hence $S(n)$ is true for $n = 1, 2, 3, \dots$ This is the principle of mathematical induction (which you meet more fully later in the book). So for all positive integers n ,

7.8: Nth root of Unity

In the following, we consider the equation

$$z^n = 1, \quad n \in \mathbb{N}, \quad (1)$$

for which we can want to determine the roots. The fundamental theorem of algebra tells that there exists exactly n roots, one of which is $z = 1$.

To find the other solutions, we write (1) in a slightly different form using the Euler representation:

$$z^n = 1 = e^{ik2\pi}, \quad \forall k \in \mathbb{Z}. \quad (2)$$

Then the solutions are $z = e^{\frac{i2k\pi}{n}}$ for $k = 0, 1, 2, \dots, n - 1$.

Geometrically, all n roots lie on the unit circle, and they form a regular polygon with n corners where the roots are $\frac{360^\circ}{\pi}$ apart, see an example in Figure 1.10.

Therefore, if we know a single root and the total number of roots, we could even geometrically find all other roots.

Example 7.8:

Cube Roots of Unity.

The 3rd roots of 1 are $z = \frac{e^{2k\pi i}}{3}$ for $k = 0, 1, 2$, i.e., $1, \frac{e^{2\pi i}}{3}, \frac{e^{4\pi i}}{3}$. These are often referred to as ω_1, ω_2 and ω_3 , and simplify to

$$\omega_1 = 1$$

$$\omega_2 = \cos \cos \frac{2\pi}{3} + i \sin \sin \frac{2\pi}{3} = \frac{(-1+i\sqrt{3})}{2}$$

$$\omega_3 = \cos \cos \frac{4\pi}{3} + i \sin \sin \frac{4\pi}{3} = \frac{(-1-i\sqrt{3})}{2}.$$

Try cubing each solution directly to validate that they are indeed cubic roots.

Solution of $z^n = a + ib$

Finding the n roots of $z^n = a + ib$ is similar to the approach discussed above. Let $a + ib = re^{i\phi}$ in the polar form. Then, for $k = 0, 1, \dots, n - 1$,

$$z^n = (a + ib)e^{2\pi ki} = re^{(\phi+2\pi k)i} \quad (3)$$

$$\Rightarrow z_k = r^{\frac{1}{n}} e^{\frac{(\phi+2\pi k)}{n}i}, \quad k=0,\dots,n-1 \quad (4)$$

Example 7.9:

Determine the cube of $1 - i$.

- 1 The polar coordinates of $1 - i$ are $r = \sqrt{2}$, $\phi = \frac{7}{4}\pi$, and the corresponding Euler representation is

$$z = \sqrt{2}e^{(i\frac{7\pi}{4})}.$$

- 2 Using (4), the cube roots of z are

$$z_1 = 2^{\frac{1}{6}} \left(\cos \cos \frac{7\pi}{12} + i \sin \sin \frac{7\pi}{12} \right) = 2^{\frac{1}{6}} e^{i \frac{7\pi}{12}}$$

$$z_2 = 2^{\frac{1}{6}} \left(\cos \cos \frac{15\pi}{12} + i \sin \sin \frac{15\pi}{12} \right) = 2^{\frac{1}{6}} \left(\cos \cos \frac{5\pi}{4} + i \sin \sin \frac{5\pi}{4} \right) = 2^{\frac{1}{6}} e^{i \frac{5\pi}{4}}$$

$$z_3 = 2^{\frac{1}{6}} \left(\cos \cos \frac{23\pi}{12} + i \sin \sin \frac{23\pi}{12} \right) = 2^{\frac{1}{6}} e^{i \frac{23\pi}{12}}.$$

Summary

1. The history of complex numbers dates back to ancient Greece when the concept of numbers satisfying $x^2 = -1$ was explored by the Greeks, but they were perplexed by the absence of such numbers.
2. Examples of problems involving complex numbers were encountered by mathematicians like Diophantus and Cardan, where they sought solutions involving numbers with real and imaginary components.
3. Complex numbers, denoted as $a+bi$, where 'a' represents the real part and 'bi' represents the imaginary part, were fully understood in the 19th century.
4. Complex numbers can be manipulated using algebraic operations such as addition, subtraction, multiplication, and division, taking into account that $i^2 = -1$.
5. The algebra of complex numbers involves adding or subtracting real and imaginary parts separately.

6. The multiplication of complex numbers follows the distributive property, and the resulting expression contains the sum of products of real and imaginary parts.
7. Division of complex numbers involves multiplying the numerator and denominator by the conjugate of the denominator, which is obtained by changing the sign of the imaginary part.
8. Complex numbers can be expressed in polar coordinates as $|z|$ (modulus) and $\arg(z)$ (argument), which represent the length and direction of the complex number on an Argand diagram.
9. The Argand diagram is a graphical representation of complex numbers, where the real part is plotted on the x-axis and the imaginary part on the y-axis.
10. The polar coordinate form of a complex number is $r(\cos\theta + i\sin\theta)$, where r is the modulus and θ is the argument of the complex number.

Study Session 8: Circular Measure.

Introduction

In mathematics, angles play a fundamental role in understanding the properties and relationships of geometric shapes. While degrees have long been familiar as the unit for measuring angles, there exists another unit known as radians, which we will delve into in this discussion. In this introduction, we will explore the concept of radians, discover the relationship between the length of an arc and the radius of a circle when measured in radians, and learn how to convert measurements between degrees and radians.

Learning Outcomes for Study Session 8

When you have studied this session, you should be able to:

- 8.1. Understand the concept of radians as a unit for measuring angles.
- 8.2. Recognize the relationship between the length of an arc and the radius of a circle when the angle is measured in radians.
- 8.3. Convert measurements from degrees to radians and vice versa using the conversion factor of $1 \text{ radian} = 180 \text{ degrees}$.
- 8.4. Apply the conversion factor to convert specific angle measurements between degrees and radians.
- 8.5. Solve conversion problems involving angles measured in radians and degrees, including both whole numbers and mixed numbers.

8.1: Radian

1. In lower secondary school, we learnt about the unit for angle in degree. In this section, we will learn one more unit for angle that is radian
2. When the value of the angle is 1 radian, then the length of the arc is equal to the length of the radius.
3. From this information, we can deduce that:

$$\frac{1 \text{ radian}}{360} = \frac{r}{2\pi r}$$

$$1 \text{ rad} = \frac{r}{2\pi r} \times 360$$

$$2\pi \text{ rad} = 360^\circ$$

4. So,

$$\pi \text{ rad} = \frac{180^\circ}{\pi}$$

$$= 57.3^\circ$$

- 5.

$$2\pi \text{ rad} = 360^\circ$$

$$1^\circ = \frac{\pi}{180} \text{ rad}$$

8.2: Converting Measurements in Degree to Radian

Example 8.1:

Convert 120° to radian.

Solution:

$$1^\circ = \frac{\pi}{180} \text{ rad}$$

$$120^\circ = 120 \times \frac{\pi}{180} \text{ rad}$$

$$= \frac{2\pi}{3} \text{ rad}$$

Example 8.2:

Convert $112^\circ 36'$

Solution:

$$112^\circ 36' = 112 + 36$$

$$= 112^\circ + \left(\frac{36}{60}\right)^\circ$$

$$= 112^\circ + 0.6^\circ$$

$$1^\circ = \frac{\pi}{180} \text{ rad}$$

$$112.6^\circ = 112.6 \times \frac{\pi}{180} \text{ rad.}$$

$$= 1.965 \text{ rad}$$

8.3: Converting Measurements in Radian to Degree

Example 8.3:

Convert $\frac{\pi}{6}$ rad to degree

Solution:

$$1 \text{ rad} = \frac{180^\circ}{\pi}$$

$$\frac{\pi}{6} \text{ rad} = \frac{\pi}{6} \times \frac{180^\circ}{\pi}$$

$$= 30^\circ$$

Example 8.4:

Convert 1.36 rad to degree

Solution:

$$1 \text{ rad} = \frac{180^\circ}{\pi}$$

$$1.36 \text{ rad} = 1.36 \times \frac{180^\circ}{\pi}$$

$$= 77.92^\circ$$

Exercise

1. Convert each of the following values to degrees (Take $\pi = 3.142$)
 - a. 0.37 rad
 - b. 2.04 rad
 - c. 1.19 rad

2. Convert each of the following values to radians, giving your answer correct to 4 significant figures (Take $\pi = 3.142$).
 - a. $248^\circ 9'$
 - b. $304^\circ 22'$
 - c. $46^\circ 14'$

Summary

1. The concept of radians as a unit for measuring angles is introduced, in addition to degrees.
2. The relationship between the length of an arc and the radius of a circle is explored, revealing that when an angle is measured in radians, the arc length is equal to the radius.
3. Conversion between degrees and radians is demonstrated using the conversion factor of $1 \text{ radian} = 180 \text{ degrees}$.

4. Several examples illustrate the conversion process, showcasing how to convert specific angle measurements from degrees to radians and vice versa.
5. The exercise section provides practice opportunities to further reinforce the conversion skills, with given values to be converted to degrees or radians.

Study Session 9: Trigonometric Functions of Angles of any Magnitude.

Introduction

Trigonometric Functions of Angles of Any Magnitude: In some areas like Astronomy, Architecture, Physics, surveying, and Medicine, we come across the usage of trigonometric functions of non-standard angles, such as, and. We can define the trigonometric functions of any angle of magnitude by observing the reference angle for the angle with respect to the quadrant it lies.

Each trigonometric function value varies from quadrant. Some of the trigonometric functions are positive in the second quadrant, whereas others are positive in another quadrant. By observing the range of the trigonometric functions, we can depict the values of some angles of various functions.

Trigonometric ratios are defined for acute angles as the ratio of the sides of a right-angled triangle. The extension of trigonometric ratios to any angle of radian measure (real numbers) are called trigonometric functions.

There is a total of six trigonometric functions sine ratio of the angle (sin), cosine ratio of the angle (cos), the tangent ratio of the angle (tan), cotangent ratio of the angle (cot), secant ratio of the angle (sec) and cosecant ratio of the angle (cosec) of the angles.

Learning Outcomes for Study Session 9

When you have studied this session, you should be able to:

- 9.1. Understand the significance and practical applications of trigonometric functions in fields such as Astronomy, Architecture, Physics, surveying, and Medicine.
- 9.2. Define and determine the trigonometric functions for non-standard angles by observing the reference angle and the quadrant in which the angle lies.
- 9.3. Identify the variations of trigonometric function values across different quadrants and recognize which functions are positive in specific quadrants.
- 9.4. Depict the values of trigonometric functions for various angles by observing their ranges and characteristics.
- 9.5. Comprehend the concept of trigonometric ratios and their definitions for acute angles as the ratio of sides in a right-angled triangle.
- 9.6. Extend the trigonometric ratios to angles measured in radians, understanding the broader concept of trigonometric functions.
- 9.7. Familiarise yourself with the six trigonometric functions: sine, cosine, tangent, cotangent, secant, and cosecant, and their respective definitions.
- 9.8. Recognize the properties and relationships of trigonometric functions, such as the Pythagorean identity $\cos^2x + \sin^2x = 1$.

9.1: Trigonometric Functions of Standard Angles

Let us take a unit circle, centre at the origin. Let $P(a,b)$ be any point on the circle.

$\angle AOB = x$ radians and the arc $AP = x$

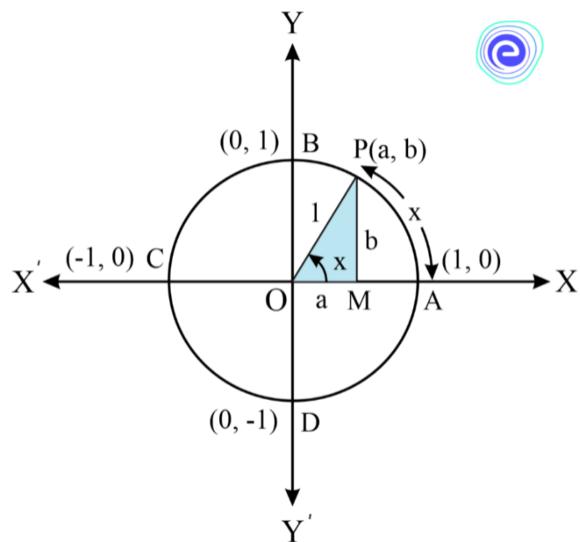


Figure 9.1: Trigonometric functions on a circle

From the figure, $\cos \cos x = a$ and $\sin \sin x = b$

In right triangle OMP , $OM^2 + MP^2 = OP^2$

Thus, $\cos^2 x + \sin^2 x = 1$

We know that, one complete revolution, the angle subtends at the centre is 2π radians. The quadrant angles $\angle AOB = \frac{\pi}{2}$, $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$ are integral multiples of $\frac{\pi}{2}$.

Consider the points A(1,0), B(0,1), C(-1,0) and D(0,-1).

Therefore, for quadrantal angles, we have

- $\cos \cos 0^0 = 1; \sin \sin 0^0 = 0$
- $\cos \cos \frac{\pi}{2} = 0; \sin \sin \frac{\pi}{2} = 1$
- $\cos \cos \pi = -1; \sin \sin \pi = 0$
- $\cos \cos \frac{3\pi}{2} = 0; \sin \sin \frac{3\pi}{2} = -1$
- $\cos \cos 2\pi = 1; \sin \sin \pi = 0$

Thus, we also observe that the values of sine and cosine functions do not change if x changes by any integral multiple of 2π . Thus

- $\sin \sin (2n\pi + x) = \sin \sin x, n \in \mathbb{Z}$
- $\cos \cos (2n\pi + x) = \cos \cos x, n \in \mathbb{Z}$

Further,

$\sin \sin x = 0$, if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$, i.e., when x is an integral multiple of π

Summary

1. Trigonometric functions are widely used in fields such as Astronomy, Architecture, Physics, surveying, and Medicine.
2. Non-standard angles can be analysed using trigonometric functions by considering the reference angle and the quadrant in which the angle lies.
3. Trigonometric function values vary across quadrants, with certain functions being positive in specific quadrants.
4. By understanding the ranges and properties of trigonometric functions, we can determine the values of angles for various functions.
5. Trigonometric ratios initially defined for acute angles can be extended to angles measured in radians.
6. The six trigonometric functions—sine, cosine, tangent, cotangent, secant, and cosecant—play distinct roles in relating angles and sides of right-angled triangles.
7. Quadrantal angles, which are integral multiples of π , have specific values for sine and cosine functions.

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Glossary

Absolute Value: The distance of a number from zero on the number line, always positive.

Area: The measure of the surface enclosed by a 2-dimensional shape, usually measured in square units.

Binomial: An algebraic expression with two terms connected by an addition or subtraction operation.

Coefficient: The numerical factor of a term in an algebraic expression or equation.

Congruent: Two figures or objects that have the same shape and size.

Derivative: The rate of change of a function with respect to its independent variable.

Exponent: The small raised number that indicates how many times a base number is multiplied by itself.

Factorial: The product of all positive integers from 1 to a given number, denoted by $n!$

Function: A relation between a set of inputs (domain) and a set of possible outputs (range) in which each input is related to exactly one output.

Hypotenuse: The side opposite the right angle in a right-angled triangle.

Irrational Number: A number that cannot be expressed as a fraction and its decimal representation goes on infinitely without repeating.

Logarithm: The inverse operation of exponentiation; it answers the question, "To what power must the base be raised to obtain a given number?"

Matrix: A rectangular array of numbers, symbols, or expressions arranged in rows and columns.

Natural Number: A positive whole number (excluding zero).

Parabola: The U-shaped curve formed by the graph of a quadratic function.

Quadratic Equation: A polynomial equation of the second degree, often written in the form $ax^2 + bx + c = 0$.

Radius: The distance from the centre of a circle or sphere to any point on its circumference or surface.

Slope: A measure of how steep a line is, calculated as the change in vertical distance divided by the change in horizontal distance.

Tangent: A line that touches a curve at one point without intersecting it, also the trigonometric function representing the ratio of the opposite side to the adjacent side in a right triangle.

Unit Circle: A circle with a radius of 1 unit, often used in trigonometry.

Variable: A symbol used to represent an unknown value in mathematical expressions or equations.

X-Axis: The horizontal number line in a coordinate system.

Y-Axis: The vertical number line in a coordinate system.

Zero: The integer denoted by 0; it serves as the additive identity in mathematics.