DSA 8205: OPTIMIZATION FOR DATA SCIENCE 06/12/2024

GROUP WORK

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i) Describe bisection method to solve the following:

$\max f(x) = x^3 + 4x^2 - 10$

$$\frac{\partial f}{\partial x} = 3x^2 + 8x$$

$$\frac{\partial^2 f}{\partial x^2} = 6x + 8$$

initial bound $\underline{x} = 1$ $\bar{x} = 2$

Let $e = 0.01$ stop at $0.015625 \leq 0.02$

$\bar{x} - \underline{x} \leq 2e$

iteration	$\frac{df}{dx}$	$\underline{x}$	$\bar{x}$	x'	$f(x)$
0	-	1	2	1.5	2.375
1	18.75	1.5	2	1.75	7.6093
2	23.1875	1.75	2	1.875	10.654
3	25.5468	1.875	2	1.9375	12.288
4	26.7617	1.9375	2	1.96875	13.1347
5	27.3780	1.96875	2	1.98438	13.5650
6	27.688	1.984375	2	1.99218	13.781929
7	27.8439	1.99218	2	1.99609	13.89081

$x^* = 1.99609$

$1.99218 \leq x^* \leq 2$

$1 \leq 1.99609 \leq 2.$

Bisection method aims to find a sequence of trial solution that leads towards an optimal solution.

ii) Describe Netwon's Method

It aims to approximate $f(x)$ within tho neighboorhood of the current trial solution by quadratic function

$f(x) = e^{2x} - x - 6$

$f'(x) = 2e^{2x} - 1$

$f''(x) = 4e^{2x}$

error $= 0.0001$ $\qquad x_{i+1} = x_i - \frac{f'_0(x_i)}{f''(x_i)}$

$x_1 = 1$

Defferentiation part e^{2x}

Let $u = 2x$ then $\frac{du}{dx} = 2$

Thus $e^{2x} = e^{u}$

Chain rule $\frac{d}{dx} 2^{2x} - \frac{d}{du} e^{u} \cdot \frac{du}{dx} = e^{u} \cdot 2 = 2e^{2x}$

iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}
1	1	0.38906	13.77811	29.55622	0.53383
2	0.53383	-3.62525	4.81173	11.63426	0.119787
3	0.119787	-4.84909	1.54142	5.08283	-0.183473
4	-0.183475	-5.123619	0.38569	2.77138	-0.32264
5	-0.32264	-5.10379	0.04903	2.09806	-0.34009
6	-0.34009	-5.15328	0.01305	2.026103	-0.34653
7	-0.34653	-5.84648	8.7184×10^{-5}	2.00017	-0.346573

Iterate stop

$x = -0.346575$ converge.

iii) Describe Gradient search procedure for a multivariate unconstrained maximization problem and give a solved example

The aim of Gradient search is to reach point where all the partial derivatives are 0.
The values of the partial derivative are used to select the specific direction.
The gradient at point $X = x$ is

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n}\right) \text{ at } X = x$$

One direction of the gradient is intepreted as the direction of the directed line segment from the origin to the point.

Example:

$\max f(x) = 2x_1x_2 + 2x_2 - x_1^2 - 2x_2^2$

Solution

$$\frac{\partial f}{\partial x_1} = 2x_2 - 2x_1$$

$$\frac{\partial f}{\partial x_2} = 2x_1 + 2 - 4x_2$$

Choose $x = (0,0)$

$\nabla f(0,0) = (0,2)$

Set

$x_1 = 0 + t(0) = 0$

$x_2 = 0 + t(2) = 2t$

$$f(x' + t\Delta f(x')) = f(0, 2t)$$

$$f(x) = 2(0)(2t) + 2(2t) - 0^2 - 2(2t)^2$$

$$= 4t - 8t^2$$

$$f(0, 2t^*) = \max_{t \geq 0} f(0, 2t)$$

$$= \max_{t \geq 0} \{4t - 8t^2\}$$

$$\frac{d}{dt}(4t - 8t^2) = 4 - 16t = 0$$

$$4 = 16t \qquad t = \tfrac{1}{4}$$

Reset $x' = (0,0) + \frac{1}{4}(0,2)$

$x' = 0, \frac{1}{2}$

The gradient $\Delta f(0, \frac{1}{2}) = (1,0)$

$x = (0, \frac{1}{2}) + t(1,0) = (t, \frac{1}{2})$

$f(x' + t\Delta f(x')) = f(0+t, \frac{1}{2} + 0t)$

$= f(t, \frac{1}{2})$

$= (2t)(\frac{1}{2}) + 2(\frac{1}{2}) - t^2 - 2(\frac{1}{2})^2$

$f(t^*, \frac{1}{2}) = \max f(t, \frac{1}{2}) \quad t \geq 0$

$$= \max_{t \geq 0} \{t - t^2 + \tfrac{1}{2}\}$$

$$\frac{d}{dt}(t - t^2 + \tfrac{1}{2}) = 1 - 2t = 0$$

$$\tfrac{2}{2}t = \tfrac{1}{2} \qquad t = \tfrac{1}{2}$$

iv) Describe karush-kuhn-Tucker (KKT) Conditions for Constrained Optimization.

This assumes that $f(x)$, $g_1(x)$, $g_2(x)$ .... $g_m(x)$ are differentiable functions.

$x^* = x_1^*, x_2^* \dots x_n^*$ can be optimal solution for the non-linear programming problem only if there exist in numbers $\mu_1, \mu_2 \dots \mu_m$.

v) Describe how to solve the KKT condition.

For a maximization problem:

Max $f(x)$

subject to $g_i(x) \leq 0$

$L = F(x) - \Sigma \lambda g_i(x)$

$\frac{\partial L}{\partial x} = 0$

$\lambda_i g_i = 0$

$g_i(x) \leq 0$

$\lambda_i \geq 0$

For a minimization problem:

Min $f(x)$

Subject to $g_i(x) \geq 0$

$\frac{\partial L}{\partial x} = 0$

$\lambda_i g_i = 0$

$g_i(x) \geq 0$

$\lambda_i \leq 0$

vi) Use KKT

Max $f(x_1, x_2) = 15x_1 + 30x_2 + 4x_1x_2 - 2x_1^2 - 4x_2^2$

Subject to: $2x_1 + 2x_2 \leq 30$

$x_1, x_2 \geq 0$

$f_i = \sum_{i=1}^{n} \lambda_i h_i$

$\lambda_i h_i = 0$

$\lambda_i \geq 0$

$h_i \geq 0$

let $f = 15x_1 + 30x_2 + 4x_1x_2 - 2x_1^2 - 4x_2^2$

$h_1(x) = 2x_1 + 2x_2 - 30$

$f = \sum_i \lambda_i h_i$

$f = \lambda_1 h_1(x)$

$= \lambda_1 [2x_1 + 2x_2 - 30]$

$15x_1 + 30x_2 + 4x_1x_2 - 2x_1^2 - 4x_2^2 = \lambda_1 [2x_1 + 2x_2 - 30]$

Differentiate

$\frac{d}{dx_1} = 15 + 4x_2 - 4x_1 = \lambda_1 [2]$ ... (i)

$\frac{d}{dx_2} = 30 + 4x_1 - 8x_2 = 2\lambda_1$ ... (ii)

$\lambda_1 = 0$

$15 + 4x_2 - 4x_1 = 0$

$4x_2 - 4x_1 = -15$

$-8x_2 + 4x_1 = -30$

$\frac{-4x_2}{4} = \frac{-45}{4}$

$x_2 = 11.25$

$4 \times 45 - 4x_1 = -15$

$-4x_1 = -15 - 45$

$x_1 = 15$

Max $z = 15(15) + 30(11.25) + 4(15 \times 11.25)$
$-2(15)^2 - 4(11.25)^2 = 281.25$

vii) Min $f(x_1, x_2) = x_1^2 - 6x_1 + x_2^3 - 3x_2$

Subject to: $x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

a) Obtain the KKt conditions

$f_i = \sum_{i} \lambda_i h_i$

$\lambda_i h_i = 0$

$\lambda_i \leq 0$

$h_i \geq 0$

Stationarity Conditions

$\nabla f(x_1, x_2) + L \nabla g(x_1, x_2) = 0$

$\nabla f(x_1, x_2) = (2x_1 - 6)$

$\nabla g(x_1, x_2) = \{ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \}$

$$\begin{pmatrix} 2x_1 - 6 \\ 3x_2^2 - 3 \end{pmatrix} + L \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

Thus the equation are:

$2x_1 - 6 + L = 0$

$3x_2^2 - 3 + L = 0$

Primal feasibility:

$x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

Dual feasibility

$L \geq 0$

Complementary

$Lg(x_1, x_2) = 0$

$L(x_1 + x_2 - 1) = 0$

b) Use the KKT to check whether $(x, x_2) = (\frac{1}{2}, \frac{1}{2})$.

$2(\frac{1}{2}) - 6 + L = 0$

$1 - 6 + L = 0$

$L = 5$

$3(\frac{1}{2})^2 - 3 + L = 0$

$\frac{3}{4} - 3 + L = 0$

$-\frac{9}{4} + L = 0$

$h\ L = 5$

$-\frac{9}{4} + 5 = 0$

$0 = \frac{11}{4}$

Thus stationary condition with (x_1, x_2) as $(\frac{1}{2}, \frac{1}{2})$ is not a satisfying the KKT condition.

c) Use KKT conditions to derive an optimal solution.

Stationarity condition $\quad 2x_1 - 6 + 1 = 0 \quad \ldots (i)$

$3x_2^2 - 3 + L = 0 \quad \ldots (ii)$

Complementary $\quad L(x_1 + x_2 + 1) = 0$

Primal feasibility $\quad x_1 + x_2 \leq 0$

$x_1 x_2 \geq 0$

Dual feasibility $\quad L \geq 0$

If $L = 0$

Then $2x_1 - 6 = 0 \quad x_1 = 3$

$3x_2^2 - 3 = 0 \quad x_2^2 = 1 \quad ; x_2 = 1 \text{ or } -1$

Thus $(x_1, x_2) = (3, 1)$ or $(3, -1)$

Substituting $\quad x_1 + x_2 \leq 1$

$3 + 1 = 4 \geq 1$

$3 - 1 = 2 \geq 1$

Therefore $L=0$ does not satisfy $x_1 + x_2 \leq 1$ making the case as not feasible.

If $L \geq 0$

$x_1 + x_2 = 1$

Through substitution

$x_1 + x_2 = 1$

$x_1 = 1 - x_2$

$2(1-x_2) - 6 + L = 0 \quad \ldots (i)$

$3x_2^2 - 3 + L = 0 \quad \ldots (ii)$

$\cancel{2x}\ 2x_2 \quad 6 + L = 0$

$-2x_2 - 4 + \cancel{6} = 0 \quad (i)$

$3x_2^2 - 3 + L = 0 \quad (ii)$

$L = 2x_2 + 4$

$3x_2^2 - 3 + 2x_2 + 4 = 0$

$3x_2^2 + 2x_2 + 1 = 0$

$$x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-2 \pm \sqrt{4-12}}{6}$$

no real values of x_2.