CS 3341 HW # 3 SOLUTION

1. Let the random variable X denote the number of heads in three independent tosses of a fair coin. Find the PMF of X. Also find the CDF of X.

   PMF of X:

x	p(x)
0	p(0) = P(X = 0) = P(TTT) = 1/8
1	p(1) = P(X = 1) = P(HTT) + P(THT) + P(TTH) = 3/8
2	p(2) = P(X = 2) = P(HHT) + P(HTH) + P(THH) = 3/8
3	p(3) = P(X = 3) = P(HHH) = 1/8

   CDF of X:

   $F(0) = P(X \le 0) = P(0 \text{ or fewer heads}) = 1/8$
   $F(1) = P(X \le 1) = P(1 \text{ or fewer heads}) = (1/8) + (3/8) = 4/8$
   $F(2) = P(X \le 2) = P(2 \text{ or fewer heads}) = (1/8) + (3/8) + (3/8) = 7/8$
   $F(3) = P(X \le 3) = P(3 \text{ or fewer heads}) = (1/8) + (3/8) + (3/8) + (1/8) = 1$

   Hence, the CDF of X is:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/8 & 0 \le x < 1 \\ 4/8 & 1 \le x < 2 \\ 7/8 & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

2. Let X denote the number of busy servers at the checkout counters in a store at 5pm. Suppose that the CDF of X is:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.20 & 0 \le x < 1 \\ 0.50 & 1 \le x < 2 \\ 0.80 & 2 \le x < 3 \\ 0.90 & 3 \le x < 4 \\ 1 & 4 \le x \end{cases}$$

   (a) Find the PMF of X.

   The possible values of X are the points at which the CDF F(x) jumps, and the sizes of the jumps are the probabilities of taking those values. Hence the PMF of X is:

x	p(x)
0	p(0) = P(X = 0) = F(0) – F(0-)* = 0.20 – 0 = 0.20
1	p(1) = P(X = 1) = F(1) – F(1-) = 0.50 – 0.20 = 0.30
2	p(2) = P(X = 2) = F(2) – F(2-) = 0.80 – 0.50 = 0.30
3	p(3) = P(X = 3) = F(3) – F(3-) = 0.90 – 0.80 = 0.10
4	p(4) = P(X = 4) = F(4) – F(4-) = 1 – 0.90 = 0.10

*F(a-) represents $P(X < a)$.

(b) Find $P(X > 3)$.

$P(X > 3) = 1 – P(X \leq 3) = 1 – F(3) = 1 – 0.90 = 0.10$

3. It is known that there is a defective chip on a computer board that contains 8 chips. A technician tests the chips one at a time until the defective chip is found. Assume that the chip to be tested is selected at random without replacement. Let the random variable X denote the number of chips tested. Find the PMF of X.

   Total of 8 chips; 7 = (G)ood, 1 = (D)efective. The PMF is:

x	P(X = x)
1	P(X = 1) = P(D) = 1/8
2	P(X = 2) = P(GD) = (7/8)(1/7) = 1/8
3	P(X = 3) = P(GGD) = (7/8)(6/7)(1/6) = 1/8
4	P(X = 4) = P(GGGD) = (7/8)(6/7)(5/6)(1/5) = 1/8
5, 6, 7, 8	P(X = 5) = P(X = 6) = P(X = 7) = P(X = 8) =1/8 [as before]

4. Let X denote the number of speeding tickets issued by Officer Smith on Monday, and let Y denote the number given on Tuesday. The PMF of X and Y are

X=k	0	1	2	3	4
P(X=k)	0.1	0.2	0.3	0.2	0.2

Y=m	0	1	2	3	4	5
P(Y=m)	0.1	0.1	0.2	0.3	0.2	0.1

   Assume that X and Y are independent. Find the joint PMF of X and Y. Also find the PMF of X + Y.

   The table of joint PMF can be obtained by using $P(X = k, Y = m) = P(X=k) P(Y=m)$, since X and Y are independent.

The possible values of $Z = X + Y$ are 0, 1, 2, …, 9. Here is the PMF of Z:

z	0	1	2	3	4	5	6	7	8	9
P(Z=z)	0.01	0.03	0.07	0.12	0.18	0.20	0.18	0.13	0.06	0.02

One example of how we get this table is the following:

$P(Z = 1) = P[(X, Y) = (0, 1) \text{ or } (1, 0)] = P[(1,0)] + P[(0,1)]$, and now substitute these probabilities from the joint PMF table.

5. The joint distribution of random variables X and Y is given in the table,

X \ Y	1	2	3	5
−1	0.04	0.02	0.06	0.12
0	0.08	0.12	0.20	0.10
1	0.16	0.02	0.04	0.04

(a) Find the marginal PMF's of X and Y.
(b) Are X and Y independent?

(a) $P(X = -1) = 0.04 + 0.02 + 0.06 + 0.12 = 0.24$,
$P(X = 0) = 0.50$, $P(X = 1) = 0.26$

$P(Y = 1) = 0.04 + 0.08 + 0.16 = 0.28$,
$P(Y = 2) = 0.16$, $P(Y = 3) = 0.30$, $P(Y = 5) = 0.26$

(b) No, since $P(X = -1, Y = 1) \neq P(X = -1)\, P(Y = 1)$

6. The probability of being able to log on to a certain computer from a remote terminal at any given time is 0.7. Let X denote the number of independent attempts that must be made to gain access to the computer. Find the PMF of X.

$P(X = 1) = 0.7$
$P(X = 2) = P(1^{st} = \text{Fail}, 2^{nd} = \text{Success}) = P(1^{st} = \text{Fail})\, P(2^{nd} = \text{Success}) = (0.3)(0.7) =$
0.21
$P(X = 3) = P(FFS) = (0.3)^2 (0.7) = 0.063$, and so on.

In general, for any positive integer k, $P(X = k) = (0.3)^{(k-1)} (0.7)$.

7. Every day, the number of network blackouts has a distribution (PMF) P(0)=0.6, P(1)=0.2, P(2)=0.2, independently of other days. What is the probability that there are more blackouts on Friday than on Thursday?

Let F = # blackouts on Friday, T = # blackouts on Thursday.

Using the theorem of total probability,

$$
\begin{aligned}
P(F > T) &= P(F > T, T = 0) + P(F > T, T = 1) + P(F > T, T = 2) \\
&= P(F > 0, T = 0) + P(F > 1, T = 1) + P(F > 2, T = 2) \\
&= \{P(F = 1, T = 0) + P(F = 2, T = 0)\} + P(F = 2, T = 1) + 0 \\
&= P(F = 1)\, P(T = 0) + P(F = 2)\, P(T = 0) + P(F = 2)\, P(T = 1) \quad \text{[independence]} \\
&= (0.2)\,(0.6) + (0.2)\,(0.6) + (0.2)\,(0.2) \\
&= 0.28
\end{aligned}
$$

Another way to think about it: Make a table of joint PMF for F and T, and observe that the event {F > T} consists of the outcomes {(F=1, T=0), (F=2, T=0), (F=2, T=1)}.

8. An internet search engine looks for a certain keyword in a sequence of independent web sites. It is believed that 20% of the sites contain this keyword.

   (a) Let X be the number of websites visited until the first keyword is found. Find the PMF of X.

$$
\begin{aligned}
P(X = x) &= P(\text{First } x - 1 \text{ sites don't have the keyword, } x\text{-th site has it}) \\
&= (0.80)^{(x-1)}\,(0.20), \text{ for } x = 1, 2, 3, \ldots.
\end{aligned}
$$

   (b) Out of the first 5 web sites, let Y be the number of sites that contain the keyword. Find the PMF of Y.

$$P(Y = 0) = P(\text{None of the sites has the keyword}) = (0.80)^5 = 0.3277$$

$$P(Y = 1) = P(\text{1 site has the keyword, 4 sites don't}) = \binom{5}{1}(0.20)(0.80)^4 = 0.4096.$$

Remember to multiply by “5 choose 1” as there are “5 choose 1 = 5” outcomes where 1 site has the keyword and the other 4 sites don’t.

$$P(Y = 2) = P(\text{2 sites have, 3 sites don't}) = \binom{5}{2}(0.20)^2(0.80)^3 = 0.2048$$

$$P(Y = 3) = P(\text{3 sites have, 2 sites don't}) = \binom{5}{3}(0.20)^3(0.80)^2 = 0.0512$$

$$P(Y = 4) = P(\text{4 sites have, 1 site doesn't}) = \binom{5}{4}(0.20)^4(0.80) = 0.0064$$

$P(Y = 5) = P(\text{All sites have}) = (0.20)^5 = 0.0003$

(c) Compute the probability that at least 3 of the first 5 websites contain the keyword.

$P(Y \geq 3) = 0.0512 + 0.0064 + 0.0003 = 0.0579$

(d) Compute the probability that the search engine had to visit at least 5 sites in order to find the first occurrence of a keyword.

$P(X \geq 5) = 1 - P(X \leq 4) = 1 - 0.5904 = 0.4096$