Heat and Mass Transfer: Fundamentals & Applications
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Chapter 3
STEADY HEAT CONDUCTION

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Objectives

- Understand the concept of thermal resistance and its limitations, and develop thermal resistance networks for practical heat conduction problems
- Solve steady conduction problems that involve multilayer rectangular, cylindrical, or spherical geometries
- Develop an intuitive understanding of thermal contact resistance, and circumstances under which it may be significant
- Identify applications in which insulation may actually increase heat transfer
- Analyze finned surfaces, and assess how efficiently and effectively fins enhance heat transfer
- Solve multidimensional practical heat conduction problems using conduction shape factors

STEADY HEAT CONDUCTION IN PLANE WALLS

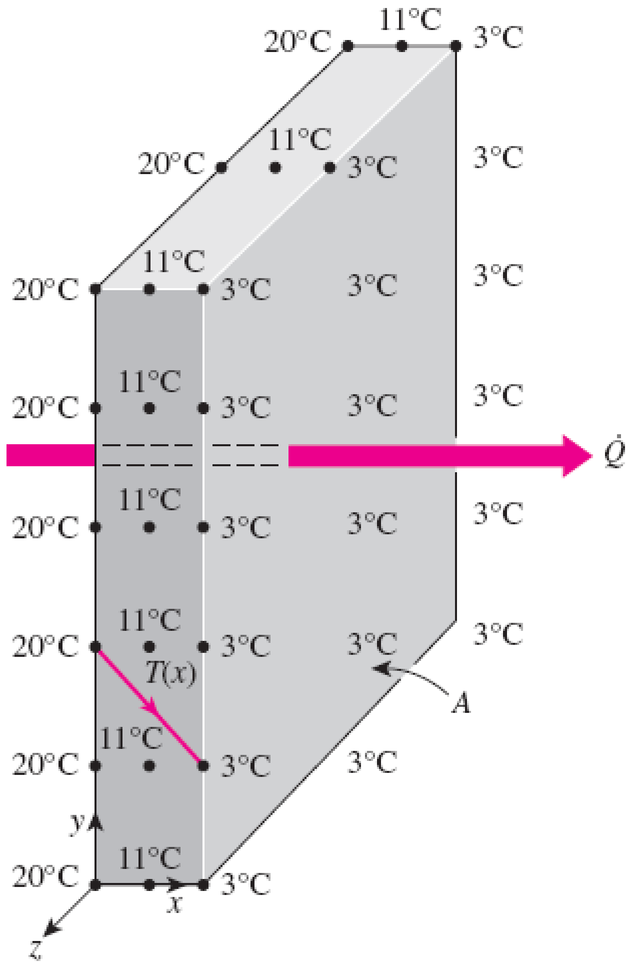


FIGURE 3–1
Heat transfer through a wall is one-dimensional when the temperature of the wall varies in one direction only.

Heat transfer through the wall of a house can be modeled as *steady* and *one-dimensional.*

The temperature of the wall in this case depends on one direction only (say the *x*-direction) and can be expressed as $T(x)$.

$$\begin{pmatrix}\text{Rate of}\\ \text{heat transfer}\\ \text{into the wall}\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\ \text{heat transfer}\\ \text{out of the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{of the wall}\end{pmatrix}$$

$$\dot{Q}_{in} - \dot{Q}_{out} = \frac{dE_{wall}}{dt}$$

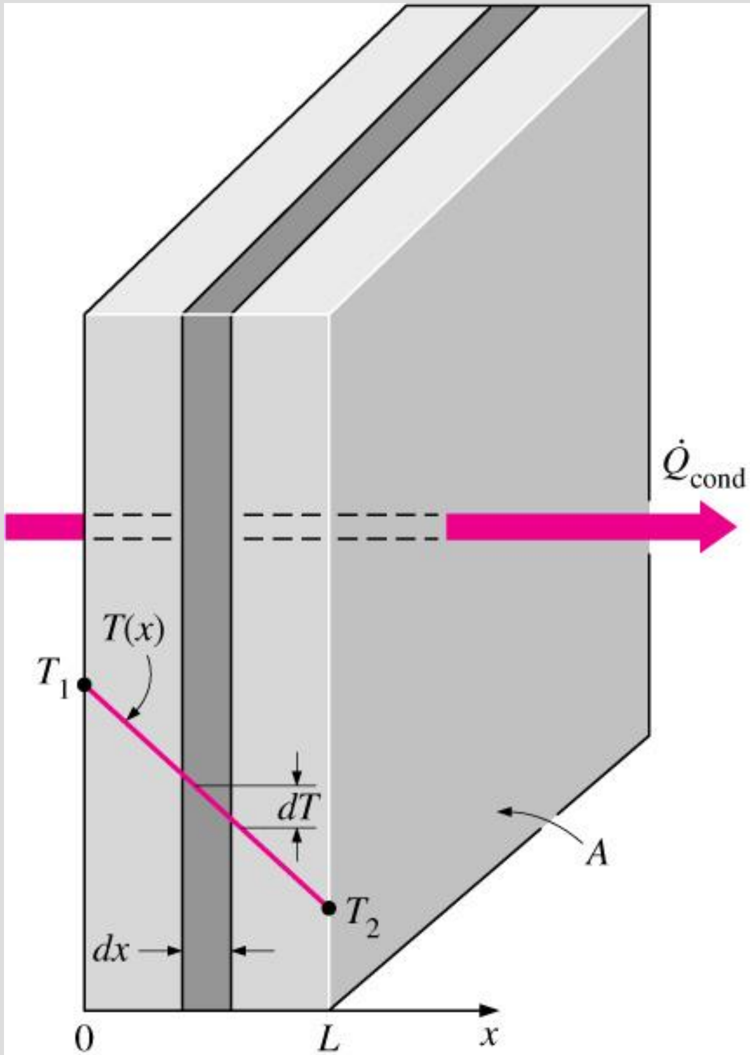
$$dE_{wall}/dt = 0$$

for *steady* operation

In steady operation, the rate of heat transfer through the wall is constant.

$$\dot{Q}_{cond,\ wall} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

Fourier's law of heat conduction



Under steady conditions, the temperature distribution in a plane wall is a straight line: ***dT*/*dx*** **= const.**

$$\dot{Q}_{\text{cond, wall}} = -kA\frac{dT}{dx}$$

$$\int_{x=0}^{L} \dot{Q}_{\text{cond, wall}}\, dx = -\int_{T=T_1}^{T_2} kA\, dT$$

$$\dot{Q}_{\text{cond, wall}} = kA\frac{T_1 - T_2}{L} \qquad \text{(W)}$$

The rate of heat conduction through a plane wall is proportional to the average thermal conductivity, the wall area, and the temperature difference, but is inversely proportional to the wall thickness.

Once the rate of heat conduction is available, the temperature $T(x)$ at any location x can be determined by replacing T_2 by T, and L by x.

Thermal Resistance Concept

$$\dot{Q}_{\text{cond, wall}} = kA\frac{T_1 - T_2}{L}$$

$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \qquad \text{(W)}$$

$$R_{\text{wall}} = \frac{L}{kA} \qquad (^\circ\text{C/W})$$

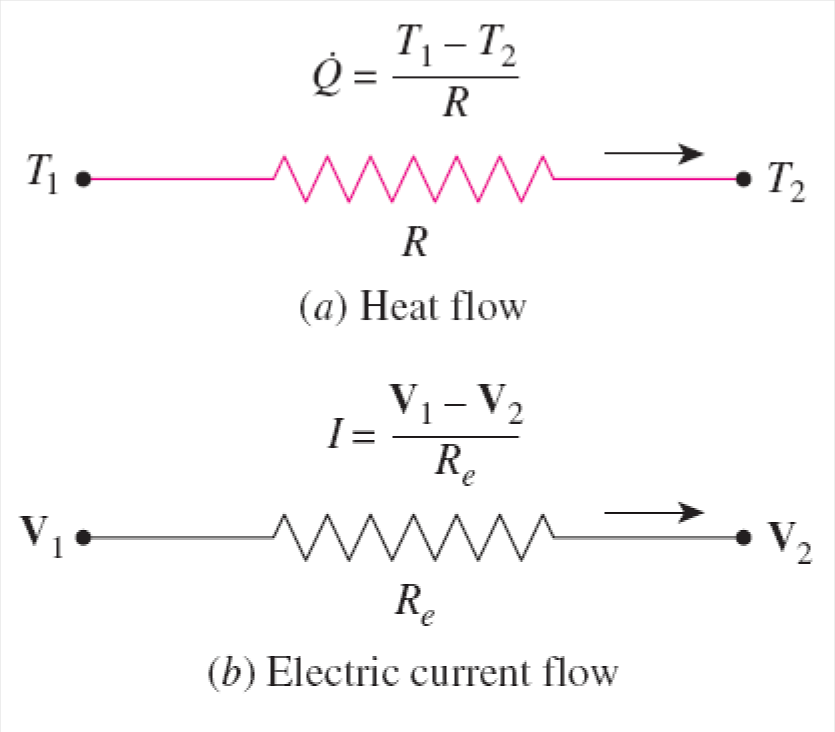
Conduction resistance of the wall: *Thermal resistance* of the wall against heat conduction.

Thermal resistance of a medium depends on the *geometry* and the *thermal properties* of the medium.

$$I = \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_e}$$

$$R_e = L/\sigma_e A$$

Electrical resistance



(*a*) Heat flow

(*b*) Electric current flow

Analogy between thermal and electrical resistance concepts.

rate of heat transfer → electric current
thermal resistance → electrical resistance
temperature difference → voltage difference

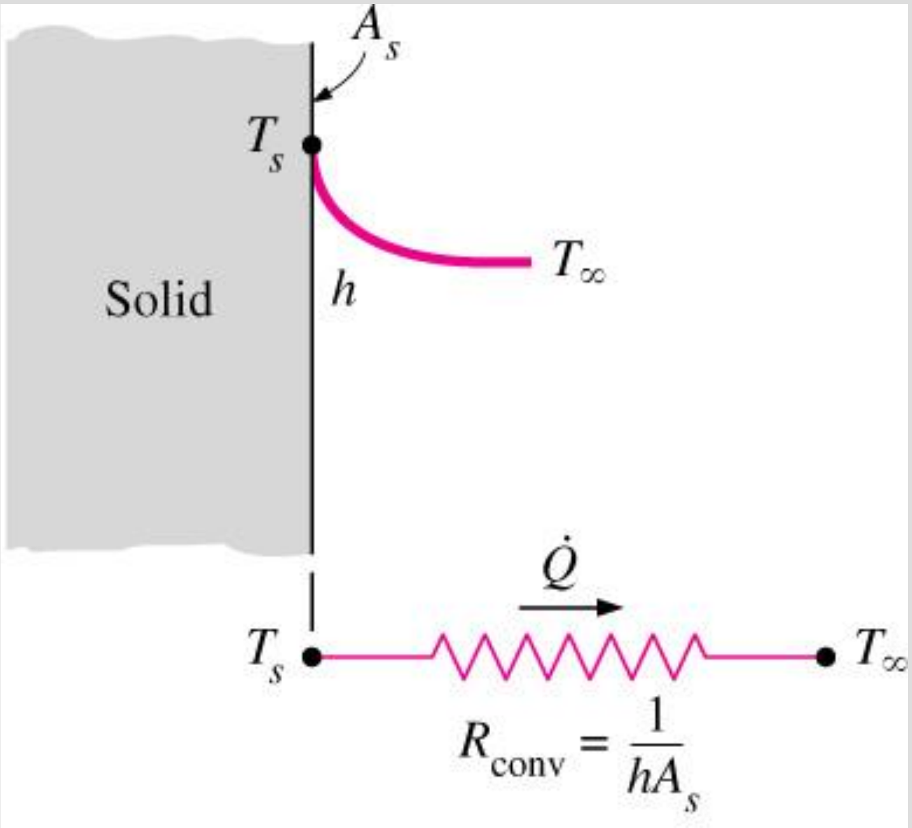
Newton's law of cooling

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$$

$$\dot{Q}_{\text{conv}} = \frac{T_s - T_\infty}{R_{\text{conv}}} \qquad \text{(W)}$$

$$R_{\text{conv}} = \frac{1}{hA_s} \qquad (^\circ\text{C/W})$$

Convection resistance of the surface: *Thermal resistance* of the surface against heat convection.



Schematic for convection resistance at a surface.

When the convection heat transfer coefficient is very large ($h \to \infty$), the convection resistance becomes *zero* and $T_s \approx T$.

That is, the surface offers *no resistance to convection*, and thus it does not slow down the heat transfer process.

This situation is approached in practice at surfaces where boiling and condensation occur.

$$\dot{Q}_{rad} = \varepsilon\sigma A_s(T_s^4 - T_{surr}^4) = h_{rad} A_s(T_s - T_{surr}) = \frac{T_s - T_{surr}}{R_{rad}}$$

$$R_{rad} = \frac{1}{h_{rad} A_s} \qquad \text{(K/W)}$$

Radiation resistance of the surface: *Thermal resistance* of the surface against radiation.

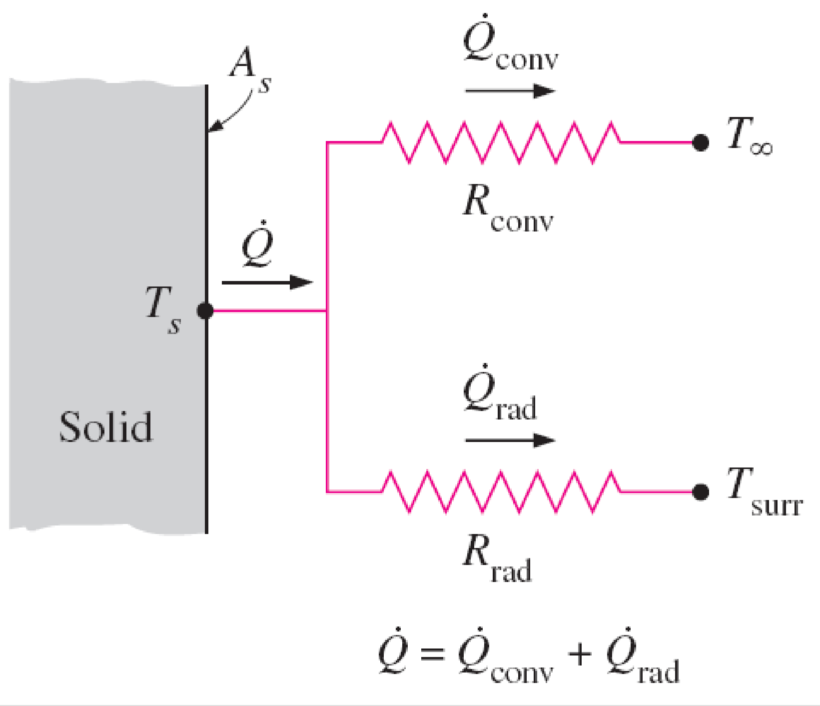
$$h_{rad} = \frac{\dot{Q}_{rad}}{A_s(T_s - T_{surr})} = \varepsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr}) \qquad (\text{W/m}^2 \cdot \text{K})$$

Radiation heat transfer coefficient

When $T_{surr} \approx T_\infty$

$$h_{combined} = h_{conv} + h_{rad}$$

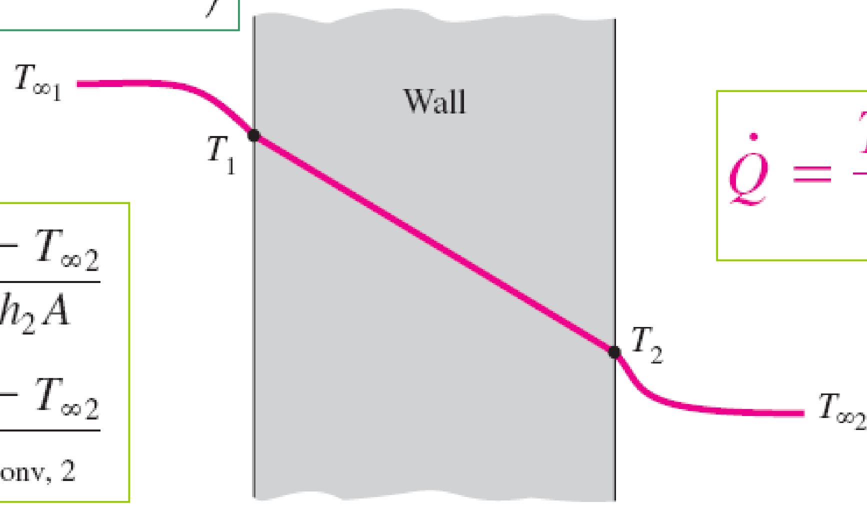
Combined heat transfer coefficient



Schematic for convection and radiation resistances at a surface.

Thermal Resistance Network

$$\begin{pmatrix}\text{Rate of}\\ \textit{heat convection}\\ \text{into the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \textit{heat conduction}\\ \text{through the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \textit{heat convection}\\ \text{from the wall}\end{pmatrix}$$



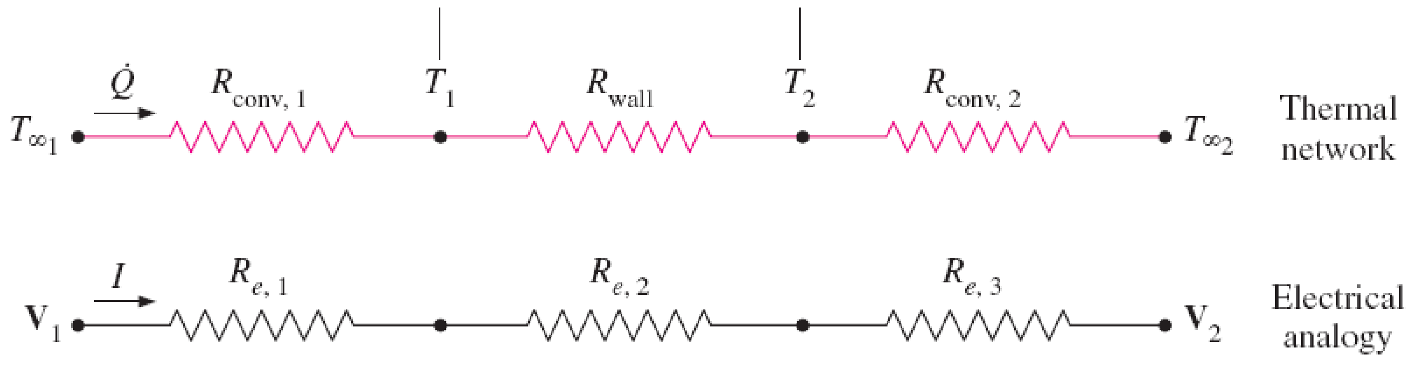
$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty 2}}{1/h_2 A}$$

$$= \frac{T_{\infty 1} - T_1}{R_{\text{conv, 1}}} = \frac{T_1 - T_2}{R_{\text{wall}}} = \frac{T_2 - T_{\infty 2}}{R_{\text{conv, 2}}}$$

$$\dot{Q} = \frac{T_{\infty_1} - T_{\infty_2}}{R_{\text{conv, 1}} + R_{\text{wall}} + R_{\text{conv, 2}}}$$

$$I = \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_{e,1} + R_{e,2} + R_{e,3}}$$



The thermal resistance network for heat transfer through a plane wall subjected to convection on both sides, and the electrical analogy.

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall}} + R_{\text{conv, 2}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \qquad (°\text{C/W})$$

Temperature drop

$$\Delta T = \dot{Q}R \qquad (°C)$$

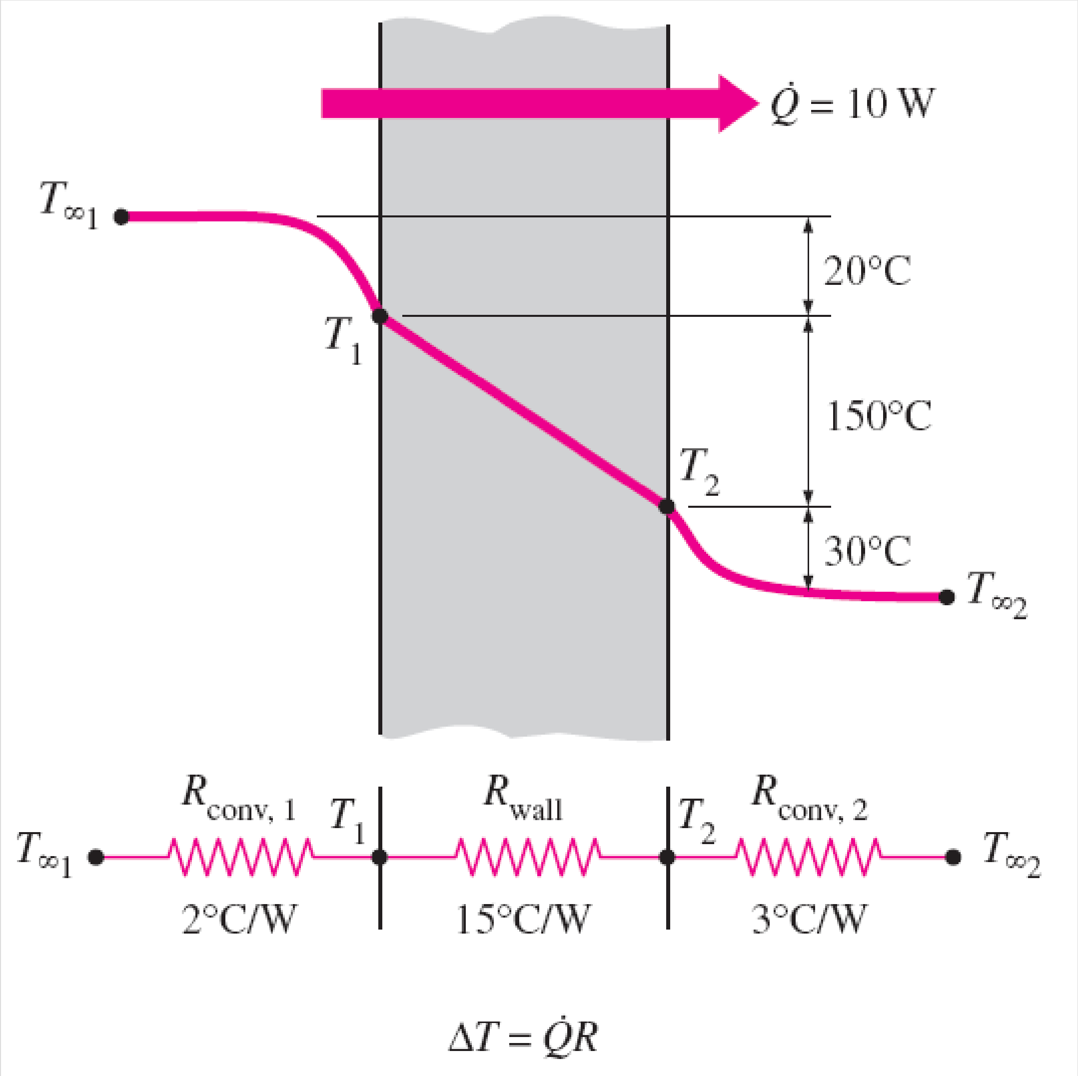
$$\dot{Q} = UA\,\Delta T \qquad (W)$$

$$UA = \frac{1}{R_{total}} \qquad (°C/K)$$

U overall heat transfer coefficient

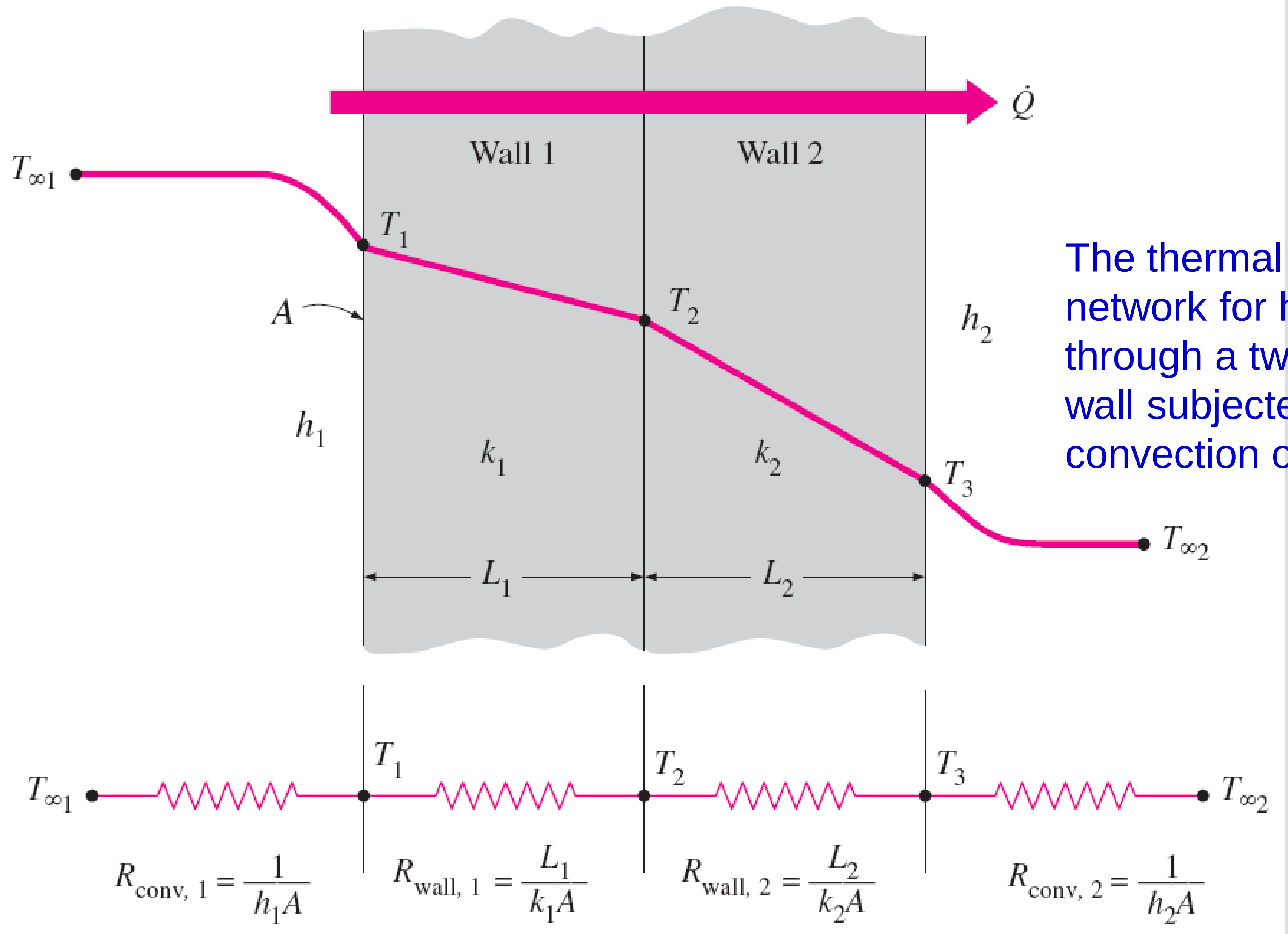
Once Q is evaluated, the surface temperature T_1 can be determined from

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{conv,\,1}} = \frac{T_{\infty 1} - T_1}{1/h_1 A}$$



The temperature drop across a layer is proportional to its thermal resistance.

Multilayer Plane Walls



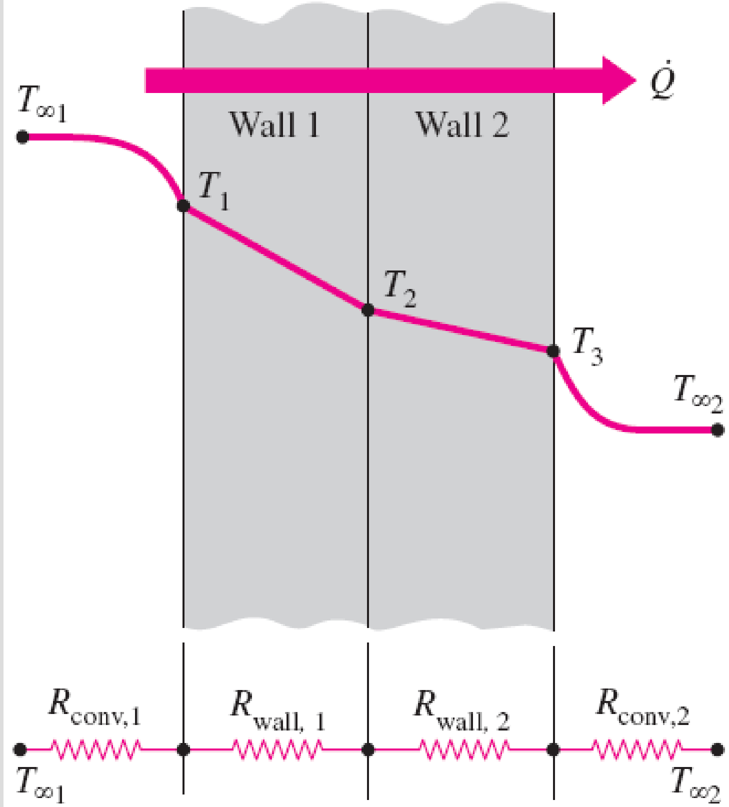
The thermal resistance network for heat transfer through a two-layer plane wall subjected to convection on both sides.

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall, 1}} + R_{\text{wall, 2}} + R_{\text{conv, 2}}$$
$$= \frac{1}{h_1 A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{1}{h_2 A}$$

$$\dot{Q} = \frac{T_i - T_j}{R_{\text{total}, i-j}}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{\text{conv}, 1} + R_{\text{wall}, 1}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1 A} + \dfrac{L_1}{k_1 A}}$$



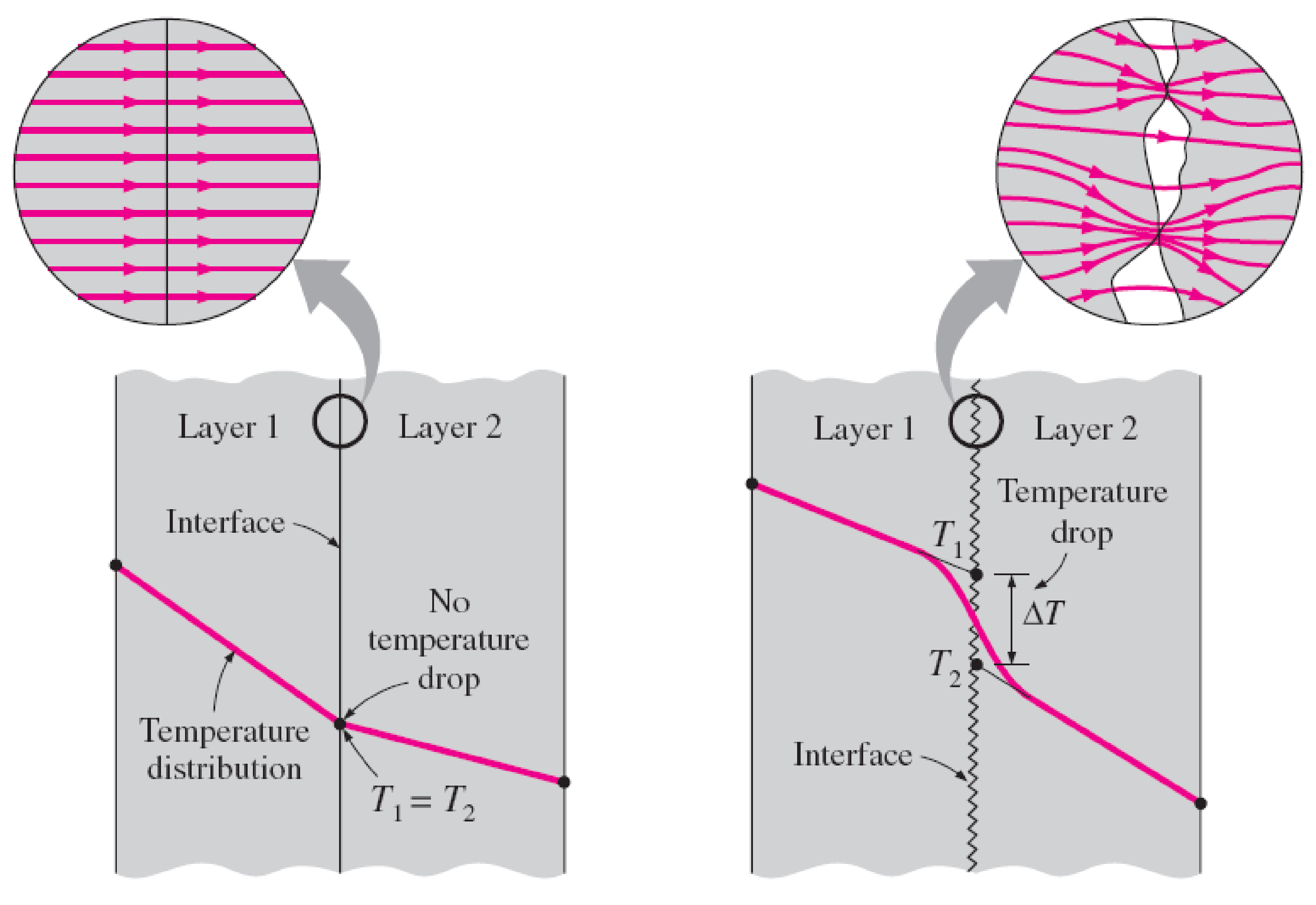
To find T_1: $\dot{Q} = \dfrac{T_{\infty1} - T_1}{R_{\text{conv},1}}$

To find T_2: $\dot{Q} = \dfrac{T_{\infty1} - T_2}{R_{\text{conv},1} + R_{\text{wall},1}}$

To find T_3: $\dot{Q} = \dfrac{T_3 - T_{\infty2}}{R_{\text{conv},2}}$

The evaluation of the surface and interface temperatures when $T_{\infty 1}$ and $T_{\infty 2}$ are given and $\dot{Q}$ is calculated.

THERMAL CONTACT RESISTANCE

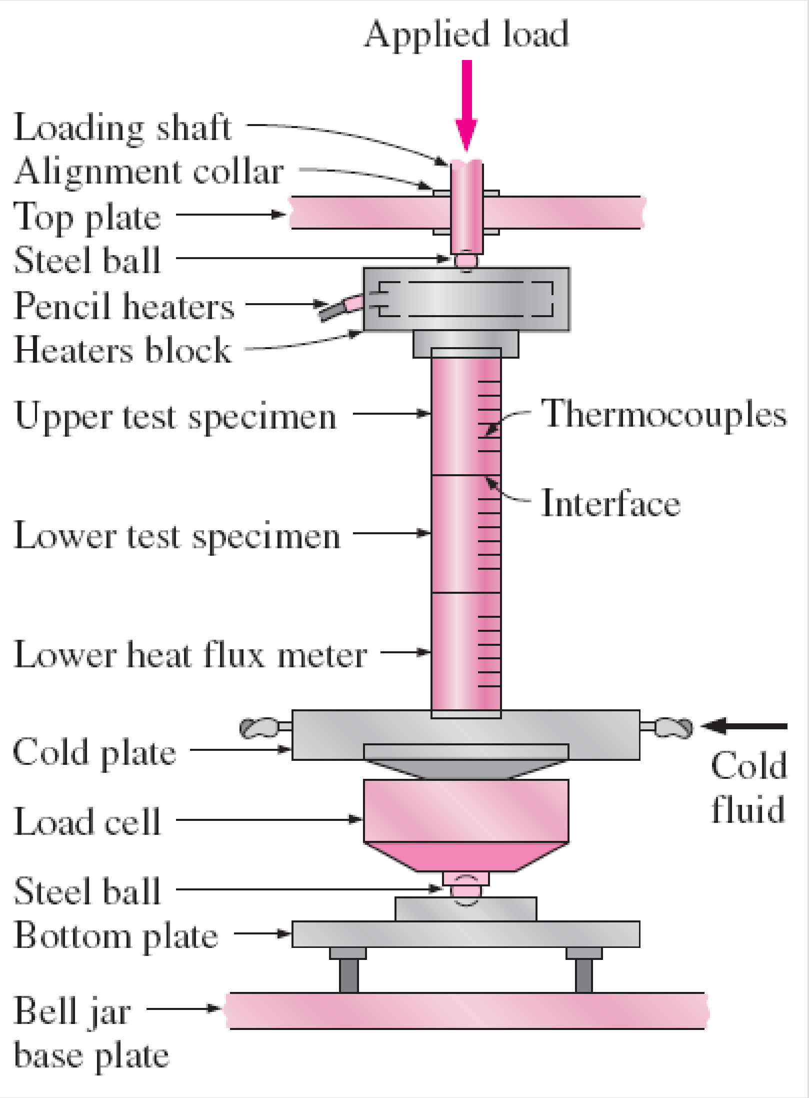


(*a*) Ideal (perfect) thermal contact

(*b*) Actual (imperfect) thermal contact

Temperature distribution and heat flow lines along two solid plates pressed against each other for the case of perfect and imperfect contact.

- When two such surfaces are pressed against each other, the peaks form good material contact but the valleys form voids filled with air.
- These numerous *air gaps* of varying sizes act as *insulation* because of the low thermal conductivity of air.
- Thus, an interface offers some resistance to heat transfer, and this resistance per unit interface area is called the **thermal contact resistance, R_c**.



A typical experimental setup for the determination of thermal contact resistance

$$\dot{Q} = \dot{Q}_{\text{contact}} + \dot{Q}_{\text{gap}}$$

$$\dot{Q} = h_c A \, \Delta T_{\text{interface}}$$

h_c thermal contact conductance

$$h_c = \frac{\dot{Q}/A}{\Delta T_{\text{interface}}} \qquad (\text{W/m}^2 \cdot {}^\circ\text{C})$$

$$R_c = \frac{1}{h_c} = \frac{\Delta T_{\text{interface}}}{\dot{Q}/A} \qquad (\text{m}^2 \cdot {}^\circ\text{C/W})$$

$$R_{c,\text{ insulation}} = \frac{L}{k} = \frac{0.01 \text{ m}}{0.04 \text{ W/m} \cdot {}^\circ\text{C}} = 0.25 \text{ m}^2 \cdot {}^\circ\text{C/W}$$

$$R_{c,\text{ copper}} = \frac{L}{k} = \frac{0.01 \text{ m}}{386 \text{ W/m} \cdot {}^\circ\text{C}} = 0.000026 \text{ m}^2 \cdot {}^\circ\text{C/W}$$

The value of thermal contact resistance depends on:

- *surface roughness,*
- *material properties,*
- *temperature* and *pressure* at the interface
- *type of fluid* trapped at the interface.

Thermal contact resistance is significant and can even dominate the heat transfer for good heat conductors such as metals, but can be disregarded for poor heat conductors such as insulations.

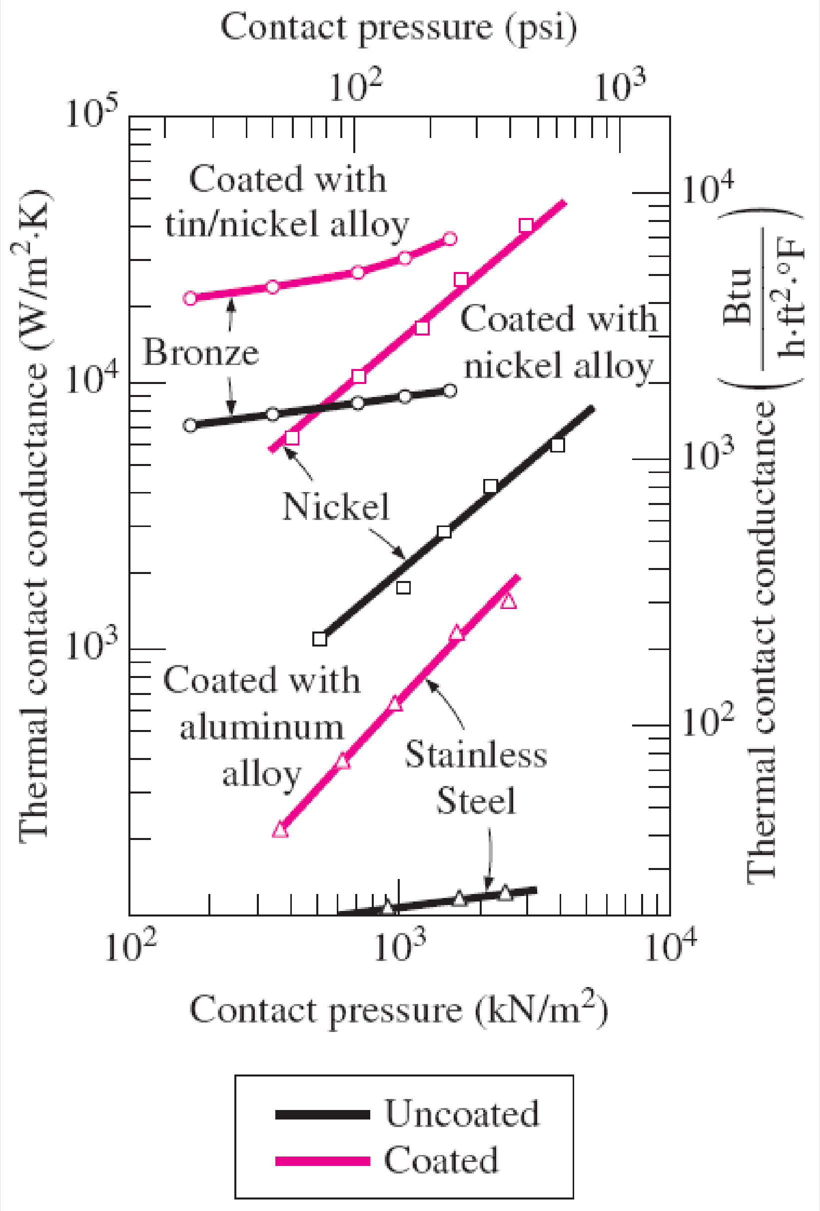
TABLE 3–1

Thermal contact conductance for aluminum plates with different fluids at the interface for a surface roughness of 10 μm and interface pressure of 1 atm (from Fried, 1969).

Fluid at the interface	Contact conductance, h_c, $W/m^2 \cdot K$
Air	3640
Helium	9520
Hydrogen	13,900
Silicone oil	19,000
Glycerin	37,700

The thermal contact resistance can be minimized by applying

- a *thermal grease* such as silicon oil
- a *better conducting gas* such as helium or hydrogen
- a *soft metallic foil* such as tin, silver, copper, nickel, or aluminum



Effect of metallic coatings on thermal contact conductance

TABLE 3–2

Thermal contact conductance of some metal surfaces in air (from various sources)

Material	Surface condition	Roughness, μm	Temperature, °C	Pressure, MPa	h_c,* W/m²·K
Identical Metal Pairs					
416 Stainless steel	Ground	2.54	90–200	0.17–2.5	3800
304 Stainless steel	Ground	1.14	20	4–7	1900
Aluminum	Ground	2.54	150	1.2–2.5	11,400
Copper	Ground	1.27	20	1.2–20	143,000
Copper	Milled	3.81	20	1–5	55,500
Copper (vacuum)	Milled	0.25	30	0.17–7	11,400
Dissimilar Metal Pairs					
Stainless steel–				10	2900
Aluminum		20–30	20	20	3600
Stainless steel–				10	16,400
Aluminum		1.0–2.0	20	20	20,800
Steel Ct-30–				10	50,000
Aluminum	Ground	1.4–2.0	20	15–35	59,000
Steel Ct-30–				10	4800
Aluminum	Milled	4.5–7.2	20	30	8300
				5	42,000
Aluminum-Copper	Ground	1.17–1.4	20	15	56,000
				10	12,000
Aluminum-Copper	Milled	4.4–4.5	20	20–35	22,000

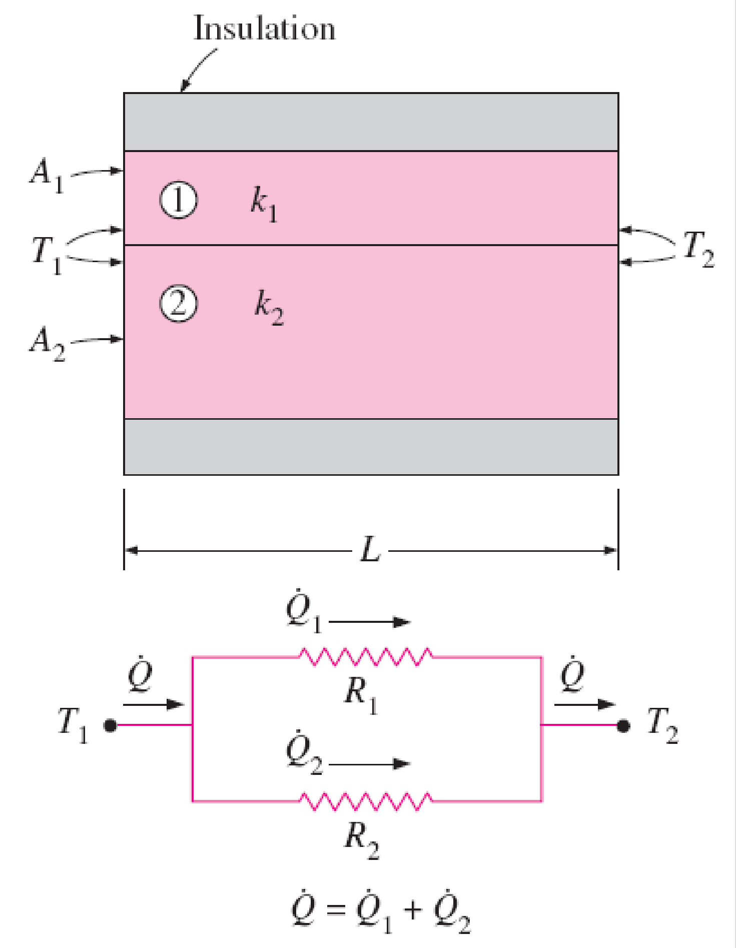
The *thermal contact conductance* is *highest* (and thus the contact resistance is lowest) for *soft metals* with *smooth surfaces* at *high pressure.*

GENERALIZED THERMAL RESISTANCE NETWORKS

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\dot{Q} = \frac{T_1 - T_2}{R_{\text{total}}}$$

$$\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2} \longrightarrow R_{\text{total}} = \frac{R_1 R_2}{R_1 + R_2}$$



Thermal resistance network for two parallel layers.

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}}$$

$$R_{\text{total}} = R_{12} + R_3 + R_{\text{conv}} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{\text{conv}}$$

$$R_1 = \frac{L_1}{k_1 A_1} \qquad R_2 = \frac{L_2}{k_2 A_2}$$

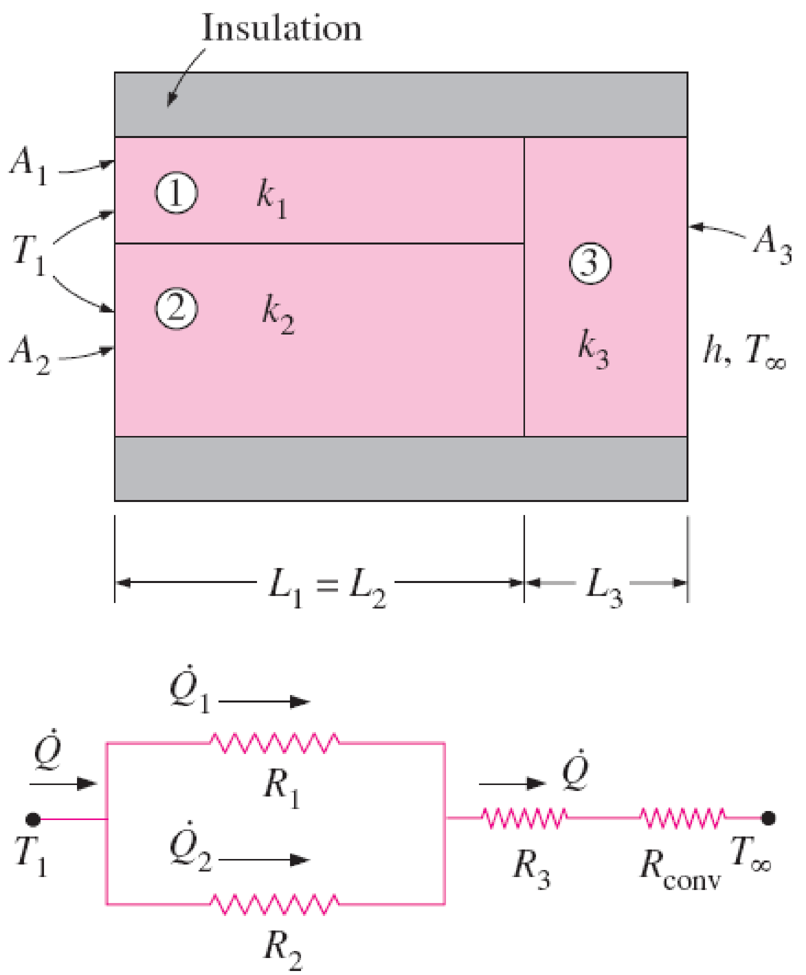
$$R_3 = \frac{L_3}{k_3 A_3} \qquad R_{\text{conv}} = \frac{1}{h A_3}$$

Two assumptions in solving complex multidimensional heat transfer problems by treating them as one-dimensional using the thermal resistance network are

(1) any plane wall normal to the *x*-axis is *isothermal* (i.e., to assume the temperature to vary in the *x*-direction only)

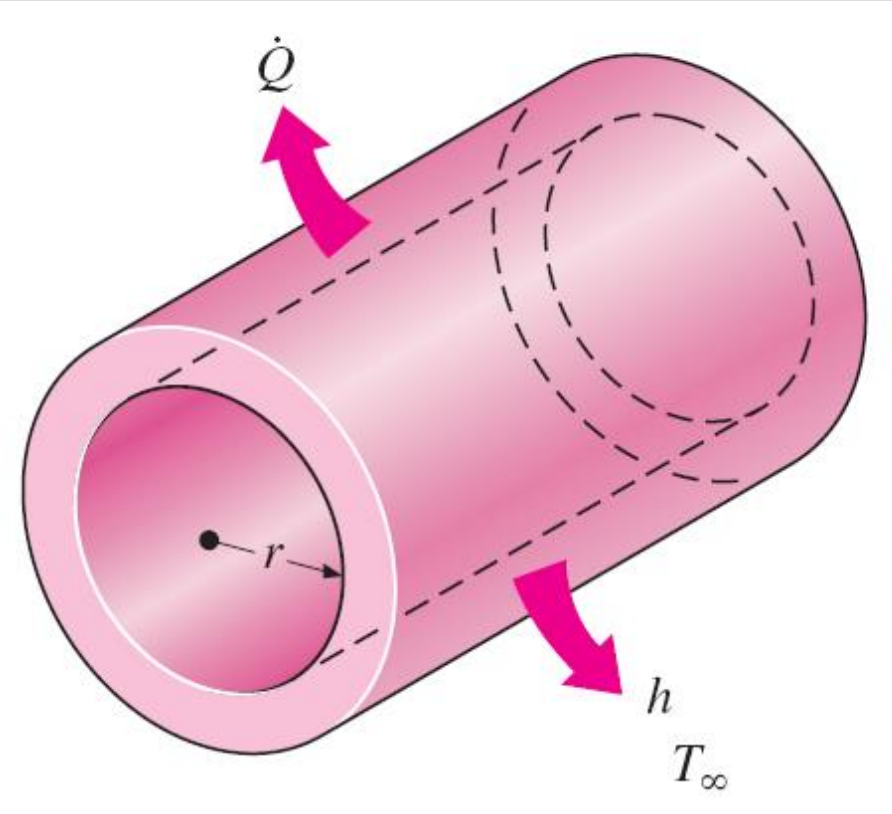
(2) any plane parallel to the *x*-axis is *adiabatic* (i.e., to assume heat transfer to occur in the *x*-direction only)

Do they give the same result?



Thermal resistance network for combined series-parallel arrangement.

HEAT CONDUCTION IN CYLINDERS AND SPHERES



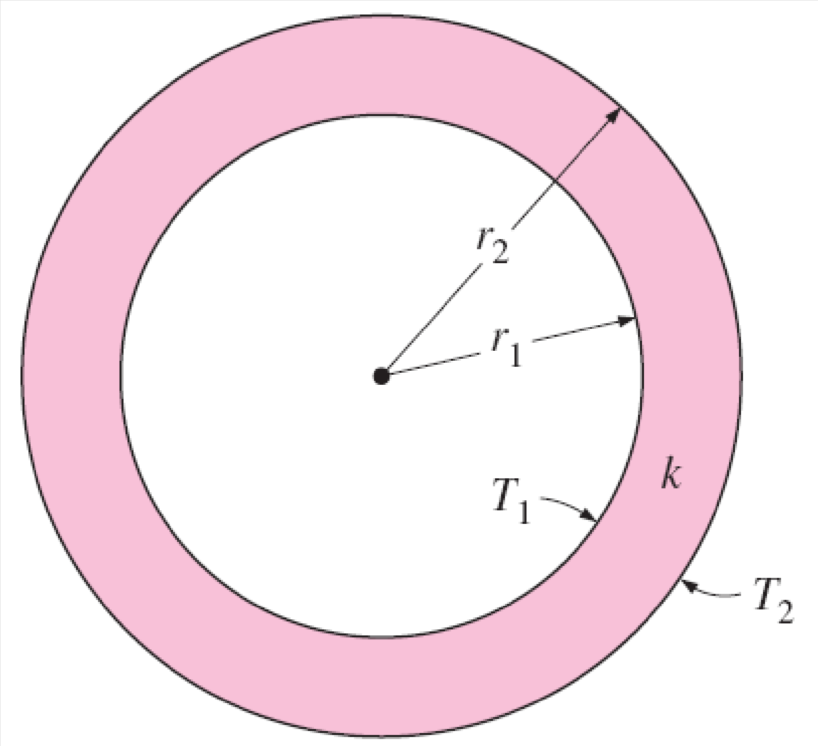
Heat is lost from a hot-water pipe to the air outside in the radial direction, and thus heat transfer from a long pipe is one-dimensional.

Heat transfer through the pipe can be modeled as *steady* and *one-dimensional.*

The temperature of the pipe depends on one direction only (the radial *r*-direction) and can be expressed as $T = T(r)$.

The temperature is independent of the azimuthal angle or the axial distance.

This situation is approximated in practice in long cylindrical pipes and spherical containers.



A long cylindrical pipe (or spherical shell) with specified inner and outer surface temperatures T_1 and T_2.

$$\dot{Q}_{\text{cond, cyl}} = -kA\frac{dT}{dr} \qquad \text{(W)}$$

$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{\text{cond, cyl}}}{A}\, dr = -\int_{T=T_1}^{T_2} k\, dT$$

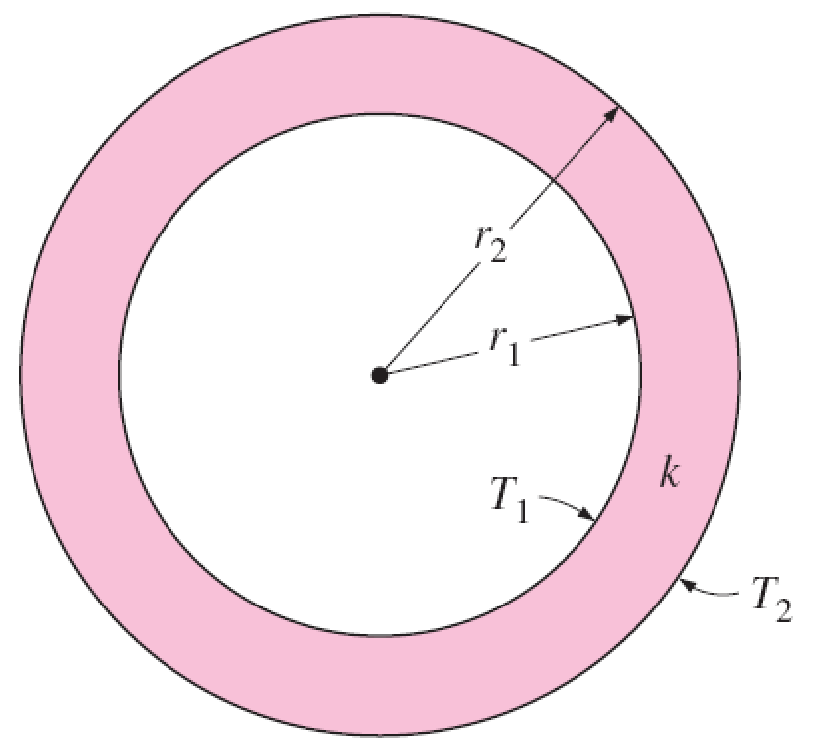
$$A = 2\pi rL$$

$$\dot{Q}_{\text{cond, cyl}} = 2\pi Lk\frac{T_1 - T_2}{\ln(r_2/r_1)} \qquad \text{(W)}$$

$$\dot{Q}_{\text{cond, cyl}} = \frac{T_1 - T_2}{R_{\text{cyl}}} \qquad \text{(W)}$$

$$R_{\text{cyl}} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{Outer radius/Inner radius})}{2\pi \times \text{Length} \times \text{Thermal conductivity}}$$

Conduction resistance of the cylinder layer

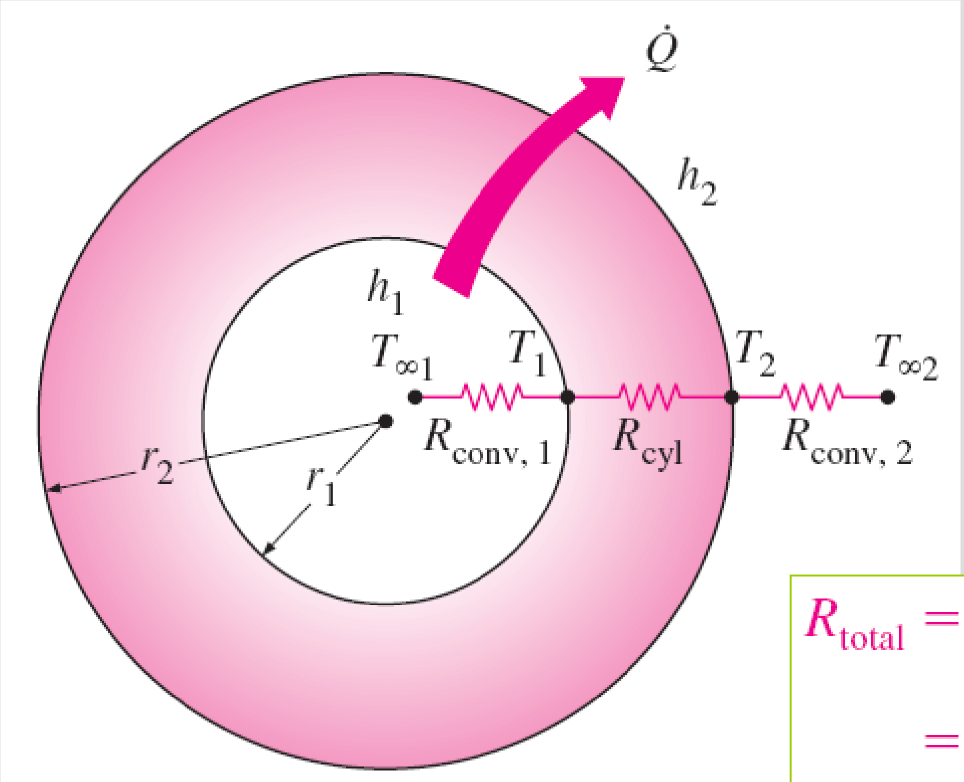


A spherical shell with specified inner and outer surface temperatures T_1 and T_2.

$$\dot{Q}_{\text{cond, sph}} = \frac{T_1 - T_2}{R_{\text{sph}}}$$

$$R_{\text{sph}} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{Outer radius} - \text{Inner radius}}{4\pi(\text{Outer radius})(\text{Inner radius})(\text{Thermal conductivity})}$$

Conduction resistance of the spherical layer



The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$

for a *cylindrical* layer

$$\begin{aligned} R_{\text{total}} &= R_{\text{conv, 1}} + R_{\text{cyl}} + R_{\text{conv, 2}} \\ &= \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2} \end{aligned}$$

for a *spherical* layer

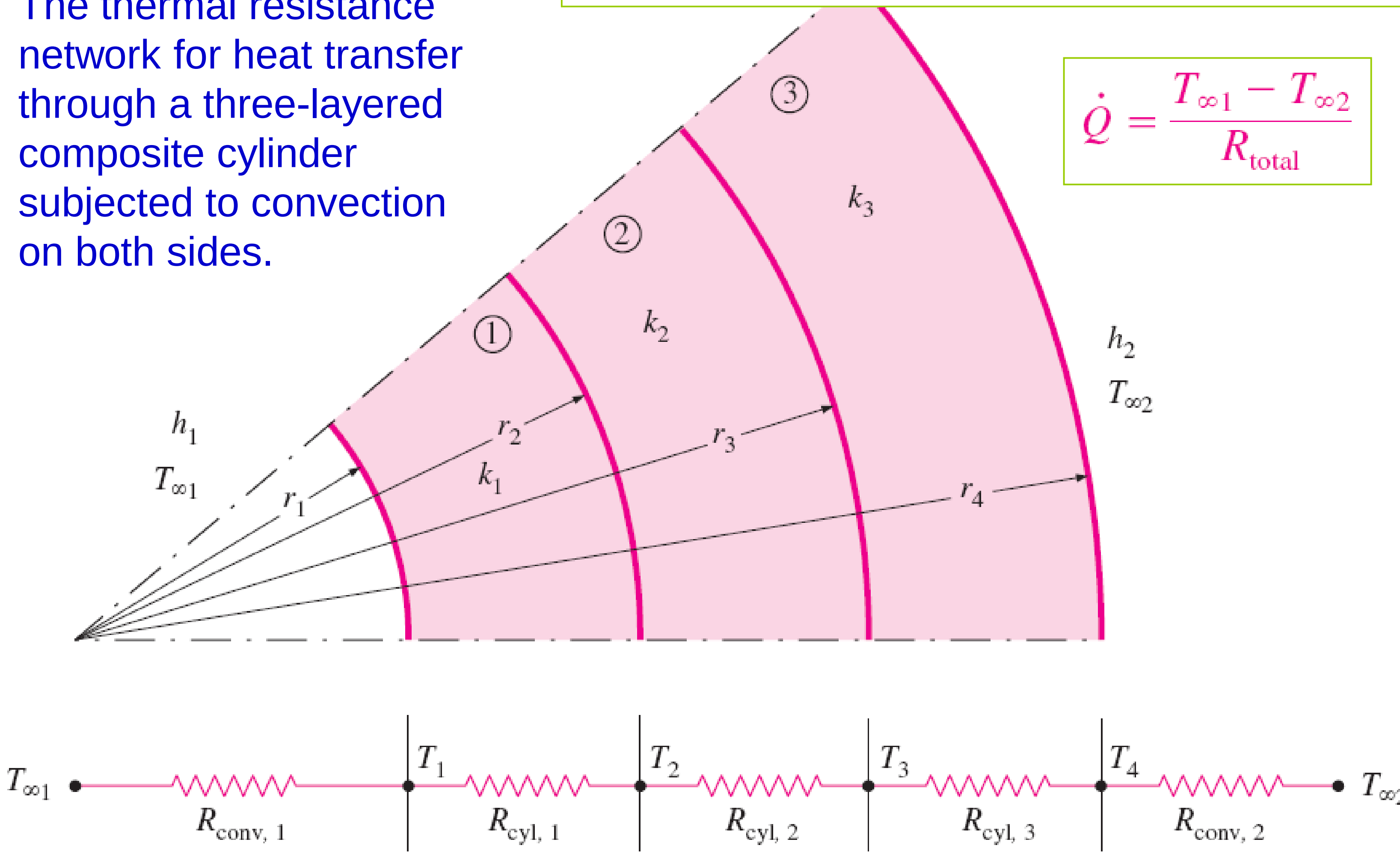
$$\begin{aligned} R_{\text{total}} &= R_{\text{conv, 1}} + R_{\text{sph}} + R_{\text{conv, 2}} \\ &= \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2} \end{aligned}$$

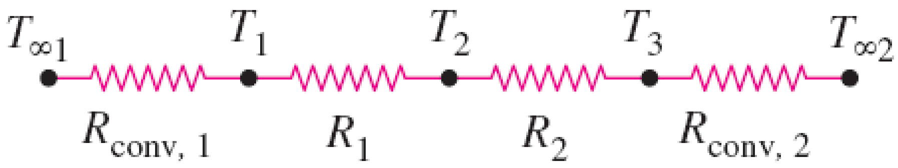
Multilayered Cylinders and Spheres

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl},1} + R_{\text{cyl},2} + R_{\text{cyl},3} + R_{\text{conv},2}$$
$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4}$$

The thermal resistance network for heat transfer through a three-layered composite cylinder subjected to convection on both sides.

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$





The ratio $\Delta T/R$ across any layer is equal to $\dot{Q}$, which remains constant in one-dimensional steady conduction.

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{\text{conv},1}}$$
$$= \frac{T_{\infty 1} - T_2}{R_{\text{conv},1} + R_1}$$
$$= \frac{T_1 - T_3}{R_1 + R_2}$$
$$= \frac{T_2 - T_3}{R_2}$$
$$= \frac{T_2 - T_{\infty 2}}{R_2 + R_{\text{conv},2}}$$
$$= \cdots$$

Once heat transfer rate Q has been calculated, the interface temperature T_2 can be determined from any of the following two relations:

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{\text{conv, 1}} + R_{\text{cyl, 1}}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1(2\pi r_1 L)} + \dfrac{\ln(r_2/r_1)}{2\pi L k_1}}$$

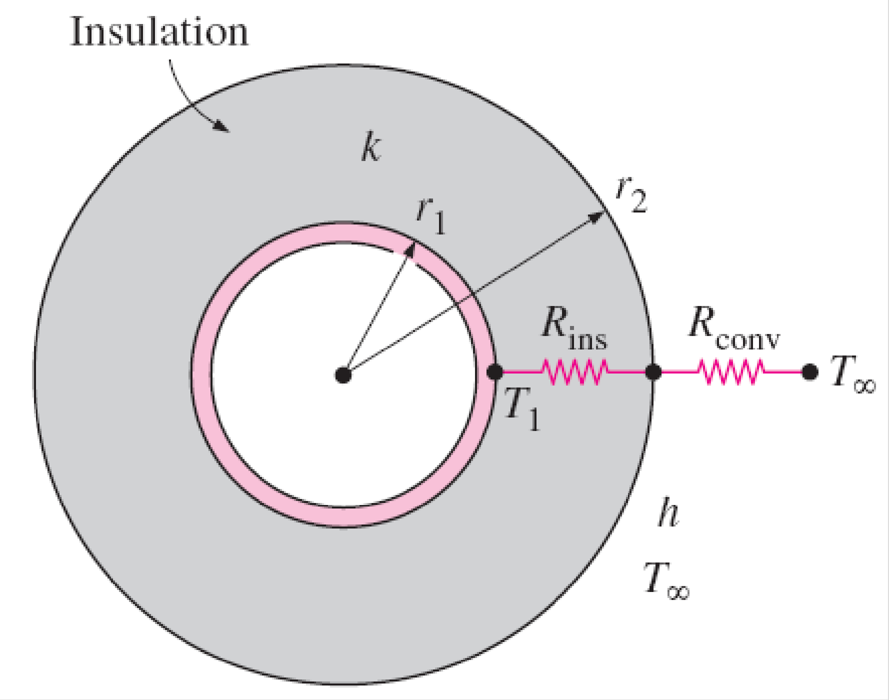
$$\dot{Q} = \frac{T_2 - T_{\infty 2}}{R_2 + R_3 + R_{\text{conv, 2}}} = \frac{T_2 - T_{\infty 2}}{\dfrac{\ln(r_3/r_2)}{2\pi L k_2} + \dfrac{\ln(r_4/r_3)}{2\pi L k_3} + \dfrac{1}{h_o(2\pi r_4 L)}}$$

CRITICAL RADIUS OF INSULATION

Adding more insulation to a wall or to the attic always decreases heat transfer since the heat transfer area is constant, and adding insulation always increases the thermal resistance of the wall without increasing the convection resistance.

In a a cylindrical pipe or a spherical shell, the additional insulation increases the conduction resistance of the insulation layer but decreases the convection resistance of the surface because of the increase in the outer surface area for convection.

The heat transfer from the pipe may increase or decrease, depending on which effect dominates.



An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{ins} + R_{conv}} = \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi Lk} + \dfrac{1}{h(2\pi r_2 L)}}$$

The critical radius of insulation for a cylindrical body:

$$r_{\text{cr, cylinder}} = \frac{k}{h} \qquad \text{(m)}$$

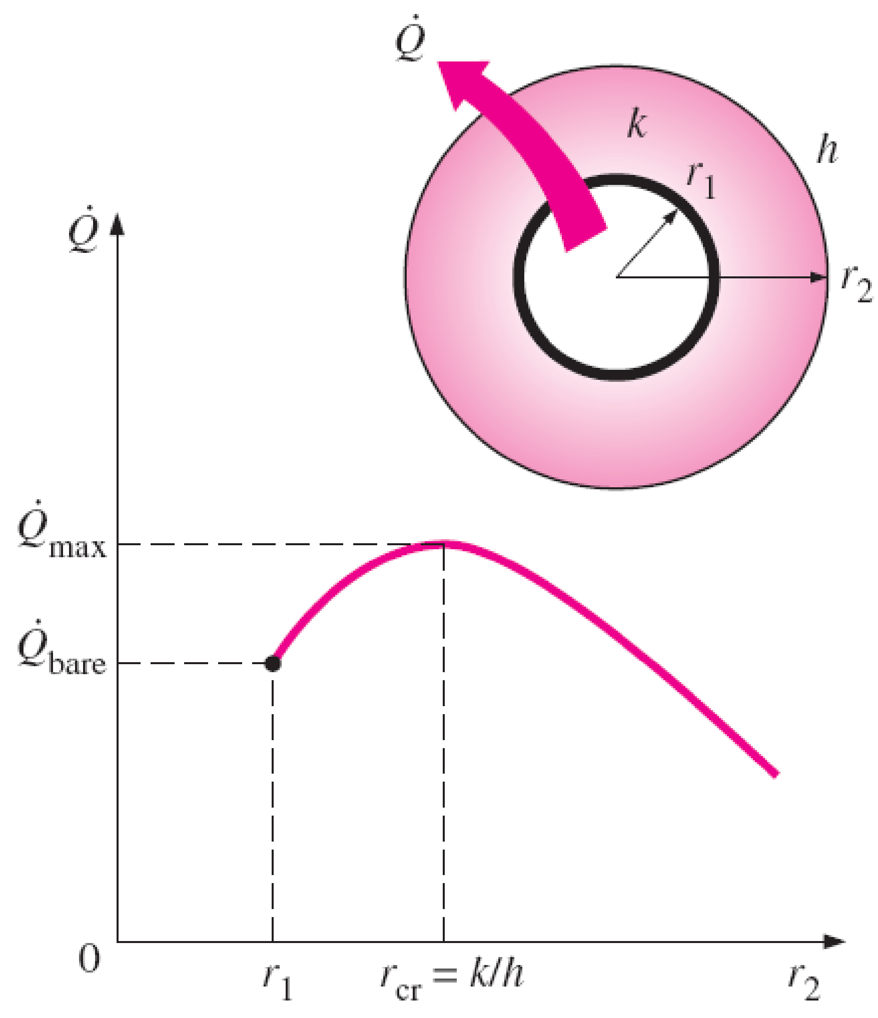
The critical radius of insulation for a spherical shell:

$$r_{\text{cr, sphere}} = \frac{2k}{h}$$

The largest value of the critical radius we are likely to encounter is

$$r_{\text{cr, max}} = \frac{k_{\text{max, insulation}}}{h_{\text{min}}} \approx \frac{0.05\ \text{W/m} \cdot {}^\circ\text{C}}{5\ \text{W/m}^2 \cdot {}^\circ\text{C}} = 0.01\ \text{m} = 1\ \text{cm}$$

We can insulate hot-water or steam pipes freely without worrying about the possibility of increasing the heat transfer by insulating the pipes.



The variation of heat transfer rate with the outer radius of the insulation r_2 when $r_1 < r_{cr}$.

HEAT TRANSFER FROM FINNED SURFACES

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$$

Newton's law of cooling: The rate of heat transfer from a surface to the surrounding medium

When T_s and T_∞ are fixed, *two ways* to increase the rate of heat transfer are

- To increase the *convection heat transfer coefficient h.* This may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate.
- To increase the *surface area A_s* by attaching to the surface *extended surfaces* called ***fins*** made of highly conductive materials such as aluminum.

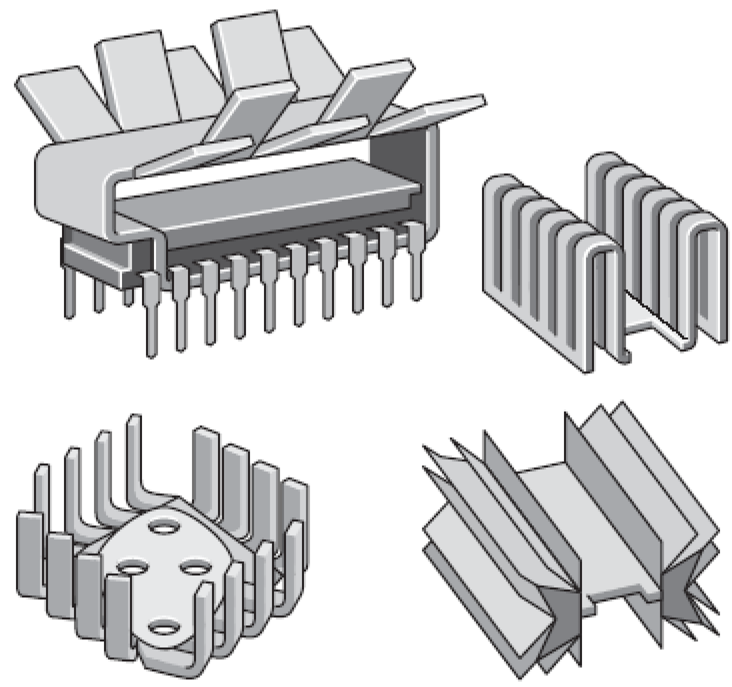
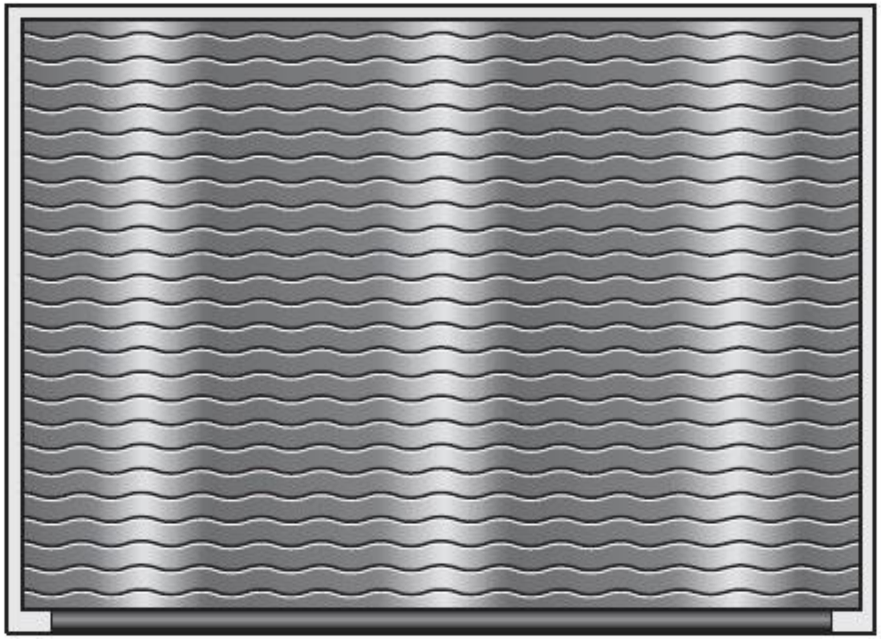


FIGURE 3–35
Some innovative fin designs.

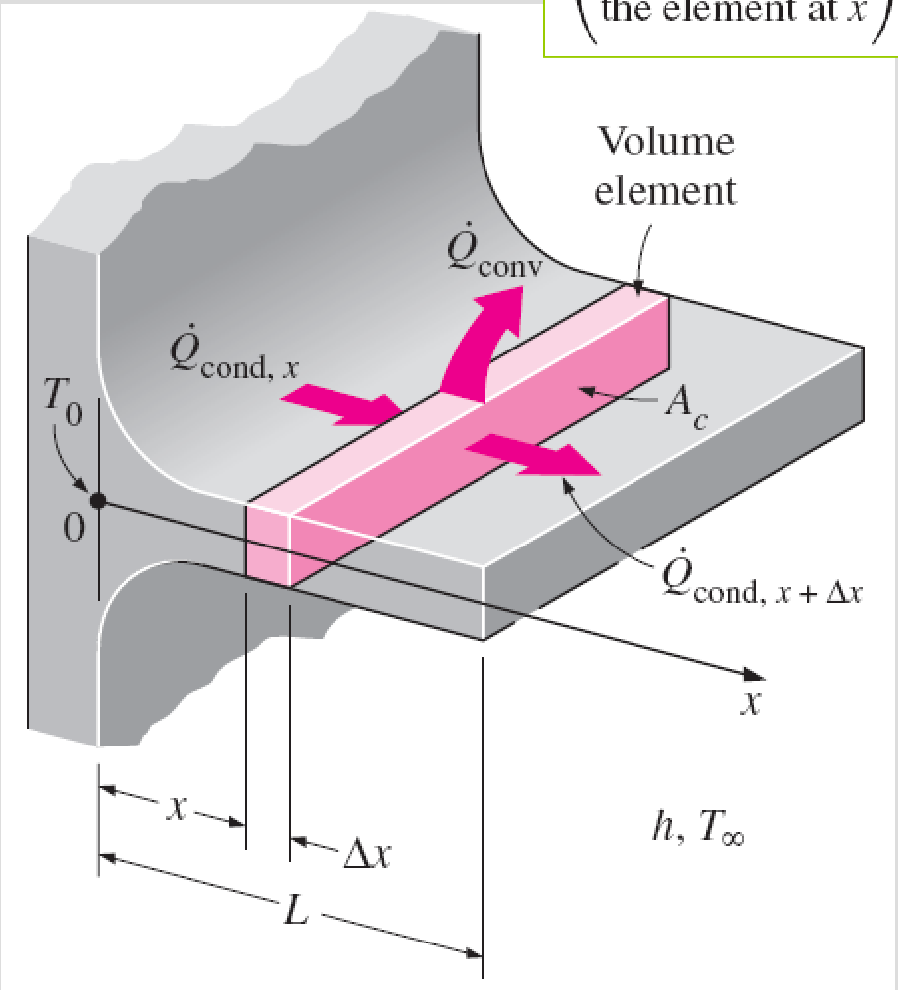
FIGURE 3–33
Presumed cooling fins on dinosaur stegosaurus. (© Alamy RF.)

The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air.



Fin Equation

$$\begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ into} \\ \text{the element at } x\end{pmatrix} = \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ from the} \\ \text{element at } x + \Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{convection} \text{ from} \\ \text{the element}\end{pmatrix}$$



Volume element of a fin at location *x* having a length of Δx, cross-sectional area of A_c, and perimeter of *p*.

$$\dot{Q}_{\text{cond, } x} = \dot{Q}_{\text{cond, } x + \Delta x} + \dot{Q}_{\text{conv}}$$

$$\dot{Q}_{\text{conv}} = h(p\,\Delta x)(T - T_\infty)$$

$$\frac{\dot{Q}_{\text{cond, } x + \Delta x} - \dot{Q}_{\text{cond, } x}}{\Delta x} + hp(T - T_\infty) = 0$$

$$\Delta x \to 0$$

$$\frac{d\dot{Q}_{\text{cond}}}{dx} + hp(T - T_\infty) = 0$$

$$\dot{Q}_{\text{cond}} = -kA_c \frac{dT}{dx}$$

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - hp(T - T_\infty) = 0$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

Differential equation

$$m^2 = \frac{hp}{kA_c}$$

$$\theta = T - T_\infty$$

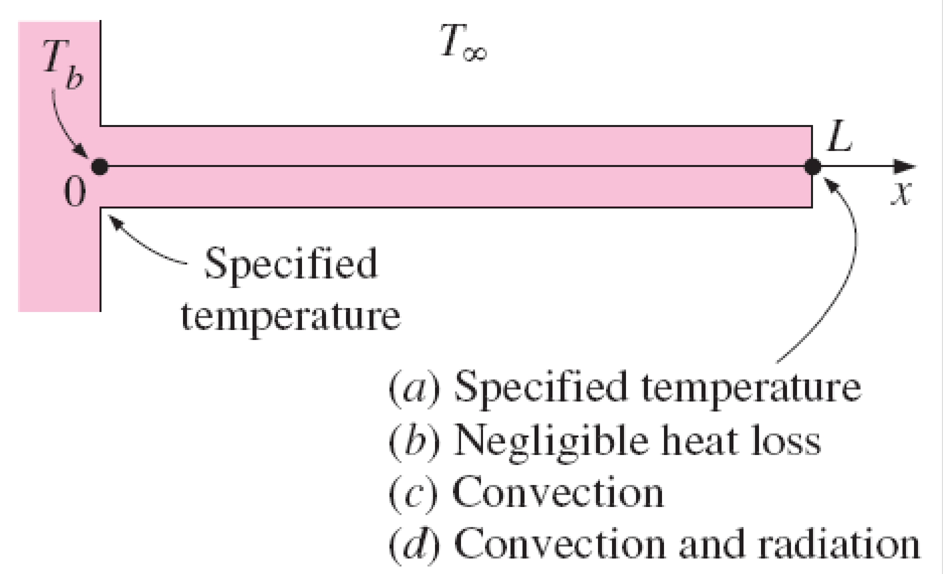
Temperature excess

The general solution of the differential equation

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Boundary condition at fin base

$$\theta(0) = \theta_b = T_b - T_\infty$$



Boundary conditions at the fin base and the fin tip.

1 Infinitely Long Fin ($T_{\text{fin tip}} = T_\infty$)

Boundary condition at fin tip

$$\theta(L) = T(L) - T_\infty = 0 \qquad L \rightarrow \infty$$

The variation of temperature along the fin

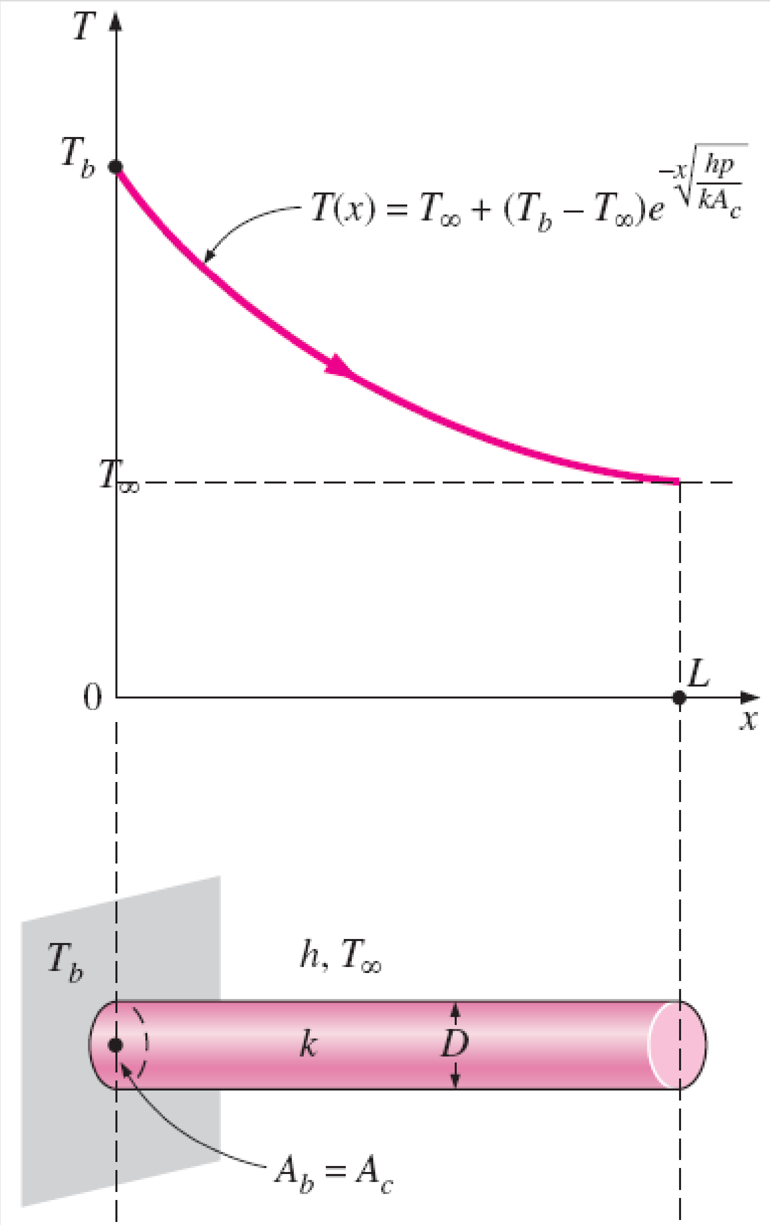
$$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx} = e^{-x\sqrt{hp/kA_c}}$$

$$\theta = T - T_\infty$$

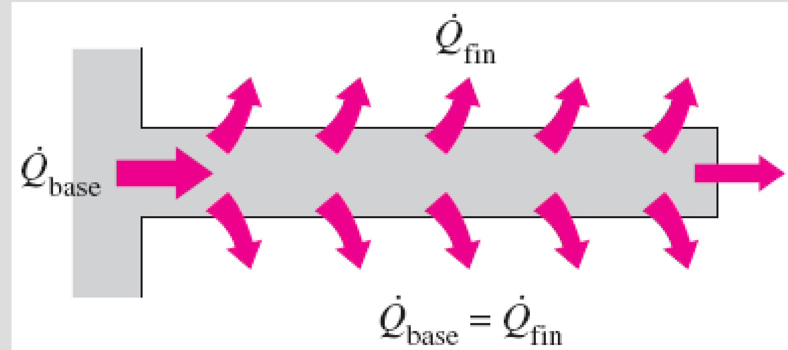
$$m = \sqrt{hp/kA_c}$$

The steady rate of *heat transfer* from the entire fin

$$\dot{Q}_{\text{long fin}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)$$



($p = \pi D$, $A_c = \pi D^2/4$ for a cylindrical fin)



Under steady conditions, heat transfer from the exposed surfaces of the fin is equal to heat conduction to the fin at the base.

The rate of heat transfer from the fin could also be determined by considering heat transfer from a differential volume element of the fin and integrating it over the entire surface of the fin:

$$\dot{Q}_{\text{fin}} = \int_{A_{\text{fin}}} h[T(x) - T_\infty]\, dA_{\text{fin}} = \int_{A_{\text{fin}}} h\theta(x)\, dA_{\text{fin}}$$

A long circular fin of uniform cross section and the variation of temperature along it.

2 Negligible Heat Loss from the Fin Tip (Adiabatic fin tip, $Q_{\text{fin tip}} = 0$)

Fins are not likely to be so long that their temperature approaches the surrounding temperature at the tip. A more realistic assumption is for heat transfer from the fin tip to be negligible since the surface area of the fin tip is usually a negligible fraction of the total fin area.

Boundary condition at fin tip

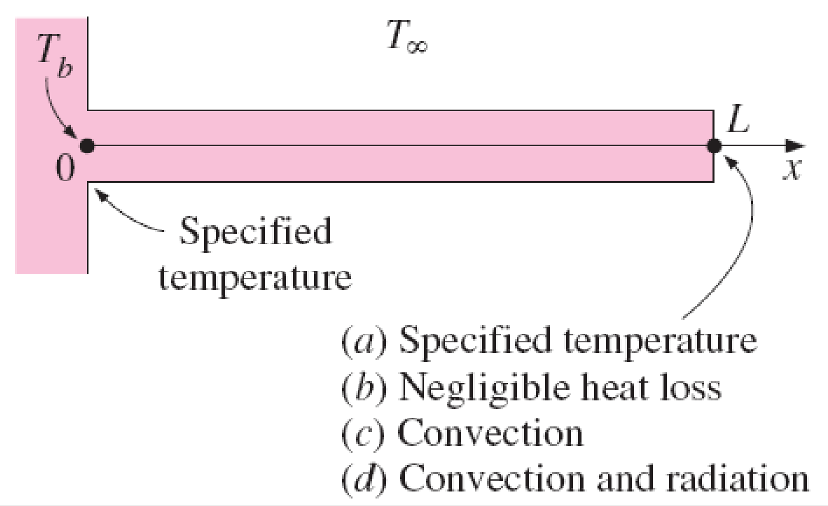
$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

The variation of temperature along the fin

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

Heat transfer from the entire fin

$$\dot{Q}_{\text{adiabatic tip}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL$$



3 Specified Temperature ($T_{\text{fin,tip}} = T_L$)

In this case the temperature at the end of the fin (the fin tip) is fixed at a specified temperature T_L.

This case could be considered as a generalization of the case of *Infinitely Long Fin* where the fin tip temperature was fixed at T_∞.

Boundary condition at fin tip: $\theta(L) = \theta_L = T_L - T_\infty$

Specified fin tip temperature:

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{[(T_L - T_\infty)/(T_b - T_\infty)]\sinh mx + \sinh m(L-x)}{\sinh mL}$$

Specified fin tip temperature:

$$\dot{Q}_{\text{specified temp.}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0}$$

$$= \sqrt{hpkA_c}(T_b - T_\infty)\frac{\cosh mL - [(T_L - T_\infty)/(T_b - T_\infty)]}{\sinh mL}$$

4 Convection from Fin Tip

The fin tips, in practice, are exposed to the surroundings, and thus the proper boundary condition for the fin tip is convection that may also include the effects of radiation. Consider the case of convection only at the tip. The condition at the fin tip can be obtained from an energy balance at the fin tip.

$$(\dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv}})$$

Boundary condition at fin tip: $$-kA_c \left.\frac{dT}{dx}\right|_{x=L} = hA_c[T(L) - T_\infty]$$

Convection from fin tip: $$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

Convection from fin tip:

$$\dot{Q}_{\text{convection}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0}$$

$$= \sqrt{hpkA_c}(T_b - T_\infty)\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

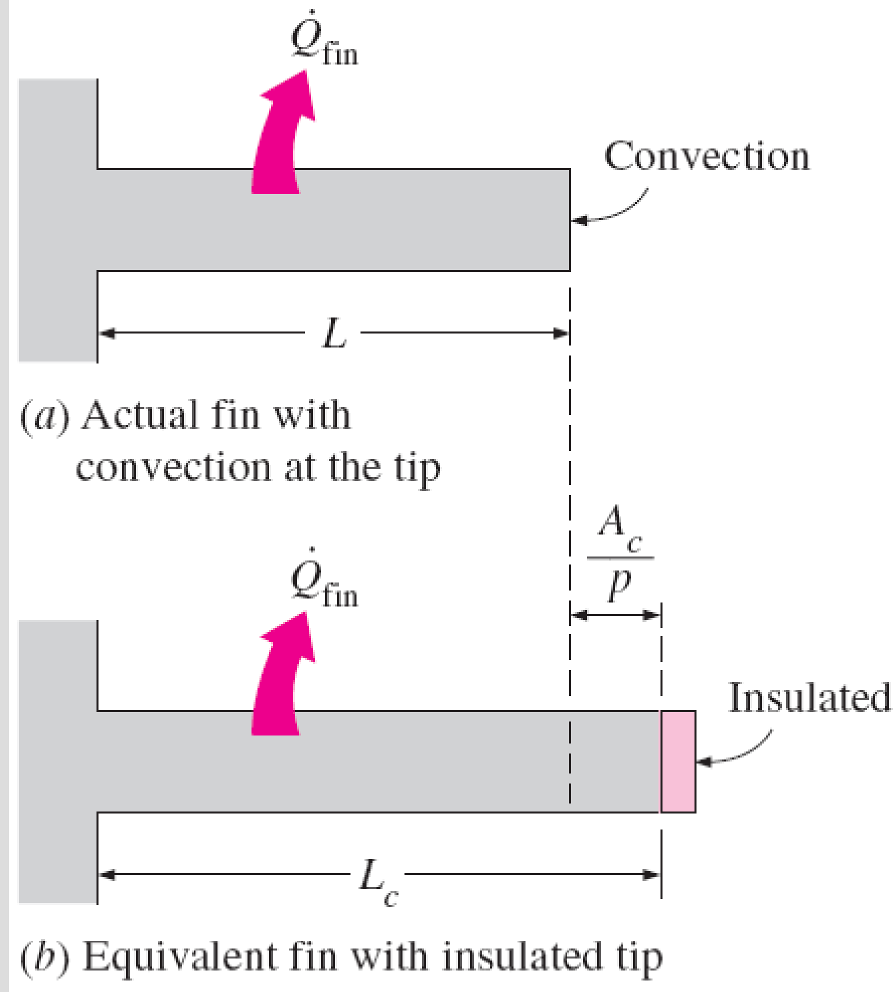
A practical way of accounting for the heat loss from the fin tip is to replace the *fin length L* in the relation for the *insulated tip* case by a **corrected length** defined as

$$L_c = L + \frac{A_c}{p}$$

$$L_{c,\text{ rectangular fin}} = L + \frac{t}{2}$$

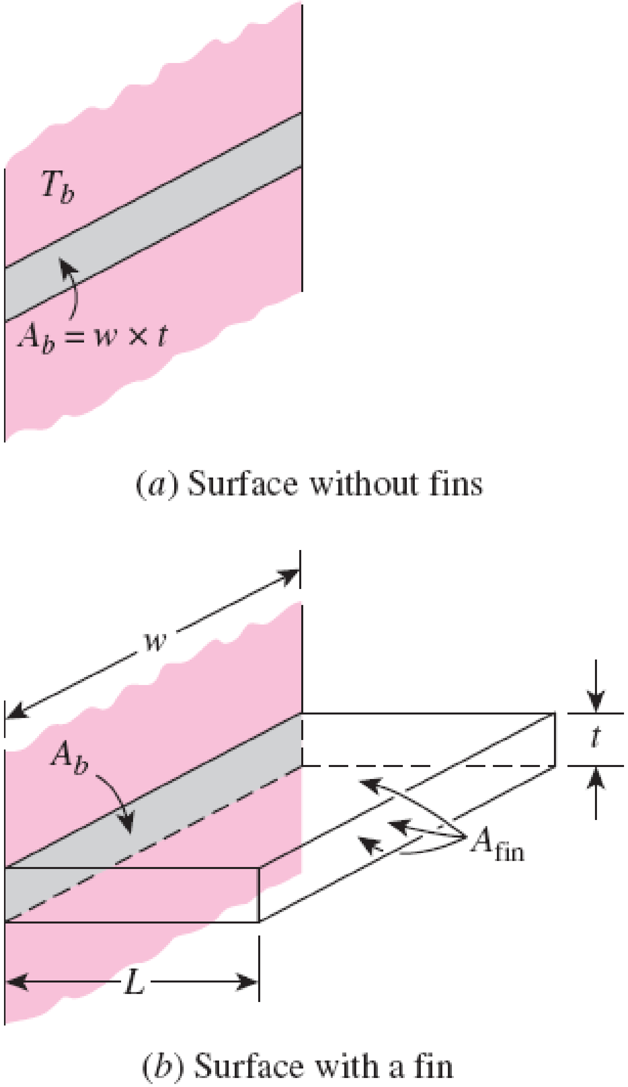
$$L_{c,\text{ cylindrical fin}} = L + \frac{D}{4}$$

t the thickness of the rectangular fins
D the diameter of the cylindrical fins



Corrected fin length L_c is defined such that heat transfer from a fin of length L_c with insulated tip is equal to heat transfer from the actual fin of length L with convection at the fin tip.

Fin Efficiency



$$A_{fin} = 2 \times w \times L + w \times t$$
$$\cong 2 \times w \times L$$

FIGURE 3–41
Fins enhance heat transfer from a surface by enhancing surface area.

$$\boxed{\dot{Q}_{fin,\,max} = hA_{fin}\,(T_b - T_\infty)}$$

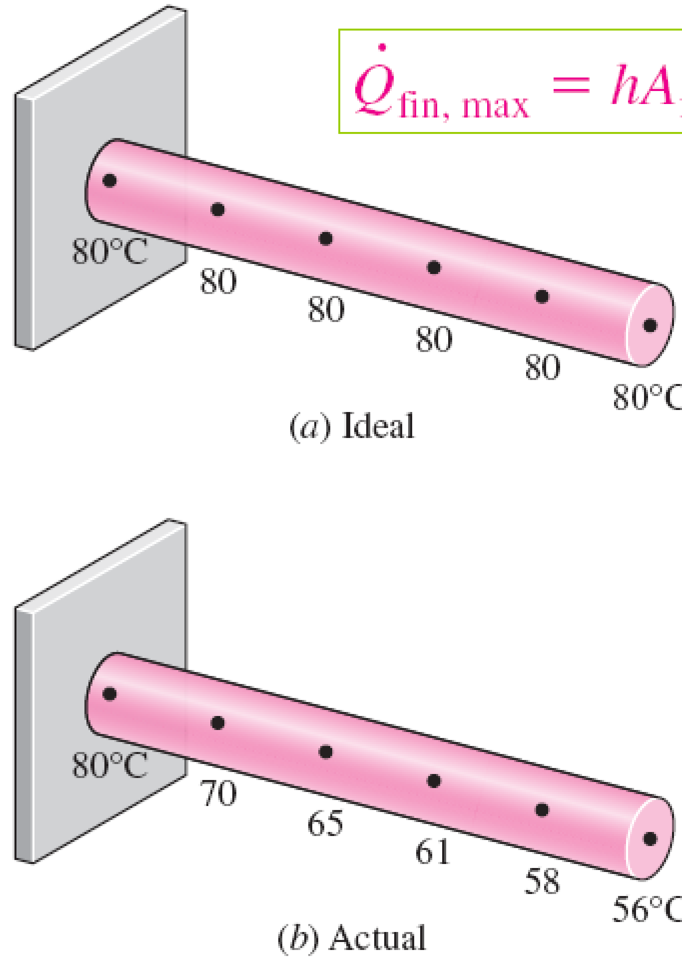


FIGURE 3–42
Ideal and actual temperature distribution along a fin.

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}\,(T_b - T_\infty)$$

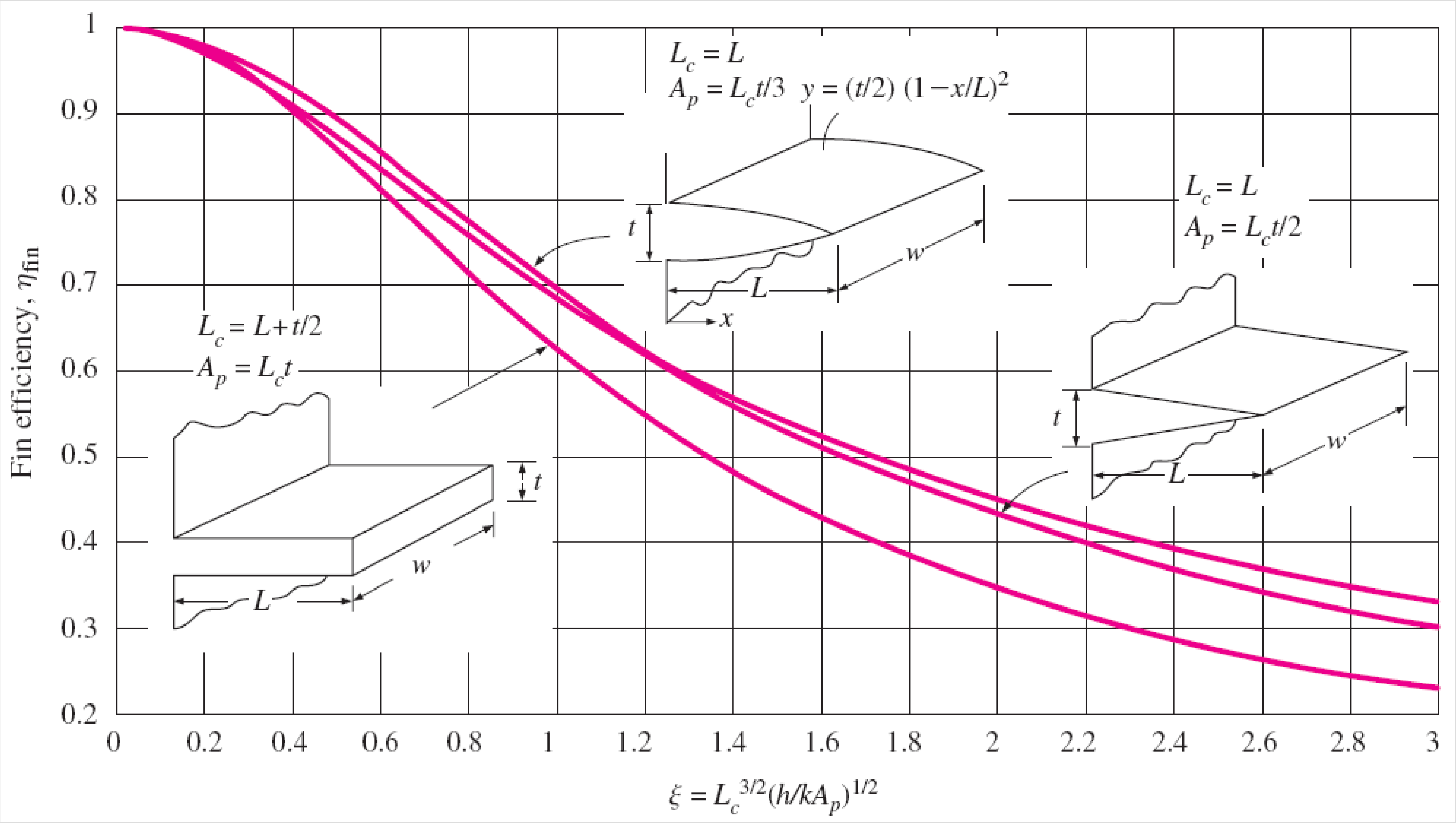
Zero thermal resistance or *infinite thermal conductivity* ($T_{\text{fin}} = T_b$)

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\begin{array}{c}\text{Ideal heat transfer rate from the fin}\\ \text{if the entire fin were at base temperature}\end{array}}$$

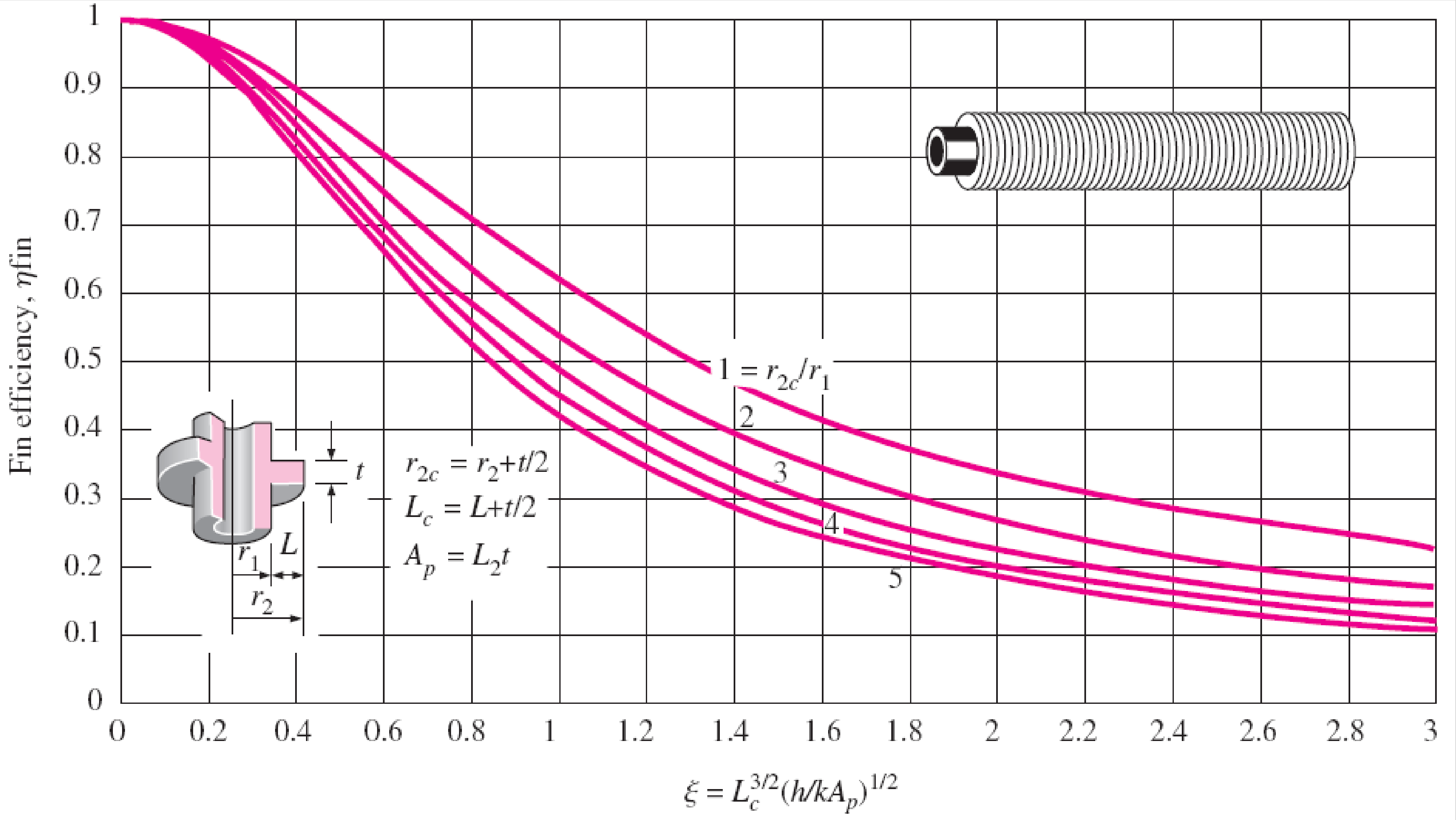
$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\,\dot{Q}_{\text{fin, max}} = \eta_{\text{fin}}\,hA_{\text{fin}}\,(T_b - T_\infty)$$

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{mL}$$

$$\eta_{\text{adiabatic tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{\tanh mL}{mL}$$



Efficiency of straight fins of rectangular, triangular, and parabolic profiles.



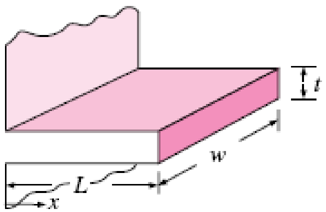
Efficiency of annular fins of constant thickness *t*.

Efficiency and surface areas of common fin configurations

Straight rectangular fins

$m = \sqrt{2h/kt}$
$L_c = L + t/2$
$A_{fin} = 2wL_c$

$$\eta_{fin} = \frac{\tanh mL_c}{mL_c}$$

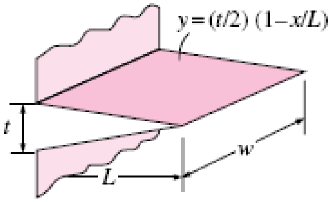


Straight triangular fins

$m = \sqrt{2h/kt}$

$A_{fin} = 2w\sqrt{L^2 + (t/2)^2}$

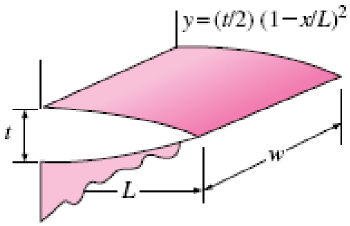
$$\eta_{fin} = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$



Straight parabolic fins

$m = \sqrt{2h/kt}$
$A_{fin} = wL[C_1 + (L/t)\ln(t/L + C_1)]$
$C_1 = \sqrt{1 + (t/L)^2}$

$$\eta_{fin} = \frac{2}{1 + \sqrt{(2mL)^2 + 1}}$$

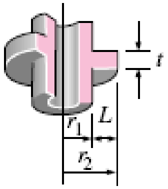


Circular fins of rectangular profile

$m = \sqrt{2h/kt}$
$r_{2c} = r_2 + t/2$
$A_{fin} = 2\pi(r_{2c}^2 - r_1^2)$

$$\eta_{fin} = C_2\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

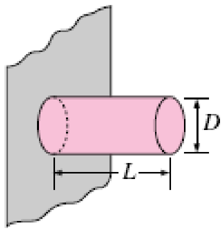
$$C_2 = \frac{2r_1/m}{r_{2c}^2 - r_1^2}$$



Pin fins of rectangular profile

$m = \sqrt{4h/kD}$
$L_c = L + D/4$
$A_{fin} = \pi DL_c$

$$\eta_{fin} = \frac{\tanh mL_c}{mL_c}$$

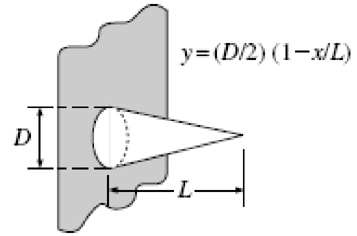


Pin fins of triangular profile

$$m = \sqrt{4h/kD}$$

$$A_{fin} = \frac{\pi D}{2}\sqrt{L^2 + (D/2)^2}$$

$$\eta_{fin} = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)}$$



Pin fins of parabolic profile

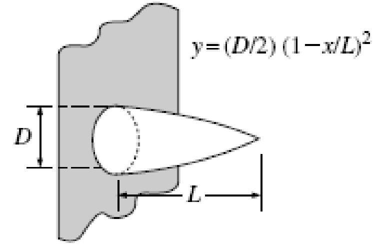
$$m = \sqrt{4h/kD}$$

$$A_{fin} = \frac{\pi L^3}{8D}\left[C_3C_4 - \frac{L}{2D}\ln(2DC_4/L + C_3)\right]$$

$$C_3 = 1 + 2(D/L)^2$$

$$C_4 = \sqrt{1 + (D/L)^2}$$

$$\eta_{fin} = \frac{2}{1 + \sqrt{(2mL/3)^2 + 1}}$$

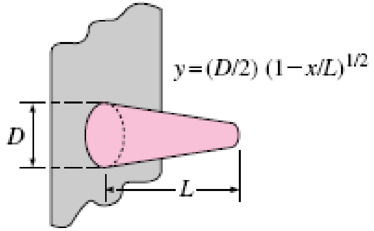


Pin fins of parabolic profile (blunt tip)

$$m = \sqrt{4h/kD}$$

$$A_{fin} = \frac{\pi D^4}{96L^2}\left\{[16(L/D)^2 + 1]^{3/2} - 1\right\}$$

$$\eta_{fin} = \frac{3}{2mL}\frac{I_1(4mL/3)}{I_0(4mL/3)}$$



- Fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles.
- The fin efficiency decreases with increasing fin length. Why?
- How to choose fin length? Increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.
- Fin lengths that cause the fin efficiency to drop below 60 percent usually cannot be justified economically.
- The efficiency of most fins used in practice is above 90 percent.

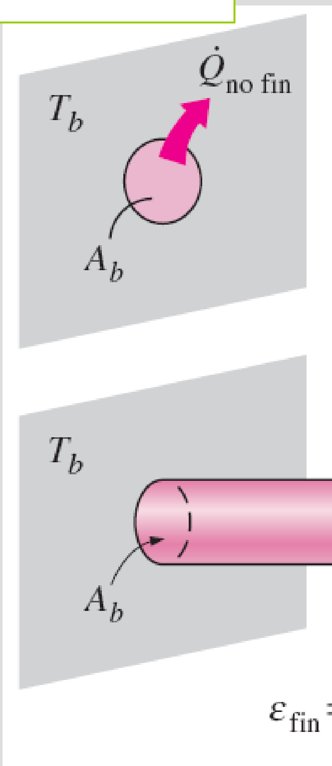
Fin Effectiveness

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } \textit{base area } A_b}{\text{Heat transfer rate from the surface of } \textit{area } A_b}$$

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{\eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\eta_{\text{fin}}$$

$$\varepsilon_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}}$$

- The *thermal conductivity* ***k*** of the fin should be as high as possible. Use aluminum, copper, iron.
- The ratio of the *perimeter* to the *cross-sectional area* of the fin ***p/A_c*** should be as high as possible. Use slender pin fins.
- *Low convection heat transfer coefficient* ***h***. Place fins on gas (air) side.



The effectiveness of a fin

The total rate of heat transfer from a finned surface

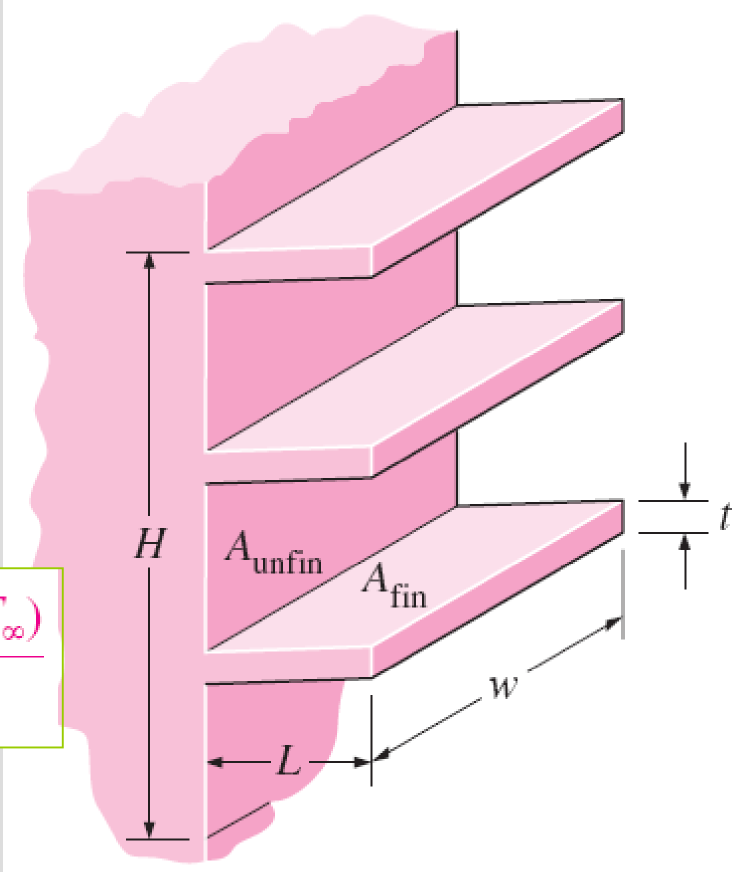
$$\dot{Q}_{\text{total, fin}} = \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}}$$
$$= hA_{\text{unfin}}(T_b - T_\infty) + \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)$$
$$= h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)$$

Overall effectiveness for a finned surface

$$\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin}}(T_b - T_\infty)}$$

The overall fin effectiveness depends on the fin density (number of fins per unit length) as well as the effectiveness of the individual fins.

The overall effectiveness is a better measure of the performance of a finned surface than the effectiveness of the individual fins.

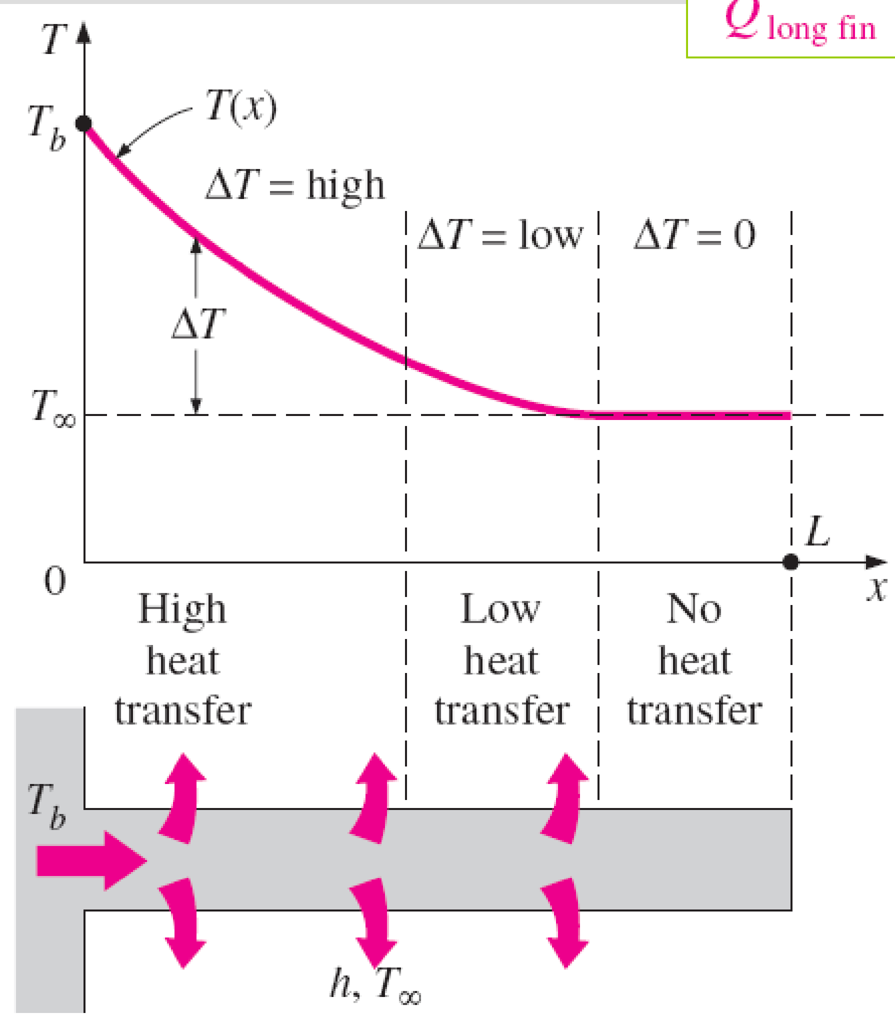


$$A_{\text{no fin}} = w \times H$$
$$A_{\text{unfin}} = w \times H - 3 \times (t \times w)$$
$$A_{\text{fin}} = 2 \times L \times w + t \times w$$
$$\cong 2 \times L \times w \text{ (one fin)}$$

Various surface areas associated with a rectangular surface with three fins.

Proper Length of a Fin

$$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh mL$$



The variation of heat transfer from a fin relative to that from an infinitely long fin

mL	$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \tanh mL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	0.987
3.0	0.995
4.0	0.999
5.0	1.000

$mL = 5 \rightarrow$ an infinitely long fin
$mL = 1$ offer a good compromise between heat transfer performance and the fin size.

Because of the gradual temperature drop along the fin, the region near the fin tip makes little or no contribution to heat transfer.

A common approximation used in the analysis of fins is to assume the fin temperature to vary in one direction only (along the fin length) and the temperature variation along other directions is negligible.

Perhaps you are wondering if this one-dimensional approximation is a reasonable one.

This is certainly the case for fins made of thin metal sheets such as the fins on a car radiator, but we wouldn’t be so sure for fins made of thick materials.

Studies have shown that the error involved in one-dimensional fin analysis is negligible (less than about 1 percent) when

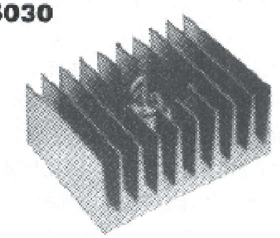
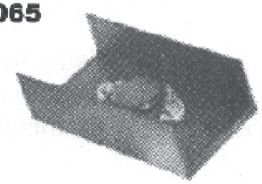
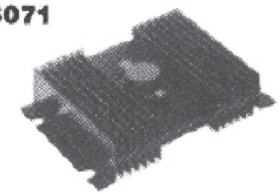
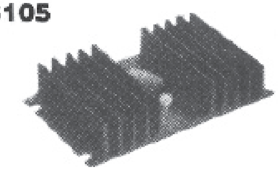
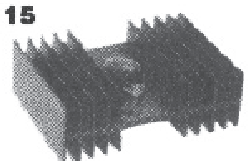
$$\frac{h\delta}{k} < 0.2$$

where δ is the characteristic thickness of the fin, which is taken to be the plate thickness *t* for rectangular fins and the diameter *D* for cylindrical ones.

- **Heat sinks:** Specially designed finned surfaces which are commonly used in the cooling of electronic equipment, and involve one-of-a-kind complex geometries.
- The heat transfer performance of heat sinks is usually expressed in terms of their *thermal resistances R.*
- A small value of thermal resistance indicates a small temperature drop across the heat sink, and thus a high fin efficiency.

$$\dot{Q}_{\text{fin}} = \frac{T_b - T_\infty}{R} = hA_{\text{fin}}\,\eta_{\text{fin}}\,(T_b - T_\infty)$$

Combined natural convection and radiation thermal resistance of various heat sinks used in the cooling of electronic devices between the heat sink and the surroundings. All fins are made of aluminum 6063T-5, are black anodized, and are 76 mm (3 in) long.

HS 5030		R = 0.9°C/W (vertical) R = 1.2°C/W (horizontal) Dimensions: 76 mm × 105 mm × 44 mm Surface area: 677 cm^2
HS 6065		R = 5°C/W Dimensions: 76 mm × 38 mm × 24 mm Surface area: 387 cm^2
HS 6071		R = 1.4°C/W (vertical) R = 1.8°C/W (horizontal) Dimensions: 76 mm × 92 mm × 26 mm Surface area: 968 cm^2
HS 6105		R = 1.8°C/W (vertical) R = 2.1°C/W (horizontal) Dimensions: 76 mm × 127 mm × 91 mm Surface area: 677 cm^2
HS 6115		R = 1.1°C/W (vertical) R = 1.3°C/W (horizontal) Dimensions: 76 mm × 102 mm × 25 mm Surface area: 929 cm^2

HEAT TRANSFER IN COMMON CONFIGURATIONS

So far, we have considered heat transfer in *simple* geometries such as large plane walls, long cylinders, and spheres.

This is because heat transfer in such geometries can be approximated as *one-dimensional*.

But many problems encountered in practice are two- or three-dimensional and involve rather complicated geometries for which no simple solutions are available.

An important class of heat transfer problems for which simple solutions are obtained encompasses those involving two surfaces maintained at *constant* temperatures T_1 and T_2.

The steady rate of heat transfer between these two surfaces is expressed as

$$Q = Sk(T_1 - T_2)$$

***S*: conduction shape factor**

k: the thermal conductivity of the medium between the surfaces

The conduction shape factor depends on the *geometry* of the system only.

Conduction shape factors are applicable only when heat transfer between the two surfaces is by ***conduction***.

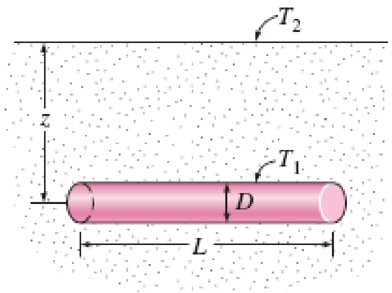
$$S = 1/kR$$

Relationship between the conduction shape factor and the thermal resistance

Conduction shape factors S for several configurations for use in $\dot{Q} = kS(T_1 - T_2)$ to determine the steady rate of heat transfer through a medium of thermal conductivity k between the surfaces at temperatures T_1 and T_2

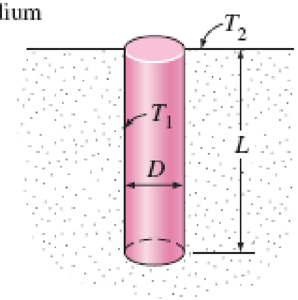
(1) Isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



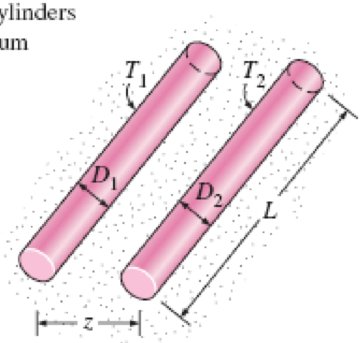
(2) Vertical isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



(3) Two parallel isothermal cylinders placed in an infinite medium ($L >> D_1, D_2, z$)

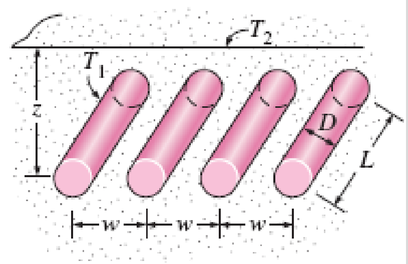
$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{4z^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$$



(4) A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L >> D, z,$ and $w > 1.5D$)

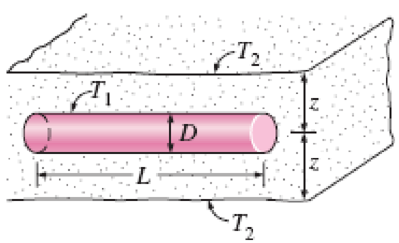
$$S = \frac{2\pi L}{\ln\left(\frac{2w}{\pi D}\sinh\frac{2\pi z}{w}\right)}$$

(per cylinder)



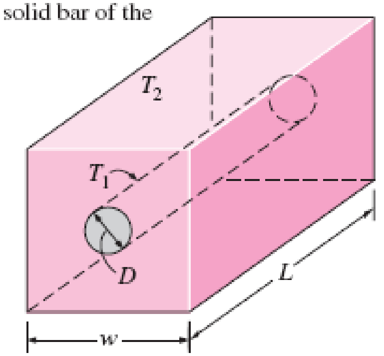
(5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



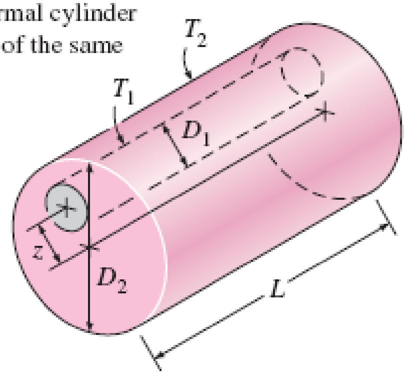
(6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08w/D)}$$



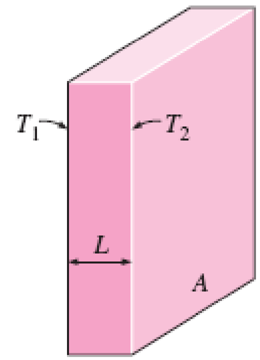
(7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{D_1^2 + D_2^2 - 4z^2}{2D_1 D_2}\right)}$$



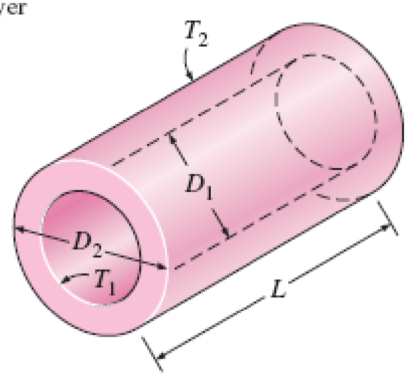
(8) Large plane wall

$$S = \frac{A}{L}$$



(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln (D_2/D_1)}$$



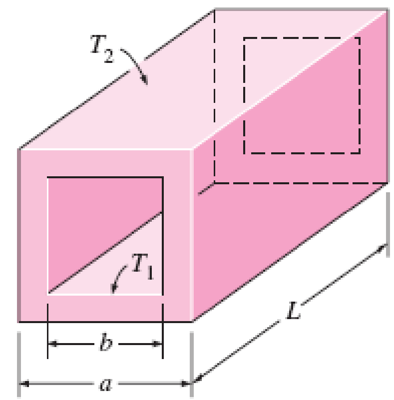
(10) A square flow passage

(a) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln (0.948a/b)}$$

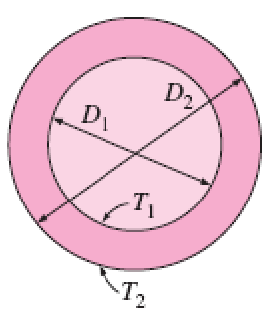
(b) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln (a/b)}$$



(11) A spherical layer

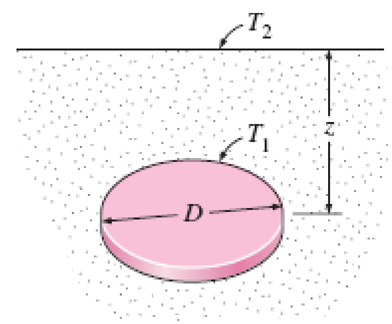
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$

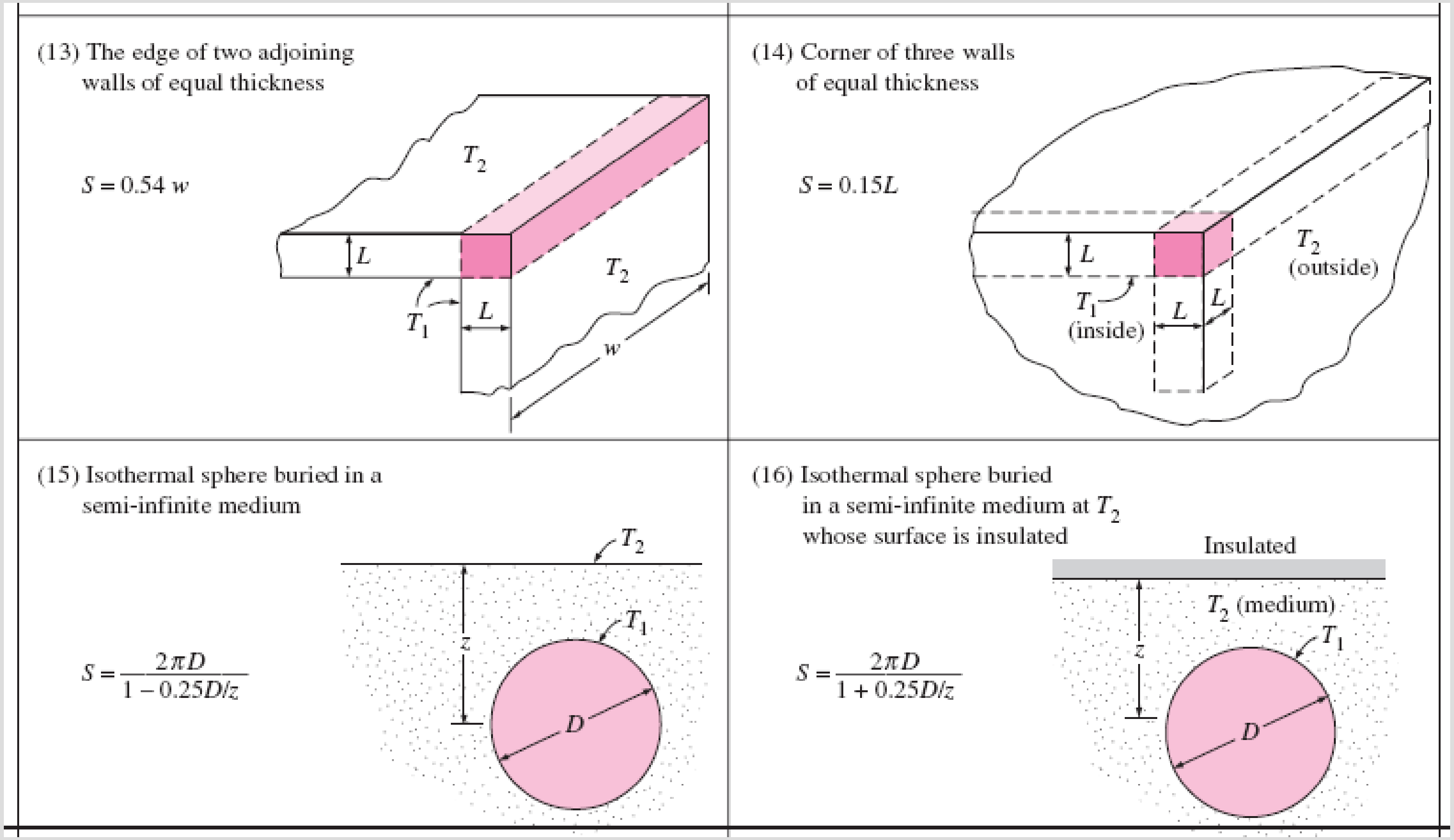


(12) Disk buried parallel to the surface in a semi-infinite medium ($z >> D$)

$S = 4D$

($S = 2D$ when $z = 0$)





Once the value of the shape factor is known for a specific geometry, the total steady heat transfer rate can be determined from the following equation using the specified two constant temperatures of the two surfaces and the thermal conductivity of the medium between them.

$$Q = Sk(T_1 - T_2)$$

Summary

- Steady Heat Conduction in Plane Walls
  - ✓ Thermal Resistance Concept
  - ✓ Thermal Resistance Network
  - ✓ Multilayer Plane Walls
- Thermal Contact Resistance
- Generalized Thermal Resistance Networks
- Heat Conduction in Cylinders and Spheres
  - ✓ Multilayered Cylinders and Spheres
- Critical Radius of Insulation
- Heat Transfer from Finned Surfaces
  - ✓ Fin Equation
  - ✓ Fin Efficiency
  - ✓ Fin Effectiveness
  - ✓ Proper Length of a Fin
- Heat Transfer in Common Configurations