Heat and Mass Transfer: Fundamentals & Applications
Fourth Edition
Yunus A. Cengel, Afshin J. Ghajar
McGraw-Hill, 2011

Chapter 4
TRANSIENT HEAT CONDUCTION

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Objectives

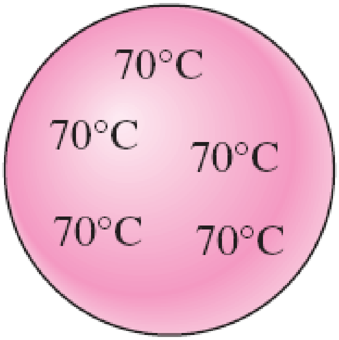
- Assess when the spatial variation of temperature is negligible, and temperature varies nearly uniformly with time, making the simplified lumped system analysis applicable
- Obtain analytical solutions for transient one-dimensional conduction problems in rectangular, cylindrical, and spherical geometries using the method of separation of variables, and understand why a one-term solution is usually a reasonable approximation
- Solve the transient conduction problem in large mediums using the similarity variable, and predict the variation of temperature with time and distance from the exposed surface
- Construct solutions for multi-dimensional transient conduction problems using the product solution approach

LUMPED SYSTEM ANALYSIS

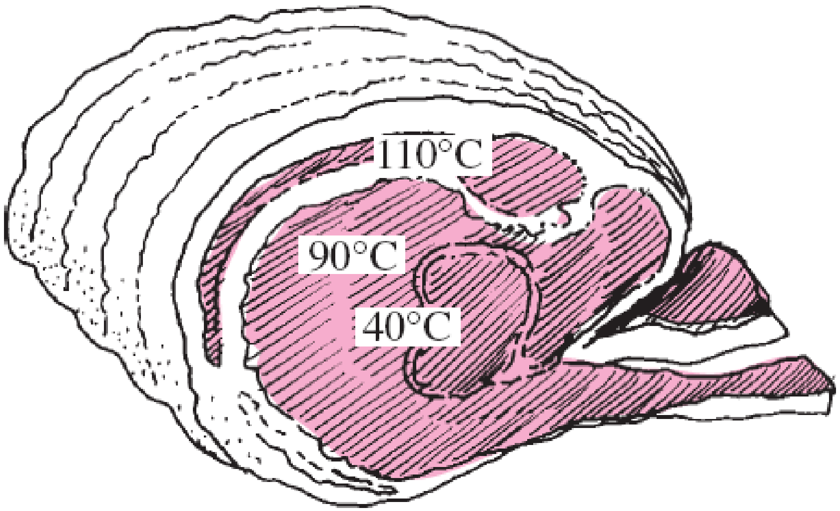
Interior temperature of some bodies remains essentially uniform at all times during a heat transfer process.

The temperature of such bodies can be taken to be a function of time only, $T(t)$.

Heat transfer analysis that utilizes this idealization is known as **lumped system analysis**.



(*a*) Copper ball



(*b*) Roast beef

A small copper ball can be modeled as a lumped system, but a roast beef cannot.

$$\begin{pmatrix}\text{Heat transfer into the body} \\ \text{during } dt\end{pmatrix} = \begin{pmatrix}\text{The increase in the} \\ \text{energy of the body} \\ \text{during } dt\end{pmatrix}$$

$$hA_s(T_\infty - T)\,dt = mc_p\,dT$$

$$m = \rho V \qquad dT = d(T - T_\infty)$$

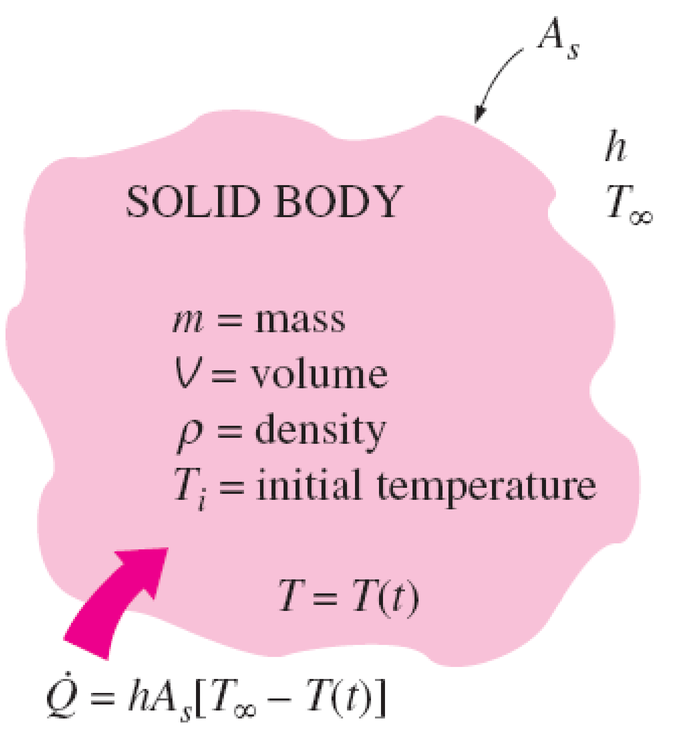
$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho V c_p}\,dt$$

Integrating with
$T = T_i$ at $t = 0$
$T = T(t)$ at $t = t$

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho V c_p}\,t$$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \qquad b = \frac{hA_s}{\rho V c_p} \quad (1/\mathrm{s})$$

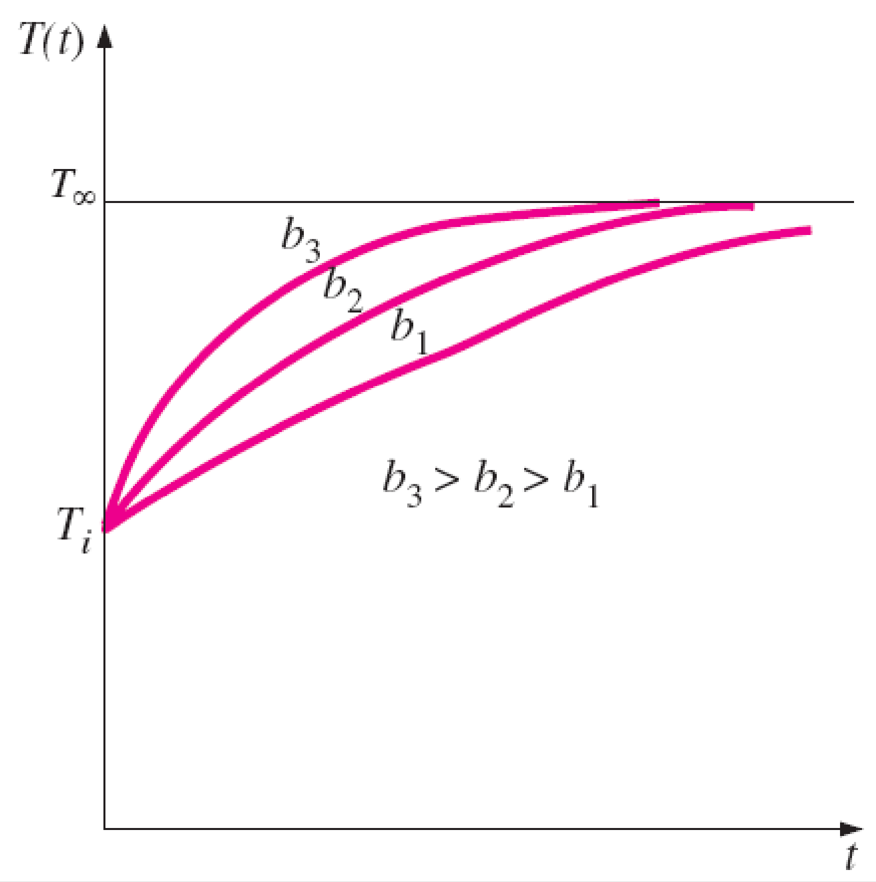
time constant



The geometry and parameters involved in the lumped system analysis.

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA_s}{\rho V c_p}$$



The temperature of a lumped system approaches the environment temperature as time gets larger.

- This equation enables us to determine the temperature $T(t)$ of a body at time t, or alternatively, the time t required for the temperature to reach a specified value $T(t)$.
- The temperature of a body approaches the ambient temperature T_∞ exponentially.
- The temperature of the body changes rapidly at the beginning, but rather slowly later on. A large value of b indicates that the body approaches the environment temperature in a short time

$$\dot{Q}(t) = hA_s[T(t) - T_\infty] \quad \text{(W)}$$

The *rate* of convection heat transfer between the body and its environment at time *t*

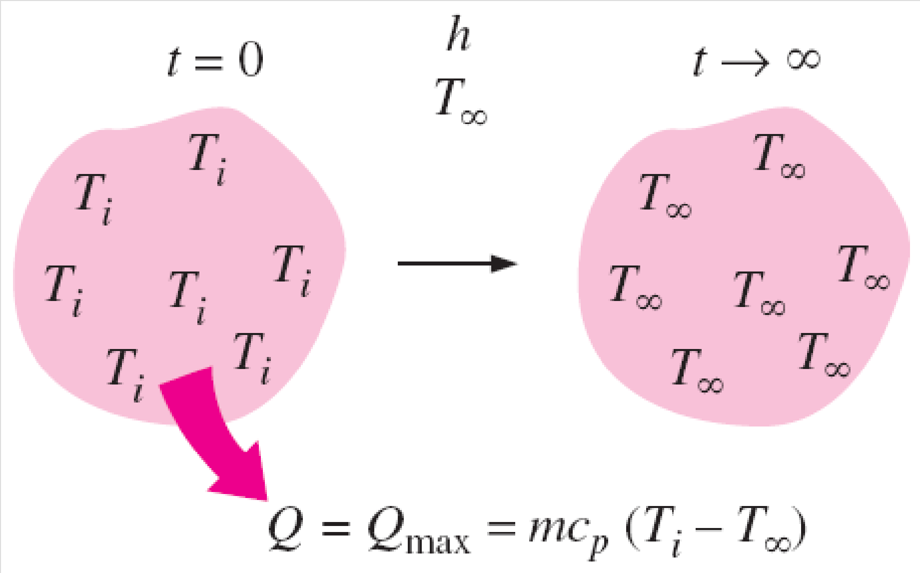
$$Q = mc_p[T(t) - T_i] \quad \text{(kJ)}$$

The *total amount* of heat transfer between the body and the surrounding medium over the time interval $t = 0$ to t

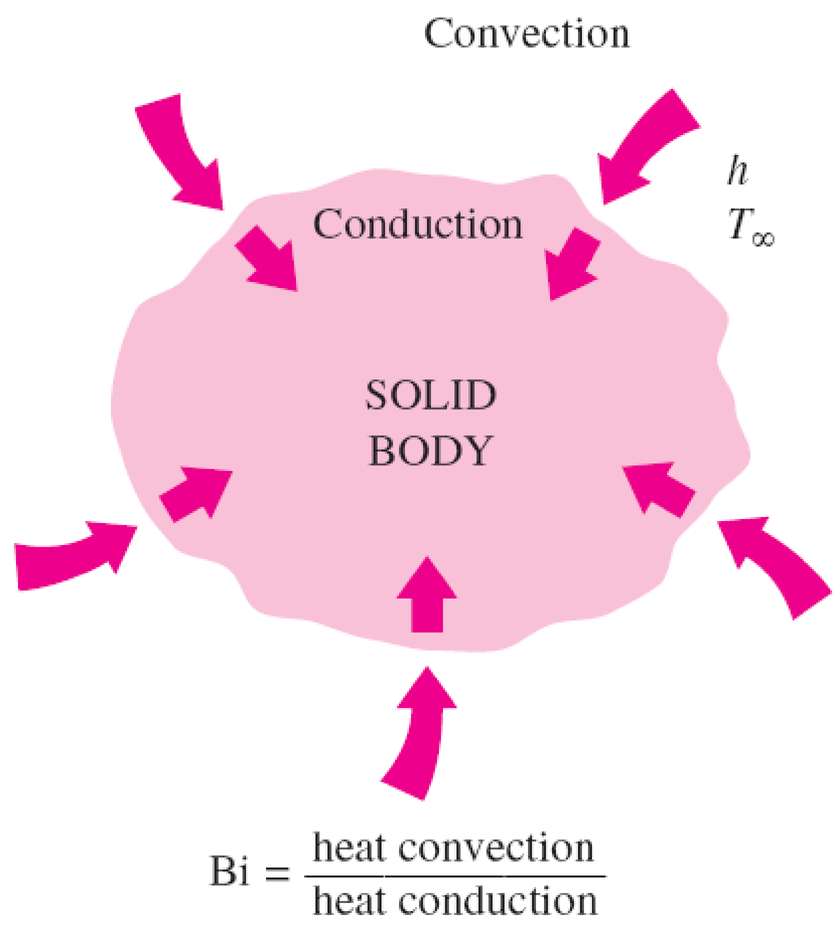
$$Q_{\max} = mc_p(T_\infty - T_i) \quad \text{(kJ)}$$

The *maximum* heat transfer between the body and its surroundings

Heat transfer to or from a body reaches its maximum value when the body reaches the environment temperature.



Criteria for Lumped System Analysis



$$L_c = \frac{V}{A_s}$$ Characteristic length

$$\text{Bi} = \frac{hL_c}{k}$$ Biot number

Lumped system analysis is *applicable* if

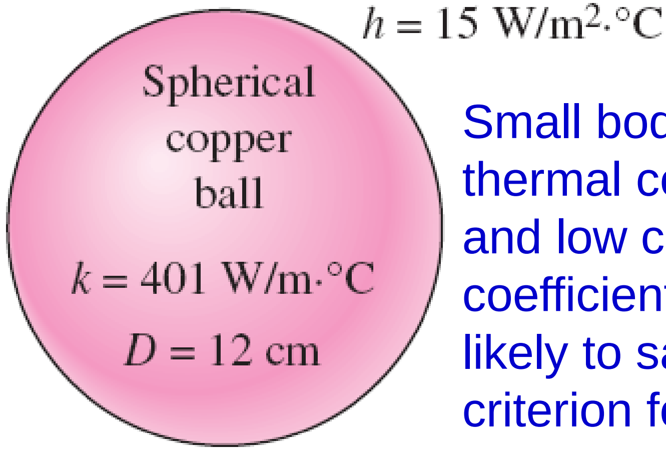
$$\text{Bi} \leq 0.1$$

When Bi ≤ 0.1, the temperatures within the body relative to the surroundings (i.e., $T - T_\infty$) remain within 5 percent of each other.

$$\text{Bi} = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

$$\text{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$

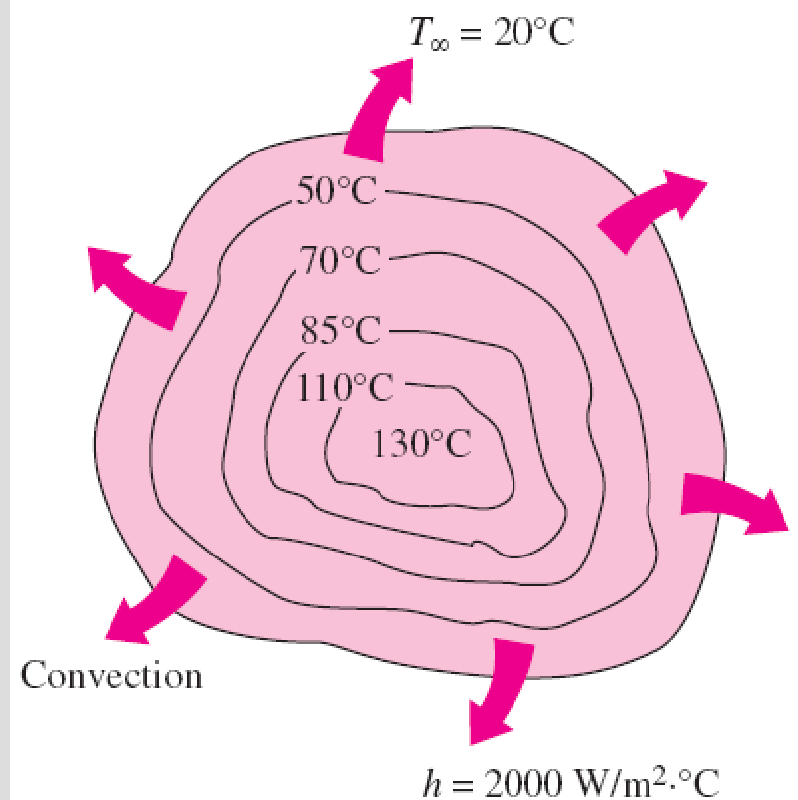
Jean-Baptiste Biot (1774–1862) was a French physicist, astronomer, and mathematician born in Paris, France. Although younger, Biot worked on the analysis of heat conduction even earlier than Fourier did (1802 or 1803) and attempted, unsuccessfully, to deal with the problem of incorporating external convection effects in heat conduction analysis. Fourier read Biot's work and by 1807 had determined for himself how to solve the elusive problem. In 1804, Biot accompanied Gay Lussac on the first balloon ascent undertaken for scientific purposes. In 1820, with Felix Savart, he discovered the law known as "Biot and Savart's Law." He was especially interested in questions relating to the polarization of light, and for his achievements in this field he was awarded the Rumford Medal of the Royal Society in 1840. The dimensionless **Biot number (Bi)** used in transient heat transfer calculations is named after him.



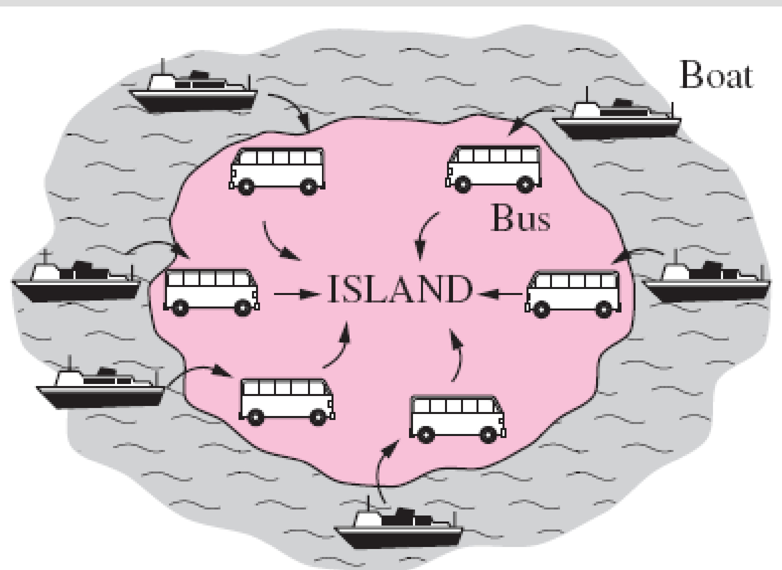
Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.

$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02\ \mathrm{m}$$

$$\mathrm{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$



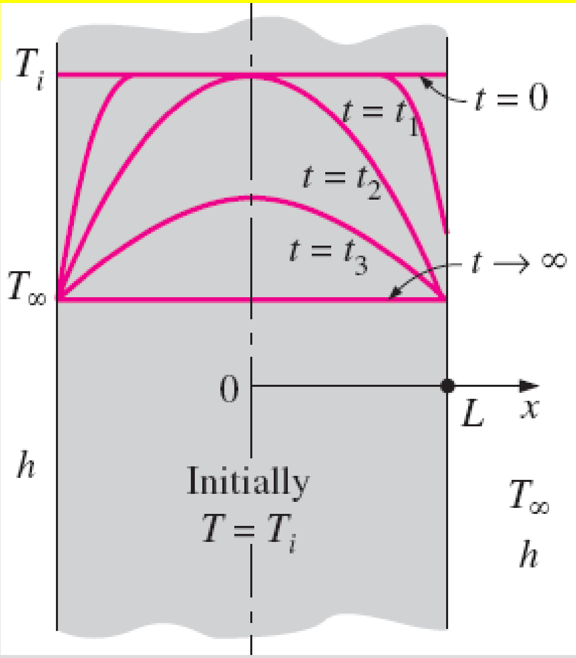
When the convection coefficient *h* is high and *k* is low, large temperature differences occur between the inner and outer regions of a large solid.



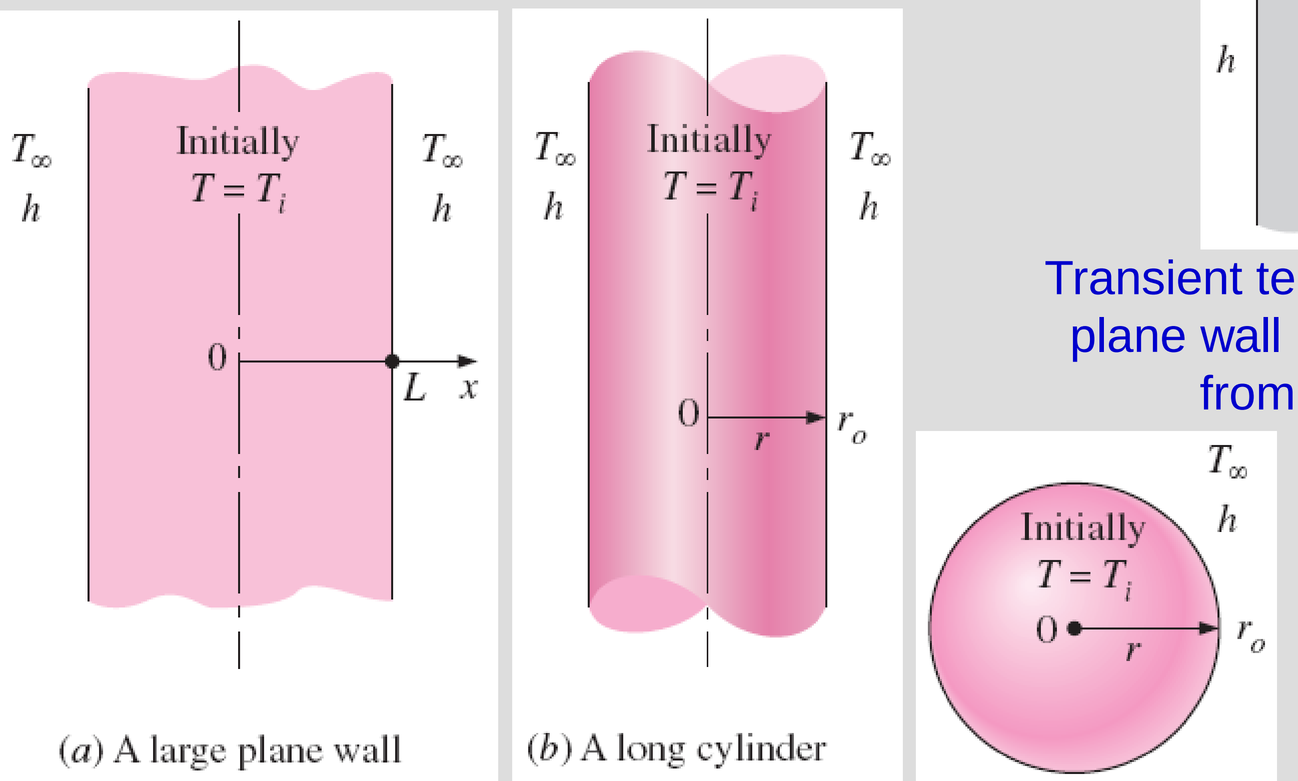
Analogy between heat transfer to a solid and passenger traffic to an island.

TRANSIENT HEAT CONDUCTION IN LARGE PLANE WALLS, LONG CYLINDERS, AND SPHERES WITH SPATIAL EFFECTS

We will consider the variation of temperature with ***time*** and ***position*** in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.



Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$.

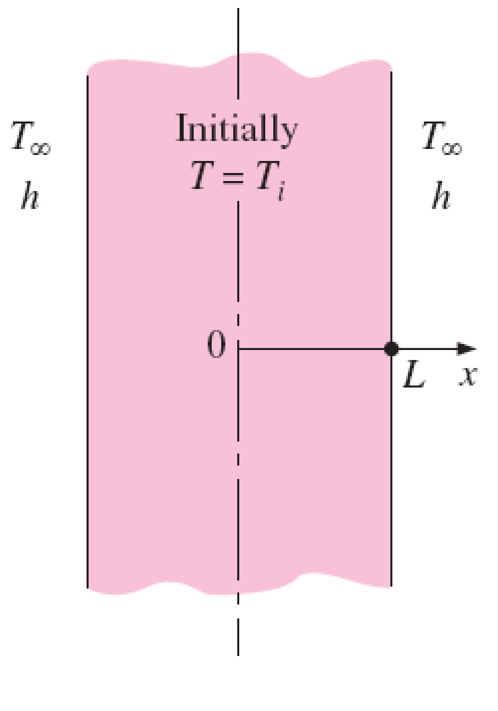


(*a*) A large plane wall

(*b*) A long cylinder

Schematic of the simple geometries in which heat transfer is one-dimensional.

Nondimensionalized One-Dimensional Transient Conduction Problem



(*a*) A large plane wall

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions:

$$\frac{\partial T(0, t)}{\partial x} = 0 \quad \text{and} \quad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

Initial condition: $T(x, 0) = T_i$

$\alpha = k/\rho c_p$ $X = x/L$ $\theta(x, t) = [T(x, t) - T_\infty]/[T_i - T_\infty]$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{and} \quad \frac{\partial \theta(1, t)}{\partial X} = \frac{hL}{k}\theta(1, t)$$

Dimensionless differential equation: $$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}$$

Dimensionless BC's: $$\frac{\partial \theta(0, \tau)}{\partial X} = 0 \quad \text{and} \quad \frac{\partial \theta(1, \tau)}{\partial X} = -\text{Bi}\,\theta(1, \tau)$$

Dimensionless initial condition: $\theta(X, 0) = 1$

$$\theta(X, \tau) = \frac{T(x, t) - T_i}{T_\infty - T_i}$$ *Dimensionless temperature*

$$X = \frac{x}{L}$$ *Dimensionless distance from the center*

$$\text{Bi} = \frac{hL}{k}$$ *Dimensionless heat transfer coefficient* (Biot number)

$$\tau = \frac{\alpha t}{L^2} = \text{Fo}$$ *Dimensionless time* (Fourier number)

(*a*) Original heat conduction problem:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad T(x, 0) = T_i$$

$$\frac{\partial T(0, t)}{\partial x} = 0, \quad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

$$T = F(x, L, t, k, \alpha, h, T_i)$$

(*b*) Nondimensionalized problem:

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}, \quad \theta(X, 0) = 1$$

$$\frac{\partial \theta(0, \tau)}{\partial X} = 0, \quad \frac{\partial \theta(1, \tau)}{\partial X} = -\text{Bi}\,\theta(1, \tau)$$

$$\theta = f(X, \text{Bi}, \tau)$$

Nondimensionalization reduces the number of independent variables in one-dimensional transient conduction problems from 8 to 3, offering great convenience in the presentation of results.

Exact Solution of One-Dimensional Transient Conduction Problem

$$\theta(X,\tau) = F(X)G(\tau)$$

$$\frac{1}{F}\frac{d^2F}{dX^2} = \frac{1}{G}\frac{dG}{d\tau}$$

$$\frac{d^2F}{dX^2} + \lambda^2 F = 0 \quad \text{and} \quad \frac{dG}{d\tau} + \lambda^2 G = 0$$

$$F = C_1\cos(\lambda X) + C_2\sin(\lambda X) \quad \text{and} \quad G = C_3 e^{-\lambda^2\tau}$$

$$\theta = FG = C_3 e^{-\lambda^2\tau}[C_1\cos(\lambda X) + C_2\sin(\lambda X)] = e^{-\lambda^2\tau}[A\cos(\lambda X) + B\sin(\lambda X)]$$

$$A = C_1C_3 \text{ and } B = C_2C_3$$

$$\frac{\partial\theta(0,\tau)}{\partial X} = 0 \rightarrow -e^{-\lambda^2\tau}(A\lambda\sin 0 + B\lambda\cos 0) = 0 \rightarrow B = 0 \rightarrow \theta = Ae^{-\lambda^2\tau}\cos(\lambda X)$$

$$\frac{\partial\theta(1,\tau)}{\partial X} = -\mathrm{Bi}\,\theta(1,\tau) \rightarrow -Ae^{-\lambda^2\tau}\lambda\sin\lambda = -\mathrm{Bi}Ae^{-\lambda^2\tau}\cos\lambda \rightarrow \lambda\tan\lambda = \mathrm{Bi}$$

$$\lambda_n \tan\lambda_n = \mathrm{Bi}$$

$$\theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau}\cos(\lambda_n X)$$

$$\theta(X,0) = 1 \rightarrow 1 = \sum_{n=1}^{\infty} A_n\cos(\lambda_n X)$$

$$\int_0^1 \cos(\lambda_n X)dX = A_n \int_0^1 \cos^2(\lambda_n X)dx \rightarrow A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

TABLE 4–1

Summary of the solutions for one-dimensional transient conduction in a plane wall of thickness $2L$, a cylinder of radius r_o and a sphere of radius r_o subjected to convention from all surfaces.*

Geometry	Solution	λ_n's are the roots of
Plane wall	$\theta = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos(\lambda_n x/L)$	$l_n \tan l_n$ 5 Bi
Cylinder	$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} e^{-\lambda_n^2 \tau} J_0(\lambda_n r/r_o)$	$\lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)} = \mathrm{Bi}$
Sphere	$\theta = \sum_{n=1}^{\infty} \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x/L)}{\lambda_n x/L}$	$1 - \lambda_n \cot \lambda_n = \mathrm{Bi}$

*Here $\theta = (T - T_\infty)/(T_i - T_\infty)$ is the dimensionless temperature, Bi $= hL/k$ or hr_o/k is the Biot number, Fo $= \tau = \alpha t / L^2$ or $\alpha\tau / r_o^2$ is the Fourier number, and J_0 and J_1 are the Bessel functions of the first kind whose values are given in Table 4–3.

$$\theta_n = A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n X)$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

$$\lambda_n \tan \lambda_n = \text{Bi}$$

For Bi = 5, $X = 1$, and $t = 0.2$:

n	λ_n	A_n	θ_n
1	1.3138	1.2402	0.22321
2	4.0336	−0.3442	0.00835
3	6.9096	0.1588	0.00001
4	9.8928	−0.876	0.00000

The analytical solutions of transient conduction problems typically involve infinite series, and thus the evaluation of an infinite number of terms to determine the temperature at a specified location and time.

The term in the series solution of transient conduction problems decline rapidly as n and thus λ_n increases because of the exponential decay function with the exponent $-\lambda_n \tau$.

Approximate Analytical and Graphical Solutions

The terms in the series solutions converge rapidly with increasing time, and for $\tau > 0.2$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2 percent.

Solution with *one-term approximation*

Plane wall: $$\theta_{\text{wall}} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2$$

Cylinder: $$\theta_{\text{cyl}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2$$

Sphere: $$\theta_{\text{sph}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2$$

Center of plane wall ($x = 0$): $$\theta_{0,\text{ wall}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of cylinder ($r = 0$): $$\theta_{0,\text{ cyl}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of sphere ($r = 0$): $$\theta_{0,\text{ sph}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

TABLE 4–2

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

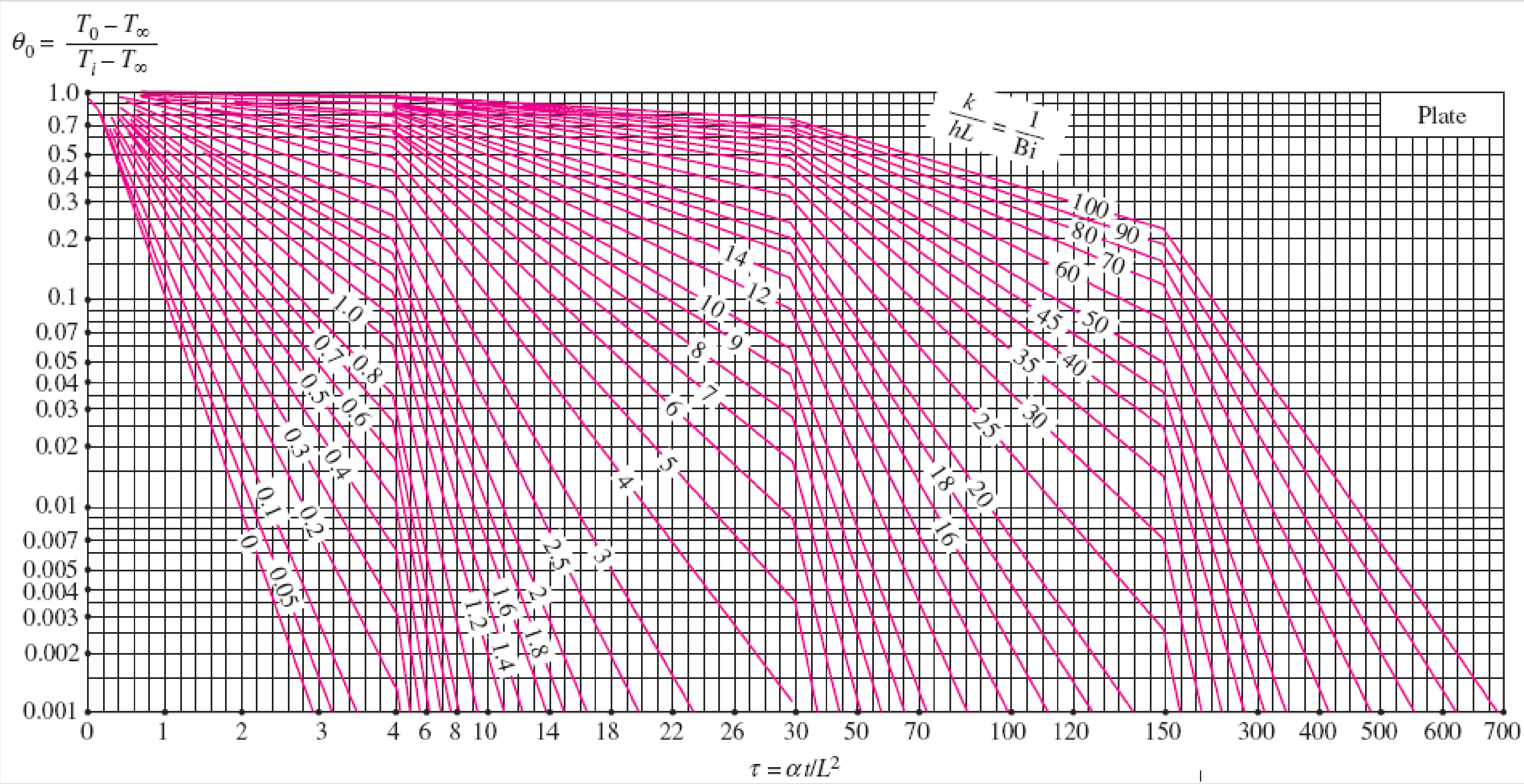
	Plane Wall		*Cylinder*		*Sphere*	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–3

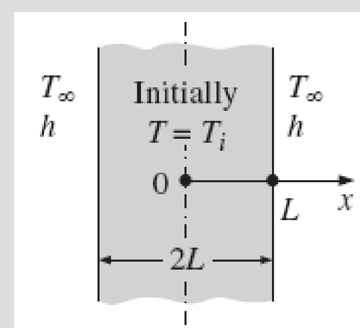
The zeroth- and first-order Bessel functions of the first kind

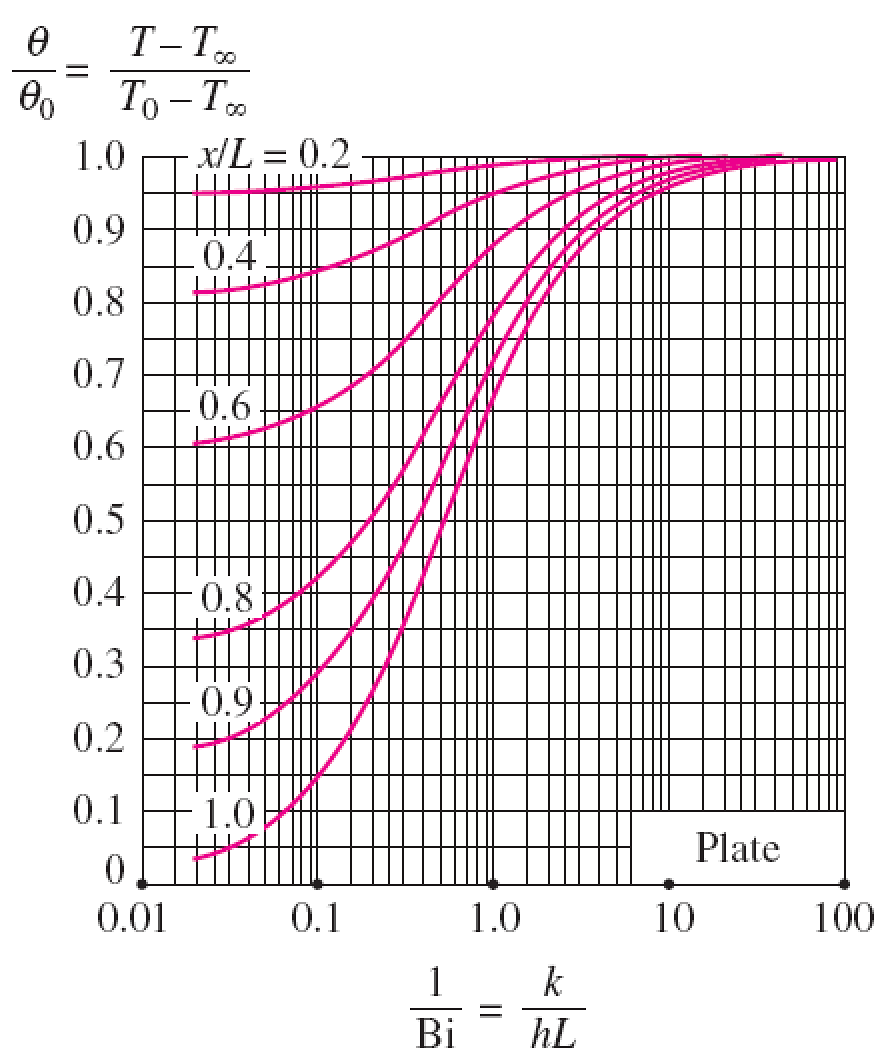
η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

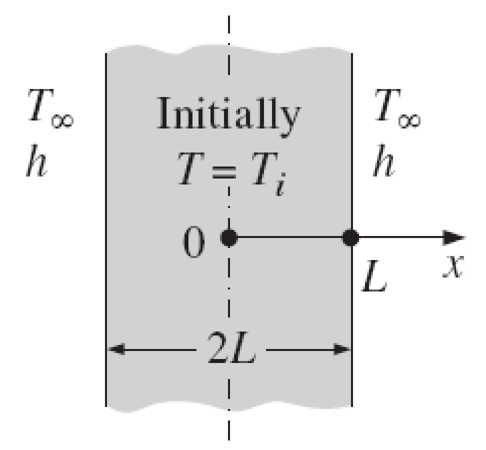
(*a*) Midplane temperature



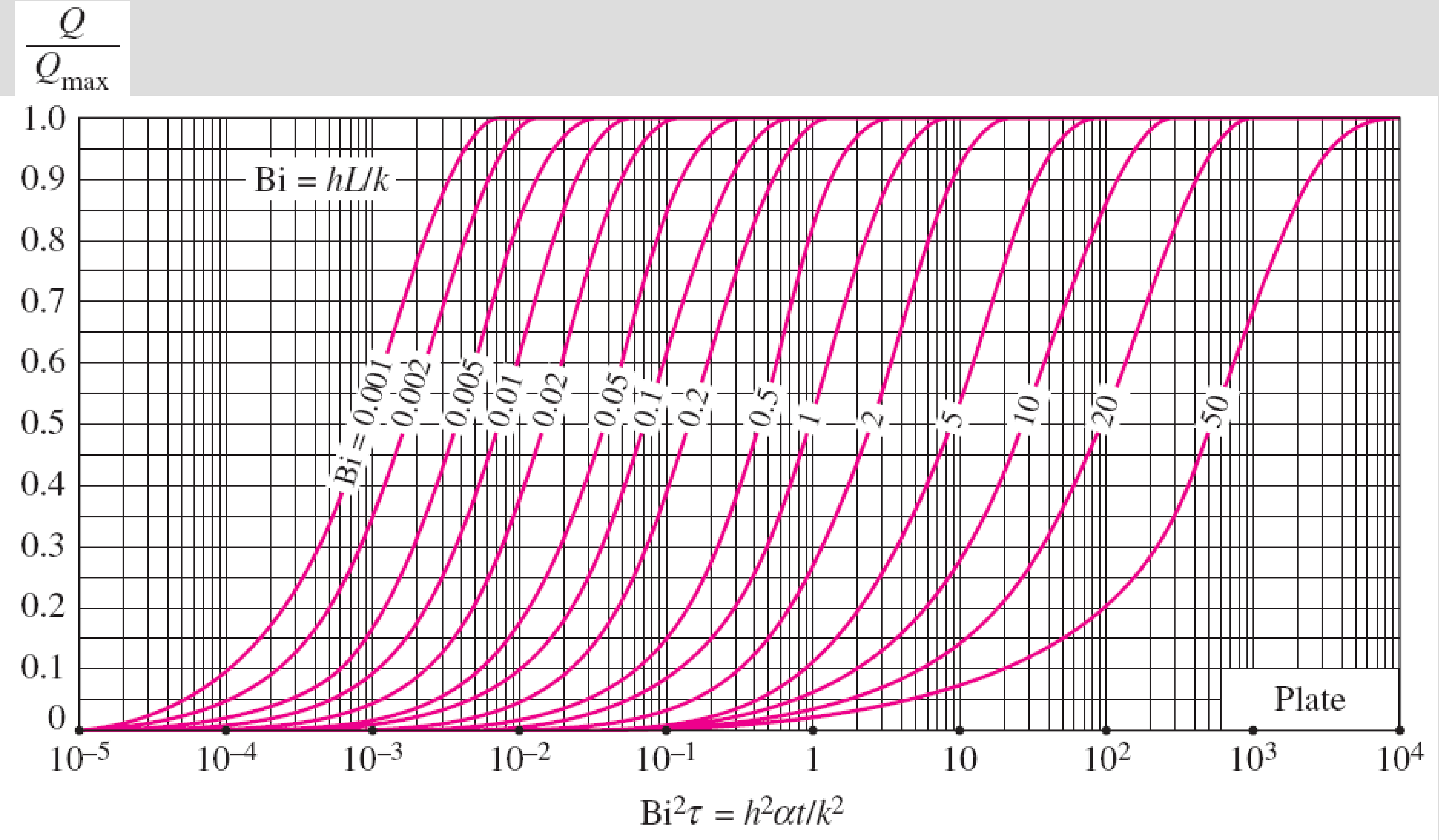
Transient temperature and heat transfer charts (Heisler and Grober charts) for a plane wall of thickness 2*L* initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of *h.*



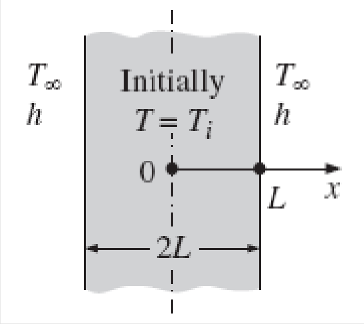




(*b*) **Temperature distribution**

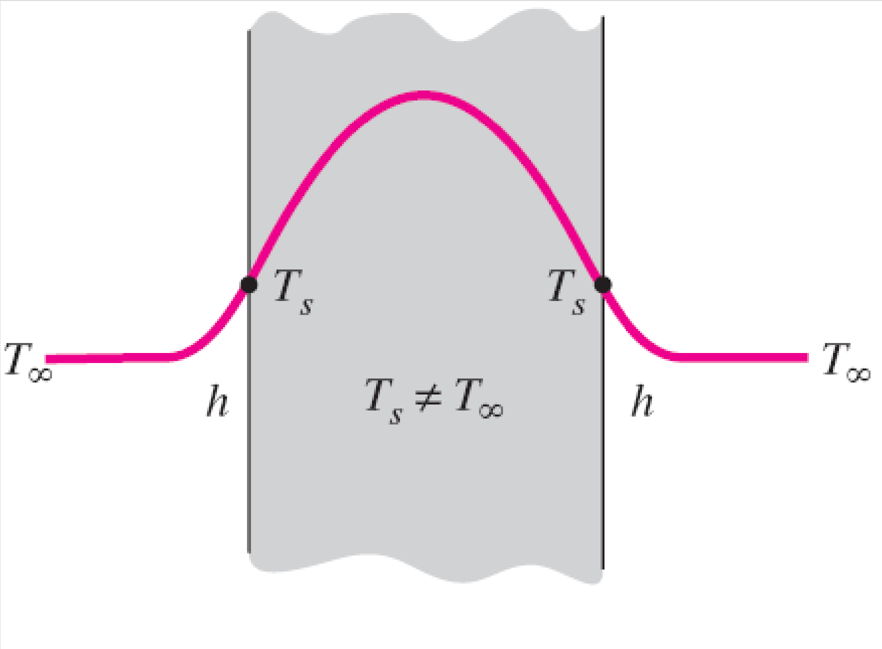


(*c*) Heat transfer

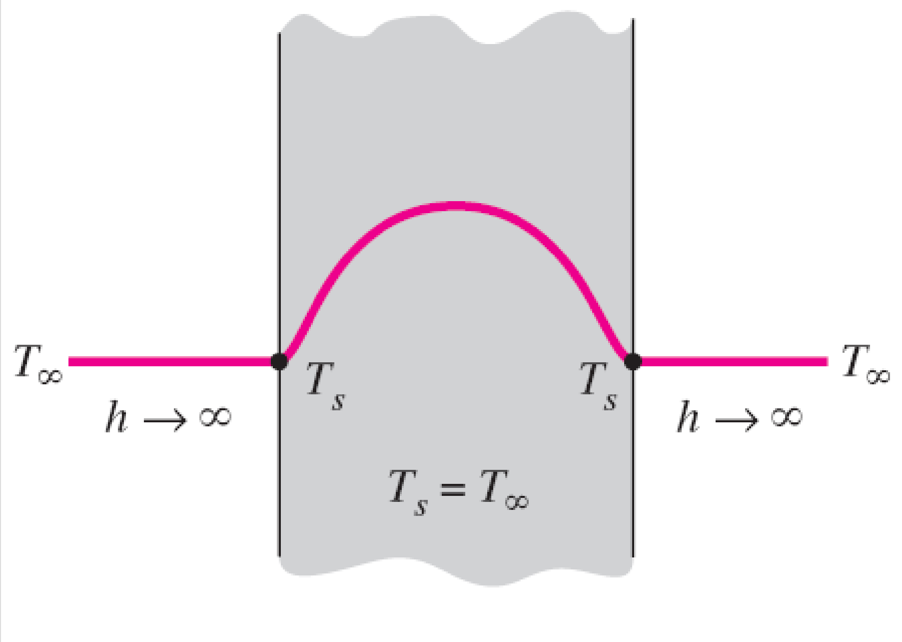


The dimensionless temperatures anywhere in a plane wall, cylinder, and sphere are related to the center temperature by

$$\frac{\theta_{\text{wall}}}{\theta_{0,\text{ wall}}} = \cos\left(\frac{\lambda_1 x}{L}\right), \quad \frac{\theta_{\text{cyl}}}{\theta_{0,\text{ cyl}}} = J_0\left(\frac{\lambda_1 r}{r_o}\right), \quad \text{and} \quad \frac{\theta_{\text{sph}}}{\theta_{0,\text{ sph}}} = \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}$$



(*a*) Finite convection coefficient



(*b*) Infinite convection coefficient

The specified surface temperature corresponds to the case of convection to an environment at T_∞ with with a convection coefficient *h* that is *infinite.*

$$Q_{max} = mc_p(T_\infty - T_i) = \rho V c_p(T_\infty - T_i) \qquad \text{(kJ)}$$

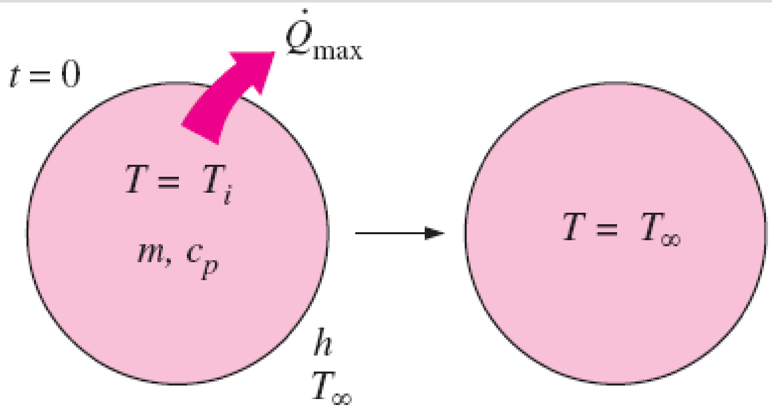
$$\frac{Q}{Q_{max}} = \frac{\int_V \rho c_p[T(x,t) - T_i]dV}{\rho c_p(T_\infty - T_i)V} = \frac{1}{V}\int_V (1-\theta)dV$$

$$Q = \int_V \rho c_p[T(x,t) - T_i]dV$$

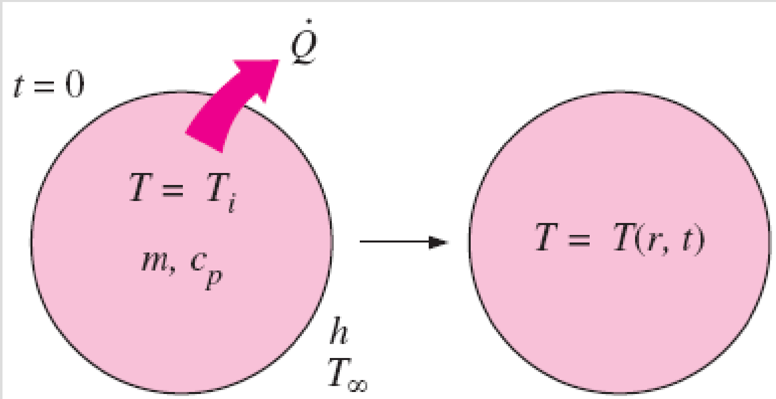
Plane wall: $$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{0,\,wall}\frac{\sin\lambda_1}{\lambda_1}$$

Cylinder: $$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - 2\theta_{0,\,cyl}\frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere: $$\left(\frac{Q}{Q_{max}}\right)_{sph} = 1 - 3\theta_{0,\,sph}\frac{\sin\lambda_1 - \lambda_1\cos\lambda_1}{\lambda_1^3}$$



(a) Maximum heat transfer ($t \to \infty$)



$$\left.\begin{aligned} &\mathrm{Bi} = \ldots \\ &\frac{h^2\alpha t}{k^2} = \mathrm{Bi}^2\tau = \cdots \end{aligned}\right\} \quad \frac{Q}{Q_{max}} = \cdots$$

(Gröber chart)

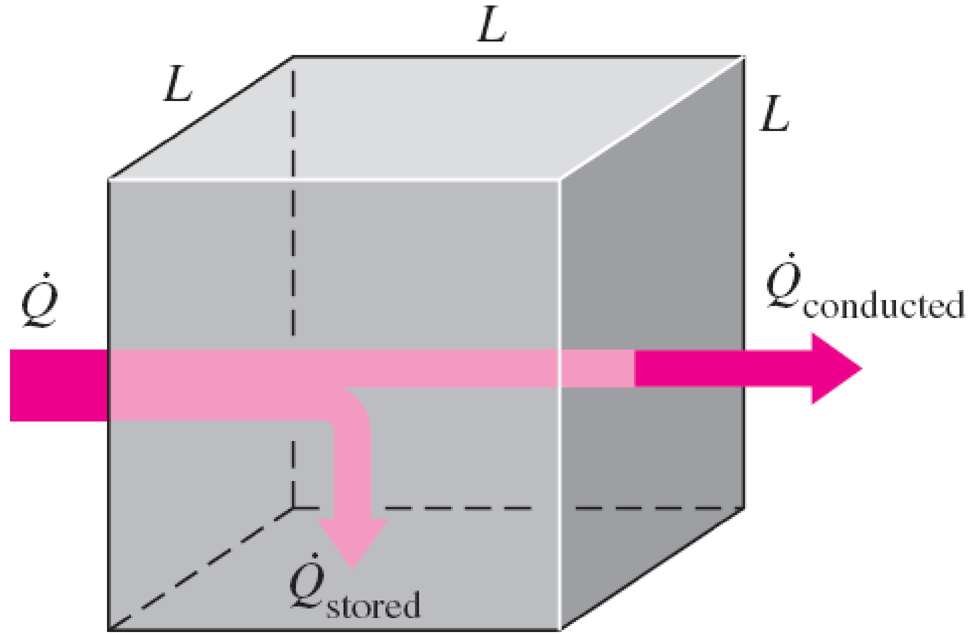
(b) Actual heat transfer for time t

The fraction of total heat transfer Q/Q_{max} up to a specified time t is determined using the Gröber charts.

The physical significance of the *Fourier number*

$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2\,(1/L)}{\rho c_p L^3/t}\,\frac{\Delta T}{\Delta T} = \frac{\text{The rate at which heat is } conducted \text{ across } L \text{ of a body of volume } L^3}{\text{The rate at which heat is } stored \text{ in a body of volume } L^3}$$

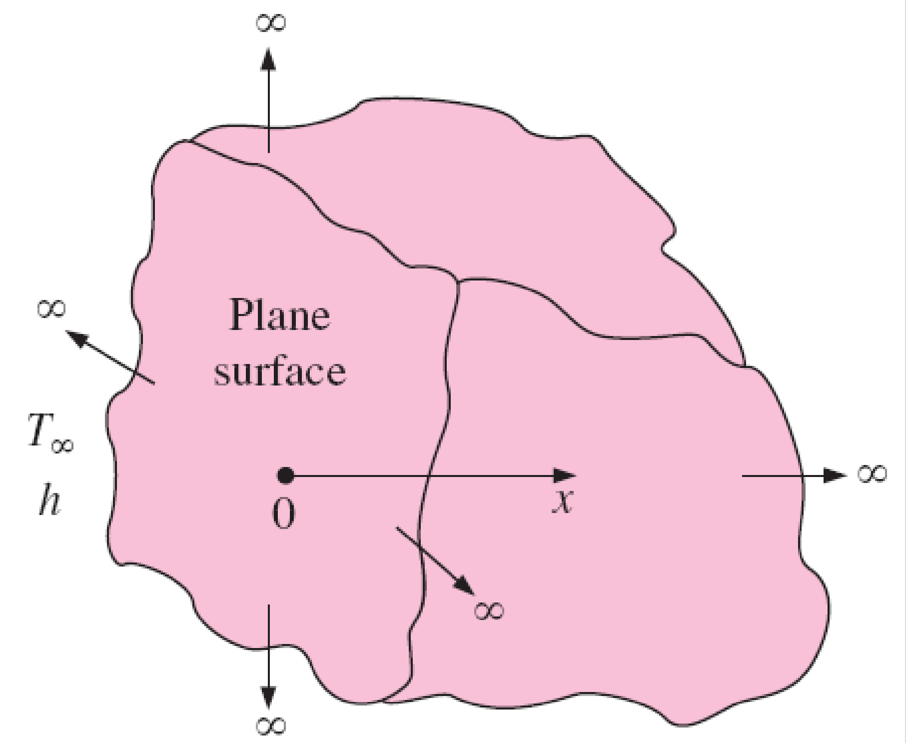
- The Fourier number is a measure of *heat conducted* through a body relative to *heat stored.*
- A large value of the Fourier number indicates faster propagation of heat through a body.



Fourier number: $\tau = \dfrac{\alpha t}{L^2} = \dfrac{\dot{Q}_{\text{conducted}}}{\dot{Q}_{\text{stored}}}$

Fourier number at time *t* can be viewed as the ratio of the rate of heat conducted to the rate of heat stored at that time.

TRANSIENT HEAT CONDUCTION IN SEMI-INFINITE SOLIDS



Schematic of a semi-infinite body.

For short periods of time, most bodies can be modeled as semi-infinite solids since heat does not have sufficient time to penetrate deep into the body.

Semi-infinite solid: An idealized body that has a *single plane surface* and extends to infinity in all directions.

The earth can be considered to be a semi-infinite medium in determining the variation of temperature near its surface.

A thick wall can be modeled as a semi-infinite medium if all we are interested in is the variation of temperature in the region near one of the surfaces, and the other surface is too far to have any impact on the region of interest during the time of observation.

Analytical solution for the case of constant temperature T_s on the surface

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions: $$T(0, t) = T_s \quad \text{and} \quad T(x \to \infty, t) = T_i$$

Initial condition: $$T(x, 0) = T_i$$

Similarity variable: $$\eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$$

$$T(0) = T_s \quad \text{and} \quad T(\eta \to \infty) = T_i$$

$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}du = \text{erf}(\eta) = 1 - \text{erfc}(\eta)$$

$$\text{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}du$$

error function

$$\text{erfc}(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta \varepsilon^{-u^2}du$$

complementary error function

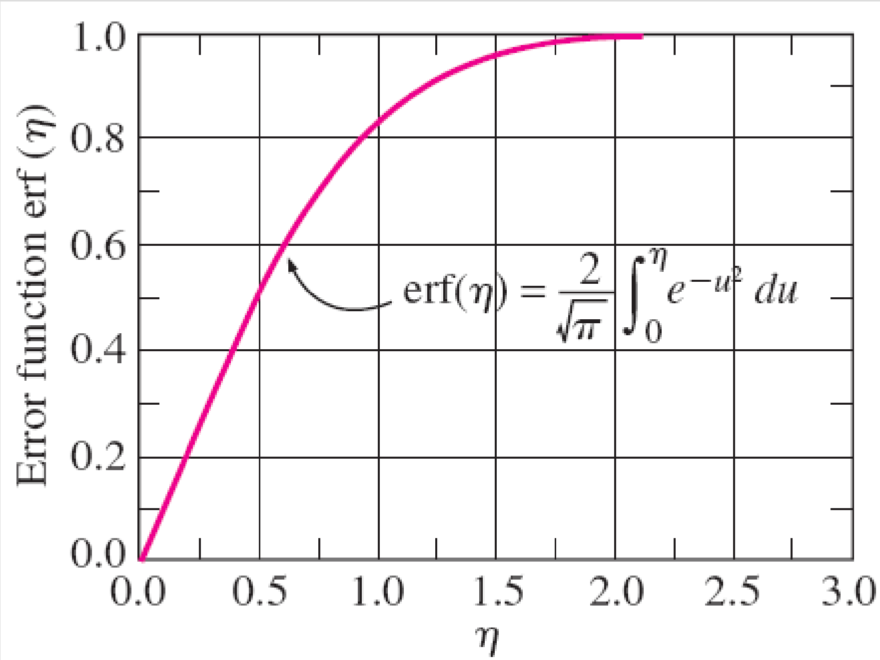
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{and} \quad \eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} = \frac{x}{2t\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{d^2T}{d\eta^2}$$

Transformation of variables in the derivatives of the heat conduction equation by the use of chain rule.



Error function is a standard mathematical function, just like the sine and cosine functions, whose value varies between 0 and 1.

TABLE 4–4

The complementary error function

η	erfc (η)	η	erfc (η)	η	erfc (η)
0.00	1.00000	0.38	0.5910	0.76	0.2825
0.02	0.9774	0.40	0.5716	0.78	0.2700
0.04	0.9549	0.42	0.5525	0.80	0.2579
0.06	0.9324	0.44	0.5338	0.82	0.2462
0.08	0.9099	0.46	0.5153	0.84	0.2349
0.10	0.8875	0.48	0.4973	0.86	0.2239
0.12	0.8652	0.50	0.4795	0.88	0.2133
0.14	0.8431	0.52	0.4621	0.90	0.2031
0.16	0.8210	0.54	0.4451	0.92	0.1932
0.18	0.7991	0.56	0.4284	0.94	0.1837
0.20	0.7773	0.58	0.4121	0.96	0.1746
0.22	0.7557	0.60	0.3961	0.98	0.1658
0.24	0.7343	0.62	0.3806	1.00	0.1573
0.26	0.7131	0.64	0.3654	1.02	0.1492
0.28	0.6921	0.66	0.3506	1.04	0.1413
0.30	0.6714	0.68	0.3362	1.06	0.1339
0.32	0.6509	0.70	0.3222	1.08	0.1267
0.34	0.6306	0.72	0.3086	1.10	0.1198
0.36	0.6107	0.74	0.2953	1.12	0.1132

$$\dot{q}_s = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = -k\frac{dT}{d\eta}\frac{\partial \eta}{\partial x}\bigg|_{\eta=0} = -kC_1 e^{-\eta^2}\frac{1}{\sqrt{4\alpha t}}\bigg|_{\eta=0} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Analytical solutions for different boundary conditions on the surface

Case 1: Specified Surface Temperature, T_s = constant

$$\frac{T(x,\ t) - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{and} \quad \dot{q}_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 2: Specified Surface Heat Flux, $\dot{q}_s$ = constant.

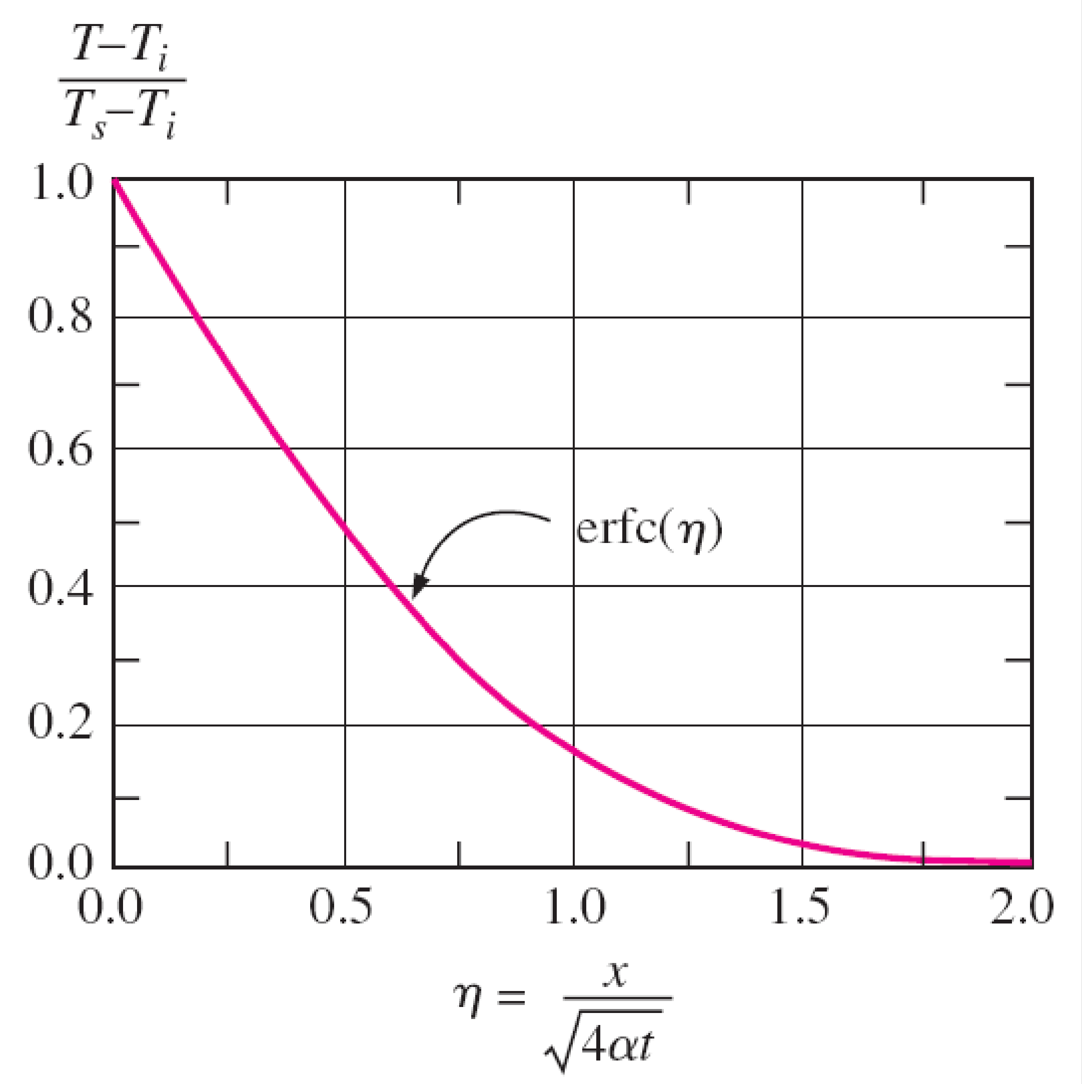
$$T(x,\ t) - T_i = \frac{\dot{q}_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - x\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

Case 3: Convection on the Surface, $\dot{q}_s(t) = h[T_\infty - T(0, t)]$.

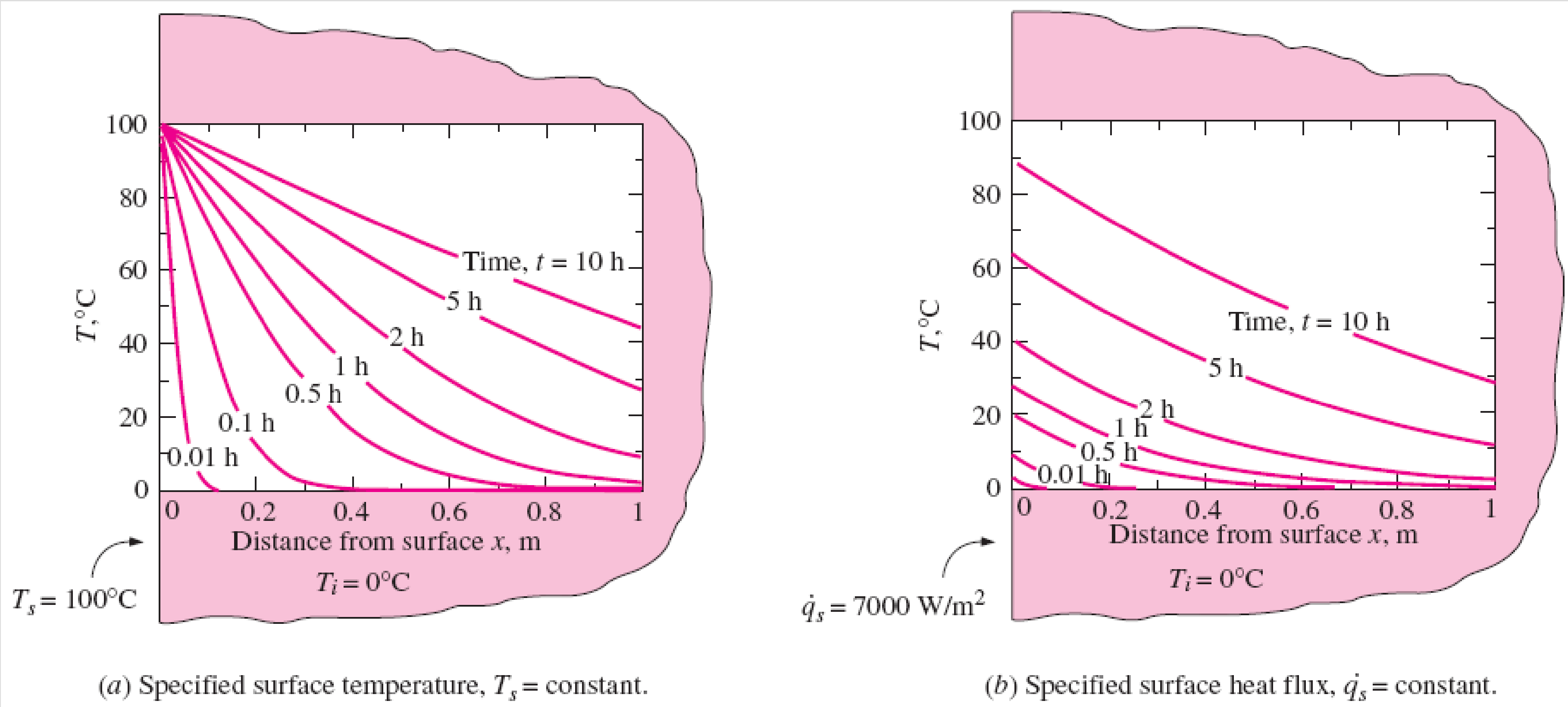
$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

Case 4: Energy Pulse at Surface, e_s = constant.

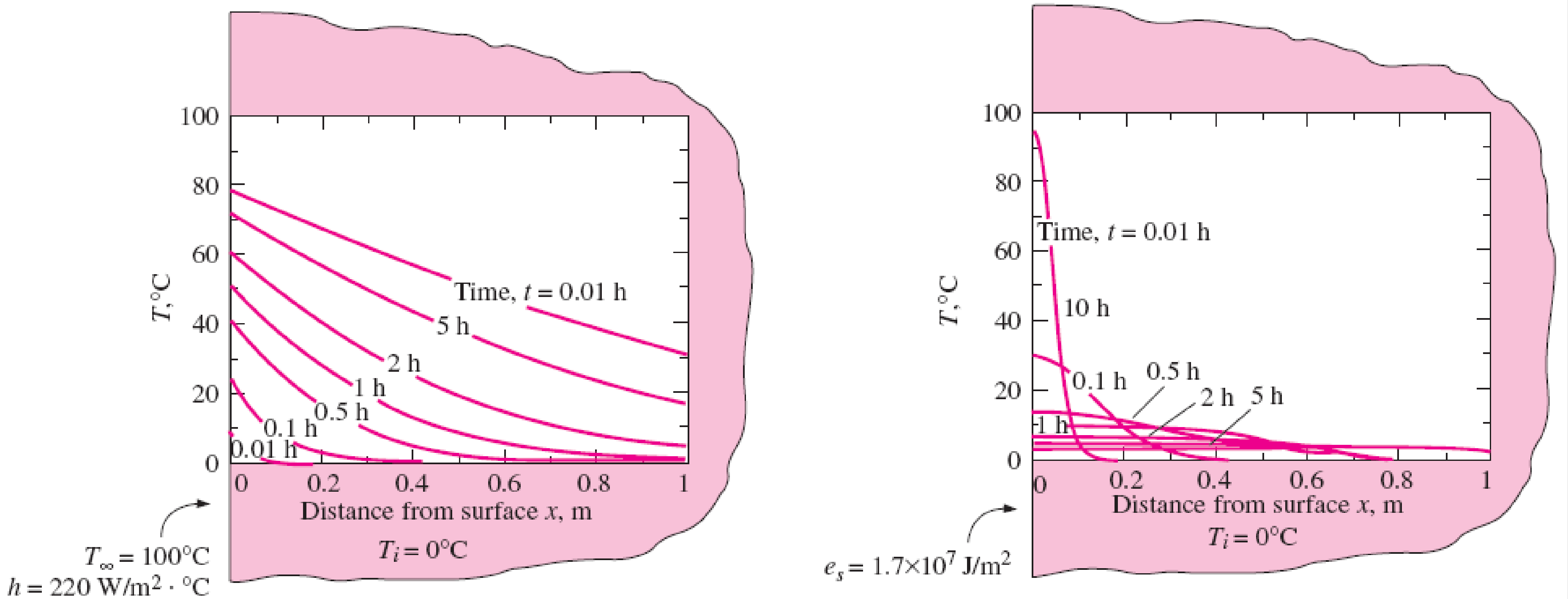
$$T(x,\ t) - T_i = \frac{e_s}{k\sqrt{\pi t/\alpha}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$



Dimensionless temperature distribution for transient conduction in a semi-infinite solid whose surface is maintained at a constant temperature T_s.



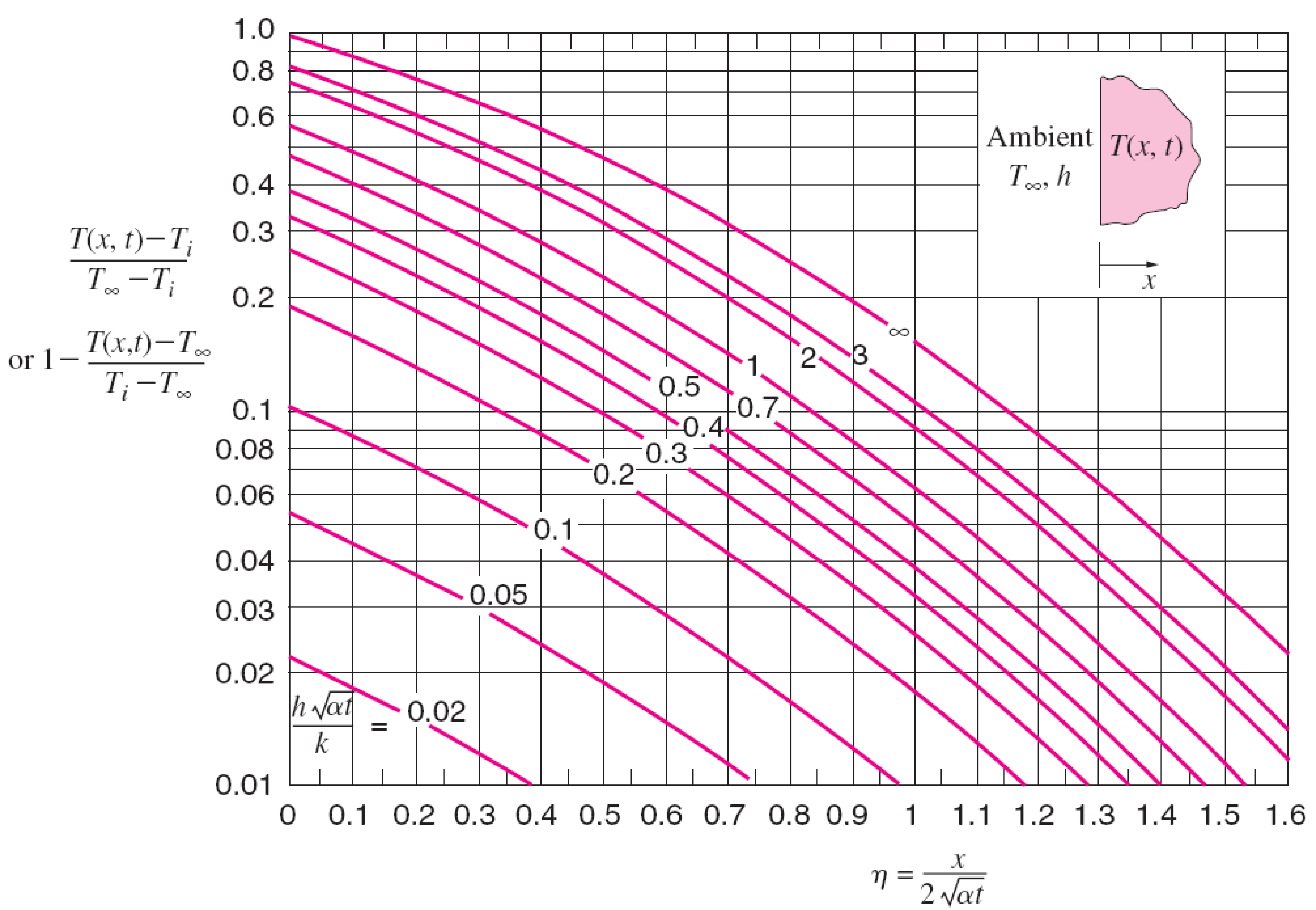
Variations of temperature with position and time in a large cast iron block ($\alpha = 2.31 \times 10^{-5}$ m^2/s, $k = 80.2$ W/m · °C) initially at 0 °C under different thermal conditions on the surface.



(c) Convection at the surface

(d) Energy pulse at the surface, e_s = constant

Variations of temperature with position and time in a large cast iron block ($\alpha = 2.31 \times 10^{-5}$ m²/s, $k = 80.2$ W/m · °C) initially at 0 °C under different thermal conditions on the surface.



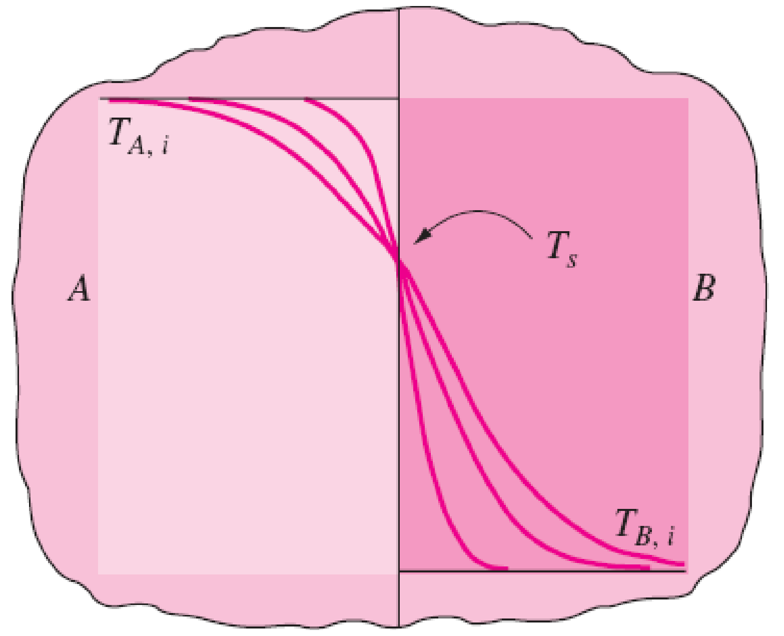
Variation of temperature with position and time in a semi-infinite solid initially at temperature T_i subjected to convection to an environment at T_∞ with a convection heat transfer coefficient of *h.*

Contact of Two Semi-Infinite Solids

When two large bodies *A* and *B*, initially at uniform temperatures $T_{A,i}$ and $T_{B,i}$ are brought into contact, they instantly achieve temperature equality at the contact surface.

If the two bodies are of the same material, the contact surface temperature is the arithmetic average, $T_s = (T_{A,i}+T_{B,i})/2$.

If the bodies are of different materials, the surface temperature T_s will be different than the arithmetic average.



Contact of two semi-infinite solids of different initial temperatures.

$$\dot{q}_{s,A} = \dot{q}_{s,B} \rightarrow -\frac{k_A(T_s - T_{A,i})}{\sqrt{\pi\alpha_A t}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi\alpha_B t}} \rightarrow \frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \sqrt{\frac{(k\rho c_p)_B}{(k\rho c_p)_A}}$$

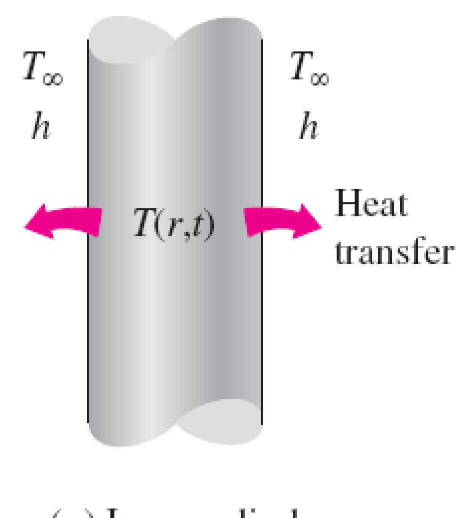
$$T_s = \frac{\sqrt{(k\rho c_p)_A}T_{A,i} + \sqrt{(k\rho c_p)_B}T_{B,i}}{\sqrt{(k\rho c_p)_A} + \sqrt{(k\rho c_p)_B}}$$

The interface temperature of two bodies brought into contact is dominated by the body with the larger $k\rho c_p$.

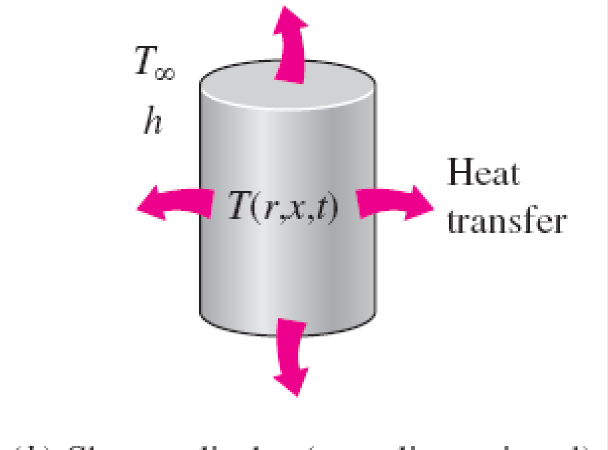
EXAMPLE: When a person with a skin temperature of 35°C touches an aluminum block and then a wood block both at 15°C, the contact surface temperature will be 15.9°C in the case of aluminum and 30°C in the case of wood.

TRANSIENT HEAT CONDUCTION IN MULTIDIMENSIONAL SYSTEMS

- Using a superposition approach called the **product solution**, the transient temperature charts and solutions can be used to construct solutions for the *two-dimensional* and *three-dimensional* transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, a rectangular prism or a semi-infinite rectangular bar, provided that *all* surfaces of the solid are subjected to convection to the *same* fluid at temperature T_∞, with the *same* heat transfer coefficient h, and the body involves no heat generation.
- The solution in such multidimensional geometries can be expressed as the ***product*** of the solutions for the one-dimensional geometries whose intersection is the multidimensional geometry.



(*a*) Long cylinder



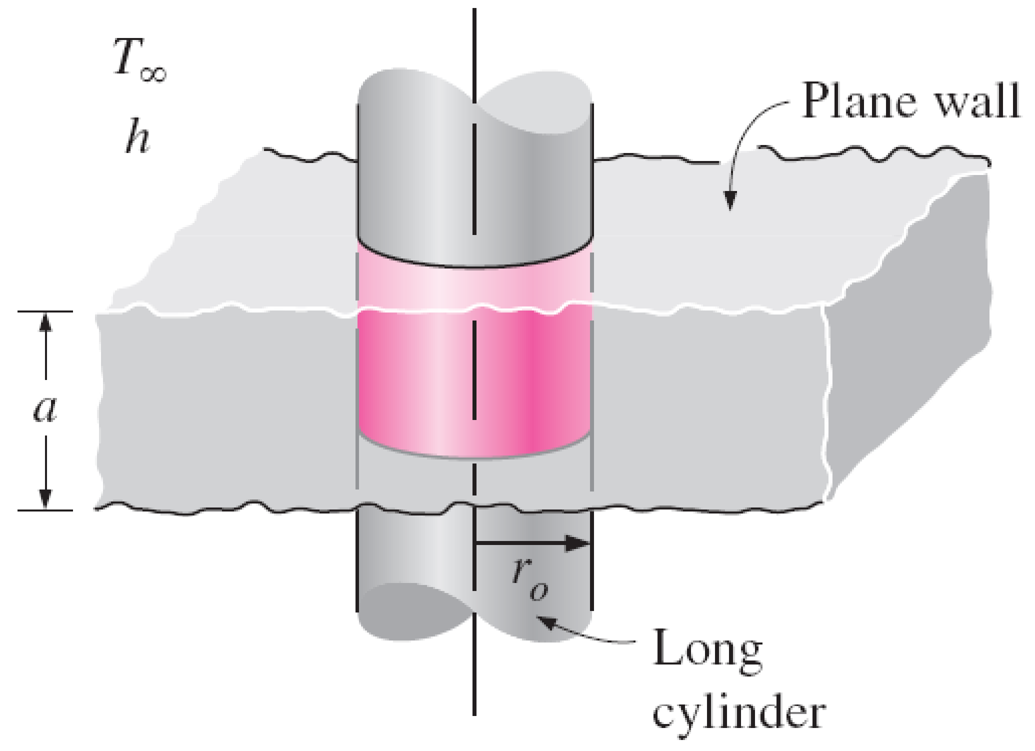
(*b*) Short cylinder (two-dimensional)

The temperature in a short cylinder exposed to convection from all surfaces varies in both the radial and axial directions, and thus heat is transferred in both directions.

The solution for a multidimensional geometry is the product of the solutions of the one-dimensional geometries whose intersection is the multidimensional body.

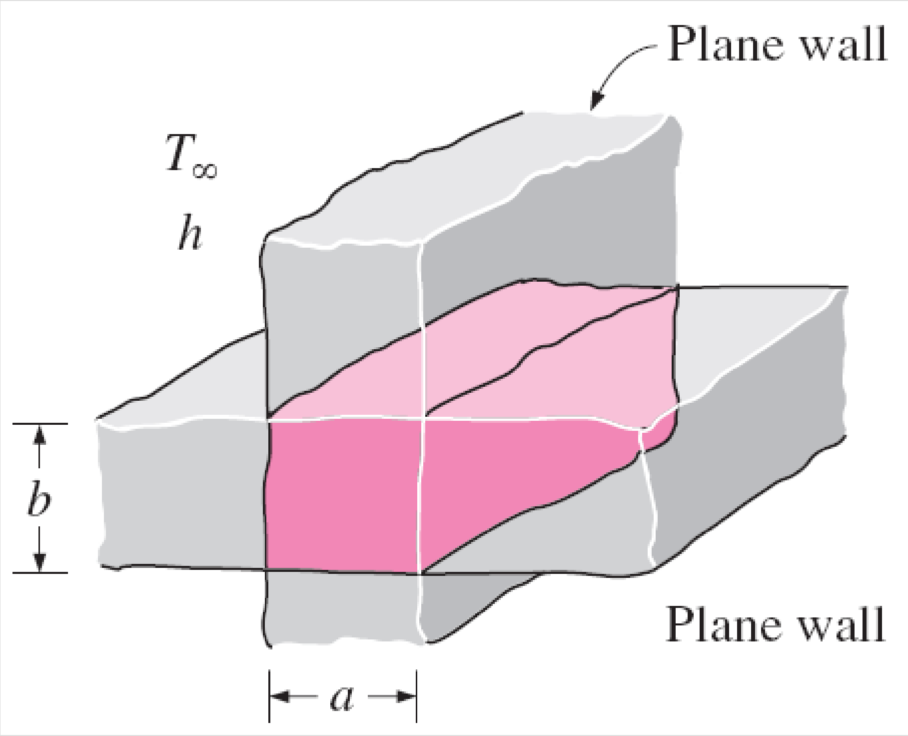
The solution for the two-dimensional short cylinder of height a and radius r_o is equal to the *product* of the nondimensionalized solutions for the one-dimensional plane wall of thickness a and the long cylinder of radius r_o.

$$\left(\frac{T(r, x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{short}\\\text{cylinder}}} = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{plane}\\\text{wall}}} \left(\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{infinite}\\\text{cylinder}}}$$



A short cylinder of radius r_o and height a is the *intersection* of a long cylinder of radius r_o and a plane wall of thickness *a.*

$$\left(\frac{T(x, y, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{rectangular}\\\text{bar}}} = \theta_{\text{wall}}(x, t)\theta_{\text{wall}}(y, t)$$



$$\theta_{\text{wall}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{plane}\\\text{wall}}}$$

$$\theta_{\text{cyl}}(r, t) = \left(\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{infinite}\\\text{cylinder}}}$$

$$\theta_{\text{semi-inf}}(x, t) = \left(\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{semi-infinite}\\\text{solid}}}$$

A long solid bar of rectangular profile $a \times b$ is the *intersection* of two plane walls of thicknesses *a* and *b.*

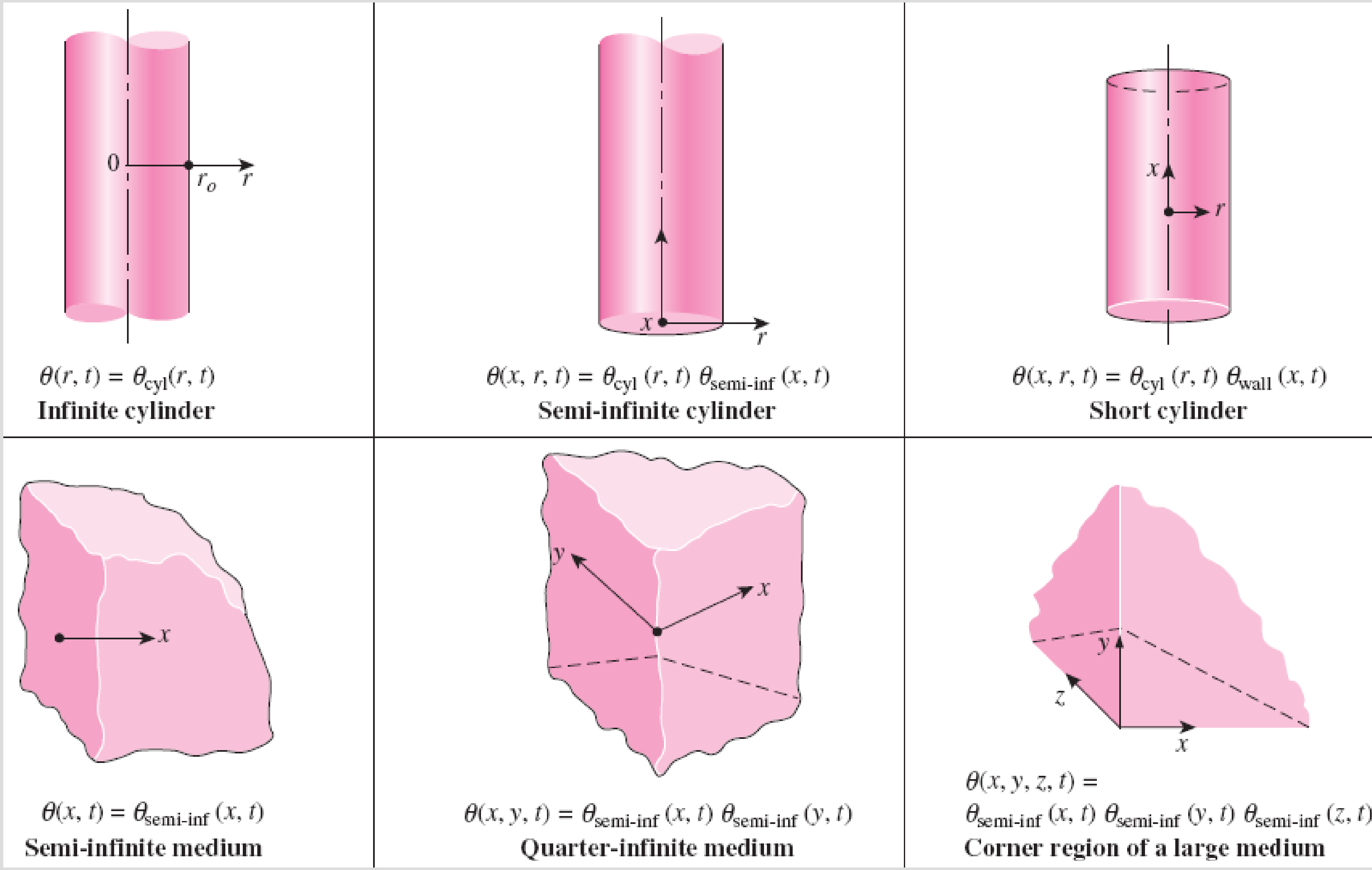
The transient heat transfer for a two-dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total, 2D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]$$

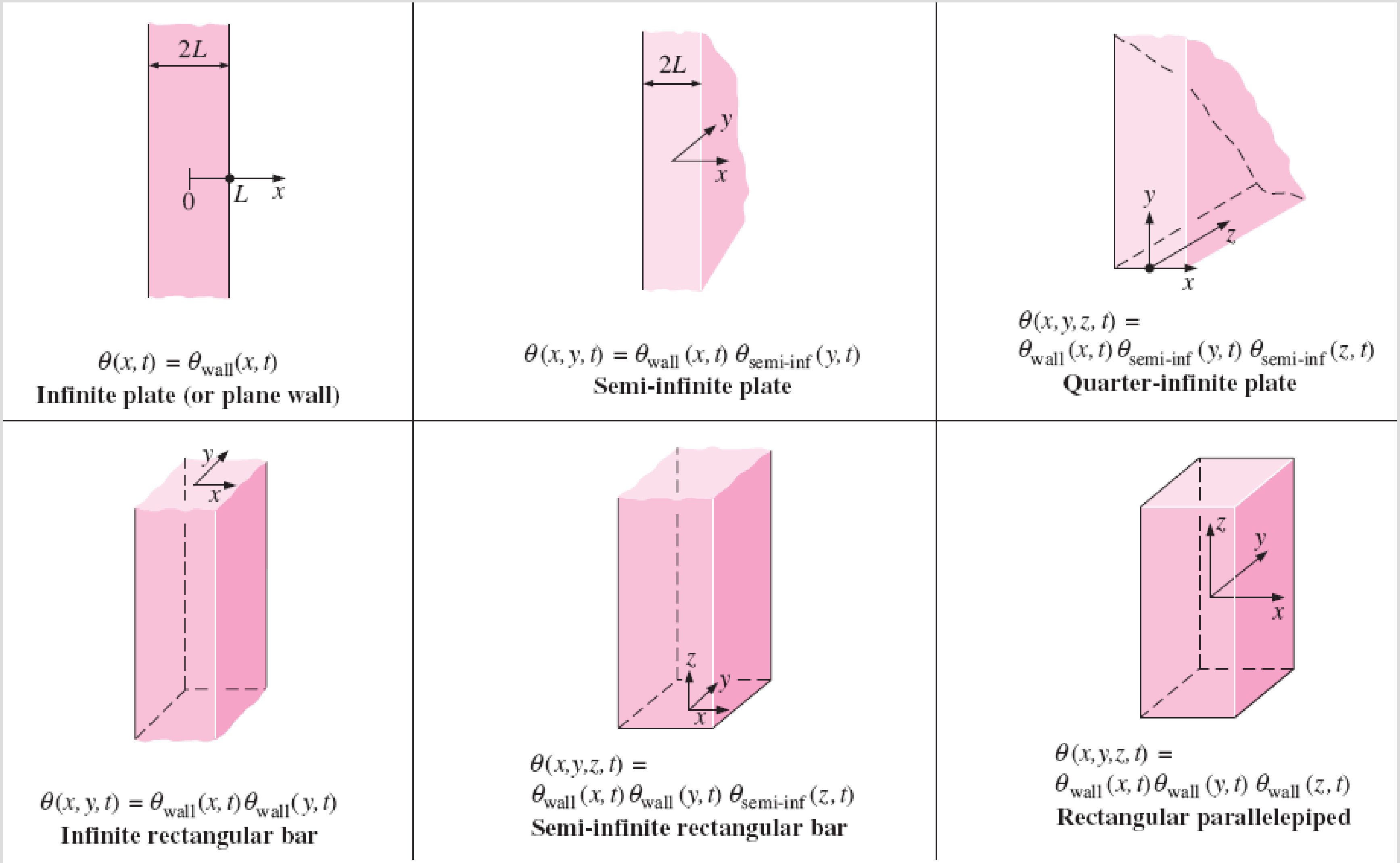
Transient heat transfer for a three-dimensional body formed by the intersection of three one-dimensional bodies 1, 2, and 3 is

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total, 3D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right] + \left(\frac{Q}{Q_{\max}}\right)_3 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]\left[1 - \left(\frac{Q}{Q_{\max}}\right)_2\right]$$

Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞



Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞



Summary

- Lumped System Analysis
  - ✓ Criteria for Lumped System Analysis
  - ✓ Some Remarks on Heat Transfer in Lumped Systems
- Transient Heat Conduction in Large Plane Walls, Long Cylinders, and Spheres with Spatial Effects
  - ✓ Nondimensionalized One-Dimensional Transient Conduction Problem
  - ✓ Exact Solution of One-Dimensional Transient Conduction Problem
  - ✓ Approximate Analytical and Graphical Solutions
- Transient Heat Conduction in Semi-Infinite Solids
  - ✓ Contact of Two Semi-Infinite Solids
- Transient Heat Conduction in Multidimensional Systems