CAT 1: Take Home

DSA 8405 Probability and Stochastic Processes. Deadline 22nd August at 11.59PM

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1) Define a *stochastic process*. Explain the difference between **discrete-time** and **continuous-time** stochastic processes, giving one practical example of each.

[2 marks]

2) Consider a symmetric simple random walk $(S_n)_{n\geq 0}$ on the integers starting at $S_0=0$, where at each step the walker moves +1 with probability 0.65 and −1 with probability 0.35.

   (a) Derive the expected value $E[S_n]$ and variance $Var(S_n)$ after n steps.

   (b) Explain why this random walk has a drift.

[5 marks]

3) A gambler starts with i dollars and plays a game where each round they win $1 with probability p and lose $1 with probability $q=1-p$. The game ends when the gambler's fortune reaches 0 or N dollars.

   (a) Derive the probability of eventual ruin when $p\neq q$.

   (b) What happens to the probability of ruin when $p=0.5$?

[5 marks]

4) For a finite-state Markov chain with transition probability matrix P:

   (a) Define *irreducibility*, *aperiodicity*, and *stationary distribution*.

   (b) Explain the conditions under which the stationary distribution exists and is unique.

[3 marks]

5) A Markov chain models the weather with states: Sunny (S), Cloudy (C), and Rainy (R). The transition matrix is:

$$P = \begin{bmatrix} 0.55 & 0.35 & 0.1 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

   a. Interpret $P_{23}=0.2$.
   b. Compute the probability that it is Sunny two days after a Cloudy day.
   c. Compute the stationary probabilities

[5 marks]

6) Suppose there are three white and three black balls in two urns distributed so that each urn contains three balls. We say the system is in state *i, i* = 0, 1, 2, 3, if there are *i* white balls in urn one. At each stage one ball is drawn at random from each urn and interchanged. Let X_n denote the state of the system after the nth draw, and compute the transition matrix for the Markov chain $\{X_n : n \geq 0\}$.

[5 marks]