Poisson's equation on an L-shaped domain

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1 Poisson's equation

Consider the following elliptic problem with Dirichlet boundary conditions

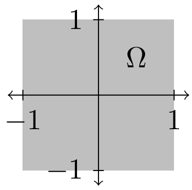
$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

and its weak formulation with solution $u \in H_0^1(\Omega)$

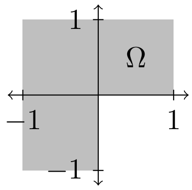
$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega),$$

abbreviated as

$$a(u, v) = (f, v) \ \ \forall v \in H_0^1(\Omega).$$



vs.



2 Shift theorem

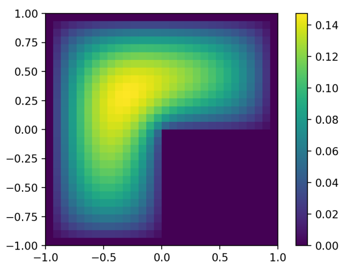
Let Ω be a convex polygonal domain in $\mathbb{R}^2$. If u is the weak solution to our problem, then

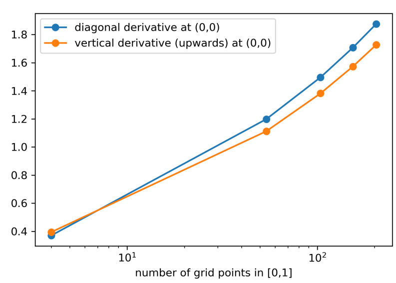
$$\|u\|_{H^2} \le C\|f\|_{H^0} \text{ for } f \in H^0(\Omega) = L^2(\Omega). \tag{1}$$

For a smooth boundary $\partial\Omega$ it even holds that

$$\|u\|_{H^{m+2}} \le C\|f\|_{H^m} \text{ for } f \in H^m(\Omega) \text{ and } m \ge 0.$$

For a square $m = 0$ and $m = 1$ are possible.
These regularity estimates never hold for non-convex polygonal domains like the L-shape.





3 Finite element error

Let u be the weak solution of our elliptic problem, $\mathcal{T}_h$ a regular triangulation of Ω and $V_h = \{u \in C(\Omega) \cap H_0^1(\Omega) : u|_T = \mathbb{P}^1 \ \forall\, T \in \mathcal{T}_h\}$ a piecewise linear finite element space. Then for the finite element solution $u_h \in V_h$,

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}. \tag{2}$$

Under the assumption that Ω is convex and u satisfies the shift theorem (1), we also get an L^2 estimation for the finite element error:

Let $z \in H_0^1(\Omega)$ be a solution of $a(z, v) = (u - u_h, v)$ for all $v \in H_0^1(\Omega)$.
For all $z_h \in V_h \subset H_0^1(\Omega)$, $a(u, z_h) = (f, z_h) = a(u_h, z_h)$. $\Rightarrow a(u - u_h, z_h) = 0$.

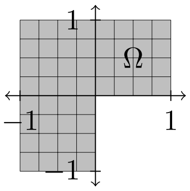
Hence, $\|u - u_h\|_{L^2}^2 = (u - u_h, u - u_h) = a(z, u - u_h) = a(z - z_h, u - u_h) \leq$
$\overset{\text{CS}}{\leq} \|\nabla(u - u_h)\|_{L^2} \cdot \|\nabla(z - z_h)\|_{L^2} \leq \|u - u_h\|_{H^1} \cdot \|z - z_h\|_{H^1} \overset{(2)}{\leq} Ch^2 \cdot \|u\|_{H^2} \cdot \|z\|_{H^2} \leq$
$\overset{(1)z}{\leq} \tilde{C}h^2 \cdot \|u\|_{H^2} \cdot \|u - u_h\|_{H^0} = \tilde{C}h^2 \cdot \|u\|_{H^2} \cdot \|u - u_h\|_{L^2}$.

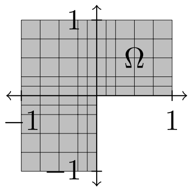
Therefore, $\|u - u_h\|_{L^2} \leq \tilde{C}h^2\|u\|_{H^2}$.

If Ω is not convex, this inequality need not hold true, too.

4 Refining the grid

Since the kink at $(0, 0)$ is causing the regularity problems in the L-shape (in comparison to the square), we now refine the grid close to this critical point and compare the number of grid points we need to get a certain residual error.





Literature: