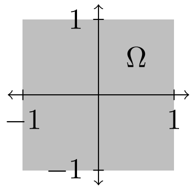
Poisson's equation on an L-shaped domain

Adéla Moravová, Barbara Präg 27 January 2020

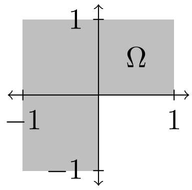
1 Poisson's equation

Consider the following elliptic problem with Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$



vs.



and its weak formulation with solution $u \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x = \int_\Omega fv \ \mathrm{d}x \quad \forall v \in H_0^1(\Omega),$$

abbreviated as

$$a(u,v) = (f,v) \ \ \forall v \in H_0^1(\Omega).$$

2 Shift theorems

Let Ω be a convex polygonal domain in $\mathbb{R}^2$. If u is the weak solution to our problem, then

$$\|u\|_{H^2} \le C\|f\|_{H^0} \text{ for } f \in H^0(\Omega) = L^2(\Omega). \tag{1}$$

Similar shift theorems exist for **fractional-order Sobolev spaces** and $0 < s < s_0 < 1$ (s_0 is a constant depending on the domain):

$$H^s(\Omega) = W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{1+s}} \in L^2(\Omega \times \Omega) \right\}$$

$$\|u\|_{H^s(\Omega)} = \|u\|_{W^{s,2}(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + \iint_{\Omega\times\Omega} \left(\frac{|u(x) - u(y)|}{|x-y|^{1+s}} \right)^2 \mathrm{d}x \ \mathrm{d}y \right)^{\frac{1}{2}}$$

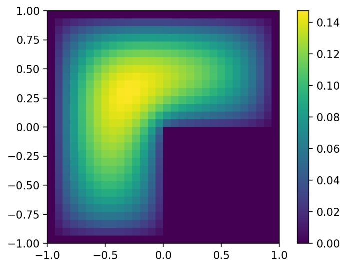
Then,

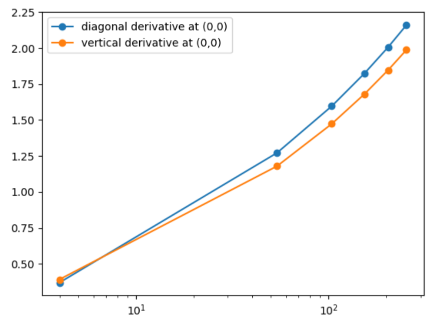
$$\|u\|_{H^{2+s}} \le C\|f\|_{H^s} \text{ for } f \in H^s(\Omega).$$

For a smooth boundary $\partial\Omega$ it even holds that

$$\|u\|_{H^{m+2}} \le C\|f\|_{H^m} \text{ for } f \in H^m(\Omega) \text{ and } m \in \mathbb{N}_0.$$

For $m = 0$ and $m = 1$ this is also true for a square. However, the regularity estimate never holds for non-convex polygonal domains like the L-shape.





3 Finite element error

Let u be the weak solution of our elliptic problem, $\mathcal{T}_h$ a regular triangulation of a polygonal domain Ω, $V_h = \{u \in C(\Omega) \cap H_0^1(\Omega) : u|_T = \mathbb{P}^1 \ \forall\, T \in \mathcal{T}_h\}$ a piecewise linear finite element space and $u_h \in V_h$ the finite element solution, Then,

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}.$$

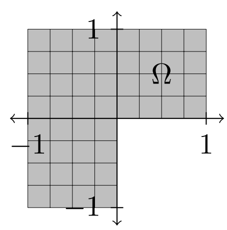
Under the assumption that Ω is convex and u satisfies the shift theorem (1), we also get an L^2 estimation for the finite element error:

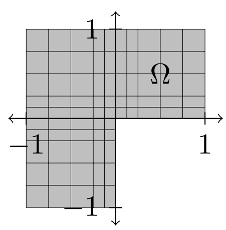
$$\|u - u_h\|_{L^2} \leq \tilde{C}h^2\|u\|_{H^2}.$$

If Ω is not convex, this inequality need not hold true.

4 Refining the grid

Since the non-convexity at $(0, 0)$ is causing the regularity problems in the L-shape (in comparison to the square), refining the grid close to this critical point should enhance the convergence of the finite element method.





Literature:
Demlow, Alan: *Notes for Math 663. Chapter 1.* Spring 2016.
Bacuta, C., Bramble, J. H., Xu, Jinchao: *Regularity estimates for elliptic boundary value problems with smooth data on polygonal domains.* January 8, 2003.