

Random Walk Process and Gambler's Ruin Problem

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Simple random walk

Let $(\varepsilon_n, n \geq 1)$ be i.i.d. (independent and identically distributed) random variables such that $P(\varepsilon_n = +1) = p$ and $P(\varepsilon_n = -1) = q = 1 - p$, where $0 < p < 1$.

Random Walk

The simple random walk on \mathbb{Z} is the random(stochastic) process $(S_n, n \in \mathbb{N})$ defined as $S_0 = 0, S_n = \varepsilon_1 + \cdots + \varepsilon_n, n \geq 1$

This type of random walk could be associated with the "walk of a drunkard". The moves are independent, $S_{n+1} = S_n + \varepsilon_{n+1}$ (i.e. at each step, we take an action that has nothing to do with the previous steps). However, the position at step $n + 1$ is highly correlated with the position at step n .

If $p = \frac{1}{2}$ the random walk is symmetric.

If $p \neq \frac{1}{2} (\Rightarrow q \neq \frac{1}{2})$ the random walk is asymmetric.

Basic Properties

Expectation (on average, where are we at time n ?)

$$\begin{aligned} E(S_n) &= E(\varepsilon_1 + \cdots + \varepsilon_n) \\ &= E(\varepsilon_1) + \cdots + E(\varepsilon_n) \end{aligned}$$

Identically distributed random variables

$$= nE(\varepsilon_i)$$

$$\begin{aligned} E(\varepsilon_i) &= 1 \cdot p + (-1) \cdot q = p + (p - 1) = 2p - 1 \Rightarrow \\ &\Rightarrow E(S_n) = n(2p - 1) \end{aligned}$$

Variance

$$\begin{aligned} \text{Var}(S_n) &= \text{Var}(\varepsilon_1 + \cdots + \varepsilon_n) \\ &= \text{Var}(\varepsilon_1) + \cdots + \text{Var}(\varepsilon_n) \\ &= n\text{Var}(\varepsilon_i) \end{aligned}$$

Basic Properties

Note:

$$\begin{aligned} \text{Var}(\varepsilon_i) &= E(\varepsilon_i^2) - [E(\varepsilon_i)]^2 \\ &= p + (1 - p) - (2p - 1)^2 \\ &= 1 - 4p^2 + 4p - 1 = 4p(1 - p) \\ &\Rightarrow \text{Var}(S_n) = 4np(1 - p) \end{aligned}$$

Note: $\text{Var}(S_n)$ reaches its maximum for $p = \frac{1}{2}$, having $\text{Var}(S_n) = n$
Standard Deviation:

$$\text{Stdev}(S_n) = \sqrt{\text{Var}(S_n)} = \sqrt{4np(1 - p)}$$

Two Important Theorems

The law of large numbers

For a sequence of i.i.d. random variables ε_i taking values in \mathbb{R} ,

$$\frac{S_n}{n} \xrightarrow{n \rightarrow \infty} E(\varepsilon_1) = 2p - 1$$

with high probability (this can be taken to mean that the probability of this event goes to 1 as $n \rightarrow \infty$), provided that $E(\varepsilon_1) < \infty$.

The Central Limit Theorem

$$\frac{Z_n}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} Z \sim N(0, 1)$$

When n goes to infinity, the distribution of Z_n will tend to a Gaussian random variable, with mean 0 and variance 1.

Gambler's Ruin Problem

Example of Gambler's Ruin Problem

Two players, A and B, play a game with independent rounds where, in each round, one of the players wins 1 USD from his opponent; A with probability p and B with probability $q = 1 - p$. A starts the game with a 100 USD and B with 50 USD. The game ends when one of the players is ruined.

- We have two gamblers, named simply A and B. There are N total dollars in the game; let A start with j dollars, which means B starts with $N - j$ dollars.
- The Gamblers play repeated, one round games. Each round, A wins with probability p , and B wins with probability $1 - p$, which we will label as q (the rounds are independent). If A wins, B gives one dollar to A, and if B wins, A gives one dollar to B.
- The game ends when one of the gamblers is ruined (they get to zero dollars), which means that the other gambler has all of the N dollars.

Discussions-Win Probability

- Naturally, some questions start to arise. Of course A had the advantage in the game above, since if $p = .6$.
- Does that mean A would win 60% of games? What about if we started A with less money, say 40 dollars (while B starts with 60)? Would A still be favored?
- Given p (the probability of A winning any one game) and N and j (the total number of dollars in the game and the total number of dollars that A starts out with) what is the probability that A wins the game and B is 'ruined?'
- We can define, for a given game (where N and p are specified), w_j as the probability that A wins given that A has j dollars. Formally:

$$w_j = P(A_{\text{wins}} | A_{\text{starts at } j \text{ dollars}})$$

The End