Bayesian Estimation.

population parameter

$\boxed{\theta}$ = proportion of those who supports a candidate (2027)

15M voters

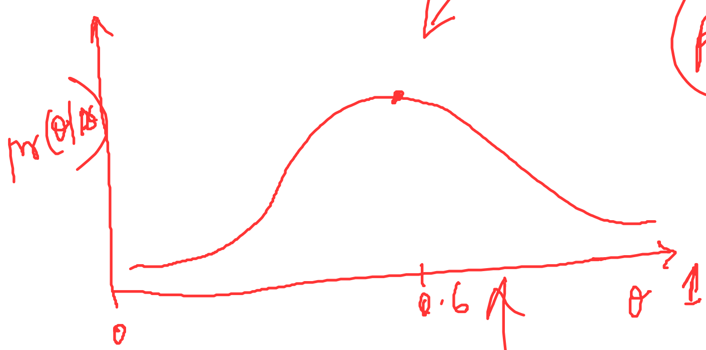
Prior distn. of θ × Data (Likelihood) $pr(D|\theta)$ ⇒ 2027 $pr(\theta|D)$

(2022) 2025

Results

pdf of θ

Normal



2027

Estimation

Bayesian Estimation.

$\hat{\theta} = 0.6$

$\theta = 0.7$

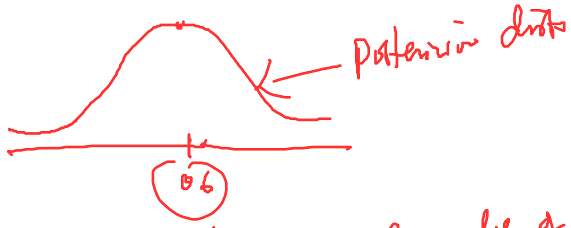
Error function
OR
Loss function

$\hat{\theta} - \theta$ ⇐ Error / optimization
minimize
maximize

Loss function

$\hat{\theta} = 0.6$

$L(\theta, a)$ — cost or penalty associated with estimating parameter θ using a



↑ Expected value of θ

$E(x) = \int x f(x)\, dx$

$\theta, a \sim \pi(\theta|x)$

θ — Not a variable; a — variable

The expected loss is defined $h(a) = E\, L(\theta, a)$

$$\boxed{h(a) = \int L(\theta, a)\, \pi(\theta/x)\, d\theta}$$

Objective

we are interested in the value of a that minimizes $h(a)$

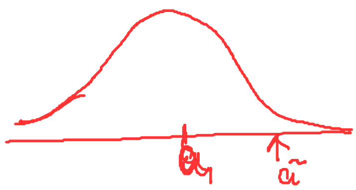
that value is called bayes estimator

Common Loss functions

① squared Error Loss.

② Absolute Error Loss

③ 0-1 Loss



$(0.9 - 0.89)$ ← small

$(0.9 - 0.6)^2$ — big

① Square Loss function $\quad L(\theta, a) = (\theta - a)^2$

$$\hat{\theta} = E(\theta/x) = \int \theta\, \pi(\theta/x)\, d\theta$$

X a_1 X a_2 a_3 ↙ mean of $\pi(\theta/x)$

① Expected posterior Loss

$$h(a) = \int L(\theta, a)\, \pi(\theta/x)\, d\theta$$

$$L(\theta, a) = (\theta - a)^2$$

$$h(a) = \int (\theta - a)^2 \pi(\theta/x)\, d\theta.$$

$$= \int (\theta^2 - 2\theta a + a^2)\, \pi(\theta/x)\, d\theta$$

$$= \int \theta^2 \pi(\theta/x)\, d\theta - 2a \int \theta\, \pi(\theta/x)\, d\theta + a^2 \underbrace{\int \pi(\theta/x)\, d\theta}_{1}$$

$$h(a) = \underline{E(\theta^2/x) - 2a\, E(\theta/x) + a^2}$$

to minimize $h(a)$

$$\frac{\partial h(a)}{\partial a} = \frac{\partial}{\partial a}\left[E(\theta^2/x) - 2a\, E(\theta/x) + a^2 \right].$$

$$0 - 2E(\theta/x) + 2a = 0$$

$$\cancel{2}a = \cancel{2}\, E(\theta/x)$$

$$a = E(\theta/x)$$ mean of $\pi(\theta/x)$

Example

suppose $\pi(\theta/x) \sim N(10, 4)$ Mean = 10

var = 4

$E(\theta/x) = 10$ under the squared loss function

$\therefore$ the $\hat{\theta}_B = E(\theta/x) = \underline{10}$.

$a = 9$ $(\theta - 9)^2 =$

$a = 11$ $(\theta - 11)^2$

Suppose $\pi(\theta/x) \sim$ Gama $(2, 5)$ θ = mean

$$\hat{\theta}_B = E(\theta/x) = \frac{\alpha}{\beta} = \left(\frac{2}{5}\right).$$

Squared Error Loss

θ = mean $[\pi(\theta/x)]$

(2) Absolute Error Loss

$$L(\theta,a) = |\theta - a|$$

Bayes estimate → posterior mean → it minimized the expected absolute error loss

$$\hat{\theta} = \text{median}\{\pi(\theta/x)\}$$

$$h(a) = \int_{-\infty}^{\infty} |\theta - a|\, \pi(\theta/x)\, d\theta . \qquad -\infty < \theta < \infty$$

$$h(a) = \int_{-\infty}^{a} (a-\theta)\, \pi(\theta/x)\, d\theta + \int_{a}^{\infty} (\theta - a)\, \pi(\theta/x)\, d\theta \qquad |\theta - \alpha| = \begin{cases} -ve \\ +ve \end{cases} \; +ve$$

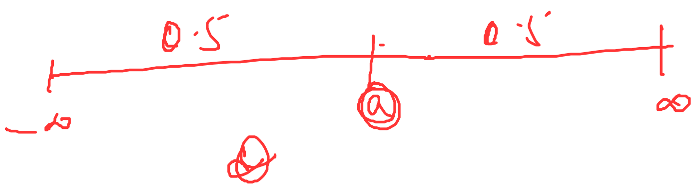
$\theta < a \; (-ve)$

$\theta > a \; (+ve)$

$$\frac{d\,h(a)}{da} = \int_{-\infty}^{a} \frac{d}{da}\left(a \cdot \pi(\cdot)\, d\theta\right) - \int_{-\infty}^{a} \frac{d}{da}\, \cancel{\theta\, \pi(\cdot)\, d\theta}$$

$$= \int_{-\infty}^{a} \pi(\theta/x)\, d\theta - \int_{a}^{\infty} \pi(\theta/x)\, d\theta = 0 \qquad \int_{-\infty}^{\infty} \pi(\theta/x)\, d\theta = 1$$

$$\int_{-\infty}^{a} \pi(\theta/x)\, d\theta = \int_{a}^{\infty} \pi(\theta/x)\, d\theta = \underline{\underline{0.5}}$$



$$\hat{\theta} = \text{median}\{\pi(\theta/x)\}$$

Example

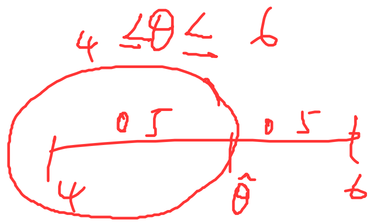
Suppose the posterior distn of θ is symmetric and rectangular defined as

$$\pi(\theta/x) = \begin{cases} 2(1 - |\theta - 5|) & 4 \le \theta \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Find the posterior median.

Sol

The posterior median $\hat{\theta}$ satisfies



$$pr(\theta \le \hat{\theta}/x) = 0.5$$

$$\int_4^{\hat{\theta}} \pi(\theta/x)\, d\theta = 0.5$$

$$\Rightarrow \int_4^{\hat{\theta}} 2(1 - |\theta - 5|)\, d\theta = 0.5$$

$$|\theta - 5| = 5 - \theta \quad \text{if } \theta \le 5$$

Therefore $4 \le \theta \le 5$

$$\pi(\theta/x) = 2(1 - (5 - \theta))$$
$$= 2(1 - 5 + \theta)$$
$$= 2(\theta - 4)$$

$$\int_4^{\hat{\theta}} 2(\theta - 4)\, d\theta = 2\int_4^{\hat{\theta}} \theta - 4\, d\theta$$

Let $\mu = \theta - 4$ | $\theta = 4$ $\quad \theta = \hat{\theta}$

$du = d\theta$ | $\mu = 0$ $\quad \mu = \hat{\theta} - 4$

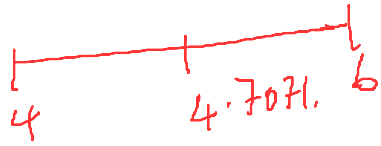
$$2\int_0^{\hat{\theta}} (\theta - 4)\, d\theta = 2\int_0^{\hat{\theta} - 4} u\, du. = \cancel{2}\left[\frac{\mu^2}{\cancel{2}}\right]_0^{\hat{\theta} - 4}$$

$$= [\hat{\theta} - 4]^2 = 0.5$$

$$\hat{\theta} - 4 = \sqrt{0.5}$$

$$\hat{\theta} = 4 + \sqrt{0.5}$$

$$\underline{\hat{\theta} = 4.7071}$$



0-1 Loss.

↳ Assigns no penalty if the estimate is exactly correct but imposes a fixed penalty if otherwise.

$$L(\hat{\theta}, \theta) = \begin{cases} 0 & \text{if } \hat{\theta} = \theta \\ 1 & \text{if } \hat{\theta} \neq \theta \end{cases}$$

$$h(a) = \int_{-\infty}^{\infty} L(\hat{\theta}, \theta)\, \pi(\theta/x)\, d\theta.$$

$$= \int_{\theta \neq \hat{\theta}} 1\, \pi(\theta/x)\, d\theta + \int_{\theta = \hat{\theta}} 0\, \pi(\theta/x)\, d\theta$$

$$h(a) = \int_{\theta \neq \theta} \pi(\theta/x)\, d\theta < 1$$

$$\text{But} \int_{\theta \in \mathbb{R}} \pi(\theta/x)\, dx = 1$$

$$\int_{\theta \in \mathbb{R}} \pi(\theta/x)\,d\theta = \int_{\theta=\hat\theta} \pi(\theta/x)\,d\theta + \int_{\theta\neq\hat\theta} \pi(\theta/x)\,d\theta$$

$$1 = \pi(\hat\theta/x) + \int_{\theta\neq\hat\theta} \pi(\theta/x)\,d\theta$$

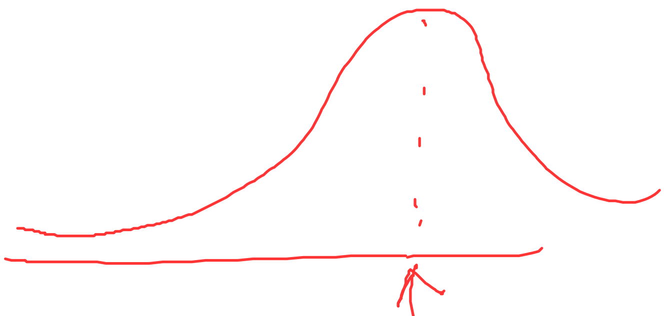
$$h(x) = \int_{\theta\neq\hat\theta} \pi(\theta/x)\,d\theta = 1 - \pi(\hat\theta/x)$$

Minize.

Minimize

$\Downarrow$

$$\text{Max } \pi(\hat\theta/x)$$



The point in the dist when its highest

$$\arg\max_\theta \pi(\theta/x) = \underline{\text{posteri Mode } (\pi(\hat\theta/x))}$$

$$\boxed{\hat\theta_{MAP}}$$

Maximum A priori

Worked Example: 0–1 Loss and MAP Estimation

Problem Setup (Discrete Parameter Case)

Suppose a diagnostic system classifies the severity level of a condition into three possible categories:

$$\theta \in \{\theta_1, \theta_2, \theta_3\} = \{\text{Low}, \text{Moderate}, \text{High}\}.$$

After observing data x, the posterior probabilities are given as:

$$\pi(\theta = \text{Low} \mid x) = 0.20, \qquad \pi(\theta = \text{Moderate} \mid x) = 0.55, \qquad \pi(\theta = \text{High} \mid x) = 0.25.$$

Solution

$$E[L(\hat{\theta}, \theta) \mid x] = \int L(\hat{\theta}, \theta)\, \pi(\theta \mid x)$$

$$= \sum_{\theta = \theta_1, \theta_2, \theta_3} L(\hat{\theta}, \theta)\, \pi(\theta \mid x)$$

$\theta = \theta_1, \theta_2, \theta_3$

=

Evaluate the risk Loss for each

$$L(\hat{\theta}, \theta) = \begin{cases} 0 & \theta = \theta_1 \\ 1 & \theta = \theta_2, \theta_3 \end{cases}$$

Case 1 $\hat{\theta} = \text{Low}$

$$h(a) = 0 \cdot \pi(\text{Low} \mid x) + 1 \cdot \pi(\text{Moderate} \mid x) + 1 \cdot \pi(\text{High} \mid x)$$

$$= 0(0.20) + 1(0.55) + 1(0.25) = 0.8$$

Case 2 $\hat{\theta} = \text{Moderate}$

$$h(a) = 0.45$$

Case 3 $\hat{\theta} = \text{high}$ $h(a) = 0.75$

Bayes Estimator is the one with the minimum risk

θ_{0-1} = Moderate ✓

θ.

$$\theta_{MAP} = \arg\max_{\theta} \pi(\theta/x)$$

$$\max[0.20, 0.55, 0.25] = 0.55$$

$\hat{\theta}_{MAP}$ = Moderate

$\hat{\theta} = [\quad]$

~~Confi~~ Confidence interval

Credible interval

100%

Confidence level $= (1-\alpha)\%$

Confidence level.

$$pr(\theta \in [a,b]) = 1-\alpha$$

90%

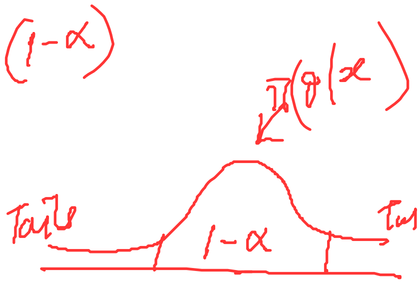
Equal tail Interval → ~~No~~ Symmetric posterior dist^n

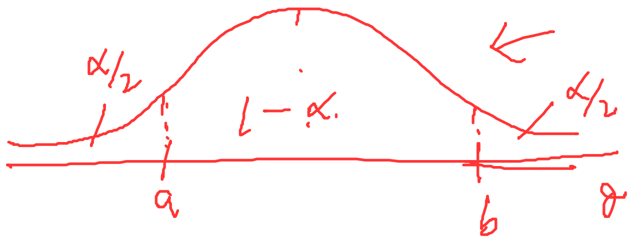
↳ Normal
↳ Beta(2,2)
↳ Triangle
↳ Uniform.

- Ensure that prob of θ fall outside ~~to~~ $[a, b]$
→ less a & more than b.

$$\frac{\alpha}{2} = pr(\theta < a | x) = pr(\theta > b | x)$$

$(1-\alpha)$





$(1-\alpha)$

$\hat{\theta} = [a, b]$

Suppose that after observing data x the posterior dist^n

$$\theta | x \sim N(\mu, \sigma^2)$$

define the Credible Interval & confidence level.

$100(1-\alpha)\%$ credible interval $[a, b]$

$$pr(a < \theta < b | x) = 1-\alpha = 95\%$$

/ Assume 95% credible interval.

$\alpha = 5\%$

$$pr(\theta < a | x) = \alpha/2 = 0.025, \quad pr(\theta > b | x) = \alpha/2 = 0.025$$

$pr(\theta < a | x)$

$$z = \frac{\theta - \mu}{\sigma}$$

$Z = \frac{\theta-\mu}{\sigma}$

~~$pr(\theta < a|x) = pr\left(\frac{\theta-\mu}{\sigma} <\right.$~~

~~Convert~~ Standardization

(1) $pr(a < \theta < b|x) = 0.95$

(2) $pr(\theta < a|x) = 0.025$

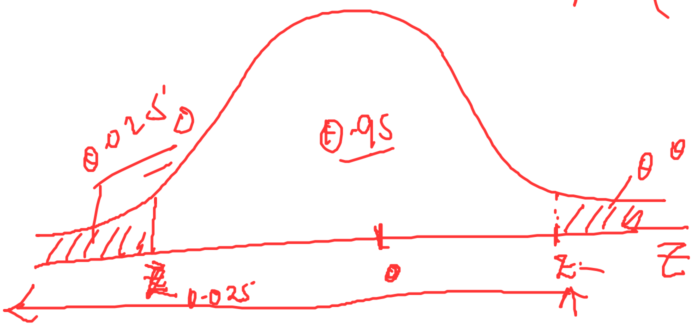
(3) $pr(\theta > b|x) = 0.025$

$\Rightarrow pr\left(\frac{a-\mu}{\sigma} < \frac{\theta-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = 0.95$

$pr\left(\overset{Z}{\frac{\theta-\mu}{\sigma}} < \frac{a-\mu}{\sigma} \middle| x\right) = 0.025$

$pr\left(\overset{Z}{\frac{\theta-\mu}{\sigma}} > \frac{b-\mu}{\sigma} \middle| x\right) = 0.025$

$pr(Z > Z_{0.025}) = 0.025$



$Z \sim N(0,1)$

$pr(\theta > b)$

$\Phi(Z_{0.025}) = 0.025 = -1.96$ $\quad$ $\Phi(Z_{0.975}) = 1.96 = 1 - pr(\theta < b).$

$pr(a \le \theta \le b) = 0.95 \Rightarrow pr(-1.96 \le Z \le 1.96) = 0.95$

$pr\left(-1.96 \le \frac{\theta-\mu}{\sigma} \le 1.96\right) = 0.95$

$pr(-1.96\sigma \le \theta - \mu \le 1.96\sigma) = 0.95$

$pr(\underbrace{\mu - 1.96\sigma}_{a} \le \theta \le \underbrace{\mu + 1.96\sigma}_{b}) = 0.95$

$[a, b] = [\mu - 1.96\sigma, \mu + 1.96\sigma]$

$\theta \sim N(10, 4)$ $\quad \mu = 10 \quad \sigma = 2$

$$[a, b] = [10 - 1.96(2), \ 10 + 1.96(2)]$$

$$= [6.08, \ 13.92]$$