

1. Stochastic Processes and filtrations

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A stochastic process $(X_t)_{t \in \mathbb{T}}$ is a collection of random variables on (Ω, \mathcal{F}) with values in a measurable space (S, \mathcal{S}) , i.e., for all t ,

$$X_t : \Omega \rightarrow S \text{ is } \mathcal{F} - \mathcal{S}\text{-measurable.}$$

In our case

1 $\mathbb{T} = \{0, 1, \dots, N\}$ or $\mathbb{T} = \mathbb{N} \cup \{0\}$ (discrete time)

2 or $\mathbb{T} = [0, \infty)$ (continuous time).

The state space will be $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (or $(S, \mathcal{S}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for some $d \geq 2$).

Example 1.1

Random walk:

Let ε_n , $n \geq 1$, be a sequence of iid random variables with

$$\varepsilon_n = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

Define $\tilde{X}_0 = 0$ and, for $k = 1, 2, \dots$,

$$\tilde{X}_k = \sum_{i=1}^k \varepsilon_i$$

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For a fixed $\omega \in \Omega$ the function

$$t \mapsto X_t(\omega)$$

is called (sample) **path** (or **realization**) associated to ω of the stochastic process X .

Let X and Y be two stochastic processes on (Ω, \mathcal{F}) .

Definition 1.2

- (i) X and Y have the same finite-dimensional distributions if, for all $n \geq 1$, for all $t_1, \dots, t_n \in [0, \infty)$ and $A \in \mathcal{B}(\mathbb{R}^n)$:

$$P((X_{t_1}, X_{t_2}, \dots, X_{t_n}) \in A) = P((Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}) \in A).$$

- (ii) X and Y are modifications of each other, if, for each $t \in [0, \infty)$,

$$P(X_t = Y_t) = 1.$$

- (iii) X and Y are indistinguishable, if

$$P(X_t = Y_t \text{ for every } t \in [0, \infty)) = 1.$$

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Example 1.3

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Definition 1.4

A stochastic process X is called continuous (right continuous), if almost all paths are continuous (right continuous), i.e.,

$$P(\{\omega : t \mapsto X_t(\omega) \text{ continuous (right continuous)}\}) = 1$$

.

Proposition 1.5

If X and Y are modifications of each other and if both processes are a.s. right continuous then X and Y are indistinguishable.

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Let (Ω, \mathcal{F}) be given and $\{\mathcal{F}_t, t \geq 0\}$ be an increasing family of σ -algebras (i.e., for $s < t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$), such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all t .
That's a

filtration.

For example:

Definition 1.6

The filtration \mathcal{F}_t^X generated by a stochastic process X is given by

$$\mathcal{F}_t^X = \sigma\{X_s, s \leq t\}.$$

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Definition 1.7

A stochastic process is adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$ if, for each $t \geq 0$

$$X_t : (\Omega, \mathcal{F}_t) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable. (Short: X_t is \mathcal{F}_t -measurable.)

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Definition 1.8 ("The usual conditions")

Let $\mathcal{F}_{t+} = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$. We say that a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions if

- 1 $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous, that is, $\mathcal{F}_{t+} = \mathcal{F}_t$, for all $t \geq 0$.
- 2 $(\mathcal{F}_t)_{t \geq 0}$ is complete. That is, \mathcal{F}_0 contains all subsets of P -nullsets.

2. Brownian motion as a weak limit of random walks

Example 2.1

Scaled random walk: for $t_k = \frac{k}{N}$, $k = 0, 1, 2, \dots$, we set

$$\tilde{X}_{t_k}^N := \frac{1}{\sqrt{N}} \tilde{X}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^k \varepsilon_i$$

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Definition 2.2 (Brownian motion)

A standard, one-dimensional Brownian motion is a continuous adapted process W on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ with $W_0 = 0$ a.s. such that, for all $0 \leq s < t$,

- 1 $W_t - W_s$ is independent of \mathcal{F}_s
- 2 $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$.

Recall the scaled random walk: for $t_k = \frac{k}{N}$, $k = 0, 1, 2, \dots$, we set

$$\tilde{X}_{t_k}^N := \frac{1}{\sqrt{N}} \tilde{X}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^k \varepsilon_i$$

By interpolation we define a continuous process on $[0, \infty)$:

$$X_t^N = \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{[Nt]} \varepsilon_i + (Nt - [Nt]) \varepsilon_{[Nt]+1} \right) \quad (2.1)$$

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Theorem 2.3 (Donsker)

Let (Ω, \mathcal{F}, P) be a probability space with a sequence of iid random variables with mean 0 and variance 1. Define $(X_t^N)_{t \geq 0}$ as in (2.1). Then $(X_t^N)_{t \geq 0}$ converges to $(W_t)_{t \geq 0}$ weakly.

Clarification of the statement:

1 $\tilde{\Omega} = C[0, \infty)$

2 On $\tilde{\Omega} = C[0, \infty)$ one can construct a probability measure P^* (the **Wiener measure**) such that the **canonical process** $(W_t)_{t \geq 0}$ is a Brownian motion. Let $\omega = \omega(t)_{t \geq 0} \in C[0, \infty)$. Then the canonical process is given as $W_t(\omega) := \omega(t)$.

3 $(X_t^N)_{t \geq 0}$ defines a measure probability P^N on $\tilde{\Omega} = C[0, \infty)$.
Indeed,

$$X^N : (\Omega, \mathcal{F}) \rightarrow (C[0, \infty), \mathcal{B}(C[0, \infty)))$$

measurable, that means we consider the random variable $\omega \mapsto X^N(\omega, \cdot)$, which maps ω to a continuous function in t (i.e. the path). Let $A \in \mathcal{B}(C[0, \infty))$, then

$$P^N(A) := P(\{\omega : X^N(\omega, \cdot) \in A\}).$$

4 So, Donsker's theorem says that on $(\tilde{\Omega}, \tilde{\mathcal{F}}) = (C[0, \infty), \mathcal{B}(C[0, \infty)))$ we have that $P^N \xrightarrow{w} P^*$.

Steps of the proof:

- 1 Show that the finite-dimensional distributions of X^N converge to the finite-dimensional distributions of W . That means, show that, for all $n \geq 1$, and all $s_1, \dots, s_n \in [0, \infty)$

$$(X_{s_1}^N, X_{s_2}^N, \dots, X_{s_n}^N) \xrightarrow{w} (W_{s_1}, W_{s_2}, \dots, W_{s_n}).$$

- 2 Show that $(P^N)_{N \geq 1}$ is a **tight** family of probability measures on $(\tilde{\Omega}, \tilde{\mathcal{F}})$.
- 3 (1) and (2) imply $P^N \xrightarrow{w} P^*$.

Easy partial result of step (1) of the proof:

By (CLT), for fixed $t \in [0, \infty)$, for $N \rightarrow \infty$,

$$X_t^N \xrightarrow{w} W_t$$

Theorem 2.4 (CLT)

Let $(\xi_n)_{n \geq 1}$ be iid, mean=0, variance= σ^2 . Then, for each $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} P \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \leq x \right) = \Phi_{\sigma}(x),$$

where $\Phi_{\sigma}(x)$ is the distribution function of $N(0, \sigma^2)$. In other words

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \xrightarrow{w} Z,$$

where $Z \sim N(0, \sigma^2)$.

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3. Brownian motion as a Gaussian process

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Definition 3.1

A stochastic process $(X_t)_{t \geq 0}$ is called Gaussian process, if for each finite family of time points $\{t_1, \dots, t_n\}$ the vector $(X_{t_1}, \dots, X_{t_n})$ has a multivariate normal distribution.

A Gaussian process is called centered if $E[X_t] = 0$ for all t . The covariance function is given by $\gamma(s, t) = \text{Cov}(X_s, X_t)$.

Theorem 3.2

A Brownian motion is a centered Gaussian process with $\gamma(s, t) = s \wedge t (= \min(s, t))$. Conversely, a centered Gaussian process with continuous paths and covariance function $\gamma(s, t) = s \wedge t$ is a Brownian motion.

4. Stopping times

Given is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.

A **random time** T is a measurable function

$$T : (\Omega, \mathcal{F}) \rightarrow ([0, \infty], \mathcal{B}([0, \infty])).$$

Definition 4.1

If X is a stochastic process and T a random time, we define the function X_T on $\{T < \infty\}$ by:

$$X_T(\omega) = X_{T(\omega)}(\omega)$$

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Definition 4.2

A random time T is a stopping time for $(\mathcal{F}_t)_{t \geq 0}$ if, for all $t \geq 0$,

$$\{T \leq t\} = \{\omega : T(\omega) \leq t\} \in \mathcal{F}_t.$$

Proposition 4.3

1 *If $T(\omega) = t$ a.s. for some constant $t \geq 0$, then T is a stopping time.*

2 *Every stopping time T satisfies that, for all t ,*

$$\{T < t\} \in \mathcal{F}_t. \tag{4.1}$$

3 *If the filtration is right continuous and a random time T satisfies (4.1) for all t , then T is a stopping time.*

Lemma 4.4

Let S and T be stopping times for $(\mathcal{F}_t)_{t \geq 0}$. Then $S \wedge T$, $S \vee T$ and $S + T$ are stopping times for $(\mathcal{F}_t)_{t \geq 0}$.

Definition 4.5

Let T be a stopping time for $(\mathcal{F}_t)_{t \geq 0}$. The σ -field \mathcal{F}_T of events determined prior to the stopping time T is given by

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t, \text{ for all } t \geq 0\}.$$

Remark:

- 1 \mathcal{F}_T is a σ -field.
- 2 T is \mathcal{F}_T -measurable.
- 3 If $T(\omega) = t$ a.s., then $\mathcal{F}_T = \mathcal{F}_t$.

Example 4.6

Let $(W_t)_{t \geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.

Let T_a be the **first passage time** of the level $a \in \mathbb{R}$, i.e.,

$$T_a(\omega) = \inf\{t > 0 : W_t(\omega) = a\}. \quad (4.2)$$

Then T_a is a stopping time for $(\mathcal{F}_t)_{t \geq 0}$.

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Remark: Let S and T be stopping times for $(\mathcal{F}_t)_{t \geq 0}$.

- 1 For $A \in \mathcal{F}_S$ we have that $A \cap \{S \leq T\} \in \mathcal{F}_T$.
- 2 If, moreover, $S \leq T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

Definition 4.7

A stochastic process $(X_t)_{t \geq 0}$ is called progressively measurable for $(\mathcal{F}_t)_{t \geq 0}$ if, for all t and all $A \in \mathcal{B}(\mathbb{R})$, the set

$$\{(s, \omega) : 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\} \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t,$$

that means

$$(s, \omega) \mapsto X_s(\omega) \text{ is } \mathcal{B}([0, t]) \otimes \mathcal{F}_t - \mathcal{B}(\mathbb{R}) - \text{measurable.}$$

Example: If $(X_t)_{t \geq 0}$ is adapted and right-continuous then it is progressively measurable.

Proposition 4.8

Let X be progressively measurable on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ and let T be a stopping time for $(\mathcal{F}_t)_{t \geq 0}$. Then

- 1** *X_T defined on $\{T < \infty\}$ is an \mathcal{F}_T -measurable random variable and*
- 2** *the **stopped** process $(X_{T \wedge t})_{t \geq 0}$ is progressively measurable.*

5. Conditional expected value

Definition 5.1

Let (Ω, \mathcal{F}, P) be a probability space and $X : \Omega \rightarrow \mathbb{R}$ be an $\mathcal{F} - \mathcal{B}(\mathbb{R})$ -measurable random variable such that $E[|X|] = \int |X| dP < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a (smaller) σ -algebra. Then there exists a random variable Y with the following properties:

- 1 Y is $\mathcal{G} - \mathcal{B}(\mathbb{R})$ -measurable.
- 2 Y is integrable, i.e., $E[|Y|] < \infty$.
- 3 For all $A \in \mathcal{G}$:

$$E[Y \mathbb{I}_A] = E[X \mathbb{I}_A].$$

The \mathcal{G} -measurable random variable Y is called conditional expected value of X given \mathcal{G} . We write $Y = E[X|\mathcal{G}]$ a.s.

Remark:

- 1 $E[X|\mathcal{G}]$ is unique with respect to equality a.s.
- 2 Suppose that, additionally, $E[X^2] < \infty$. Then $E[X|\mathcal{G}]$ is the orthogonal projection of $X \in L^2(\Omega, \mathcal{F})$ onto the subspace $L^2(\Omega, \mathcal{G})$.
- 3 If $\mathcal{G} = \{\emptyset, \Omega\}$ then $E[X|\mathcal{G}] = E[X]$ a.s.

Definition 5.2

Let Z be a function on Ω . Then $E[X|Z] := E[X|\sigma(Z)]$.

Example 5.3

Example 5.4

$X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Suppose $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Let $\mathcal{G} := \sigma\{\Omega_k, k \geq 1\}$.

$$E[X|\mathcal{G}] =$$

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Example 5.5

Suppose (X, Z) has a bivariate density $f(x, z)$.

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Theorem 5.6 (A list of properties of conditional expectation)

In the following let $X, Y, X_n, n \geq 1$, be integrable random variables on (Ω, \mathcal{F}, P) . The following holds

(a) *If X is \mathcal{G} -measurable, then $E[X|\mathcal{G}] = X$ a.s.*

(b) *Linearity: $E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$ a.s.*

(c) *Positivity: If $X \geq 0$ a.s. then $E[X|\mathcal{G}] \geq 0$ a.s.*

(d) *Monotone convergence (MON): If $0 \leq X_n$ and $X_n \uparrow X$ a.s. then*

$$E[X_n|\mathcal{G}] \uparrow E[X|\mathcal{G}] \quad \text{a.s..}$$

(e) *Fatou: If $0 \leq X_n$, for all n , then*

$$E[\liminf_n X_n | \mathcal{G}] \leq \liminf_n E[X_n | \mathcal{G}].$$

(f) *Dominated convergence (DOM): If $|X_n| \leq Y$, for all $n \geq 1$ and $Y \in L^1$ (i.e. integrable) and $\lim X_n = X$ a.s. then*

$$\lim_n E[X_n | \mathcal{G}] = E[X | \mathcal{G}] \quad \text{a.s.}$$

(g) *Jensen's inequality: Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function such that $\varphi(X)$ is integrable. Then*

$$\varphi(E[X | \mathcal{G}]) \leq E[\varphi(X) | \mathcal{G}]$$

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(h) *Tower Property: If $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ then*

$$E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[X|\mathcal{G}_1] \quad a.s.$$

(i) *Taking out what is known: Let Y be \mathcal{G} -measurable and let $E[|XY|] < \infty$. Then*

$$E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$$

(j) *Role of independence: If X is independent of \mathcal{G} , then*

$$E[X|\mathcal{G}] = E[X]$$

6. Martingales and the discrete time stochastic integral

Definition 6.1

A stochastic process $(X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a martingale if X is adapted and $E[|X_t|] < \infty$, for all $t \geq 0$, and, for $s < t$,

$$E[X_t | \mathcal{F}_s] = X_s \quad \text{a.s.} \quad (6.1)$$

If the $=$ in (6.1) is replaced by \leq (\geq) the process is called supermartingale (submartingale).

Remark:

- 1 (6.1) is equivalent to $E[X_t - X_s | \mathcal{F}_s] = 0$ a.s.
- 2 Suppose $(X_t)_{n=0}^\infty$ is a stochastic process in discrete time adapted to $(\mathcal{F}_n)_{n=0}^\infty$ then the martingale property reads as: for all $n \geq 1$

$$E[X_n | \mathcal{F}_{n-1}] = X_{n-1} \quad \text{a.s.}$$

Example 6.2 (Random walk: sum of independent r.v.)

Let ε_k , $k \geq 1$, be independent with $E[\varepsilon_k] = 0$. Let $\mathcal{F}_n = \sigma\{\varepsilon_1, \dots, \varepsilon_n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Define $X_0 = 0$ and, for $n \geq 1$,

$$X_n = \sum_{k=1}^n \varepsilon_k.$$

Then $(X_n)_{n \geq 0}$ is a martingale for $(\mathcal{F}_n)_{n \geq 0}$.

Example 6.3 (Product of independent r.v.)

Let ε_k , $k \geq 1$, be non-negative and independent with $E[\varepsilon_k] = 1$. Let $\mathcal{F}_n = \sigma\{\varepsilon_1, \dots, \varepsilon_n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Define $X_0 = 1$ and, for $n \geq 1$,

$$X_n = \prod_{k=1}^n \varepsilon_k.$$

Then $(X_n)_{n \geq 0}$ is a martingale for $(\mathcal{F}_n)_{n \geq 0}$.

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Example 6.4

Let X be a r.v. on (Ω, \mathcal{F}, P) such that $E[|X|] < \infty$. Let $\mathcal{F}_t)_{t \geq 0}$ be any filtration with $\mathcal{F}_t \subseteq \mathcal{F}$, for each $t \geq 0$. Define

$$X_t = E[X | \mathcal{F}_t].$$

Then $(X_t)_{t \geq 0}$ is a martingale for $(\mathcal{F}_t)_{t \geq 0}$.

7. Stochastic integral in discrete time

Given is a discrete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$.

Definition 7.1

A discrete stochastic process $(H_n)_{n=1}^{\infty}$ is called predictable if

$$H_n : (\Omega, \mathcal{F}_{n-1}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable. (Short: H_n is \mathcal{F}_{n-1} -measurable.)

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9. Quad.var.,
path prop.

10. Ito integral

11. Ito's
formula

Definition 7.2

Let $(X_n)_{n=0}^{\infty}$ be an adapted stochastic process and $(H_n)_{n=1}^{\infty}$ be a predictable stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$. Then the stochastic integral in discrete time at time $n \geq 1$ is given by

$$(H \cdot X)_n = \sum_{k=1}^n H_k (X_k - X_{k-1}).$$

We define $(H \cdot X)_0 = 0$. The stochastic integral process is given by $((H \cdot X)_n)_{n=0}^{\infty}$.

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian p.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral**
- 8. Refl. princ., pass.times
- 9. Quad.var., path prop.
- 10. Ito integral
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Theorem 7.3

Let H be bounded and predictable and X be a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$. Then the stochastic integral process $((H \cdot X)_n)_{n=0}^{\infty}$ is a martingale.

Theorem 7.4 (Doob's Optional Stopping Theorem)

Let $(M_t)_{t \geq 0}$ be a martingale in continuous time on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Then, for a bounded stopping time T it holds that

$$E[M_T] = M_0.$$

More generally, if $S \leq T$ are bounded stopping times, then

$$E[M_T | \mathcal{F}_S] = M_S \quad \text{a.s.}$$

8. Reflection principle, passage times, running maximum/minimum

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian p.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass.times**
- 9. Quad.var., path prop.
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Definition 8.1 (Markov property)

Brownian motion satisfies the Markov property, that means, for all $t \geq 0$ and $s > 0$:

$$P(W_{t+s} \leq y | \mathcal{F}_t) = P(W_{t+s} \leq y | W_t) \quad \text{a.s.}$$

Reflection principle: Recall that $T_a = \inf\{t > 0 : W_t = a\}$.

$$P(T_a < t) =$$

$$\boxed{P(T_a < t) = 2P(W_t > a)} \quad (8.1)$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
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In all the following $(W_t)_{t \geq 0}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.

Theorem 8.2 (Strong Markov property)

The BM $(W_t)_{t \geq 0}$ satisfies the strong Markov property: for each stopping time τ with $P(\tau < \infty) = 1$ it holds that

$$\hat{W}_t = W_{\tau+t} - W_\tau, \quad t \geq 0$$

is a standard Brownian motion independent of \mathcal{F}_τ .

We will see later that $P(T_a < \infty) = 1$ for all $a \in \mathbb{R}$.

Theorem 8.3 (Reflection principle)

Let $a \neq 0$ and T_a be the passage time for a .

$$\tilde{W}_t = \begin{cases} W_t & \text{for } t \leq T_a \\ 2W_{T_a} - W_t = 2a - W_t & \text{for } t \geq T_a \end{cases}$$

Then $(\tilde{W}_t)_{t \geq 0}$ is again a standard Brownian motion.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian pr.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
- 8. Refl. princ., pass.times**
9. Quad.var., path prop.
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11. Ito's formula

Lemma 8.4

$$P(\sup_{t \geq 0} W_t = +\infty \text{ and } \inf_{t \geq 0} W_t = -\infty) = 1$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
- 8. Refl. princ., pass.times**
9. Quad.var., path prop.
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- 1. Stoch. pr., filtrations
- 2. BM as weak limit
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- 4. Stopping times
- 5. Cond. expectation
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- 8. Refl. princ., pass.times**
- 9. Quad.var., path prop.
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Theorem 8.5 (Recurrence)

$P(T_a < \infty) = 1$ for all $a \in \mathbb{R}$.

Theorem 8.6 (Distribution of passage time)

Let $a \neq 0$. The density of T_a is given by

$$f_{T_a}(t) = \frac{|a|}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{a^2}{2t}},$$

which is the density of an invers-gamma-distribution with parameters $\frac{1}{2}$ and $\frac{a^2}{2}$. In particular, we have that $E[T_a] = +\infty$.

Running maximum and minimum of Brownian motion:

Let $M_t = \max_{0 \leq s \leq t} W_s$ and $m_t = \min_{0 \leq s \leq t} W_s$.

Theorem 8.7

- 1 For each $a > 0$:
$$P(M_t \geq a) = P(T_a \leq t) = P(T_a < t) = 2P(W_t > a).$$
- 2 For each $a < 0$: $P(m_t \leq a) = 2P(W_t < a).$

Theorem 8.8 (Joint distribution)

Let $y \geq x \geq 0$. Then

$$P(W_t \leq x, M_t \geq y) = P(W_t \geq 2y - x).$$

This implies that the joint distribution of (W_t, M_t) has the following density: for $y \geq 0, x \leq y$

$$f_{W,M}(x, y) = \sqrt{\frac{2}{\pi}} \frac{(2y - x)}{t^{\frac{3}{2}}} e^{\frac{-(2y-x)^2}{2t}}.$$

1. Stoch. pr., filtrations
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Let $a < 0 < b$ and let τ be the first time to leave an interval (a, b) (exit time), i.e.,

$$\tau = \inf\{t > 0 : W_t \notin (a, b)\}.$$

We have that $\tau = \min(T_a, T_b) = T_a \wedge T_b$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
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Theorem 8.9 (Exit time)

Let $a < 0 < b$ and $\tau = T_a \wedge T_b$ as above. Then $P(\tau < \infty) = 1$ and $E[\tau] = |ab|$. Moreover

$$P(W_\tau = b) = \frac{|a|}{|a| + b} = \frac{-a}{-a + b}$$

$$P(W_\tau = a) = \frac{b}{|a| + b} = \frac{b}{-a + b}$$

1. Stoch. pr., filtrations
2. BM as weak limit
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6. Martingales
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9. Quadratic variation and path properties of BM

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
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Definition 9.1

A stochastic process has finite quadratic variation if there exists an a.s. finite stochastic process $([X, X]_t)_{t \geq 0}$ such that for all t and partitions $\pi_n = \{t_0^n, \dots, t_{m_n}^n\}$ with $0 = t_0^n < t_1^n < \dots < t_{m_n}^n = t$ and $\delta_n = \max_{i=1, \dots, m_n} (t_i^n - t_{i-1}^n) \rightarrow 0$ we have that, for $n \rightarrow \infty$,

$$V_t^2(\pi_n) := \sum_{i=1}^{m_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 \rightarrow [X, X]_t \quad \text{in probability.} \quad (9.1)$$

Theorem 9.2

Let $(W_t)_{t \geq 0}$ be a Brownian motion. Then $V_t^2(\pi_n) \rightarrow t$ in L^2 , that means

$$\lim_{n \rightarrow \infty} E[(V_t^2(\pi_n) - t)^2] = 0.$$

Therefore it follows that $V_t^2(\pi_n) \rightarrow t$ in probability. Hence $[W, W]_t = t$ a.s.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian pr.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
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- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian p.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass.times
- 9. Quad.var., path prop.
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Corollary 9.3

$(W_t)_{t \geq 0}$ has a.s. paths of infinite variation on each interval.

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian p.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass.times
- 9. Quad.var., path prop.
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Corollary 9.4

For almost all ω the path $t \mapsto W_t(\omega)$ is not monotone in each interval.

Remark 9.5

Almost all paths of $(W_t)_{t \geq 0}$ are not differentiable on each interval.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
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- 9. Quad.var., path prop.**
10. Ito integral
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10. Stochastic integral by Ito

For the whole chapter let $(W_t)_{t \geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Everything will be based on this filtered probability space.

Recall:

Riemann integral:

$$\int_a^b f(t) dt = \lim_{\delta_n \rightarrow 0} \sum_{i=1}^n f(\xi_i^n)(t_i^n - t_{i-1}^n)$$

where $a = t_0^n < t_1^n < \dots < t_n^n = b$, $t_{i-1}^n \leq \xi_i^n \leq t_i^n$ and $\delta_n = \max_{i=1, \dots, n}(t_i^n - t_{i-1}^n)$.

1. Stoch. pr., filtrations
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Riemann-Stieltjes-integral: Let f be a continuous function and g be a function of bounded variation. Then the following limit exists and is called R-S-integral of f with respect to g :

$$\int_a^b f(t)dg(t) := \lim_{\delta_n \rightarrow 0} \sum_{i=1}^n f(t_{i-1}^n)(g(t_i^n) - g(t_{i-1}^n)),$$

where $a = t_0^n < t_1^n < \dots < t_n^n = b$ and $\delta_n = \max_{i=1, \dots, n}(t_i^n - t_{i-1}^n)$.

Theorem 10.1

Fix $[0, t]$ and let, for each n , $0 = t_0^n < t_1^n < \dots < t_n^n = t$. If g is a function such that

$$\lim_{\delta_n \rightarrow 0} \sum_{i=1}^n f(t_{i-1}^n)(g(t_i^n) - g(t_{i-1}^n))$$

exists for each continuous function $f : [0, t] \rightarrow \mathbb{R}$, then g is of bounded variation on $[0, t]$.

Remark 10.2

No pathwise R-S-integral for Brownian motion!!!

Ito integral with simple integrands:

Definition 10.3

A stochastic process $(X_t)_{0 \leq t \leq T}$ is called **simple predictable integrand** if

$$X_t = \xi_0 \mathbb{I}_{[t_0, t_1]}(t) + \sum_{i=1}^{n-1} \xi_i \mathbb{I}_{(t_i, t_{i+1}]}(t),$$

where $0 = t_0 < t_1 < \dots < t_n = T$ and ξ_0, \dots, ξ_{n-1} are random variables such that ξ_i is \mathcal{F}_{t_i} -measurable and $E[\xi_i^2] < \infty$, for $i = 1, \dots, n-1$.

For each $t \leq T$ the Ito integral at time t for a simple integrand X is given by:

$$\int_0^t X_s dW_s =: (X \cdot W)_t = \sum_{i=0}^{n-1} \xi_i (W_{t_{i+1} \wedge t} - W_{t_i \wedge t}).$$

1. Stoch. pr.,
filtrations

2. BM as
weak limit

3. Gaussian pr.

4. Stopping
times

5. Cond.
expectation

6. Martingales

7. Discrete
stoch. integral

8. Refl. princ.,
pass.times

9. Quad.var.,
path prop.

10. Ito integral

11. Ito's
formula

Remark 10.4

If $t < T$ is such that $t_k \leq t < t_{k+1} \leq t_n = T$, this means

$$(X \cdot W)_t = \sum_{i=0}^{k-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_k (W_t - W_{t_k}).$$

For T this means

$$(X \cdot W)_T = \sum_{i=0}^{n-1} \xi_i (W_{t_{i+1}} - W_{t_i})$$

Theorem 10.5 (Properties of Ito integral for simple integrands)

1 *Linearity: X, Y simple, α, β constants, then*

$$\int_0^T (\alpha X_t + \beta Y_t) dW_t = \alpha \int_0^T X_t dW_t + \beta \int_0^T Y_t dW_t.$$

2 *Let $0 \leq a < b$. Then $\int_0^T \mathbb{I}_{(a,b]}(t) dW_t = W_b - W_a$ and $\int_0^T \mathbb{I}_{(a,b]}(t) X_t dW_t = \int_a^b X_t dW_t$.*

3 *Zero mean: $E[\int_0^T X_t dW_t] = 0$*

4 *Isometry: $E \left[\left(\int_0^T X_t dW_t \right)^2 \right] = \int_0^T E[X_t^2] dt$*

1. Stoch. pr.,
filtrations

2. BM as
weak limit

3. Gaussian pr.

4. Stopping
times

5. Cond.
expectation

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7. Discrete
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8. Refl. princ.,
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9. Quad.var.,
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10. Ito integral

11. Ito's
formula

(More) general integrands:

Lemma 10.6

Let $(X_t)_{0 \leq t \leq T}$ be a **bounded** and progressively measurable stochastic process. Then there exists a sequence of simple predictable processes $(X_t^m)_{0 \leq t \leq T}$, $m \geq 1$, such that

$$\lim_{m \rightarrow \infty} E \left[\int_0^T |X_t^m - X_t|^2 dt \right] = 0. \quad (10.1)$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Lemma 10.7

Let $(X_t)_{0 \leq t \leq T}$ be a progressively measurable stochastic process such that

$$E \left[\int_0^T X_t^2 dt \right] < \infty. \quad (10.2)$$

Then there exists a sequence of simple predictable processes $(X_t^m)_{0 \leq t \leq T}$, $m \geq 1$, such that (10.1) holds, i.e.,

$$\lim_{m \rightarrow \infty} E \left[\int_0^T |X_t^m - X_t|^2 dt \right] = 0.$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian pr.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
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Theorem 10.8 (Ito integral for general integrands with (10.2))

Let $(X_t)_{0 \leq t \leq T}$ be a progressively measurable stochastic process such that (10.2) holds, i.e., $E[\int_0^T X_t^2 dt] < \infty$. Then the stochastic integral $\int_0^T X_t dW_t$ is defined and has the following properties:

1 **Linearity:** X, Y as above, then

$$\int_0^T (\alpha X_t + \beta Y_t) dW_t = \alpha \int_0^T X_t dW_t + \beta \int_0^T Y_t dW_t.$$

2 Let $0 \leq a < b$. Then $\int_0^T \mathbb{I}_{(a,b]}(t) dW_t = W_b - W_a$ and $\int_0^T \mathbb{I}_{(a,b]}(t) X_t dW_t = \int_a^b X_t dW_t$.

3 **Zero mean:** $E[\int_0^T X_t dW_t] = 0$

4 **Isometry:** $E \left[\left(\int_0^T X_t dW_t \right)^2 \right] = \int_0^T E[X_t^2] dt$

Theorem 10.9 (Ito integral for even more general integrands)

Let $(X_t)_{0 \leq t \leq T}$ be a progressively measurable stochastic process such that

$$P \left(\int_0^T X_t^2 dt < \infty \right) = 1. \quad (10.3)$$

Then the stochastic integral $\int_0^T X_t dW_t$ is defined and has satisfies (1) and (2) of Theorem 10.8.

Remark 10.10

Attention: in this case (3) Zero-Mean and (4) Isometry are *not satisfied* in general!!! If, additionally the stronger condition (10.2) holds, then we are in the case of Theorem 10.8 and Zero-Mean and Isometry hold.

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian p.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass.times
- 9. Quad.var., path prop.
- 10. Ito integral
- 11. Ito's formula

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian p.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass.times
- 9. Quad.var., path prop.
- 10. Ito integral
- 11. Ito's formula

Corollary 10.11

Let X be continuous and adapted. Then $\int_0^T X_t dW_t$ exists. In particular, for any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ the stochastic integral $\int_0^T f(W_t) dW_t$ exists.

Remark 10.12

The Ito integral is not monotone.

Ito integral process:

Let X be progressively measurable such that (10.3) holds, i.e.,

$$P\left(\int_0^T X_s^2 ds < \infty\right) = 1.$$

Then $\int_0^t X_s dW_s$ is defined for all $t \leq T$. This gives a stochastic process $(I_t)_{0 \leq t \leq T}$ with

$$I_t = \int_0^t X_s dW_s.$$

There exists a modification with continuous paths: we always take this modification.

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian pr.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass.times
- 9. Quad.var., path prop.
- 10. Ito integral
- 11. Ito's formula

Definition 10.13

A martingale $(M_t)_{0 \leq t \leq T}$ is called quadratic integrable on $[0, T]$ if

$$\sup_{t \in [0, T]} E[M_t^2] < \infty.$$

Theorem 10.14

Let X be progressively measurable such that (10.2) holds, i.e., $E[\int_0^T X_s^2 ds] < \infty$. Then $(I_t)_{0 \leq t \leq T}$, where $I_t = \int_0^t X_s dW_s$, is a quadratic integrable martingale.

1. Stoch. pr.,
filtrations

2. BM as
weak limit

3. Gaussian p.

4. Stopping
times

5. Cond.
expectation

6. Martingales

7. Discrete
stoch. integral

8. Refl. princ.,
pass.times

9. Quad.var.,
path prop.

10. Ito integral

11. Ito's
formula

Corollary 10.15

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and bounded function. Then $(I_t)_{0 \leq t \leq T}$, where $I_t = \int_0^t f(W_s) dW_s$, is a quadratic integrable martingale.

Quadratic variation and covariation of Ito integrals:

Theorem 10.16

Let X be progressively measurable such that (10.3) holds. The quadratic variation of $\int_0^t X_s dW_s$ satisfies

$$\left[\int_0^t X_s dW_s, \int_0^t X_s dW_s \right] = \int_0^t X_s^2 ds \quad a.s.$$

for all $0 \leq t \leq T$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
- 10. Ito integral**
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1. Stoch. pr.,
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2. BM as
weak limit

3. Gaussian p.

4. Stopping
times

5. Cond.
expectation

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7. Discrete
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Covariation: Let $I_t = \int_0^t X_s dW_s$ and $\tilde{I}_t = \int_0^t \tilde{X}_s dW_s$, for $0 \leq t \leq T$, and progressively measurable processes X, \tilde{X} such that (10.3) holds.

The covariation of I and \tilde{I} is given by

$$[I, \tilde{I}]_t := \frac{1}{2} \left([I + \tilde{I}, I + \tilde{I}]_t - [I, I]_t - [\tilde{I}, \tilde{I}]_t \right).$$

Corollary 10.17

It holds that $[I, \tilde{I}]_t = \int_0^t X_s \tilde{X}_s ds$ a.s.

Remark 10.18

Analogously to the quadratic variation the covariation of two stochastic processes Y and X can be defined as follows:

$$[Y, X]_t = \lim_{n \rightarrow \infty} \sum_{i=1}^{m_n} (Y_{t_i^n} - Y_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}),$$

in probability where $0 = t_0^n < t_1^n < \dots < t_{m_n}^n = t$ and $\max_{i=1, \dots, m_n} (t_i^n - t_{i-1}^n) \rightarrow 0$ for $n \rightarrow \infty$.

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian pr.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass.times
- 9. Quad.var., path prop.
- 10. Ito integral
- 11. Ito's formula

Integration by parts:

Theorem 10.19 (Integration by parts)

Let X, Y be continuous and adapted processes such that (10.3) holds for both processes. Then the following holds, for each $0 \leq t \leq T$,

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

Notation: $d(X_t Y_t) = X_t dY_t + Y_t dX_t + d[X, Y]_t$

11. Ito's formula: change of variables

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
- 11. Ito's formula**

Theorem 11.1

Let $(W_t)_{t \geq 0}$ be a Brownian motion and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a twice continuously differentiable function ($f \in C^2(\mathbb{R})$). Then, for each t ,

$$f(W_t) - f(W_0) = \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds. \quad (11.1)$$

In differential notation (11.1) reads as follows:

$$df(W_t) = f'(W_t) dW_t + \frac{1}{2} f''(W_t) dt$$

- 1. Stoch. pr., filtrations
- 2. BM as weak limit
- 3. Gaussian pr.
- 4. Stopping times
- 5. Cond. expectation
- 6. Martingales
- 7. Discrete stoch. integral
- 8. Refl. princ., pass. times
- 9. Quad. var., path prop.
- 10. Ito integral
- 11. Ito's formula**