Poisson's equation on an L-shaped domain

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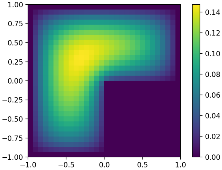

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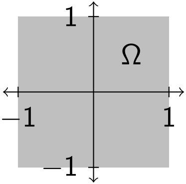
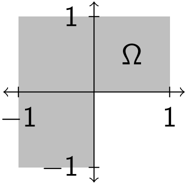
Poisson's equation

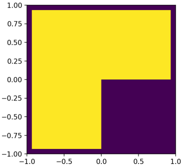
$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \quad u = 0 \text{ on } \partial\Omega \end{cases}$$

Weak formulation with solution $u \in H_0^1(\Omega)$:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \; \mathrm{d}x = \int_\Omega fv \; \mathrm{d}x = (f, v) \quad \forall v \in H_0^1(\Omega)$$







Reminder: Sobolev spaces and their norms

Let $\Omega \subset \mathbb{R}^n$ be open.

$H^k(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega)$ for all $|a| \leq k\}$

$(u, v)_{H^k} = \sum_{|a|\leq k} \int_\Omega D^\alpha u \; D^\alpha v \; \mathrm{d}x$

$\|u\|_{H^k(\Omega)} = \sqrt{(u, u)_{H^k}}$

$u \in H^1_0(\Omega)$ if $u \in H^1(\Omega)$ and "$u|_{\partial\Omega} = 0$"

Shift theorems

Let Ω be a *convex polygonal* domain in $\mathbb{R}^2$ and u the weak solution to our problem. Then

$$\|u\|_{H^2} \leq C\|f\|_{H^0} \text{ for } f \in H^0(\Omega) = L^2(\Omega). \tag{1}$$

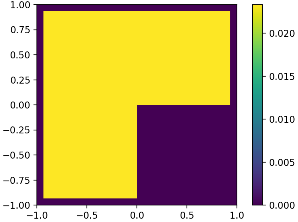
For a smooth boundary $\partial\Omega$ it even holds that

$$\|u\|_{H^{m+2}} \leq C\|f\|_{H^m} \text{ for } f \in H^m(\Omega) \text{ and } m \geq 0.$$

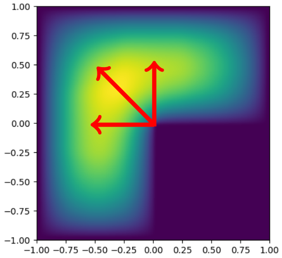
These regularity estimates never hold for non-convex polygonal domains like the L-shape.

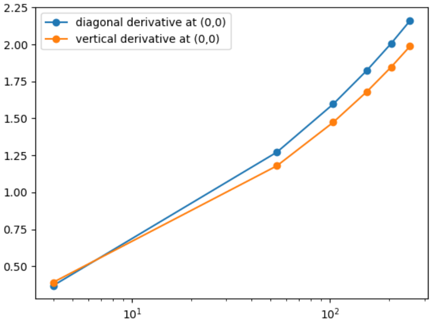
Numerical solution of Poisson's equation on an L-shaped domain

regular triangulation with right-angled triangles
finite element method
conjugate gradients solver



Regularity problem of the L





Finite element error

Let u be the weak solution of our elliptic problem, $\mathcal{T}_h$ a regular triangulation of Ω and $V_h = \{u \in C(\Omega) \cap H_0^1(\Omega) : u|_T = \mathbb{P}^1 \ \forall \ T \in \mathcal{T}_h\}$ a piecewise linear finite element space. Then for the finite element solution $u_h \in V_h$,

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}. \tag{2}$$

Under the assumption that Ω is convex and u satisfies the shift theorem (**??**), we also get an L^2 estimation for the finite element error:

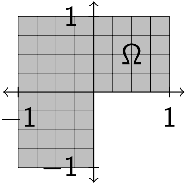
Finite element error

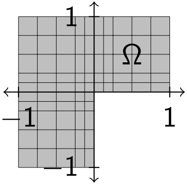
Let $z \in H_0^1(\Omega)$ be a solution of $a(z,v) = (u - u_h, v)$ for all $v \in H_0^1(\Omega)$.
For all $z_h \in V_h \subset H_0^1(\Omega)$, $a(u, z_h) = (f, z_h) = a(u_h, z_h)$.
$\Rightarrow a(u - u_h, z_h) = 0$.

Hence,
$$\|u - u_h\|_{L^2}^2 = (u - u_h, u - u_h) = a(z, u - u_h) = a(z - z_h, u - u_h) \leq$$
$$\overset{CS}{\leq} \|\nabla(u - u_h)\|_{L^2} \cdot \|\nabla(z - z_h)\|_{L^2} \leq \|u - u_h\|_{H^1} \cdot \|z - z_h\|_{H^1} \overset{(2)}{\leq}$$
$$Ch^2 \cdot \|u\|_{H^2} \cdot \|z\|_{H^2} \leq$$
$$\overset{(??)z}{\leq} \tilde{C}h^2 \cdot \|u\|_{H^2} \cdot \|u - u_h\|_{H^0} = \tilde{C}h^2 \cdot \|u\|_{H^2} \cdot \|u - u_h\|_{L^2}.$$
Therefore, $\|u - u_h\|_{L^2} \leq \tilde{C}h^2 \|u\|_{H^2}$.

If Ω is not convex, this inequality need not hold true, too.

Since the kink at $(0,0)$ is causing the regularity problems in the L-shape (in comparison to the square), we now refine the grid close to this critical point and compare the number of grid points we need to get a certain residual error.





Vielen Dank für eure Aufmerksamkeit!

Literatur und Graphiken: `https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/MIT16_07F09_Lec30.pdf` Zugriff: January 24, 2020