1 Gambler's Ruin Problem

Consider a gambler who starts with an initial fortune of \$1 and then on each successive gamble either wins \$1 or loses \$1 independent of the past with probabilities p and $q = 1-p$ respectively. Let R_n denote the total fortune after the n^{th} gamble. The gambler's objective is to reach a total fortune of \$$N$, without first getting *ruined* (running out of money). If the gambler succeeds, then the gambler is said to *win* the game. In any case, the gambler stops playing after winning or getting ruined, whichever happens first. There is nothing special about starting with \$1, more generally the gambler starts with \$$i$ where $0 < i < N$.

While the game proceeds, $\{R_n : n \geq 0\}$ forms a simple random walk

$$R_n = \Delta_1 + \cdots + \Delta_n, \ R_0 = i,$$

where $\{\Delta_n\}$ forms an i.i.d. sequence of r.v.s. distributed as $P(\Delta = 1) = p, \ P(\Delta = -1) = q = 1 - p$, and represents the earnings on the succesive gambles.

Since the game stops when either $R_n = 0$ or $R_n = N$, let

$$\tau_i = \min\{n \geq 0 : R_n \in \{0, N\} | R_0 = i\},$$

denote the time at which the game stops when $R_0 = i$. If $R_{\tau_i} = N$, then the gambler wins, if $R_{\tau_i} = 0$, then the gambler is ruined.

Let $P_i = P(R_{\tau_i} = N)$ denote the probability that the gambler wins when $R_0 = i$. Clearly $P_0 = 0$ and $P_N = 1$ by definition, and we next proceed to compute $P_i, \ 1 \leq i \leq N-1$.

The key idea is to condition on the outcome of the first gamble, $\Delta_1 = 1$ or $\Delta_1 = -1$, yielding

$$P_i = pP_{i+1} + qP_{i-1}. \tag{1}$$

The derivation of this recursion is as follows: If $\Delta_1 = 1$, then the gambler's total fortune increases to $R_1 = i+1$ and so by the Markov property the gambler will now win with probability P_{i+1}. Similarly, if $\Delta_1 = -1$, then the gambler's fortune decreases to $R_1 = i - 1$ and so by the Markov property the gambler will now win with probability P_{i-1}. The probabilities corresponding to the two outcomes are p and q yielding (1). Since $p + q = 1$, (1) can be re-written as $pP_i + qP_i = pP_{i+1} + qP_{i-1}$, yielding

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}).$$

In particular, $P_2 - P_1 = (q/p)(P_1 - P_0) = (q/p)P_1$ (since $P_0 = 0$), so that
$P_3 - P_2 = (q/p)(P_2 - P_1) = (q/p)^2 P_1$, and more generally

$$P_{i+1} - P_i = (\frac{q}{p})^i P_1, \ 0 < i < N.$$

Thus

$$\begin{aligned} P_{i+1} - P_1 &= \sum_{k=1}^{i} (P_{k+1} - P_k) \\ &= \sum_{k=1}^{i} (\frac{q}{p})^k P_1, \end{aligned}$$

yielding

$$
\begin{aligned}
P_{i+1} &= P_1 + P_1 \sum_{k=1}^{i} (\frac{q}{p})^k = P_1 \sum_{k=0}^{i} (\frac{q}{p})^k \\
&= \begin{cases} P_1 \frac{1-(\frac{q}{p})^{i+1}}{1-(\frac{q}{p})}, & \text{if } p \neq q; \\ P_1(i+1), & \text{if } p = q = 0.5. \end{cases} \qquad (2)
\end{aligned}
$$

(Here we are using the "geometric series" equation $\sum_{n=0}^{i} a^i = \frac{1-a^{i+1}}{1-a}$, for any number a and any integer $i \geq 1$.)

Choosing $i = N-1$ and using the fact that $P_N = 1$ yields

$$
1 = P_N = \begin{cases} P_1 \frac{1-(\frac{q}{p})^N}{1-(\frac{q}{p})}, & \text{if } p \neq q; \\ P_1 N, & \text{if } p = q = 0.5, \end{cases}
$$

from which we conclude that

$$
P_1 = \begin{cases} \frac{1-\frac{q}{p}}{1-(\frac{q}{p})^N}, & \text{if } p \neq q; \\ \frac{1}{N}, & \text{if } p = q = 0.5, \end{cases}
$$

thus obtaining from (2) (after algebra) the solution

$$
P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}, & \text{if } p \neq q; \\ \frac{i}{N}, & \text{if } p = q = 0.5. \end{cases} \qquad (3)
$$

(Note that $1 - P_i$ is the probability of ruin.)

1.1 Becoming infinitely rich or getting ruined

If $p > 0.5$, then $\frac{q}{p} < 1$ and thus from (3)

$$
\lim_{N\to\infty} P_i = 1 - (\frac{q}{p})^i > 0, \; p > 0.5. \qquad (4)
$$

If $p \leq 0.5$, then $\frac{q}{p} \geq 1$ and thus from (3)

$$
\lim_{N\to\infty} P_i = 0, \; p \leq 0.5. \qquad (5)
$$

To interpret the meaning of (4) and (5), suppose that the gambler starting with $X_0 = i$ wishes to continue gambling forever until (if at all) ruined, with the intention of earning as much money as possible. So there is no winning value N; the gambler will only stop if ruined. What will happen?

(4) says that if $p > 0.5$ (each gamble is in his favor), then there is a positive probability that the gambler will never get ruined but instead will become infinitely rich.

(5) says that if $p \leq 0.5$ (each gamble is not in his favor), then with probability one the gambler will get ruined.

Examples

1. John starts with \$2, and $p = 0.6$: What is the probability that John obtains a fortune of $N = 4$ without going broke?

   SOLUTION $i = 2$, $N = 4$ and $q = 1 - p = 0.4$, so $q/p = 2/3$, and we want

   $$P_2 = \frac{1 - (2/3)^2}{1 - (2/3)^4} = 0.91$$

2. What is the probability that John will become infinitely rich?

   SOLUTION

   $1 - (q/p)^i = 1 - (2/3)^2 = 5/9 = 0.56$

3. If John instead started with $i = \$1$, what is the probability that he would go broke?

   SOLUTION

   The probability he becomes infinitely rich is $1-(q/p)^i = 1-(q/p) = 1/3$, so the probability of ruin is $2/3$.

1.2 Applications

Risk insurance business

Consider an insurance company that earns \$1 per day (from interest), but on each day, independent of the past, might suffer a *claim* against it for the amount \$2 with probability $q = 1 - p$. Whenever such a claim is suffered, \$2 is removed from the reserve of money. Thus on the n^{th} day, the net income for that day is exactly Δ_n as in the gamblers' ruin problem: 1 with probability p, -1 with probability q.

If the insurance company starts off initially with a reserve of $\$i \geq 1$, *then what is the probability it will eventually get ruined (run out of money)?*

The answer is given by (4) and (5): If $p > 0.5$ then the probability is given by $(\frac{q}{p})^i > 0$, whereas if $p \leq 0.5$ ruin will always ocurr. This makes intuitive sense because if $p > 0.5$, then the average net income per day is $E(\Delta) = p - q > 0$, whereas if $p \leq 0.5$, then the average net income per day is $E(\Delta) = p - q \leq 0$. So the company can not expect to stay in business unless earning (on average) more than is taken away by claims.

1.3 Random walk hitting probabilities

Let $a > 0$ and $b > 0$ be integers, and let R_n denote a simple random walk with $R_0 = 0$. Let

$$p(a) = P(R_n \text{ hits level } a \text{ before hitting level } -b).$$

By letting $a = N - i$ and $b = i$ (so that $N = a + b$), we can imagine a gambler who starts with $i = b$ and wishes to reach $N = a+b$ before going broke. So we can compute $p(a)$ by casting the problem into the framework of the gamblers ruin problem: $p(a) = P_i$ where $N = a + b$, $i = b$. Thus

$$p(a) = \begin{cases} \frac{1-(\frac{q}{p})^b}{1-(\frac{q}{p})^{a+b}}, & \text{if } p \neq q; \\ \frac{b}{a+b}, & \text{if } p = q = 0.5. \end{cases} \tag{6}$$

Examples

1. Ellen bought a share of stock for \$10, and it is believed that the stock price moves (day by day) as a simple random walk with $p = 0.55$. What is the probability that Ellen's stock reaches the high value of \$15 before the low value of \$5?

   SOLUTION

   We want "the probability that the stock goes up by 5 before going down by 5." This is equivalent to starting the random walk at 0 with $a = 5$ and $b = 5$, and computing $p(a)$.

   $$p(a) = \frac{1-(\frac{q}{p})^b}{1-(\frac{q}{p})^{a+b}} = \frac{1-(0.82)^5}{1-(0.82)^{10}} = 0.73$$

2. What is the probability that Ellen will become infinitely rich?

   SOLUTION

   Here we keep $i = 10$ in the Gambler's ruin problem and let $N \to \infty$ in the formula for P_{10} as in (4);

   $$\lim_{N\to\infty} P_{10} = 1 - (q/p)^{10} = 1 - (.82)^{10} = 0.86.$$

1.4 Markov chain approach

When we restrict the random walk to remain within the set of states $\{0, 1, \dots, N\}$, $\{R_n\}$ yields a Markov chain (MC) on the state space $\mathcal{S} = \{0, 1, \dots, N\}$. The transition probabilities are given by $P(R_{n+1} = i+1|R_n = i) = p_{i,i+i} = p$, $P(R_{n+1} = i-1|R_n = i) = p_{i,i-i} = q$, $0 < i < N$, and both 0 and N are absorbing states, $p_{00} = p_{NN} = 1$.[1]

For example, when $N = 4$ the transition matrix is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the gambler's ruin problem can be viewed as a special case of a *first passage time* problem: Compute the probability that a Markov chain, initially in state i, hits state j_1 before state j_2.

[1] There are three communication classes: $C_1 = \{0\}$, $C_2 = \{1, \dots, N-1\}$, $C_3 = \{N\}$. C_1 and C_3 are recurrent whereas C_2 is transient.