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DSA 8505: Bayesian Statistics

CAT 1

Question 1.

A researcher is studying the probability θ that a patient responds to a new treatment.

Prior distribution: $\theta \sim Beta(\alpha, \beta)$, $\alpha = 3$, $\beta = 7$

Observed data:

Number of patients $n = 20$

Number of successful responses $k = 14$

a) Likelihood function $p(D|\theta)$ under the Binomial model

Each patient outcome is binary that is response / no response

Hence, the number of successful responses follows a Binomial model:

$$Y|\theta \sim Binomial(n, \theta)$$

The likelihood function for observing k successes out of n trials is:

$$p(D|\theta) = \binom{n}{k} \theta^k (1-\theta)^{n-k}$$

Substituting values $n = 20$ and $k = 14$.

$$P(D|\theta) = \binom{20}{14} \theta^{14} (1-\theta)^{20-14}$$

$$P(D|\theta) = \binom{20}{14} \theta^{14} (1-\theta)^{6}$$

b) Using Bayes' theorem, derive the posterior distribution $p(\theta|D)$ showing all steps.

Step 1: Prior distribution

The prior distribution is $\theta \sim Beta(3,7)$

The density function is

$$P(\theta) = \frac{1}{B(3,7)} \theta^{3-1} (1-\theta)^{7-1}$$

$$P(\theta) = \frac{1}{B(3,7)} \theta^{2} (1-\theta)^{6}$$

step 2 : Apply Bayes' theorem

it states : $P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$

Since the marginal likelihood $P(D)$ does not depend on θ

hence $P(\theta|D) \propto P(D|\theta)\,p(\theta)$

step 3 : Substituting the likelihood and prior.

$$P(\theta|D) \propto \left[\underbrace{\binom{20}{14}}_{\text{constant}} \theta^{14}(1-\theta)^{6}\right]\left[\underbrace{\frac{1}{B(3,7)}}_{\text{constant}} \theta^{2}(1-\theta)^{6}\right]$$

$\binom{20}{14}$ and $\frac{1}{B(3-7)}$ are constants with respect to θ and can be ignored

Therefore $p(\theta|D) \propto \theta^{14}+\theta^{2}\,(1-\theta)^{6+6}$

$= P(\theta|D) \propto \theta^{16}(1-\theta)^{12}$

step 4: The posterior distribution.

The kernel $\theta^{16}(1-\theta)^{12}$ that should match the form of a beta distribution $\theta^{\alpha'-1}(1-\theta)^{\beta'-1}$ is $\alpha' = 17$, $\beta' = 13$

c) state the posterior distribution clearly in the form:

$\theta|D \sim Beta(\alpha, \beta)$

The posterior distribution is $\theta|D \sim Beta(17, 13)$

By updating the prior parameters with the observed data:

where α posterior $= \alpha + k = 3 + 14 = 17$

β posterior $= \beta + n - k = 7 + (13-7) = 13$

When using Beta prior and a Binomial likelihood, the posterior distribution remains in the Beta family because of conjugacy. The observed clinical data shifts the prior belief toward higher response probabilities while still retaining uncertainty informed by the prior structure.

Question 2.

The number of emergency cases arriving per night follows a poisson model

$$Y|\lambda \sim \text{Poisson}(\lambda)$$

Prior distribution : $\lambda \sim \Gamma(\alpha, \beta)$, $\alpha = 4$, $\beta = 2$

Observed data : $y = 7$

a) Derive posterior distribution for λ

Step 1: Likelihood function

For Poisson random variable, likelihood of observing y cases given λ is :

$$p(y/\lambda) = \frac{\lambda^y e^{-\lambda}}{y!}$$

replacing $y = 7$

$$P(y|\lambda) = \frac{\lambda^7 e^{-\lambda}}{7!}$$

Step 2: Prior distribution

The Gamma prior with shape α and rate β has density:

$$p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \quad \lambda > 0$$

Substituting $\alpha = 4$, $\beta = 2$

$$p(\lambda) = \frac{2^4}{\Gamma(4)} \lambda^3 e^{-2\lambda}$$

$$P(\lambda) = \frac{2^4}{\Gamma(4)} \lambda^3 e^{-2\lambda}$$

Step 3: Applying Bayes' theorem

It state : $$P(\lambda|y) = \frac{P(y/\lambda) p(\lambda)}{P(y)}$$

Ignoring the marginal likelihood $p(y)$, which does not depend on λ:

$$p(\lambda|y) \propto p(y/\lambda) p(\lambda)$$

Step 4: Combining likelihood and prior

$$p(\lambda|y) \propto p(\lambda^7 e^{-\lambda})(\lambda^3 e^{-2\lambda})$$

$$p(\lambda|y) \propto \lambda^{10} e^{-3\lambda}$$

Step 5: The posterior distribution

The kernel $\lambda^{10} \quad e^{-3\lambda}$

that matches the Gamma distribution form:

$$\lambda^{\alpha'-1} e^{-\beta'\lambda}$$

Hence $\alpha'' \, , \beta' = 3$

$\lambda | y \sim \Gamma(11, 3)$

This follows the conjugate updating rule for the Gamma-Poisson model:

α posterior $= \alpha + y$, βposterior $= \beta + 1$

b) Find the posterior mean $E(\lambda|y)$

For a Gamma distribution with shape α and rate β

hence $E(\lambda) = \alpha/\beta$

using the posterior parameters

$E(\lambda|y) = 11/3 \approx 3.67$

After observing 7 emergency cases in one night, the prior belief about the arrival rate is updated from $\Gamma(4,2)$ to $\Gamma(11,3)$.

The posterior mean is approximately 3.67 represents the updated expected number of emergency cases per night, balancing prior information with observed data.

Question 3:

A single observation is recorded from the model:

$$x|\theta \sim N(\theta, \sigma^2), \quad \sigma^2 = 9$$

Observed value: $x = 42$

Prior distribution: $\theta \sim$ Uniform $(0, 100)$

a) Write Bayes' theorem for $p(\theta|x)$

For a continuous parameter is given by:

$$p(\theta|x) = \frac{p(x|\theta)\, p(\theta)}{p(x)}$$

where:

$p(x|\theta)$ is the likelihood

$p(\theta)$ is the prior

$p(x) = \int p(x|\theta)\, p(\theta)\, d\theta$ - is the marginal likelihood

Since $p(x)$ does not depend on θ, then

$$p(\theta|x) \propto p(x|\theta)\, p(\theta)$$

b) Show that the posterior has a Normal kernel

Step 1. Likelihood function

Normal likelihood function for a single observation is:

$$p(x|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)$$

Substitute $x = 42$ and $\sigma^2 = 9$.

$$P(x|\theta) = \underbrace{\frac{1}{\sqrt{18\pi}}}_{\text{constant}} \exp\left(-\frac{(42-\theta)^2}{18}\right)$$

Ignoring constant because it not involving θ:

$$p(x|\theta) \propto \exp\left(-\frac{(42-\theta)^2}{18}\right)$$

Step 2 Prior distribution

The uniform prior on (0, 100) is:

$$P(\theta) = \begin{cases} \frac{1}{100}, & 0 \leq \theta \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

Within the interval (0,100), the prior is constant.

Step 3: Combine likelihood and Prior

Using Bayes' theorem:

$$P(\theta|x) \propto \exp\left(-\frac{(42-\theta)^2}{18}\right), \quad 0 \leq \theta \leq 100$$

The exponential term is exactly the kernel of a Normal distribution with mean 42 and variance 9.

c) State the posterior distribution form and explain why it is a truncated Normal distribution.

By ignoring the bounds temporarily, the kernel corresponds to:

$$\theta|x \sim N(42, 9)$$

However, because the prior restricts θ to the interval (0,100), the posterior density is $\theta|x \sim N(42, 9)$ truncated to (0, 100)

Why the posterior is truncated.

- The prior assigns zero probability to any value of θ outside (0, 100)
- As a result the posterior must also be zero outside this interval.
- The likelihood alone suggests a Normal distribution centered at 42.
- The posterior follows a Normal shape within the interval (0,100), but is cut off at the boundaries.
- The posterior is a truncated Normal distribution, obtained by restricting a N(42, 9) distribution to the support imposed by the prior.

With a uniform prior and a normal likelihood, the posterior retains a Normal kernel. The bounded support of the prior truncates this distribution, resulting to a truncated Normal posterior centered at the observed value.