Contents

1 Sets **3**
- 1.1 Extensionality . . . 3
- 1.2 Subsets . . . 3
- 1.3 The empty set . . . 4
- 1.4 Disjointness of sets . . . 4
- 1.5 Unordered pairing and set adjunction . . . 4
- 1.6 Union and intersection . . . 5
  - 1.6.1 Union of a set . . . 5
  - 1.6.2 Intersection of a set . . . 5
  - 1.6.3 Binary union . . . 5
  - 1.6.4 Binary intersection . . . 7
  - 1.6.5 Interaction of union and intersection . . . 8
- 1.7 Set difference . . . 9
- 1.8 Tuples . . . 10
- 1.9 Additional results about cons . . . 11
- 1.10 Successor . . . 12
- 1.11 Symmetric difference . . . 13
- 1.12 Powerset . . . 14
- 1.13 Bipartitions . . . 15
- 1.14 Partitions . . . 16
- 1.15 Cantor's theorem . . . 16

2 Filters **16**

3 Regularity **17**
- 3.1 Fixpoints . . . 18

4 Relations **18**
- 4.1 Converse of a relation . . . 19
  - 4.1.1 Domain of a relation . . . 20
  - 4.1.2 Range of a relation . . . 21
  - 4.1.3 Domain and range of converse . . . 21
  - 4.1.4 Field of a relation . . . 22
- 4.2 Image . . . 23
- 4.3 Preimage . . . 24
- 4.4 Upward and downward closure . . . 24
- 4.5 Relation (and later also function) composition . . . 24
- 4.6 Restriction . . . 25
- 4.7 Set of relations . . . 26
- 4.8 Identity relation . . . 28
- 4.9 Membership relation . . . 28
- 4.10 Subset relation . . . 28

4.11 Properties of relations . . . 29
4.12 Quasiorders . . . 30
4.13 Equivalences . . . 31
4.13.1 Equivalence classes . . . 31
4.13.2 Quotients . . . 32
4.14 Closure operations on relations . . . 33
4.15 Injective relations . . . 33
4.16 Right-unique relations . . . 33
4.17 Left-total relations . . . 33
4.18 Right-total relations . . . 34

5 Functions **34**
5.1 Image of a function . . . 35
5.2 Families of functions . . . 36
5.3 The empty function . . . 36
5.4 Function composition . . . 36
5.5 Injections . . . 38
5.6 Surjections . . . 38
5.7 Bijections . . . 39
5.8 Converse as a function . . . 39
5.8.1 Inverses of a function . . . 39
5.9 Identity function . . . 40

6 Transitive sets **40**
6.0.1 Closure properties of $\in$-transitive sets . . . 41

7 Ordinals **41**
7.0.1 Construction of ordinals . . . 43
7.0.2 Limit and successor ordinals . . . 44
7.1 Natural numbers as ordinals . . . 45

8 Natural numbers **45**

9 Cardinality **45**

10 Magmas **46**

11 Semigroups **47**

12 Regular semigroups **47**

13 Inverse semigroups **47**

14 Topological spaces **49**
14.1 Closed sets . . . 50

14.2 Topological basis 51
14.3 Disconnections 52

1 Sets

Abbreviation 1. $A \ni a$ iff $a \in A$.

1.1 Extensionality

The axiom of set extensionality says that sets are determined by their *extension*, that is, two sets are equal iff they have the same elements.

Axiom 2. **(Set extensionality)** Suppose for all a we have $a \in A$ iff $a \in B$. Then $A = B$.

This axiom is also available as the justification "… by set extensionality", which applies it to goals of the form "$A = B$" and "$A \neq B$".

Proposition 3. **(Witness for disequality)** Suppose $A \neq B$. Then there exists c such that either $c \in A$ and $c \notin B$ or $c \notin A$ and $c \in B$.

Proof. Suppose not. Then $A = B$ by set extensionality. Contradiction. □

1.2 Subsets

Definition 4. $A \subseteq B$ iff for all $a \in A$ we have $a \in B$.

Abbreviation 5. A is a subset of B iff $A \subseteq B$.

Abbreviation 6. $B \supseteq A$ iff $A \subseteq B$.

Proposition 7. $A \subseteq A$.

Proposition 8. Suppose $A \subseteq B \subseteq A$. Then $A = B$.

Proof. Follows by set extensionality. □

Proposition 9. Suppose $a \in A \subseteq B$. Then $a \in B$.

Proposition 10. Suppose $A \subseteq B$ and $c \notin B$. Then $c \notin A$.

Proposition 11. Suppose $A \subseteq B \subseteq C$. Then $A \subseteq C$.

Definition 12. $A \subset B$ iff $A \subseteq B$ and $A \neq B$.

Proposition 13. $A \not\subset A$.

Proposition 14. Suppose $A \subseteq B \subseteq C$. Then $A \subseteq C$.

Proposition 15. Suppose $A \subset B$. Then there exists $b \in B$ such that $b \notin A$.

Proof. $A \subseteq B$ and $A \neq B$. □

Abbreviation 16. F is a family of subsets of X iff for all $A \in F$ we have $A \subseteq X$.

1.3 The empty set

Axiom 17. For all a we have $a \notin \emptyset$.

Definition 18. A is inhabited iff there exists a such that $a \in A$.

Abbreviation 19. A is empty iff A is not inhabited.

Proposition 20. If x and y are empty, then $x = y$.

Proposition 21. For all a we have $\emptyset \subseteq a$.

Proposition 22. $A \subseteq \emptyset$ iff $A = \emptyset$.

1.4 Disjointness of sets

Definition 23. A is disjoint from B iff there exists no a such that $a \in A, B$.

Abbreviation 24. $A \not\between B$ iff A is disjoint from B.

Abbreviation 25. $A \between B$ iff A is not disjoint from B.

Proposition 26. If A is disjoint from B, then B is disjoint from A.

1.5 Unordered pairing and set adjunction

Finite set expressions are desugared to iterated application of cons to $\emptyset$. Thus $\{x, y, z\}$ is an abbreviaton of $\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \emptyset)))$. The cons operation is determined by the following axiom:

Axiom 27. $x \in \mathsf{cons}(y, X)$ iff $x = y$ or $x \in X$.

Proposition 28. $x \in \mathsf{cons}(x, X)$.

Proposition 29. If $y \in X$, then $y \in \mathsf{cons}(x, X)$.

Proposition 30. $a \in \{a, b\}$.

Proposition 31. $b \in \{a, b\}$.

Proposition 32. Suppose $c \in \{a, b\}$. Then $a = c$ or $b = c$.

Proposition 33. $c \in \{a, b\}$ iff $a = c$ or $b = c$.

Proposition 34. $a \in \{a\}$.

Proposition 35. If $a \in \{b\}$, then $a = b$.

Proposition 36. $a \in \{b\}$ iff $a = b$.

Abbreviation 37. A is a subsingleton iff for all $a, b \in A$ we have $a = b$.

Proposition 38. $\{a\}$ is inhabited.

Proposition 39. Let A be a subsingleton. Let $a \in A$. Then $A = \{a\}$.

Proof. Follows by set extensionality. □

Proposition 40. Suppose $a \in C$. Then $\{a\} \subseteq C$.

Proposition 41. Suppose $\{a\} \subseteq C$. Then $a \in C$.

1.6 Union and intersection

1.6.1 Union of a set

Axiom 42. $z \in \bigcup X$ iff there exists $Y \in X$ such that $z \in Y$.

Proposition 43. Suppose $A \in B \in C$. Then $A \in \bigcup C$.

Proof. There exists $B \in C$ such that $A \in B$. □

Proposition 44. $\bigcup \emptyset = \emptyset$.

Proposition 45. Let F be a family of subsets of X. Then $\bigcup F \subseteq X$.

Abbreviation 46. T is closed under arbitrary unions iff for every subset M of T we have $\bigcup M \in T$.

1.6.2 Intersection of a set

Definition 47. $\bigcap A = \{x \in \bigcup A \mid \text{for all } a \in A \text{ we have } x \in a\}$.

Proposition 48. $z \in \bigcap X$ iff X is inhabited and for all $Y \in X$ we have $z \in Y$.

Proposition 49. Suppose C is inhabited. Suppose for all $B \in C$ we have $A \in B$. Then $A \in \bigcap C$.

Proposition 50. Suppose $A \in \bigcap C$. Suppose $B \in C$. Then $A \in B$.

Proposition 51. Suppose A is inhabited. Suppose for all $a \in A$ we have $C \subseteq a$. Then $C \subseteq \bigcap A$.

Proposition 52. Suppose A is inhabited. Then $C \subseteq \bigcap A$ iff for all $a \in A$ we have $C \subseteq a$.

Proposition 53. Let $B \in A$. Then $\bigcap A \subseteq B$.

Proposition 54. $\bigcap\{a\} = a$.

Proof. Every element of a is an element of $\bigcap\{a\}$ by propositions [36], [38] and [48]. Follows by set extensionality. □

Proposition 55. $\bigcap\{\emptyset\} = \emptyset$.

Proof. Follows by set extensionality. □

1.6.3 Binary union

Axiom 56. Let A, B be sets. $a \in A \cup B$ iff $a \in A$ or $a \in B$.

Proposition 57. If $c \in A$, then $c \in A \cup B$.

Proposition 58. If $c \in B$, then $c \in A \cup B$.

Proposition 59. **(Commutativity of union)** $A \cup B = B \cup A$.

Proof. Follows by set extensionality. □

Proposition 60. **(Associativity of union)** $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof. Follows by set extensionality. $\square$

Proposition 61. **(Idempotence of union)** $A \cup A = A$.

Proof. Follows by set extensionality. $\square$

Proposition 62. $A \cup B \subseteq C$ iff $A \subseteq C$ and $B \subseteq C$.

Proposition 63. $A \subseteq A \cup B$.

Proposition 64. $B \subseteq A \cup B$.

Proposition 65. Suppose $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

Proposition 66. $A \cup \emptyset = A$.

Proof. Follows by set extensionality. $\square$

Proposition 67. Suppose $A = \emptyset$ and $B = \emptyset$. Then $A \cup B = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 68. Suppose $A \cup B = \emptyset$. Then $A = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 69. Suppose $A \cup B = \emptyset$. Then $B = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 70. Suppose $A \subseteq B$. Then $A \cup B = B$.

Proof. Follows by set extensionality. $\square$

Proposition 71. Suppose $A \subseteq B$. Then $B \cup A = B$.

Proof. Follows by set extensionality. $\square$

Proposition 72. If $A \cup B = B$, then $A \subseteq B$.

Proposition 73. $\bigcup \mathsf{cons}(b, A) = b \cup \bigcup A$.

Proof. Follows by set extensionality. $\square$

Proposition 74. $\mathsf{cons}(b, A) \cup C = \mathsf{cons}(b, A \cup C)$.

Proof. Follows by set extensionality. $\square$

Proposition 75. $A \cup (A \cup B) = A \cup B$.

Proof. Follows by set extensionality. $\square$

Proposition 76. $(A \cup B) \cup B = A \cup B$.

Proof. Follows by set extensionality. □

Proposition 77. $A \cup (B \cup C) = B \cup (A \cup C)$.

Proof. Follows by set extensionality. □

Abbreviation 78. T is closed under binary unions iff for every $U, V \in T$ we have $U \cup V \in T$.

1.6.4 Binary intersection

Definition 79. $A \cap B = \{a \in A \mid a \in B\}$.

Proposition 80. If $c \in A, B$, then $c \in A \cap B$.

Proposition 81. If $c \in A \cap B$, then $c \in A$.

Proposition 82. If $c \in A \cap B$, then $c \in B$.

Proposition 83. $\bigcap\{A, B\} = A \cap B$.

Proof. $\{A, B\}$ is inhabited. Thus for all c we have $c \in \bigcap\{A, B\}$ iff $c \in A \cap B$ by propositions [33] and [48] and definition [79]. Follows by extensionality. □

Proposition 84. **(Commutativity of intersection)** $A \cap B = B \cap A$.

Proof. Follows by set extensionality. □

Proposition 85. **(Associativity of intersection)** $(A \cap B) \cap C = A \cap (B \cap C)$.

Proof. Follows by set extensionality. □

Proposition 86. **(Idempotence of intersection)** $A \cap A = A$.

Proof. Follows by set extensionality. □

Proposition 87. $A \cap B \subseteq A$.

Proposition 88. $A \cap \emptyset = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 89. Suppose $A \subseteq B$. Then $A \cap B = A$.

Proof. Follows by set extensionality. □

Proposition 90. Suppose $A \subseteq B$. Then $B \cap A = A$.

Proof. Follows by set extensionality. □

Proposition 91. Suppose $A \cap B = A$. Then $A \subseteq B$.

Proposition 92. $C \subseteq A \cap B$ iff $C \subseteq A$ and $C \subseteq B$.

Proposition 93. $A \cap B \subseteq A$.

Proposition 94. $A \cap B \subseteq B$.

Proposition 95. $A \cap (A \cap B) = A \cap B$.

Proof. Follows by set extensionality. $\square$

Proposition 96. $(A \cap B) \cap B = A \cap B$.

Proof. Follows by set extensionality. $\square$

Proposition 97. $A \cap (B \cap C) = B \cap (A \cap C)$.

Proof. Follows by set extensionality. $\square$

Abbreviation 98. T is closed under binary intersections iff for every $U, V \in T$ we have $U \cap V \in T$.

1.6.5 Interaction of union and intersection

Proposition 99. **(Binary intersection over binary union)** $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

Proof. Follows by set extensionality. $\square$

Proposition 100. **(Binary union over binary intersection)** $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

Proof. Follows by set extensionality. $\square$

Proposition 101. Suppose $C \subseteq A$. Then $(A \cap B) \cup C = A \cap (B \cup C)$.

Proof. Follows by set extensionality. $\square$

Proposition 102. Suppose $(A \cap B) \cup C = A \cap (B \cup C)$. Then $C \subseteq A$.

Proposition 103. $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$.

Proof. Follows by set extensionality. $\square$

Proposition 104. **(Intersection over binary union)** Suppose A and B are inhabited. Then $\bigcap A \cup B = (\bigcap A) \cap \bigcap B$.

Proof. $A \cup B$ is inhabited. Thus for all c we have $c \in \bigcap A \cup B$ iff $c \in (\bigcap A) \cap \bigcap B$ by definition [79], axiom [56], and proposition [48]. Follows by set extensionality. $\square$

1.7 Set difference

Definition 105. $A \setminus B = \{a \in A \mid a \notin B\}$.

Proposition 106. If $a \in A$ and $a \notin B$, then $a \in A \setminus B$.

Proposition 107. If $a \in A \setminus B$, then $a \in A$.

Proposition 108. If $a \in A \setminus B$, then $a \notin B$.

Proposition 109. $x \setminus \emptyset = x$.

Proof. Follows by set extensionality. $\square$

Proposition 110. $\emptyset \setminus x = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 111. $x \setminus x = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 112. $x \setminus (x \setminus y) = x \cap y$.

Proof. Follows by set extensionality. $\square$

Proposition 113. Suppose $y \subseteq x$. $x \setminus (x \setminus y) = y$.

Proof. Follows by propositions [90] and [112]. $\square$

Proposition 114. $x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z)$.

Proof. Follows by set extensionality. $\square$

Proposition 115. $x \setminus (y \cup z) = (x \setminus y) \cap (x \setminus z)$.

Proof. Follows by set extensionality. $\square$

Proposition 116. $x \cap (y \setminus z) = (x \cap y) \setminus (x \cap z)$.

Proof. Follows by set extensionality. $\square$

Proposition 117. Let A, B be sets. Suppose $A \subset B$. Then $B \setminus A$ is inhabited.

Proof. Take b such that $b \in B$ and $b \notin A$. Then $b \in B \setminus A$. $\square$

Proposition 118. $B \setminus A \subseteq B$.

Proposition 119. Suppose $C \subseteq A$. Suppose $C \cap B = \emptyset$. Then $C \subseteq A \setminus B$.

Proposition 120. Suppose $A \subseteq B$. Then $C \setminus A \supseteq C \setminus B$.

Proposition 121. Suppose $A \cap B = \emptyset$. Then $A \setminus B = A$.

Proposition 122. $A \setminus B = \emptyset$ iff $A \subseteq B$.

Proposition 123. Suppose $B \subseteq A \setminus C$ and $c \notin B$. Then $B \subseteq A \setminus \mathsf{cons}(c, C)$.

Proposition 124. Suppose $B \subseteq A \setminus \mathsf{cons}(c, C)$. Then $B \subseteq A \setminus C$ and $c \notin B$.

Proposition 125. $A \setminus \mathsf{cons}(a, B) = (A \setminus \{a\}) \setminus B$.

Proof. Follows by set extensionality. □

Proposition 126. $A \setminus \mathsf{cons}(a, B) = (A \setminus B) \setminus \{a\}$.

Proof. Follows by set extensionality. □

Proposition 127. $A \cap (B \setminus A) = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 128. Suppose $A \subseteq B$. $A \cup (B \setminus A) = B$.

Proof. Follows by set extensionality. □

Proposition 129. $A \subseteq B \cup (A \setminus B)$.

Proposition 130. Suppose $A \subseteq B \subseteq C$. Then $B \setminus (C \setminus A) = A$.

Proof. Follows by set extensionality. □

Proposition 131. Then $(A \cup B) \setminus (B \setminus A) = A$.

Proof. Follows by set extensionality. □

Proposition 132. Suppose $A, B \subseteq C$. Then $A \setminus B = A \cap (C \setminus B)$.

Proof. Follows by set extensionality. □

1.8 Tuples

As with unordered pairs, orderd pairs are a primitive construct and n-tuples desugar to iterated applications of the primitive operator $(-,-)$. For example (a, b, c, d) equals $(a, (b, (c, d)))$ by definition. While ordered pairs could be encoded set-theoretically, we simply postulate the defining property to prevent misguiding proof automation.

Axiom 133. $(a, b) = (a', b')$ iff $a = a' \land b = b'$.

Axiom 134. $(a, b) \neq \emptyset$.

Axiom 135. $(a, b) \neq a$.

Axiom 136. $(a, b) \neq b$.

Repeated application of the defining property of pairs yields the defining property of all tuples.

Proposition 137. $(a, b, c) = (a', b', c')$ iff $a = a' \land b = b' \land c = c'$.

There are primitive projections fst and snd that satisfy the following axioms.

Axiom 138. $\mathsf{fst}(a, b) = a$.

Axiom 139. $\mathsf{snd}(a, b) = b$.

Proposition 140. $(a, b) = (\mathsf{fst}(a, b), \mathsf{snd}(a, b))$.

Definition 141. $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Proposition 142. Suppose $(x, y) \in X \times Y$. Then $x \in X$ and $y \in Y$.

Proof. Take x', y' such that $x' \in X \wedge y' \in Y \wedge (x, y) = (x', y')$ by definition [141]. Then $x = x'$ and $y = y'$ by axiom [133]. □

Proposition 143. Suppose $x \in X$ and $y \in Y$. Then $(x, y) \in X \times Y$.

Proposition 144. $\emptyset \times Y = \emptyset$.

Proposition 145. $X \times \emptyset = \emptyset$.

Proposition 146. $X \times Y$ is empty iff X is empty or Y is empty.

Proof. Follows by definitions [18] and [141]. □

Proposition 147. Suppose $c \in A \times B$. Then $\mathsf{fst}\, c \in A$.

Proof. Take a, b such that $c = (a, b)$ and $a \in A$ by definition [141]. $a = \mathsf{fst}\, c$ by axiom [138]. □

Proposition 148. Suppose $c \in A \times B$. Then $\mathsf{snd}\, c \in B$.

Proof. Take a, b such that $c = (a, b)$ and $b \in B$ by definition [141]. $b = \mathsf{snd}\, c$ by axiom [139]. □

Proposition 149. Suppose $p \in X \times Y$. Then there exist x, y such that $x \in X$ and $y \in Y$ and $p = (x, y)$.

Proposition 150. Suppose $p \in X \times Y$. Then $\mathsf{fst}\, p \in X$ and $\mathsf{snd}\, p \in Y$.

1.9 Additional results about cons

Proposition 151. Suppose $x \in X$. Suppose $Y \subseteq X$. Then $\mathsf{cons}(x, Y) \subseteq X$.

Proposition 152. Suppose $\mathsf{cons}(x, Y) \subseteq X$. Then $x \in X$ and $Y \subseteq X$.

Proposition 153. $\mathsf{cons}(x, Y) \subseteq X$ iff $x \in X$ and $Y \subseteq X$.

Proposition 154. If $C \subseteq B$, then $C \subseteq \mathsf{cons}(a, B)$.

Corollary 155. $X \subseteq \mathsf{cons}(y, X)$.

Abbreviation 156. $B \setminus \{a\} = B \setminus \{a\}$.

Proposition 157. Suppose $a \in C \wedge C \setminus \{a\} \subseteq B$. Then $C \subseteq \mathsf{cons}(a, B)$.

Proof. Follows by propositions [122] and [125]. □

Proposition 158. Suppose $C \subseteq B$. Then $C \subseteq \mathsf{cons}(a, B)$.

Proposition 159. Suppose $C \subseteq \mathsf{cons}(a, B)$. Then $C \subseteq B \vee (a \in C \wedge C \setminus \{a\} \subseteq B)$.

Proof. Follows by propositions [122] and [125], definition [4], and axiom [27]. ☐

Proposition 160. $C \subseteq \mathsf{cons}(a, B)$ iff $C \subseteq B \vee (a \in C \wedge C \setminus \{a\} \subseteq B)$.

Proposition 161. $B \setminus \{a\} = B \setminus \{a\}$.

Proof. Follows by set extensionality. ☐

Proposition 162. $\{a\} \cup B = \mathsf{cons}(a, B)$.

Proof. Follows by set extensionality. ☐

Proposition 163. $\mathsf{cons}(a, \mathsf{cons}(b, C)) = \mathsf{cons}(b, \mathsf{cons}(a, C))$.

Proof. Follows by set extensionality. ☐

Proposition 164. Suppose $a \in A$. Then $\mathsf{cons}(a, A) = A$.

Proof. Follows by set extensionality. ☐

Proposition 165. Suppose $a \in A$. Then $\mathsf{cons}(a, A \setminus \{a\}) = A$.

Proof. Follows by set extensionality. ☐

Proposition 166. Then $\mathsf{cons}(a, \mathsf{cons}(a, B)) = \mathsf{cons}(a, B)$.

Proof. Follows by set extensionality. ☐

Proposition 167. Suppose B is inhabited. Then $\bigcap \mathsf{cons}(a, B) = a \cap \bigcap B$.

Proof. $\mathsf{cons}(a, B)$ is inhabited. Thus for all c we have $c \in \bigcap \mathsf{cons}(a, B)$ iff $c \in a \cap \bigcap B$ by proposition [48], axiom [27], and definition [79]. Follows by extensionality. ☐

1.10 Successor

Definition 168. $x^+ = \mathsf{cons}(x, x)$.

Proposition 169. $x \in x^+$.

Proposition 170. Suppose $x \in y$. Then $x \in y^+$.

Proposition 171. Suppose $x \in y^+$. Then $x = y$ or $x \in y$.

Proposition 172. $x \in y^+$ iff $x = y$ or $x \in y$.

Proposition 173. $x^+ \neq \emptyset$.

Proposition 174. Suppose $x^+ \subseteq y$. Then $x \in y$.

Proposition 175. $x^+ \neq x$.

Proposition 176. Suppose $x^+ = y^+$. Then $x = y$.

Proof. Suppose not. $x^+ \subseteq y^+$. Hence $x \in y^+$. Then $x \in y$. $y^+ \subseteq x^+$. Hence $y \in x^+$. Then $y \in x$. Contradiction. ☐

Proposition 177. $x \subseteq x^+$.

Proposition 178. Suppose $x \in y$ and $x \subseteq y$. Then $x^+ \subseteq y$.

Proposition 179. Suppose $x^+ \subseteq y$. Then $x \in y$ and $x \subseteq y$.

Proposition 180. There exists no z such that $x \subset z \subset x^+$.

Proof. Follows by definitions [4], [12] and [168] and propositions [15] and [171]. □

1.11 Symmetric difference

Definition 181. $x \triangle y = (x \setminus y) \cup (y \setminus x)$.

Proposition 182. $x \triangle y = (x \cup y) \setminus (y \cap x)$.

Proof. Follows by set extensionality. □

Proposition 183. If $z \in x \triangle y$, then either $z \in x$ or $z \in y$.

Proposition 184. If either $z \in x$ or $z \in y$, then $z \in x \triangle y$.

Proof. If $z \in x$ and $z \notin y$, then $z \in x \setminus y$. If $z \notin x$ and $z \in y$, then $z \in y \setminus x$. □

Proposition 185. $x \triangle (y \triangle z) = (x \triangle y) \triangle z$.

Proof. Follows by set extensionality. □

Proposition 186. $x \triangle y = y \triangle x$.

Proof. Follows by set extensionality. □

Proposition 187. Suppose $A \subseteq C$. Then $A \times B \subseteq C \times B$.

Proof. It suffices to show that for all $w \in A \times B$ we have $w \in C \times B$. □

Proposition 188. Suppose $B \subseteq D$. Then $A \times B \subseteq A \times D$.

Proof. It suffices to show that for all $w \in A \times B$ we have $w \in A \times D$. □

Proposition 189. Suppose $w \in (A \cap B) \times (C \cap D)$. Then $w \in (A \times C) \cap (B \times D)$.

Proof. Take a, c such that $w = (a, c)$ by proposition [149]. Then $a \in A, B$ and $c \in C, D$ by proposition [142] and definition [79]. Thus $w \in (A \times C), (B \times D)$. □

Proposition 190. Suppose $w \in (A \times C) \cap (B \times D)$. Then $w \in (A \cap B) \times (C \cap D)$.

Proof. $w \in A \times C$. Take a, c such that $w = (a, c)$. $a \in A, B$ by definition [79] and proposition [142]. $c \in C, D$ by definition [79] and proposition [142]. Thus $(a, c) \in (A \cap B) \times (C \cap D)$ by definition [141] and proposition [80]. □

Proposition 191. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Proof. Follows by set extensionality. □

Proposition 192. $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$.

Proof. Follows by set extensionality. $\square$

Proposition 193. $X \times (Y \cap Z) = (X \times Y) \cap (X \times Z)$.

Proof. Follows by set extensionality. $\square$

Proposition 194. Suppose $w \in (A \cup B) \times (C \cup D)$. Then $w \in (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$.

Proof. Take a, c such that $w = (a, c)$. $a \in A$ or $a \in B$ by axiom [56] and proposition [142]. $c \in C$ or $c \in D$ by axiom [56] and proposition [142]. Thus $(a, c) \in (A \times C)$ or $(a, c) \in (B \times D)$ or $(a, c) \in (A \times D)$ or $(a, c) \in (B \times C)$. Thus $(a, c) \in (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$. $\square$

Proposition 195. Suppose $w \in (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$. Then $w \in (A \cup B) \times (C \cup D)$.

Proof. Case: $w \in (A \times C)$. Straightforward. Case: $w \in (B \times D)$. Straightforward. Case: $w \in (A \times D)$. Straightforward. Case: $w \in (B \times C)$. Straightforward. $\square$

Proposition 196. $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$.

Proof. Follows by set extensionality. $\square$

Proposition 197. $(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z)$.

Proof. Follows by set extensionality. $\square$

Proposition 198. $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

Proof. Follows by set extensionality. $\square$

1.12 Powerset

Abbreviation 199. The powerset of X denotes $\mathsf{Pow}(X)$.

Axiom 200. $B \in \mathsf{Pow}(A)$ iff $B \subseteq A$.

Proposition 201. Suppose $A \subseteq B$. Then $A \in \mathsf{Pow}(B)$.

Proposition 202. Let $A \in \mathsf{Pow}(B)$. Then $A \subseteq B$.

Proposition 203. $\emptyset \in \mathsf{Pow}(A)$.

Proposition 204. $A \in \mathsf{Pow}(A)$.

Proposition 205. Let A be a set. Let B be a subset of $\mathsf{Pow}(A)$. Then $\bigcup B \subseteq A$.

Proof. Follows by definition [4], proposition [202], and axiom [42]. $\square$

Proposition 206. $\bigcup \mathsf{Pow}(A) = A$.

Proof. Follows by set extensionality. $\square$

Proposition 207. $\bigcap \mathsf{Pow}(A) = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 208. $\mathsf{Pow}(A) \cup \mathsf{Pow}(B) \subseteq \mathsf{Pow}(A \cup B)$.

Proof. $\mathsf{Pow}(A) \subseteq \mathsf{Pow}(A) \cup \mathsf{Pow}(B)$ by proposition [63]. $\mathsf{Pow}(B) \subseteq \mathsf{Pow}(A) \cup \mathsf{Pow}(B)$ by proposition [64]. Follows by definition [4], axioms [56] and [200], and propositions [14] and [202]. $\square$

Proposition 209. $\mathsf{Pow}(\emptyset) = \{\emptyset\}$.

Proposition 210. $\mathsf{Pow}(A) \cup \mathsf{Pow}(B) \subseteq \mathsf{Pow}(A \cup B)$.

Proposition 211. $A \subseteq \mathsf{Pow}(\bigcup A)$.

Proof. Follows by definition [4], axiom [200], and proposition [43]. $\square$

Proposition 212. $\bigcup \mathsf{Pow}(A) = A$.

Proposition 213. $\bigcup A \in \mathsf{Pow}(B)$ iff $A \in \mathsf{Pow}(\mathsf{Pow}(B))$.

Proposition 214. $\mathsf{Pow}(A \cap B) = \mathsf{Pow}(A) \cap \mathsf{Pow}(B)$.

Proof. Follows by axioms [2] and [200], definition [79], and proposition [92]. $\square$

1.13 Bipartitions

Abbreviation 215. C is partitioned by A and B iff $A, B \neq \emptyset$ and A is disjoint from B and $A \cup B = C$.

Definition 216. $\mathsf{Bipartitions}\, X = \{p \in \mathsf{Pow}(X) \times \mathsf{Pow}(X) \mid X \text{ is partitioned by } \mathsf{fst}\, p \text{ and } \mathsf{snd}\, p\}$.

Abbreviation 217. P is a bipartition of X iff $P \in \mathsf{Bipartitions}\, X$.

Proposition 218. Suppose C is partitioned by A and B. Then (A, B) is a bipartition of C.

Proof. $(A, B) \in \mathsf{Pow}(C) \times \mathsf{Pow}(C)$. C is partitioned by $\mathsf{fst}(A, B)$ and $\mathsf{snd}(A, B)$. Thus (A, B) is a bipartition of C by definition [216]. $\square$

Proposition 219. Suppose (A, B) is a bipartition of C. Then C is partitioned by A and B.

Proof. $\mathsf{fst}(A, B) = A$. $\mathsf{snd}(A, B) = B$. $\square$

Proposition 220. $\mathsf{Bipartitions}\, \emptyset$ is empty.

Proposition 221. Suppose $d \notin C$. Suppose $A \cup B = \mathsf{cons}(d, C)$. Suppose $A, B \neq \{d\}$. Then $A \setminus \{d\} \cup B \setminus \{d\} = C$.

Proof. Follows by set extensionality. $\square$

Proposition 222. Suppose $d \notin C$. Suppose $\mathsf{cons}(d, C)$ is partitioned by A and B. Suppose $A, B \neq \{d\}$. Then C is partitioned by $A \setminus \{d\}$ and $B \setminus \{d\}$.

Proof. $A \setminus \{d\}, B \setminus \{d\} \neq \emptyset$. $A \setminus \{d\} \cup B \setminus \{d\} = C$ by proposition [221]. $\square$

1.14 Partitions

Definition 223. P is a partition iff $\emptyset \notin P$ and for all $B, C \in P$ such that $B \neq C$ we have B is disjoint from C.

Abbreviation 224. P is a partition of A iff P is a partition and $\bigcup P = A$.

Proposition 225. $\emptyset$ is a partition of $\emptyset$.

Definition 226. P' is a refinement of P iff for every $A' \in P'$ there exists $A \in P$ such that $A' \subseteq A$.

Abbreviation 227. $P' \leq P$ iff P' is a refinement of P.

Proposition 228. Suppose $P'' \leq P' \leq P$. Then $P'' \leq P$.

Proof. It suffices to show that for all $A'' \in P''$ there exists $A \in P$ such that $A'' \subseteq A$. Fix $A'' \in P''$. Take $A' \in P'$ such that $A'' \subseteq A'$ by definition [226]. Take $A \in P$ such that $A' \subseteq A$. Then $A'' \subseteq A$. Follows by definition. $\square$

1.15 Cantor's theorem

Theorem 229. (Cantor) There exists no surjection from A to $\mathsf{Pow}(A)$.

Proof. Suppose not. Consider a surjection f from A to $\mathsf{Pow}(A)$. Let $B = \{a \in A \mid a \notin f(a)\}$. Then $B \in \mathsf{Pow}(A)$. There exists $a' \in A$ such that $f(a') = B$ by the definition of surjectivity. Now $a' \in B$ iff $a' \notin f(a') = B$. Contradiction. $\square$

2 Filters

Abbreviation 230. F is upward-closed in S iff for all A, B such that $A \subseteq B \subseteq S$ and $A \in F$ we have $B \in F$.

Definition 231. F is a filter on S iff F is a family of subsets of S and S is inhabited and $S \in F$ and $\emptyset \notin F$ and F is closed under binary intersections and F is upward-closed in S.

Definition 232. $\uparrow_S A = \{X \in \mathsf{Pow}(S) \mid A \subseteq X\}$.

Proposition 233. Suppose $A \subseteq S$. Suppose A is inhabited. Then $\uparrow_S A$ is a filter on S.

Proof. S is inhabited. $\uparrow_S A$ is a family of subsets of S. $S \in \uparrow_S A$. $\emptyset \notin \uparrow_S A$. $\uparrow_S A$ is closed under binary intersections. $\uparrow_S A$ is upward-closed in S. Follows by definition [231]. $\square$

Proposition 234. Suppose $A \subseteq S$. $A \in \uparrow_S A$.

Proof. $A \in \mathsf{Pow}(S)$. $\square$

Proposition 235. Let $X \in \mathsf{Pow}(S)$. Suppose $X \notin {\uparrow_S} A$. Then $A \not\subseteq X$.

Proof. ☐

Definition 236. F is a maximal filter on S iff F is a filter on S and there exists no filter F' on S such that $F \subset F'$.

Proposition 237. Suppose $a \in S$. Then ${\uparrow_S}\{a\}$ is a filter on S.

Proof. $\{a\} \subseteq S$. $\{a\}$ is inhabited. Follows by proposition [233]. ☐

Proposition 238. Suppose $a \in S$. Then ${\uparrow_S}\{a\}$ is a maximal filter on S.

Proof. $\{a\} \subseteq S$. $\{a\}$ is inhabited. Thus ${\uparrow_S}\{a\}$ is a filter on S by proposition [233]. It suffices to show that there exists no filter F' on S such that ${\uparrow_S}\{a\} \subset F'$. Suppose not. Take a filter F' on S such that ${\uparrow_S}\{a\} \subset F'$. Take $X \in F'$ such that $X \notin {\uparrow_S}\{a\}$. $X \in \mathsf{Pow}(S)$. Thus $\{a\} \not\subseteq X$ by proposition [235]. Thus $a \notin X$. $\{a\} \in F'$ by definitions [12] and [232] and propositions [7], [9], [57], [70] and [201]. Thus $\emptyset = X \cap \{a\}$. Hence $\emptyset \in F'$ by definition [231]. Follows by contradiction to the definition of a filter. ☐

3 Regularity

Abbreviation 239. a is an $\in$-minimal element of A iff $a \in A$ and $a \not\between A$.

Lemma 240. For all a, A such that $a \in A$ there exists $b \in A$ such that $b \not\between A$.

Proof by $\in$-induction on a. Case: $a \not\between b$. Straightforward. Case: $a \between b$. Take a' such that $a' \in a, b$. Straightforward. ☐

Proposition 241. **(Regularity)** Let A be an inhabited set. Then there exists a $\in$-minimal element of A.

Proof. Follows by lemma [240] and definition [18]. ☐

Theorem 242. **(Foundation)** Let A be a set. Then $A = \emptyset$ or there exists $a \in A$ such that for all $x \in a$ we have $x \notin A$.

Proof. Case: $A = \emptyset$. Straightforward. Case: A is inhabited. Take a such that a is a $\in$-minimal element of A. Then for all $x \in a$ we have $x \notin A$. ☐

.

Proposition 243. For all sets A we have $A \notin A$.

Proof by $\in$-induction. Straightforward. ☐

Proposition 244. If $a \in A$, then $a \neq A$.

Proposition 245. For all sets a, b such that $a \in b$ we have $b \notin a$.

Proof by $\in$-induction on a. Straightforward. ☐

3.1 Fixpoints

Definition 246. a is a fixpoint of f iff $a \in \mathsf{dom}\, f$ and $f(a) = a$.

Definition 247. f is $\subseteq$-preserving iff for all $A, B \in \mathsf{dom}\, f$ such that $A \subseteq B$ we have $f(A) \subseteq f(B)$.

Theorem 248. **(Knaster–Tarski)** Let f be a $\subseteq$-preserving function from $\mathsf{Pow}(A)$ to $\mathsf{Pow}(A)$. Then there exists a fixpoint of f.

Proof. $\mathsf{dom}\, f = \mathsf{Pow}(A)$. Let $P = \{a \in \mathsf{Pow}(A) \mid a \subseteq f(a)\}$. $P \subseteq \mathsf{Pow}(A)$. Thus $\bigcup P \in \mathsf{Pow}(A)$. Hence $f(\bigcup P) \in \mathsf{Pow}(A)$.

Show $\bigcup P \subseteq f(\bigcup P)$. *Subproof.* It suffices to show that every element of $\bigcup P$ is an element of $f(\bigcup P)$. Fix $u \in \bigcup P$. Take $p \in P$ such that $u \in p$. Then $u \in f(p)$. $p \subseteq \bigcup P$. $f(p) \subseteq f(\bigcup P)$ by definition [247]. Thus $u \in f(\bigcup P)$. □

Now $f(\bigcup P) \subseteq f(f(\bigcup P))$ by definition [247]. Thus $f(\bigcup P) \in P$ by definition.

Hence $f(\bigcup P) \subseteq \bigcup P$.

Thus $f(\bigcup P) = \bigcup P$ by proposition [8]. Follows by definition [246]. □

4 Relations

Definition 249. R is a relation iff for all $w \in R$ there exists x, y such that $w = (x, y)$.

Definition 250. a is comparable with b in R iff $a\ R\ b$ or $b\ R\ a$.

Proposition 251. Let R, S be relations. Suppose for all x, y we have $x\ R\ y$ iff $x\ S\ y$. Then $R = S$.

Proof. Follows by set extensionality. □

Abbreviation 252. F is a family of relations iff every element of F is a relation.

Proposition 253. Let F be a family of relations. Then $\bigcup F$ is a relation.

Proposition 254. Let F be a family of relations. Then $\bigcap F$ is a relation.

Proposition 255. Let R, S be relations. Then $R \cup S$ is a relation.

Proposition 256. Suppose $R \subseteq A \times B$. Suppose $S \subseteq C \times D$. Then $R \cup S \subseteq (A \cup C) \times (B \cup D)$.

Proof. Follows by definition [4], propositions [65] and [195], and axiom [56]. □

Proposition 257. Let R, S be relations. Then $R \cap S$ is a relation.

Proposition 258. Let R, S be relations. Then $R \setminus S$ is a relation.

4.1 Converse of a relation

Definition 259. $R^{\mathsf{T}} = \{z \mid \exists w \in R. \exists x, y. w = (x, y) \wedge z = (y, x)\}$.

Proposition 260. If $y \, R \, x$, then $x \, R^{\mathsf{T}} \, y$.

Proposition 261. If $x \, R^{\mathsf{T}} \, y$, then $y \, R \, x$.

Proposition 262. $x \, R^{\mathsf{T}} \, y$ iff $y \, R \, x$.

Proposition 263. R^{T} is a relation.

Proposition 264. $x \, R^{\mathsf{T}^{\mathsf{T}}} \, y$ iff $x \, R \, y$.

Proposition 265. Let R be a relation. Then $R^{\mathsf{T}^{\mathsf{T}}} = R$.

Proof. Follows by set extensionality. □

Proposition 266. Suppose $R \subseteq A \times B$. Then $R^{\mathsf{T}} \subseteq B \times A$.

Proof. Follows by definitions [4] and [259] and propositions [142] and [143]. □

Proposition 267. Then $B \times A^{\mathsf{T}} = A \times B$.

Proof. For all w we have $w \in B \times A^{\mathsf{T}}$ iff $w \in A \times B$ by definitions [141] and [259] and propositions [142] and [149]. Follows by extensionality. □

Proposition 268. Then $\emptyset^{\mathsf{T}} = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 269. Let R be a relation. If $R \subseteq S$, then $R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$.

Proof. Follows by definitions [4], [249] and [259]. □

Proposition 270. Let R be a relation. If $R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$, then $R \subseteq S$.

Proof. Follows by definitions [4], [249] and [259] and propositions [264] and [269]. □

Proposition 271. Let R be a relation. $R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$ iff $R \subseteq S$.

Proof. Follows by propositions [269] and [270]. □

Proposition 272. $(R \cup S)^{\mathsf{T}} = R^{\mathsf{T}} \cup S^{\mathsf{T}}$.

Proof. $(R \cup S)^{\mathsf{T}}$ is a relation by proposition [263]. $R^{\mathsf{T}} \cup S^{\mathsf{T}}$ is a relation by propositions [255] and [263]. For all a, b we have $(a, b) \in (R \cup S)^{\mathsf{T}}$ iff $(a, b) \in R^{\mathsf{T}} \cup S^{\mathsf{T}}$ by axiom [56] and proposition [262]. Follows by extensionality. □

Proposition 273. $(R \cap S)^{\mathsf{T}} = R^{\mathsf{T}} \cap S^{\mathsf{T}}$.

Proof. $(R \cap S)^{\mathsf{T}}$ is a relation by proposition [263]. $R^{\mathsf{T}} \cap S^{\mathsf{T}}$ is a relation by propositions [257] and [263]. For all a, b we have $(a, b) \in (R \cap S)^{\mathsf{T}}$ iff $(a, b) \in R^{\mathsf{T}} \cap S^{\mathsf{T}}$ by definition [79] and proposition [262]. Follows by extensionality. □

Proposition 274. $(R \setminus S)^{\mathsf{T}} = R^{\mathsf{T}} \setminus S^{\mathsf{T}}$.

Proof. $(R \setminus S)^{\mathsf{T}}$ is a relation by proposition [263]. $R^{\mathsf{T}} \setminus S^{\mathsf{T}}$ is a relation by propositions [258] and [263]. For all a, b we have $(a, b) \in (R \setminus S)^{\mathsf{T}}$ iff $(a, b) \in R^{\mathsf{T}} \setminus S^{\mathsf{T}}$. Follows by extensionality. □

4.1.1 Domain of a relation

Definition 275. $\mathsf{dom}\, R = \{x \mid \exists w \in R. \exists y. w = (x, y)\}$.

Proposition 276. $a \in \mathsf{dom}\, R$ iff there exists b such that $a \mathrel{R} b$.

Proposition 277. Suppose $a \mathrel{R} b$. Then $a \in \mathsf{dom}\, R$.

Proof. Follows by proposition [276]. □

Proposition 278. $\mathsf{dom}\, \emptyset = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 279. $\mathsf{dom}(A \times B) \subseteq A$.

Proposition 280. Suppose $b \in B$. $\mathsf{dom}(A \times B) = A$.

Proof. Follows by set extensionality. □

Proposition 281. $\mathsf{dom}\, \mathsf{cons}((a, b), R) = \mathsf{cons}(a, \mathsf{dom}\, R)$.

Proof. Follows by set extensionality. □

Proposition 282. $\mathsf{dom}(A \cup B) = \mathsf{dom}\, A \cup \mathsf{dom}\, B$.

Proof. Follows by set extensionality. □

Proposition 283. $\mathsf{dom}(A \cap B) \subseteq \mathsf{dom}\, A \cap \mathsf{dom}\, B$.

Proof. Follows by definitions [4] and [79] and proposition [276]. □

Proposition 284. $\mathsf{dom}(A \setminus B) \supseteq \mathsf{dom}\, A \setminus \mathsf{dom}\, B$.

4.1.2 Range of a relation

Definition 285. $\mathsf{ran}\, R = \{y \mid \exists w \in R. \exists x. w = (x, y)\}$.

Proposition 286. $b \in \mathsf{ran}\, R$ iff there exists a such that $a \; R \; b$.

Proposition 287. Suppose $a \; R \; b$. Then $b \in \mathsf{ran}\, R$.

Proof. Follows by proposition [286]. □

Proposition 288. $\mathsf{ran}\, \emptyset = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 289. $\mathsf{ran}(A \times B) \subseteq B$.

Proposition 290. Suppose $a \in A$. $\mathsf{ran}(A \times B) = B$.

Proof. Follows by set extensionality. □

Proposition 291. $\mathsf{ran}(\mathsf{cons}((a, b), R)) = \mathsf{cons}(b, \mathsf{ran}\, R)$.

Proof. Follows by set extensionality. □

Proposition 292. $\mathsf{ran}(A \cup B) = \mathsf{ran}\, A \cup \mathsf{ran}\, B$.

Proof. Follows by set extensionality. □

Proposition 293. $\mathsf{ran}(A \cap B) \subseteq \mathsf{ran}\, A \cap \mathsf{ran}\, B$.

Proof. Follows by definitions [4] and [79] and proposition [286]. □

Proposition 294. $\mathsf{ran}(A \setminus B) \supseteq \mathsf{ran}\, A \setminus \mathsf{ran}\, B$.

Proof. Follows by definitions [4] and [105] and proposition [286]. □

4.1.3 Domain and range of converse

Proposition 295. $\mathsf{dom}\, R^{\mathsf{T}} = \mathsf{ran}\, R$.

Proof. Follows by set extensionality. □

Proposition 296. $\mathsf{ran}\, R^{\mathsf{T}} = \mathsf{dom}\, R$.

Proof. Follows by set extensionality. □

4.1.4 Field of a relation

Definition 297. $\mathsf{field}\, R = \mathsf{dom}\, R \cup \mathsf{ran}\, R$.

Proposition 298. $c \in \mathsf{field}\, R$ iff there exists d such that $c\ R\ d$ or $d\ R\ c$.

Proof. Follows by definition [297], propositions [276] and [286], and axiom [56]. ☐

Proposition 299. Suppose $(a, b) \in R$. Then $a \in \mathsf{field}\, R$.

Proof. Follows by definitions [275] and [297] and axiom [56]. ☐

Proposition 300. Suppose $(a, b) \in R$. Then $b \in \mathsf{field}\, R$.

Proof. Follows by definitions [285] and [297] and axiom [56]. ☐

Proposition 301. Then $\mathsf{dom}\, R \subseteq \mathsf{field}\, R$.

Proof. Follows by definition [297] and proposition [63]. ☐

Proposition 302. Then $\mathsf{ran}\, R \subseteq \mathsf{field}\, R$.

Proof. Follows by definition [297] and proposition [64]. ☐

Proposition 303. $\mathsf{field}(A \times B) \subseteq A \cup B$.

Proof. Follows by definition [297] and propositions [65], [279] and [289]. ☐

Proposition 304. Let R be a relation. Suppose $w \in R$. Then $w \in \mathsf{field}\, R \times \mathsf{field}\, R$.

Proof. Take a, b such that $w = (a, b)$ by definition [249]. Then $a, b \in \mathsf{field}\, R$ by proposition [298]. Thus $(a, b) \in \mathsf{field}\, R \times \mathsf{field}\, R$ by proposition [143]. ☐

Proposition 305. Let R be a relation. Then $R \subseteq \mathsf{field}\, R \times \mathsf{field}\, R$.

Proof. Follows by proposition [304] and definition [4]. ☐

Proposition 306. $\mathsf{field}(A \times A) = A$.

Proposition 307. $\mathsf{field}\, \emptyset = \emptyset$.

Proposition 308. $\mathsf{field}(\mathsf{cons}((a, b), R)) = \mathsf{cons}(a, \mathsf{cons}(b, \mathsf{field}\, R))$.

Proposition 309. $\mathsf{field}(A \cup B) = \mathsf{field}\, A \cup \mathsf{field}\, B$.

Proof.

$$\begin{aligned}
\mathsf{field}(A \cup B) &= \mathsf{dom}(A \cup B) \cup \mathsf{ran}(A \cup B) \quad \text{[by definition [297]]} \\
&= (\mathsf{dom}\, A \cup \mathsf{dom}\, B) \cup (\mathsf{ran}\, A \cup \mathsf{ran}\, B) \quad \text{[by propositions [282] and [292]]} \\
&= (\mathsf{dom}\, A \cup \mathsf{ran}\, A) \cup (\mathsf{dom}\, B \cup \mathsf{ran}\, B) \quad \text{[by propositions [59] and [60]]} \\
&= \mathsf{field}\, A \cup \mathsf{field}\, B \quad \text{[by definition [297]]}
\end{aligned}$$

☐

Proposition 310. $\mathsf{field}(A \cap B) \subseteq \mathsf{field}\, A \cap \mathsf{field}\, B$.

Proof. Follows by definition [4] and propositions [92] and [298]. □

Proposition 311. $\mathsf{field}(A \setminus B) \supseteq \mathsf{field}\, A \setminus \mathsf{field}\, B$.

Proof. Follows by propositions [118] and [298] and definitions [4] and [105]. □

Proposition 312. $\mathsf{field}\, R^{\mathsf{T}} = \mathsf{field}\, R$.

Proof. Follows by definition [297] and propositions [59], [295] and [296]. □

4.2 Image

Definition 313. $R^{\rightarrow}(A) = \{b \in \mathsf{ran}\, R \mid \exists a \in A. a \mathrel{R} b\}$.

Proposition 314. Suppose $a \in A$ and $a \mathrel{R} b$. Then $b \in R^{\rightarrow}(A)$.

Proof. Follows by definitions [285] and [313]. □

Proposition 315. $b \in R^{\rightarrow}(A)$ iff there exists $a \in A$ such that $a \mathrel{R} b$.

Proposition 316. Suppose $A \subseteq B$. Then $R^{\rightarrow}(A) \subseteq R^{\rightarrow}(B)$.

Proof. Follows by definition [4] and proposition [315]. □

Proposition 317. Then $R^{\rightarrow}(A) \subseteq \mathsf{ran}\, R$.

Proposition 318. Then $R^{\rightarrow}(\mathsf{dom}\, R) = \mathsf{ran}\, R$.

Proposition 319. $R^{\rightarrow}(A \cup B) = R^{\rightarrow}(A) \cup R^{\rightarrow}(B)$.

Proof. Follows by axioms [2] and [56] and proposition [315]. □

Proposition 320. $R^{\rightarrow}(A \cap B) \subseteq R^{\rightarrow}(A) \cap R^{\rightarrow}(B)$.

Proof. Follows by proposition [315] and definitions [4] and [79]. □

Proposition 321. $R^{\rightarrow}(A \setminus B) \supseteq R^{\rightarrow}(A) \setminus R^{\rightarrow}(B)$.

Proof. Follows by proposition [315] and definitions [4] and [105]. □

Proposition 322. $b \in R^{\rightarrow}(\{a\})$ iff $a \mathrel{R} b$.

Proposition 323. Suppose $b \in R^{\rightarrow}(\{a\})$. Then $b \in \mathsf{ran}\, R$ and $(a, b) \in R$.

Proof. Follows by propositions [9], [36], [315] and [317]. □

Proposition 324. $R^{\rightarrow}(\{a\}) = \{b \in \mathsf{ran}\, R \mid (a, b) \in R\}$.

Proposition 325. $R^{\rightarrow}(\emptyset) = \emptyset$.

Proof. Follows by set extensionality. □

4.3 Preimage

Definition 326. $R^{\leftarrow}(B) = \{a \in \mathsf{dom}\, R \mid \exists b \in B. a\ R\ b\}$.

Proposition 327. $a \in R^{\leftarrow}(B)$ iff there exists $b \in B$ such that $a\ R\ b$.

Proposition 328. $R^{\leftarrow}(B) = R^{\mathsf{T}\rightarrow}(B)$.

Proof. Follows by set extensionality. □

Proposition 329. Suppose $A \subseteq B$. Then $R^{\leftarrow}(A) \subseteq R^{\leftarrow}(B)$.

Proposition 330. Then $R^{\leftarrow}(A) \subseteq \mathsf{dom}\, R$.

Proposition 331. $R^{\leftarrow}(A \cup B) = R^{\leftarrow}(A) \cup R^{\leftarrow}(B)$.

Proof. Follows by set extensionality. □

Proposition 332. $R^{\leftarrow}(A \cap B) \subseteq R^{\leftarrow}(A) \cap R^{\leftarrow}(B)$.

Proposition 333. $R^{\leftarrow}(A \setminus B) \supseteq R^{\leftarrow}(A) \setminus R^{\leftarrow}(B)$.

4.4 Upward and downward closure

Definition 334. $a^{\uparrow R} = \{b \in \mathsf{ran}\, R \mid a\ R\ b\}$.

Definition 335. $b^{\downarrow R} = \{a \in \mathsf{dom}\, R \mid a\ R\ b\}$.

Proposition 336. $a \in b^{\downarrow R}$ iff $a\ R\ b$.

4.5 Relation (and later also function) composition

Composition ignores the non-relational parts of sets. Note that the order is flipped from usual relation composition. This lets us use the same symbol for composition of functions.

Definition 337. $S \circ R = \{(x, z) \mid x \in \mathsf{dom}\, R, z \in \mathsf{ran}\, S \mid \exists y.\ x\ R\ y\ S\ z\}$.

Proposition 338. $S \circ R$ is a relation.

Proposition 339. Suppose $x\ R\ y\ S\ z$. Then $x\ (S \circ R)\ z$.

Proof. $x \in \mathsf{dom}\, R$ and $z \in \mathsf{ran}\, S$. Then $(x, z) \in S \circ R$ by definition [337]. □

Proposition 340. Suppose $x\ (S \circ R)\ z$. Then there exists y such that $x\ R\ y\ S\ z$.

Proof. There exists y such that $x\ R\ y\ S\ z$ by definition [337] and axiom [133]. □

Proposition 341. $x\ (S \circ R)\ z$ iff there exists y such that $x\ R\ y\ S\ z$.

Proposition 342. $(T \circ S) \circ R = T \circ (S \circ R)$.

Proof. For all a, b we have $(a, b) \in (T \circ S) \circ R$ iff $(a, b) \in T \circ (S \circ R)$ by proposition [341]. Now $(T \circ S) \circ R$ is a relation and $T \circ (S \circ R)$ is a relation by proposition [338]. Follows by relation extensionality. □

Proposition 343. Suppose $(a,c) \in R^\top \circ S^\top$. Then $(a,c) \in (S \circ R)^\top$.

Proof. Take b such that $a\ S^\top\ b\ R^\top\ c$. Now $c\ R\ b\ S\ a$ by proposition [262]. Hence $c\ (S \circ R)\ a$. Thus $a\ (S \circ R)^\top\ c$. □

Proposition 344. Suppose $(a,c) \in (S \circ R)^\top$. Then $(a,c) \in R^\top \circ S^\top$.

Proof. $c\ (S \circ R)\ a$. Take b such that $c\ R\ b\ S\ a$. Now $a\ S^\top\ b\ R^\top\ c$. □

Proposition 345. $(S \circ R)^\top = R^\top \circ S^\top$.

Proof. $(S \circ R)^\top$ is a relation. $R^\top \circ S^\top$ is a relation. For all x, y we have $(x,y) \in (S \circ R)^\top$ iff $(x,y) \in R^\top \circ S^\top$. Thus $(S \circ R)^\top = R^\top \circ S^\top$ by proposition [251]. □

4.6 Restriction

Definition 346. $R\big|_X = \{w \in R \mid \exists x, y. x \in X \land w = (x,y)\}$.

Proposition 347. $a\ R\big|_X\ b$ iff $a\ R\ b$ and $a \in X$.

Proposition 348. $R\big|_X \subseteq R$.

Proposition 349. Suppose $x \in \operatorname{\mathsf{dom}} R\big|_X$. Then $x \in \operatorname{\mathsf{dom}} R, X$.

Proof. Take y such that $x \in X$ and $(x,y) \in R\big|_X$. Then $(x,y) \in R$. Thus $x \in \operatorname{\mathsf{dom}} R$. □

Proposition 350. Suppose $x \in \operatorname{\mathsf{dom}} R, X$. Then $x \in \operatorname{\mathsf{dom}} R\big|_X$.

Proof. Take y such that $(x,y) \in R$ by proposition [276]. Then $(x,y) \in R\big|_X$. Thus $x \in \operatorname{\mathsf{dom}} R\big|_X$. □

Proposition 351. Suppose R is a relation. $R\big|_X = R \cap (X \times \operatorname{\mathsf{ran}} R)$.

Proof. For all a we have $a \in R \cap (X \times \operatorname{\mathsf{ran}} R)$ iff $a \in R\big|_X$ by definitions [79] and [346] and propositions [143], [149] and [286]. Follows by extensionality. □

Corollary 352. Suppose R is a relation. $\operatorname{\mathsf{dom}} R\big|_X = \operatorname{\mathsf{dom}} R \cap X$.

Proof. Follows by set extensionality. □

Proposition 353. Suppose $V \subseteq U$. Then $R\big|_U\big|_V = R\big|_V$.

Proof. For all w we have $w \in R\big|_U\big|_V$ iff $w \in R\big|_V$ by definitions [4] and [346]. Follows by extensionality. □

Proposition 354. Let R be a relation. Then $R\big|_{\operatorname{\mathsf{dom}} R} = R$.

Proof. For all w we have $w \in R\big|_{\operatorname{\mathsf{dom}} R}$ iff $w \in R$ by definitions [249], [275] and [346]. Follows by extensionality. □

Proposition 355. Then $\operatorname{\mathsf{dom}} R\big|_X \subseteq X$.

Proposition 356. Suppose $X \subseteq \mathsf{dom}\, R$. Let $b \in \mathsf{ran}\, R\big|_X$. Then $b \in R^{\rightarrow}(X)$.

Proof. Take $a \in X$ such that $(a, b) \in R\big|_X$ by definitions [4], [275] and [285] and proposition [355]. Then $a \mathrel{R} b$ and $b \in \mathsf{ran}\, R$. Thus $b \in R^{\rightarrow}(X)$ by definition [313]. ☐

Proposition 357. Suppose $X \subseteq \mathsf{dom}\, R$. Let $b \in R^{\rightarrow}(X)$. Then $b \in \mathsf{ran}\, R\big|_X$.

Proof. Follows by definition [313] and propositions [287] and [347]. ☐

Proposition 358. Suppose $X \subseteq \mathsf{dom}\, R$. Then $\mathsf{ran}\, R\big|_X = R^{\rightarrow}(X)$.

Proof. Follows by set extensionality. ☐

Proposition 359. Suppose $X \subseteq \mathsf{dom}\, R$. Then $R\big|_X{}^{\rightarrow}(A) = R^{\rightarrow}(X \cap A)$.

Proof. For all b we have $b \in R\big|_X{}^{\rightarrow}(A)$ iff $b \in R^{\rightarrow}(X \cap A)$ by propositions [315] and [347] and definition [79]. Follows by extensionality. ☐

4.7 Set of relations

Abbreviation 360. R is a binary relation on X iff $R \subseteq X \times X$.

Proposition 361. Let R be a relation. Suppose $\mathsf{ran}\, R \subseteq B$. Suppose $\mathsf{dom}\, R \subseteq A$. Suppose $w \in R$. Then $w \in A \times B$.

Proof. Take a, b such that $(a, b) = w$. Then $a \in \mathsf{dom}\, R$ and $b \in \mathsf{ran}\, R$. Thus $a \in A$ and $b \in B$. Thus $(a, b) \in A \times B$. ☐

Proposition 362. Let R be a relation. Suppose $\mathsf{ran}\, R \subseteq B$. Suppose $\mathsf{dom}\, R \subseteq A$. Then $R \subseteq A \times B$.

Proposition 363. Suppose $R \subseteq A \times B$. Suppose $a \in \mathsf{dom}\, R$. Then $a \in A$.

Proof. Take w, b such that $w \in R$ and $w = (a, b)$. Follows by definition [275] and propositions [9] and [142]. ☐

Proposition 364. Suppose $R \subseteq A \times B$. Then $\mathsf{dom}\, R \subseteq A$.

Proof. Follows by definition [4] and proposition [363]. ☐

Proposition 365. Suppose $R \subseteq A \times B$. Suppose $b \in \mathsf{ran}\, R$. Then $b \in B$.

Proof. Take w, a such that $w \in R$ and $w = (a, b)$. Follows by definition [285] and propositions [9] and [142]. ☐

Proposition 366. Suppose $R \subseteq A \times B$. Then $\mathsf{ran}\, R \subseteq B$.

Proof. Follows by definition [4] and proposition [365]. ☐

Definition 367. $\mathsf{Rel}(A, B) = \mathsf{Pow}(A \times B)$.

Proposition 368. Suppose $R \subseteq A \times B$. Then $R \in \mathsf{Rel}(A, B)$.

Proposition 369. Let R be a relation. Suppose $\mathsf{dom}\, R \subseteq A$. Suppose $\mathsf{ran}\, R \subseteq B$. Then $R \in \mathsf{Rel}(A, B)$.

Proof. $R \subseteq A \times B$. $\square$

Proposition 370. Suppose $R \in \mathsf{Rel}(A, B)$. Then $R \subseteq A \times B$.

Proposition 371. Suppose $R \in \mathsf{Rel}(A, B)$. Then $\mathsf{dom}\, R \subseteq A$.

Proof. Follows by propositions [364] and [370]. $\square$

Proposition 372. Suppose $R \in \mathsf{Rel}(A, B)$. Then $\mathsf{ran}\, R \subseteq B$.

Proof. Follows by propositions [366] and [370]. $\square$

Proposition 373. Let $R \in \mathsf{Rel}(A, B)$. Then R is a relation.

Proof. It suffices to show that for all $w \in R$ there exists x, y such that $w = (x, y)$. Fix $w \in R$. Now $R \subseteq A \times B$ by proposition [370]. Thus $w \in A \times B$. $\square$

Proposition 374. Let $R \in \mathsf{Rel}(A, B)$. Suppose $A \subseteq C$. Then $R \in \mathsf{Rel}(C, B)$.

Proof. $R \subseteq A \times B \subseteq C \times B$. Thus $R \subseteq C \times B$. $\square$

Proposition 375. Let $R \in \mathsf{Rel}(A, B)$. Suppose $B \subseteq D$. Then $R \in \mathsf{Rel}(A, D)$.

Proof. $R \subseteq A \times B \subseteq A \times D$. Thus $R \subseteq A \times D$. $\square$

Proposition 376. Let $R \in \mathsf{Rel}(A, B)$. Suppose $(a, b) \in R$. Then $(a, b) \in A \times B$.

Proof. $R \subseteq A \times B$ by proposition [370]. $\square$

Proposition 377. Let $R \in \mathsf{Rel}(A, B)$. Suppose $(a, b) \in R$. Then $a \in A$.

Proof. $(a, b) \in A \times B$ by proposition [376]. $\square$

Proposition 378. Let $R \in \mathsf{Rel}(A, B)$. Suppose $(a, b) \in R$. Then $b \in B$.

Proof. $(a, b) \in A \times B$ by proposition [376]. $\square$

Proposition 379. Let $R \in \mathsf{Rel}(A, B)$. Then $R \in \mathsf{Rel}(\mathsf{dom}\, R, B)$.

Proof. R is a relation by proposition [373]. $\mathsf{dom}\, R \subseteq \mathsf{dom}\, R$ by proposition [7]. $\mathsf{ran}\, R \subseteq B$. Follows by proposition [369]. $\square$

Proposition 380. Let $R \in \mathsf{Rel}(A, B)$. Then $R \in \mathsf{Rel}(A, \mathsf{ran}\, R)$.

Proof. R is a relation by proposition [373]. $\mathsf{dom}\, R \subseteq A$. $\mathsf{ran}\, R \subseteq \mathsf{ran}\, R$ by proposition [7]. Follows by proposition [369]. $\square$

4.8 Identity relation

Definition 381. $\mathsf{id}_A = \{(a, a) \mid a \in A\}$.

Proposition 382. $a \mathrel{\mathsf{id}_A} b$ iff $a = b \in A$.

Proof. Follows by definition [381] and axiom [133]. ☐

Proposition 383. Suppose $a \in A$. Then $(a, a) \in \mathsf{id}_A$.

Proof. Follows by definition [381]. ☐

Proposition 384. Suppose $w \in \mathsf{id}_A$. Then there exists $a \in A$ such that $w = (a, a)$.

Proof. Follows by definition [381]. ☐

Proposition 385. id_A is a relation.

Proposition 386. $\mathsf{dom}\,\mathsf{id}_A = A$.

Proof. For every $a \in A$ we have $(a, a) \in \mathsf{id}_A$. $\mathsf{dom}\,\mathsf{id}_A = A$ by set extensionality. ☐

Proposition 387. $\mathsf{ran}\,\mathsf{id}_A = A$.

Proof. For every a we have $a \in \mathsf{ran}\,\mathsf{id}_A$ iff $a \in A$ by propositions [286] and [382]. For every $a \in A$ we have $(a, a) \in \mathsf{id}_A$. $\mathsf{ran}\,\mathsf{id}_A = A$ by set extensionality. ☐

Proposition 388. $\mathsf{id}_A^{\rightarrow}(B) = A \cap B$.

Proof. Follows by set extensionality. ☐

Proposition 389. $\mathsf{id}_A \in \mathsf{Rel}(A, A)$.

4.9 Membership relation

Definition 390. $\in_A = \{(a, b) \mid a \in A, b \in A \mid a \in b\}$.

Proposition 391. Suppose $a, b \in A$. Suppose $a \in b$. Then $(a, b) \in \in_A$.

Proposition 392. Suppose $w \in \in_A$. Then there exists $a, b \in A$ such that $w = (a, b)$ and $a \in b$.

Proof. Follows by definition [390]. ☐

Proposition 393. $\in_A$ is a relation.

4.10 Subset relation

Definition 394. $\subseteq_A = \{(a, b) \mid a \in A, b \in A \mid a \subseteq b\}$.

Proposition 395. $\subseteq_A$ is a relation.

4.11 Properties of relations

Definition 396. R is left quasireflexive iff for all x, y such that $x \mathrel{R} y$ we have $x \mathrel{R} x$.

Definition 397. R is right quasireflexive iff for all x, y such that $x \mathrel{R} y$ we have $y \mathrel{R} y$.

Definition 398. R is quasireflexive iff for all x, y such that $x \mathrel{R} y$ we have $x \mathrel{R} x$ and $y \mathrel{R} y$.

Definition 399. R is coreflexive iff for all x, y such that $x \mathrel{R} y$ we have $x = y$.

Definition 400. R is reflexive on X iff for all $x \in X$ we have $x \mathrel{R} x$.

Definition 401. R is irreflexive iff for all x we have $(x, x) \notin R$.

Proposition 402. Suppose R is quasireflexive. Then R is reflexive on $\mathsf{field}\, R$.

Proposition 403. Suppose R is reflexive on $\mathsf{field}\, R$. Then R is quasireflexive.

Proposition 404. Let F be an inhabited family of relations. Suppose every element of F is reflexive on A. Then $\bigcap F$ is reflexive on A.

Proof. For all $a \in A$ we have for all $R \in F$ we have $a \mathrel{R} a$. Thus for all $a \in A$ we have $a \mathrel{(\bigcap F)} a$. □

Definition 405. R is antisymmetric iff for all x, y such that $x \mathrel{R} y \mathrel{R} x$ we have $x = y$.

Definition 406. (Symmetry) R is symmetric iff for all x, y we have $x \mathrel{R} y \iff y \mathrel{R} x$.

Definition 407. R is asymmetric iff for all x, y such that $x \mathrel{R} y$ we have $y \not\mathrel{R} x$.

Proposition 408. Suppose R is asymmetric. Then R is irreflexive.

Proposition 409. Suppose R is asymmetric. Then R is antisymmetric.

Proposition 410. Suppose R is antisymmetric. Suppose R is irreflexive. Then R is asymmetric.

Definition 411. (Transitivity) R is transitive iff for all x, y, z such that $x \mathrel{R} y \mathrel{R} z$ we have $x \mathrel{R} z$.

Proposition 412. Suppose R is transitive. Suppose $a \in b^{\downarrow R}$. Suppose $c \in a^{\downarrow R}$. Then $c \in b^{\downarrow R}$.

Proof. $c \mathrel{R} a \mathrel{R} b$. Thus $c \mathrel{R} b$ by transitivity. □

Proposition 413. Suppose R is transitive. Suppose $a \in b^{\downarrow R}$. Then $a^{\downarrow R} \subseteq b^{\downarrow R}$.

Definition 414. R is dense iff for all x, z such that $x \mathrel{R} z$ there exists y such that $x \mathrel{R} y \mathrel{R} z$.

Definition 415. R is quasiconnex iff for all $x, y \in \mathsf{field}\, R$ such that $x \neq y$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Definition 416. R is connex on X iff for all $x, y \in X$ such that $x \neq y$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Definition 417. R is strongly quasiconnex iff for all $x, y \in \mathsf{field}\, R$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Definition 418. R is strongly connex on X iff for all $x, y \in X$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Proposition 419. R is strongly quasiconnex iff R is quasiconnex and quasireflexive.

Proof. Follows by definitions [297], [400], [415] and [417] and propositions [402] and [403]. $\square$

Proposition 420. Suppose R is connex on A. Let $a, b \in A \setminus \mathsf{ran}\, R$. Then $a = b$.

Proof. Suppose not. $a, b \in A$. Then $(a, b) \in R$ or $(b, a) \in R$ by definition [416]. $(a, b) \notin R$. $(b, a) \notin R$. Thus $a = b$. $\square$

Definition 421. R is right Euclidean iff for all a, b, c such that $a \mathrel{R} b, c$ we have $b \mathrel{R} c$.

Definition 422. R is left Euclidean iff for all a, b, c such that $a, b \mathrel{R} c$ we have $a \mathrel{R} b$.

4.12 Quasiorders

Abbreviation 423. R is a quasiorder iff R is quasireflexive and transitive.

Abbreviation 424. R is a quasiorder on A iff R is a binary relation on A and R is reflexive on A and transitive.

Struct 425. A quasiordered set X is a onesorted structure equipped with

1. $\leq$

such that

1. $\leq_X$ is a binary relation on X.
2. $\leq_X$ is reflexive on X.
3. $\leq_X$ is transitive.

Lemma 426. Let X be a quasiordered set. Let $a, b, c, d \in X$. Suppose $a \leq_X b \leq_X c \leq_X d$. Then $a \leq_X d$.

Proof. $\leq_X$ is transitive. Thus $a \leq_X c \leq_X d$ by transitivity. Hence $a \leq_X d$ by transitivity. $\square$

Proposition 427. $\subseteq_A$ is a quasiorder on A.

Proof. $\subseteq_A$ is reflexive on A. $\subseteq_A$ is transitive. $\square$

4.13 Equivalences

Abbreviation 428. E is a partial equivalence iff E is transitive and symmetric.

Proposition 429. Let E be a partial equivalence. Then E is quasireflexive.

Abbreviation 430. E is an equivalence iff E is a symmetric quasiorder.

Abbreviation 431. E is an equivalence on A iff E is a symmetric quasiorder on A.

Proposition 432. Let F be a family of relations. Suppose every element of F is an equivalence. Then $\bigcap F$ is an equivalence.

Proof. $\bigcap F$ is quasireflexive by definition [398] and propositions [48] and [50]. $\bigcap F$ is symmetric by definition [406] and propositions [48] and [50]. $\bigcap F$ is transitive by definition [411] and propositions [48] and [50]. ☐

Proposition 433. Let F be an inhabited family of relations. Suppose every element of F is an equivalence on A. Then $\bigcap F$ is an equivalence on A.

Proof. $\bigcap F$ is reflexive on A by proposition [404]. $\bigcap F$ is symmetric. $\bigcap F$ is transitive. ☐

4.13.1 Equivalence classes

Abbreviation 434. $[a]_E = a^{\downarrow E}$.

Abbreviation 435. The E-equivalence class of a is $[a]_E$.

Proposition 436. Let E be an equivalence. Let $a \in \mathsf{field}\, E$. Then $a \in [a]_E$.

Proof. $a\ E\ a$ by definition [400] and proposition [402]. ☐

Proposition 437. Let E be an equivalence on A. Let $a \in A$. Then $a \in [a]_E$.

Proof. $a\ E\ a$ by definition [400]. ☐

Proposition 438. Let E be an equivalence on A. Let $a, b \in A$. Suppose $a\ E\ b$. Then $[a]_E = [b]_E$.

Proof. Follows by set extensionality. ☐

Proposition 439. Let E be an equivalence on A. Let $a, b \in A$. Suppose $[a]_E = [b]_E$. Then $a\ E\ b$.

Proposition 440. Let E be an equivalence on A. Let $a, b \in A$. Then $a\ E\ b$ iff $[a]_E = [b]_E$.

Proposition 441. Let E be a partial equivalence. Suppose $[a]_E \neq [b]_E$. Then $[a]_E$ is disjoint from $[b]_E$.

Proof. Suppose not. Take c such that $c \in [a]_E, [b]_E$. Then $c \ E \ a$ and $c \ E \ b$. E is symmetric. Thus $a \ E \ c$ by symmetry. E is transitive. Thus $a \ E \ b$ by transitivity. Then $b \ E \ a$ by symmetry. Thus $a \in [b]_E$ and $b \in [a]_E$ by proposition [336]. Hence $[a]_E \subseteq [b]_E \subseteq [a]_E$ by proposition [413]. Contradiction by proposition [8]. $\square$

Corollary 442. Let E be an equivalence. Suppose $[a]_E \neq [b]_E$. Then $[a]_E$ is disjoint from $[b]_E$.

Proof. Follows by proposition [441]. $\square$

Corollary 443. Let E be an equivalence on A. Suppose $[a]_E \neq [b]_E$. Then $[a]_E$ is disjoint from $[b]_E$.

Proof. Follows by proposition [441]. $\square$

4.13.2 Quotients

Definition 444. $A/E = \{[a]_E \mid a \in A\}$.

Proposition 445. $\emptyset/\emptyset = \emptyset$.

Proposition 446. Let E be an equivalence on A. Suppose $B, C \in A/E$ and $B \neq C$. Then B is disjoint from C.

Proof. Take b such that $B = [b]_E$. Take c such that $C = [c]_E$. Then B is disjoint from C by corollary [443]. $\square$

Proposition 447. Let E be an equivalence on A. Suppose $C \in A/E$. Then C is inhabited.

Proof. Take $a \in A$ such that $C = [a]_E$. Then $a \in [a]_E$. C is inhabited by definitions [18] and [444] and proposition [436]. $\square$

Proposition 448. Let E be an equivalence on A. Suppose $a \in C \in A/E$. Then $a \in A$.

Proof. Take $b \in A$ such that $C = [b]_E$ by definition [444]. Then $a \ E \ b$. Thus $a \in A$ by proposition [142] and definition [4]. $\square$

Corollary 449. Let E be an equivalence on A. $\emptyset \notin A/E$.

Proposition 450. Let E be an equivalence on A. A/E is a partition.

Proof. $\emptyset \notin A/E$. For all $B, C \in A/E$ such that $B \neq C$ we have B is disjoint from C. $\square$

Proposition 451. Let E be an equivalence on A. A/E is a partition of A.

Proof. $\bigcup(A/E) = A$ by set extensionality. $\square$

Definition 452. $E_P = \{(a, b) \mid a \in A, b \in A \mid \exists C \in P.\ a, b \in C\}$.

Proposition 453. Let P be a partition of A. Let $a, b \in A$. Suppose $a, b \in C \in P$. Then $a \ E_P \ b$.

Proposition 454. Let P be a partition of A. E_P is reflexive on A.

Proposition 455. Let P be a partition. E_P is symmetric.

Proof. Follows by definitions [406] and [452] and axiom [17]. $\square$

Proposition 456. Let P be a partition. E_P is transitive.

Proposition 457. Let P be a partition of A. E_P is an equivalence on A.

Proposition 458. Let E be an equivalence on A. Then $E_{A/E} = E$.

Proof. Follows by set extensionality. $\square$

Proposition 459. Let P be a partition of A. Then $A/E_P = P$.

Proof. Follows by set extensionality. $\square$

4.14 Closure operations on relations

Definition 460. $\mathsf{ReflCl}_X(R) = R \cup \mathsf{id}_X$.

Proposition 461. $\mathsf{ReflCl}_X(R)$ is reflexive on X.

Definition 462. $\mathsf{ReflReduc}_X(R) = R \setminus \mathsf{id}_X$.

Definition 463. $\mathsf{SymCl}(R) = R \cup R^{\mathsf{T}}$.

4.15 Injective relations

Definition 464. R is injective iff for all a, a', b such that $a, a' \mathrel{R} b$ we have $a = a'$.

Abbreviation 465. R is left-unique iff R is injective.

Proposition 466. Suppose $S \subseteq R$. Suppose R is injective. Then S is injective.

Proposition 467. Suppose R is injective. Then $R\big|_A$ is injective.

Proof. $R\big|_A \subseteq R$. $\square$

Proposition 468. Suppose R and S are injective. Then $S \circ R$ is injective.

Proposition 469. Then id_A is injective.

4.16 Right-unique relations

Definition 470. R is right-unique iff for all a, b, b' such that $a \mathrel{R} b, b'$ we have $b = b'$.

Abbreviation 471. R is one-to-one iff R is right-unique and injective.

Proposition 472. Suppose $S \subseteq R$. Suppose R is right-unique. Then S is right-unique.

Proposition 473. Suppose R and S are right-unique. Then $S \circ R$ is right-unique.

4.17 Left-total relations

Definition 474. R is left-total on A iff for all $a \in A$ there exists b such that $a \mathrel{R} b$.

4.18 Right-total relations

Definition 475. R is right-total on B iff for all $b \in B$ there exists a such that $a \mathrel{R} b$.

Abbreviation 476. R is surjective on B iff R is right-total on B.

5 Functions

Abbreviation 477. f is a function iff f is right-unique and f is a relation.

Definition 478. $f(x) = \bigcup f^{\rightarrow}(\{x\})$.

Proposition 479. Let f be a function. Suppose $(a, b), (a, b') \in f$. Then $b = b'$.

Proof. Follows by right-uniqueness. $\square$

Proposition 480. Let f be a function. Suppose $(a, b) \in f$. Then $f(a) = b$.

Proof. Let $B = f^{\rightarrow}(\{a\})$. $B = \{b' \in \mathsf{ran}\, f \mid (a, b') \in f\}$ by proposition [324]. $b \in \mathsf{ran}\, f$. For all $b' \in B$ we have $(a, b') \in f$. For all $b', b'' \in B$ we have $b' = b''$ by right-uniqueness. Then $B = \{b\}$ by proposition [39]. Then $\bigcup B = b$. Thus $f(a) = b$ by definition [478]. $\square$

Proposition 481. Let f be a function. Suppose $w \in f$. Then there exists $x \in \mathsf{dom}\, f$ such that $w = (x, f(x))$.

Proof. Follows by definitions [249], [275] and [478] and proposition [480]. $\square$

Proposition 482. Let f be a function. Suppose $x \in \mathsf{dom}\, f$. Then $(x, f(x)) \in f$.

Proof. Follows by propositions [276] and [480]. $\square$

Proposition 483. Let f be a function. $(a, b) \in f$ iff $a \in \mathsf{dom}\, f$ and $f(a) = b$.

Proposition 484. Let f, g be functions. Suppose $\mathsf{dom}\, f \subseteq \mathsf{dom}\, g$. Suppose for all $x \in \mathsf{dom}\, f$ we have $f(x) = g(x)$. Then $f \subseteq g$.

Proof. For all x, y such that $(x, y) \in f$ we have $(x, y) \in g$. Follows by definitions [4] and [249]. $\square$

Proposition 485. **(Function extensionality)** Let f, g be functions. Suppose $\mathsf{dom}\, f = \mathsf{dom}\, g$. Suppose for all x we have $f(x) = g(x)$. Then $f = g$.

Proof. $\mathsf{dom}\, f \subseteq \mathsf{dom}\, g \subseteq \mathsf{dom}\, f$. For all $x \in \mathsf{dom}\, f$ we have $f(x) = g(x)$. Thus $f \subseteq g$. For all $x \in \mathsf{dom}\, g$ we have $f(x) = g(x)$. Thus $g \subseteq f$. $\square$

Abbreviation 486. f is a function on X iff f is a function and $X = \mathsf{dom}\, f$.

Abbreviation 487. f is a function to Y iff f is a function and for all $x \in \mathsf{dom}\, f$ we have $f(x) \in Y$.

Proposition 488. Let f be a function to B. Suppose $B \subseteq C$. Then f is a function to C.

Proposition 489. Let f be a function to B. Then $\mathsf{ran}\, f \subseteq B$.

Proof. Follows by definitions [4], [275], [285] and [478], proposition [481], and axiom [133]. □

Definition 490. $\mathsf{Fun}(A, B) = \{f \in \mathsf{Rel}(A, B) \mid A = \mathsf{dom}\, f$ and f is right-unique$\}$.

Abbreviation 491. f is a function from X to Y iff $f \in \mathsf{Fun}(X, Y)$.

Proposition 492. Let $f \in \mathsf{Fun}(A, B)$. Then f is a relation.

Proof. Follows by definition [490] and proposition [373]. □

Proposition 493. Let $f \in \mathsf{Fun}(A, B)$. Then f is a function.

Proposition 494. $\mathsf{Fun}(A, B) \subseteq \mathsf{Rel}(A, B)$.

Proof. Follows by definitions [4] and [490]. □

Proposition 495. Let f be a function to B such that $A = \mathsf{dom}\, f$. Then $f \in \mathsf{Fun}(A, B)$.

Proof. $\mathsf{dom}\, f \subseteq A$ by proposition [7]. $\mathsf{ran}\, f \subseteq B$ by proposition [489]. Thus $f \in \mathsf{Rel}(A, B)$ by proposition [369]. Thus $f \in \mathsf{Fun}(A, B)$ by definition [490]. □

Proposition 496. Let $f \in \mathsf{Fun}(A, B)$. Then f is a function to B such that $A = \mathsf{dom}\, f$.

Proof. f is a function by proposition [493]. It suffices to show that for all $a \in \mathsf{dom}\, f$ we have $f(a) \in B$. Fix $a \in \mathsf{dom}\, f$. Take b such that $f(a) = b$. Thus $(a, b) \in f$ by proposition [482]. Now $b \in \mathsf{ran}\, f$ by proposition [286]. Finally $\mathsf{ran}\, f \subseteq B$ by definition [490] and proposition [372]. □

Proposition 497. Let $f \in \mathsf{Fun}(A, B)$. Suppose $B \subseteq D$. Then $f \in \mathsf{Fun}(A, D)$.

Proof. $f \in \mathsf{Rel}(A, D)$ by definition [490] and proposition [375]. Follows by definition [490]. □

Proposition 498. Let $f \in \mathsf{Fun}(A, B)$. Let $a \in A$. Then $f(a) \in B$.

Proof. $(a, f(a)) \in f$. Thus $f(a) \in B$ by definition [490] and proposition [378]. □

5.1 Image of a function

Proposition 499. Let f be a function. Suppose $x \in \mathsf{dom}\, f \cap X$. Then $f(x) \in f^{\rightarrow}(X)$.

Proof. $x \in X$ by proposition [82]. Thus $(x, f(x)) \in f$ by propositions [81] and [482]. □

Proposition 500. Let f be a function. Suppose $y \in f^{\rightarrow}(X)$. Then there exists $x \in \mathsf{dom}\, f \cap X$ such that $y = f(x)$.

Proof. Take $x \in X$ such that $(x, y) \in f$. Then $x \in \mathsf{dom}\, f$ and $y = f(x)$ by propositions [277] and [483]. □

Proposition 501. Suppose f is a function. $f^{\rightarrow}(X) = \{f(x) \mid x \in \mathsf{dom}\, f \cap X\}$.

Proof. Follows by propositions [499] and [500]. $\square$

5.2 Families of functions

Abbreviation 502. F is a family of functions iff every element of F is a function.

Proposition 503. Let F be a family of functions. Suppose that for all $f, g \in F$ we have $f \subseteq g$ or $g \subseteq f$. Then $\bigcup F$ is a function.

Proof. $\bigcup F$ is a relation by proposition [253]. For all x, y, y' such that $(x, y), (x, y') \in \bigcup F$ there exists $f \in F$ such that $(x, y), (x, y') \in f$ by axiom [42] and definition [4]. Thus $\bigcup F$ is right-unique by definition [470]. $\square$

5.3 The empty function

Proposition 504. $\emptyset$ is a function.

Proposition 505. $\emptyset$ is a function on $\emptyset$.

Proposition 506. $\emptyset$ is a function to X.

Proposition 507. $\emptyset$ is injective.

5.4 Function composition

Abbreviation 508. g is composable with f iff $\mathsf{ran}\, f \subseteq \mathsf{dom}\, g$.

Proposition 509. Suppose f and g are right-unique. Then $g \circ f$ is a function.

Proposition 510. Let f, g be functions. Suppose g is composable with f. Let $x \in \mathsf{dom}\, f$. Then $(g \circ f)(x) = g(f(x))$.

Proof. $(x, g(f(x))) \in g \circ f$ by definitions [4], [285] and [337] and proposition [482]. $g \circ f$ is a function by proposition [509]. Thus $(g \circ f)(x) = g(f(x))$ by proposition [480]. $\square$

Proposition 511. Let f, g be functions. Suppose g is composable with f. $\mathsf{dom}\, g \circ f = f^{\leftarrow}(\mathsf{dom}\, g)$.

Proof. Every element of $\mathsf{dom}\, g \circ f$ is an element of $f^{\leftarrow}(\mathsf{dom}\, g)$ by definitions [275], [326] and [337] and axiom [133]. Follows by set extensionality. $\square$

Proposition 512. Let f, g be functions. Suppose $\mathsf{ran}\, f = \mathsf{dom}\, g$. $\mathsf{dom}\, g \circ f = \mathsf{dom}\, f$.

Proof. Every element of $\mathsf{dom}\, g \circ f$ is an element of $\mathsf{dom}\, f$. Follows by set extensionality. $\square$

Proposition 513. Let f, g be functions. Suppose g is composable with f. Suppose $y \in g^{\rightarrow}(\mathsf{ran}\, f)$. Then $y \in \mathsf{ran}\, g \circ f$.

Proof. Take $x \in \operatorname{ran} f$ such that $(x, y) \in g$. Take $x' \in \operatorname{dom} f$ such that $(x', x) \in f$. Then $(x', y) \in g \circ f$. Follows by proposition [287]. $\square$

Proposition 514. Let f, g be functions. Suppose g is composable with f. Suppose $y \in \operatorname{ran} g \circ f$. Then $y \in g^{\rightarrow}(\operatorname{ran} f)$.

Proof. Take $x \in \operatorname{dom} f$ such that $(x, y) \in g \circ f$ by definitions [275], [285] and [337] and proposition [341]. $f(x) \in \operatorname{ran} f$. $(f(x), y) \in g$ by propositions [341] and [480] and definition [478]. Follows by proposition [315]. $\square$

Proposition 515. Let f, g be functions. Suppose g is composable with f. Then $\operatorname{ran} g \circ f = g^{\rightarrow}(\operatorname{ran} f)$.

Proof. Follows by set extensionality. $\square$

Proposition 516. Let f, g be functions. Suppose $\operatorname{ran} f = \operatorname{dom} g$. Then $\operatorname{ran} g \circ f = \operatorname{ran} g$.

Proof.

$$\begin{aligned} \operatorname{ran} g \circ f &= g^{\rightarrow}(\operatorname{ran} f) \quad \text{[by propositions [7] and [515]]} \\ &= g^{\rightarrow}(\operatorname{dom} g) \\ &= \operatorname{ran} g \quad \text{[by proposition [318]]} \end{aligned}$$

$\square$

Proposition 517. Let f, g be functions. Let A be a set. Suppose $\operatorname{ran} f \subseteq \operatorname{dom} g$. Suppose $c \in g \circ f^{\rightarrow}(A)$. Then $c \in g^{\rightarrow}(f^{\rightarrow}(A))$.

Proof. Take $a \in A$ such that $(a, c) \in g \circ f$. Take b such that $(a, b) \in f$ and $(b, c) \in g$. Then $b \in f^{\rightarrow}(A)$. Follows by proposition [315]. $\square$

Proposition 518. Let f, g be functions. Let A be a set. Suppose $\operatorname{ran} f \subseteq \operatorname{dom} g$. Then $g \circ f^{\rightarrow}(A) = g^{\rightarrow}(f^{\rightarrow}(A))$.

Proof. For all c we have $c \in g^{\rightarrow}(f^{\rightarrow}(A))$ iff $c \in g \circ f^{\rightarrow}(A)$ by propositions [315] and [341]. Follows by extensionality. $\square$

Proposition 519. Let f be a function. Let A be a set. $f\big|_A$ is a function.

Proposition 520. Let f be a function. Suppose $A \subseteq \operatorname{dom} f$. Let $a \in A$. Then $(f\big|_A)(a) = f(a)$.

Proof. Then $(a, f(a)) \in f$. Then $(a, f(a)) \in f\big|_A$ by proposition [347]. Thus $(f\big|_A)(a) = f(a)$. $\square$

Proposition 521. Suppose $x \notin \operatorname{dom} f$. Then $f(x) = \emptyset$.

Proof. $f^{\rightarrow}(\{x\}) = \emptyset$ by axioms [2] and [17] and propositions [277] and [322]. Follows by definition [478] and proposition [44]. $\square$

5.5 Injections

Proposition 522. Suppose f is a function. f is injective iff for all $x, y \in \mathsf{dom}\, f$ we have $f(x) = f(y) \implies x = y$.

Proof. Follows by definition [464] and proposition [483]. $\square$

Abbreviation 523. f is an injection iff f is an injective function.

Definition 524. $\mathsf{Inj}(A, B) = \{f \in \mathsf{Fun}(A, B) \mid \text{for all } x, y \in A \text{ such that } f(x) = f(y) \text{ we have } x = y\}$.

5.6 Surjections

Abbreviation 525. f is a surjection onto Y iff f is a function such that f is surjective on Y.

Definition 526. $\mathsf{Surj}(A, B) = \{f \in \mathsf{Fun}(A, B) \mid \text{for all } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b\}$.

Abbreviation 527. f is a surjection from A to B iff $f \in \mathsf{Surj}(A, B)$.

Lemma 528. Let f be a function. Then f is surjective on $\mathsf{ran}\, f$.

Proof. It suffices to show that for all $y \in \mathsf{ran}\, f$ there exists $x \in \mathsf{dom}\, f$ such that $f(x) = y$. Fix $y \in \mathsf{ran}\, f$. Take x such that $(x, y) \in f$. Then $x \in \mathsf{dom}\, f$ and $f(x) = y$. $\square$

Lemma 529. Let $f \in \mathsf{Surj}(A, B)$. Then $f \in \mathsf{Fun}(A, B)$.

Lemma 530. Let $f \in \mathsf{Fun}(A, B)$. Then $f \in \mathsf{Surj}(A, \mathsf{ran}\, f)$.

Proof. $f \in \mathsf{Rel}(A, \mathsf{ran}\, f)$ by definition [490] and proposition [380]. Thus $f \in \mathsf{Fun}(A, \mathsf{ran}\, f)$ by definition [490]. It suffices to show that for all $b \in \mathsf{ran}\, f$ there exists $a \in A$ such that $f(a) = b$ by definition [526]. Fix $b \in \mathsf{ran}\, f$. Take a such that $(a, b) \in f$. Then $a \in \mathsf{dom}\, f = A$. $\square$

Definition 531. f surjects onto Y iff $Y = \{f(x) \mid x \in \mathsf{dom}\, f\}$.

Proposition 532. f surjects onto $f^{\to}(\mathsf{dom}\, f)$.

Proof. Omitted. $\square$

Proposition 533. Suppose f surjects onto Y. Then $Y \subseteq f^{\to}(\mathsf{dom}\, f)$.

Proof. Omitted. $\square$

Proposition 534. Let f be a function. Suppose f surjects onto Y. Then $\mathsf{ran}\, f = Y$.

Proof. $Y \subseteq \mathsf{ran}\, f$ by definitions [4] and [531] and propositions [286] and [483]. $\mathsf{ran}\, f \subseteq Y$ by definition [531] and proposition [489]. Follows by antisymmetry. $\square$

Proposition 535. Let f be a function. Suppose $\mathsf{ran}\, f = Y$. Then f surjects onto Y.

Proof. Omitted. $\square$

Proposition 536. Let f be a function. f surjects onto Y iff $\mathsf{ran}\, f = Y$.

Proof. Omitted. $\square$

5.7 Bijections

Definition 537. f is a bijection from X to Y iff $\mathsf{dom}\, f = X$ and f surjects onto Y and f is an injection.

Proposition 538. Let f be a bijection from A to B. Let g be a bijection from B to C. Then $g \circ f$ is a bijection from A to C.

Proof. $\mathsf{dom}\, f = A$. $\mathsf{dom}\, g = B = \mathsf{ran}\, f$ by definition [537] and proposition [536]. $\mathsf{dom}\, g \circ f = A$ by definition [537] and proposition [512]. $g \circ f$ surjects onto C. $g \circ f$ is an injection. $\square$

5.8 Converse as a function

Proposition 539. Let f be a function. Then f^{T} is injective.

Proposition 540. Suppose f is injective. Then f^{T} is a function.

Proposition 541. Let f be a bijection from A to B. Then f^{T} is a function.

Proof. Follows by definition [537] and proposition [540]. $\square$

Proposition 542. Let f be a bijection from A to B. Then f^{T} is a bijection from B to A.

Proof. f^{T} is a function by proposition [541]. f^{T} is injective by definition [537] and proposition [539]. f^{T} surjects onto A. $\mathsf{dom}\, f^{\mathsf{T}} = \mathsf{ran}\, f = B$ by definition [537] and propositions [295] and [536]. Follows by definition [537]. $\square$

5.8.1 Inverses of a function

Abbreviation 543. g is a left inverse of f iff for all $x \in \mathsf{dom}\, f$ we have $g(f(x)) = x$.

Abbreviation 544. g is a right inverse of f iff $f \circ g = \mathsf{id}_{\mathsf{dom}\, g}$.

Abbreviation 545. g is a right inverse of f on B iff $f \circ g = \mathsf{id}_B$.

Proposition 546. Let f be an injection. Then f^{T} is a left inverse of f.

Proof. f^{T} is a function by proposition [540].
Omitted. $\square$

5.9 Identity function

Proposition 547. id_A is right-unique.

Proof. Follows by definitions [381] and [470] and axiom [133]. □

Proposition 548. id_A is a function.

Proposition 549. id_A is a function on A.

Proposition 550. id_A is a function to A.

Proposition 551. id_A is a function from A to A.

Proposition 552. $\mathsf{id}_A \in \mathsf{Fun}(A, A)$.

Proof. id_A is a function. $\mathsf{id}_A \in \mathsf{Rel}(A, A)$. $\mathsf{dom}\,\mathsf{id}_A \subseteq A$. □

Proposition 553. Suppose $a \in A$. Suppose $f = \mathsf{id}_A$. Then $f(a) = a$.

Proof. $(a, a) \in \mathsf{id}_A$ by proposition [382]. Follows by propositions [480] and [548]. □

Proposition 554. id_A is a bijection from A to A.

Proof. id_A is an injection by propositions [469] and [548]. $\mathsf{dom}\,\mathsf{id}_A = A$ by proposition [386]. id_A surjects onto A by propositions [387] and [536]. Follows by definition [537]. □

6 Transitive sets

We use the word *transitive* to talk about sets as relations, so we will explicitly talk about *$\in$-transitivity* here.

Definition 555. A set A is $\in$-transitive iff for all x, y such that $x \in y \in A$ we have $x \in A$.

Proposition 556. A is $\in$-transitive iff for all $a \in A$ we have $a \subseteq A$.

Proposition 557. A is $\in$-transitive iff $A \subseteq \mathsf{Pow}(A)$.

Proof. For all $a \in A$ we have $a \subseteq A \iff a \in \mathsf{Pow}(A)$. Follows by propositions [9] and [556], definition [4], and axiom [200]. □

Proposition 558. A is $\in$-transitive iff $\bigcup A^+ = A$.

Proof. Follows by definitions [4], [168] and [555], propositions [8], [178], [204], [205] and [557], and axiom [42]. □

Proposition 559. A is $\in$-transitive iff $\bigcup A \subseteq A$.

Proposition 560. Suppose A is $\in$-transitive. Suppose $\{a, b\} \in A$. Then $a, b \in A$.

6.0.1 Closure properties of $\in$-transitive sets

Proposition 561. $\emptyset$ is $\in$-transitive.

Proposition 562. Suppose A and B are $\in$-transitive. Then $A \cup B$ is $\in$-transitive.

Proposition 563. Let A, B be $\in$-transitive sets. Then $A \cap B$ is $\in$-transitive.

Proposition 564. Let A be an $\in$-transitive set. Then A^+ is $\in$-transitive.

Proposition 565. Let A be an $\in$-transitive set. Then $\bigcup A$ is $\in$-transitive.

Proposition 566. Suppose every element of A is an $\in$-transitive set. Then $\bigcup A$ is $\in$-transitive.

Proof. Follows by definition [555] and axiom [42]. □

Proposition 567. Suppose every element of A is an $\in$-transitive set. Then $\bigcap A$ is $\in$-transitive.

Proof. Follows by definitions [47] and [555] and proposition [566]. □

7 Ordinals

Definition 568. α is an ordinal iff α is $\in$-transitive and every element of α is $\in$-transitive.

Proposition 569. Suppose α is $\in$-transitive. Suppose every element of α is $\in$-transitive. Then α is an ordinal.

Proposition 570. Let α be an ordinal. Then α is $\in$-transitive.

Proposition 571. Let α be an ordinal. Suppose $A \in \alpha$. Then A is $\in$-transitive.

Proposition 572. Let α be an ordinal. Suppose $\beta \in \alpha$. Then β is an ordinal.

Proposition 573. Suppose α^+ is an ordinal. Then α is an ordinal.

Proposition 574. Let α be an ordinal. Suppose $\beta \subseteq \alpha$. Suppose β is $\in$-transitive. Then β is an ordinal.

Proof. Follows by definitions [4] and [568]. □

Proposition 575. Let α, β be ordinals. Suppose $\alpha \in \beta$. Then $\alpha \subseteq \beta$.

Proposition 576. Let α be an ordinal. Suppose $\gamma \in \beta \in \alpha$. Then $\gamma \in \alpha$.

Proof. Follows by definitions [555] and [568]. □

Proposition 577. Let β be an ordinal. Suppose $\alpha \in \beta$. Then $\alpha^+ \subseteq \beta$.

Abbreviation 578. $\alpha < \beta$ iff β is an ordinal and $\alpha \in \beta$.

Abbreviation 579. $\alpha \leq \beta$ iff β is an ordinal and $\alpha \subseteq \beta$.

Lemma 580. Let α, β be sets. Suppose $\alpha < \beta$. Then α is an ordinal.

Proof. Follows by proposition [572]. $\square$

We already have global irreflexivity and asymmetry of $\in$. $\in$ is transitive on ordinals by definition. To show that $\in$ is a strict total order it only remains to show that $\in$ is connex.

Proposition 581. For all ordinals α, β we have $\alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta$.

Proof by $\in$-induction on α. Assume α is an ordinal. Show for all ordinals γ we have $\alpha \in \gamma \vee \gamma \in \alpha \vee \alpha = \gamma$. *Subproof.* [Proof by $\in$-induction on γ] Assume γ is an ordinal. Follows by axiom [2] and definitions [555] and [568]. $\square$ $\square$

Proposition 582. Let α, β be ordinals. Suppose $\alpha \subset \beta$. Then $\alpha \in \beta$.

Proof. $\beta \setminus \alpha$ is inhabited. Take γ such that γ is an $\in$-minimal element of $\beta \setminus \alpha$. Now $\gamma \in \beta$ by proposition [107]. Hence $\gamma \subseteq \beta$ by definition [568] and proposition [556]. For all $\delta \in \beta \setminus \alpha$ we have $\delta \notin \gamma$. Thus $\gamma \setminus \alpha = \emptyset$. Hence $\gamma \subseteq \alpha$. It suffices to show that for all $\delta \in \alpha$ we have $\delta \in \gamma$. Suppose not. Take $\delta \in \alpha$ such that $\delta \notin \gamma$. Now if $\delta = \gamma$ or $\gamma \in \delta$, then $\gamma \in \alpha$ by definition [568] and propositions [9], [556], [572] and [581]. $\square$

Proposition 583. Let α, β be ordinals. Suppose $\alpha \in \beta$. Then $\alpha \subset \beta$.

Proof. $\alpha \subseteq \beta$. $\square$

Proposition 584. Let α, β be ordinals. Suppose $\alpha \leq \beta$. Then $\alpha \subseteq \beta$.

Proof. Case: $\alpha = \beta$. Trivial. Case: $\alpha < \beta$. $\alpha \subset \beta$. $\square$

Proposition 585. Let α, β be ordinals. Then $\alpha \in \beta$ or $\beta \subseteq \alpha$.

Proposition 586. Let α, β be ordinals. Then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Proposition 587. Let α, β be ordinals. Suppose $\alpha \subseteq \beta$. Then $\alpha \in \beta$ or $\alpha = \beta$.

Corollary 588. Let α, β be ordinals. Then $(\alpha \subset \beta \vee \beta \subset \alpha) \vee \alpha = \beta$.

Proposition 589. Let α, β be ordinals. Suppose neither $\alpha \in \beta$ nor $\beta \in \alpha$. Then $\alpha = \beta$.

Proof. Neither $\alpha \subset \beta$ nor $\beta \subset \alpha$. $\square$

Proposition 590. Let α, β be ordinals. Then $(\alpha \in \beta \vee \beta \in \alpha) \vee \alpha = \beta$.

Proof. Suppose not. Then neither $\alpha \in \beta$ nor $\beta \in \alpha$. Thus $\alpha = \beta$ by proposition [589]. Contradiction. $\square$

Corollary 591. Let α, β be ordinals. Suppose neither $\alpha < \beta$ nor $\beta < \alpha$. Then $\alpha = \beta$.

Proof. Follows by proposition [589]. $\square$

Corollary 592. Let α, β be ordinals. Then $\alpha \in \beta$ or $\beta \subseteq \alpha$.

7.0.1 Construction of ordinals

Proposition 593. $\emptyset$ is an ordinal.

Proposition 594. Let α be an ordinal. α^+ is an ordinal.

Proof. α^+ is $\in$-transitive by definition [568] and proposition [564]. For every $\beta \in \alpha$ we have that β is $\in$-transitive. $\square$

Proposition 595. α is an ordinal iff α^+ is an ordinal.

Proposition 596. Let α be an ordinal. Then $\alpha \in \alpha^+$.

Corollary 597. Let α be an ordinal. Then $\alpha < \alpha^+$.

Proposition 598. Let α, β be ordinals. Suppose $\alpha \in \beta$. Then $\alpha \subseteq \beta^+$.

Proof. $\alpha \subset \beta$. In particular, $\alpha \subseteq \beta$. Hence $\alpha \subseteq \mathsf{cons}(\beta, \beta)$. $\square$

Proposition 599. Let α be an ordinal. Then $\bigcup \alpha$ is an ordinal.

Proof. For all x, y such that $x \in y \in \bigcup \alpha$ we have $x \in \bigcup \alpha$ by proposition [43], axiom [42], and definitions [555] and [568]. Thus $\bigcup \alpha$ is $\in$-transitive. Every element of $\bigcup \alpha$ is $\in$-transitive. $\square$

Lemma 600. Let α be an ordinal. Then $\bigcup \alpha \subseteq \alpha$.

Proof. Follows by definition [568] and proposition [559]. $\square$

Proposition 601. Let α, β be ordinals. Then $\alpha \cup \beta$ is an ordinal.

Proof. $\alpha \cup \beta$ is $\in$-transitive by proposition [562] and definition [568]. Every element of $\alpha \cup \beta$ is $\in$-transitive by definitions [555] and [568] and axiom [56]. Follows by definition [568]. $\square$

Proposition 602. For all ordinals α we have $\alpha = \emptyset$ or $\emptyset \in \alpha$.

Proof by $\in$-induction. Straightforward. $\square$

Proposition 603. Let A be a set. Suppose that for every $\alpha \in A$ we have α is an ordinal. Suppose that A is $\in$-transitive. Then A is an ordinal.

Theorem 604. (Burali-Forti antimony) There exists no set Ω such that for all α we have $\alpha \in \Omega$ iff α is an ordinal.

Proof. Suppose not. Take Ω such that for all α we have $\alpha \in \Omega$ iff α is an ordinal. For all x, y such that $x \in y \in \Omega$ we have $x \in \Omega$. Thus Ω is $\in$-transitive. Thus Ω is an ordinal. Therefore $\Omega \in \Omega$. Contradiction. $\square$

Proposition 605. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then $\bigcap A$ is an ordinal.

Proof. It suffices to show that $\bigcap A$ is $\in$-transitive. $\square$

Proposition 606. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then for all $\alpha \in A$ we have $\bigcap A \subseteq \alpha$.

Proposition 607. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then $\bigcap A \in A$.

Proof. Follows by propositions [48], [53], [244], [587] and [605]. □

Proposition 608. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then $\bigcap A$ is an $\in$-minimal element of A.

Proof. For all $\alpha \in A$ we have $\bigcap A \subseteq \alpha$. □

Proposition 609. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then for all $\alpha \in A$ we have $\bigcap A = \alpha$ or $\bigcap A \in \alpha$.

Proof. For all $\alpha \in A$ we have $\bigcap A \subseteq \alpha$. □

Proposition 610. Let α, β be ordinals. Then $\alpha \cap \beta$ is an ordinal.

Proof. $\alpha \cap \beta$ is $\in$-transitive by definitions [79], [555] and [568]. Every element of $\alpha \cap \beta$ is $\in$-transitive by definitions [79], [555] and [568]. Follows by definition [568]. □

7.0.2 Limit and successor ordinals

Definition 611. λ is a limit ordinal iff $\emptyset < \lambda$ and for all $\alpha \in \lambda$ we have $\alpha^+ \in \lambda$.

Definition 612. α is a successor ordinal iff there exists an ordinal β such that $\alpha = \beta^+$.

Lemma 613. Let α be an ordinal such that $\emptyset < \alpha$. Then α is a limit ordinal or α is a successor ordinal.

Proof. Case: α is a limit ordinal. Trivial. Case: α is not a limit ordinal. Take β such that $\beta \in \alpha$ and $\beta^+ \notin \alpha$ by definition [611]. □

Lemma 614. $\emptyset$ is not a successor ordinal.

Lemma 615. $\emptyset$ is not a limit ordinal.

Proof. Suppose not. Then $\emptyset < \emptyset$ by axiom [17] and definition [611]. Thus $\emptyset \in \emptyset$. Contradiction. □

Lemma 616. Let λ be a limit ordinal. Let $\alpha \in \lambda$. Then $\alpha^+ \in \lambda$.

Proof. Follows by definition [611]. □

Lemma 617. Let λ be a limit ordinal. Then $\bigcup \lambda = \lambda$.

Proof. $\bigcup \lambda \subseteq \lambda$ by definition [611] and lemma [600]. For all $\alpha \in \lambda$ we have $\alpha \in \alpha^+ \in \lambda$ by proposition [169] and lemma [616]. Thus $\lambda \subseteq \bigcup \lambda$ by definition [4] and proposition [43]. Follows by proposition [8]. □

7.1 Natural numbers as ordinals

Lemma 618. Let $n \in \mathcal{N}$. Suppose $n \neq \emptyset$. Then n is a successor ordinal.

Proof. Let $M = \{m \in \mathcal{N} \mid m = \emptyset \text{ or } m \text{ is a successor ordinal}\}$. M is an inductive set by propositions [593] and [594], axiom [624], and definition [612]. Now $M \subseteq \mathcal{N} \subseteq M$ by definition [4] and axiom [625]. Thus $M = \mathcal{N}$. Follows by definition [4]. ☐

Lemma 619. $\mathcal{N}$ is $\in$-transitive.

Proof. Let $M = \{m \in \mathcal{N} \mid \text{for all } n \in m \text{ we have } n \in \mathcal{N}\}$. $\emptyset \in M$. For all $n \in M$ we have $n^+ \in M$ by axiom [624] and definition [168]. Thus M is an inductive set. Now $M \subseteq \mathcal{N} \subseteq M$ by definition [4] and axiom [625]. Hence $\mathcal{N} = M$. ☐

Lemma 620. Every natural number is an ordinal.

Proof. Follows by propositions [173], [573] and [594], axiom [624], lemma [618], and definition [612]. ☐

Lemma 621. $\mathcal{N}$ is an ordinal.

Proof. Follows by lemmas [619] and [620] and proposition [603]. ☐

Lemma 622. $\mathcal{N}$ is a limit ordinal.

Proof. $\emptyset < \mathcal{N}$. If $n \in \mathcal{N}$, then $n^+ \in \mathcal{N}$. ☐

8 Natural numbers

Abbreviation 623. A is an inductive set iff $\emptyset \in A$ and for all $a \in A$ we have $a^+ \in A$.

Axiom 624. $\mathcal{N}$ is an inductive set.

Axiom 625. Let A be an inductive set. Then $\mathcal{N} \subseteq A$.

Abbreviation 626. n is a natural number iff $n \in \mathcal{N}$.

9 Cardinality

Definition 627. X is finite iff there exists a natural number k such that there exists a bijection from k to X.

Abbreviation 628. X is infinite iff X is not finite.

10 Magmas

Struct 629. A magma A is a onesorted structure equipped with

1. mul

such that

1. for all $a, b \in A$ we have $\mathsf{mul}_A(a, b) \in A$.

Abbreviation 630. $a \cdot b = \mathsf{mul}(a, b)$.

Abbreviation 631. a is an idempotent element of A iff $a \in A$ and $\mathsf{mul}_A(a, a) = a$.

Definition 632. $\mathsf{Idempotent}(A) = \{a \in A \mid \mathsf{mul}_A(a, a) = a\}$.

Abbreviation 633. a commutes with b iff $a \cdot b = b \cdot a$.

Definition 634. A is a submagma of B iff A is a magma and B is a magma and $A \subseteq B$ and $\mathsf{mul}_A \subseteq \mathsf{mul}_B$.

Proposition 635. Suppose A is a submagma of B. Suppose B is a submagma of C. Then A is a submagma of C.

Proof. Follows by definition [634] and proposition [11]. $\square$

Struct 636. A unital magma A is a magma equipped with

1. e

such that

1. $\mathsf{e}_A \in A$.
2. for all $a \in A$ we have $\mathsf{mul}_A(a, \mathsf{e}_A) = a$.
3. for all $a \in A$ we have $\mathsf{mul}_A(\mathsf{e}_A, a) = a$.

Proposition 637. Let A be a unital magma. Then $\mathsf{mul}(\mathsf{e}, \mathsf{e}) = \mathsf{e}$.

Proposition 638. Let A be a unital magma. Let e be a set such that $e \in A$ and for all $x \in A$ we have $\mathsf{mul}(x, e) = x = \mathsf{mul}(e, x)$. Then $e = \mathsf{e}$.

Proof. Follows by items [1] and [3]. $\square$

Definition 639. **(Left orbit)** $A \cdot x = \{\mathsf{mul}_A(a, x) \mid a \in A\}$.

Proposition 640. Let A be a magma. Let $e, f \in A$. Suppose $A \cdot e = A \cdot f$. Let $x \in A$. Then there exists $y \in A$ such that $x \cdot e = y \cdot f$.

Proof. We have $x \cdot e \in A \cdot e$ by definition [639]. Thus $x \cdot e \in A \cdot f$ by assumption. Take $y \in A$ such that $x \cdot e = y \cdot f$ by definition [639]. $\square$

11 Semigroups

Struct 641. A semigroup A is a magma such that

1. for all a, b, c we have $\mathsf{mul}_A(a, \mathsf{mul}_A(b, c)) = \mathsf{mul}_A(\mathsf{mul}_A(a, b), c)$.

12 Regular semigroups

Struct 642. A regular semigroup A is a semigroup such that

1. for all a there exists $b \in A$ such that $\mathsf{mul}_A(a, \mathsf{mul}_A(b, a)) = a$.

13 Inverse semigroups

Struct 643. An inverse semigroup A is a regular semigroup such that

1. for all $a, b \in \mathsf{Idempotent}(A)$ we have $\mathsf{mul}_A(a, b) = \mathsf{mul}_A(b, a)$.

Proposition 644. Suppose A is an inverse semigroup. Then A is a semigroup.

Proposition 645. Suppose A is an inverse semigroup. Then A is a regular semigroup.

Proposition 646. Let A be an inverse semigroup. Let $e, f \in \mathsf{Idempotent}(A)$. Suppose for all $x \in A$ there exists $y \in A$ such that $x \cdot e = y \cdot f$. Suppose for all $x \in A$ there exists $y \in A$ such that $x \cdot f = y \cdot e$. Then $e = f$.

Proof. Take $x, y \in A$ such that $e = x \cdot f$ and $f = y \cdot e$ by definition [632].

$$\begin{aligned}
e &= x \cdot f \\
&= x \cdot (f \cdot f) \quad \text{[by definition [632]]} \\
&= (x \cdot f) \cdot f \quad \text{[by item [1] and proposition [644]]} \\
&= e \cdot f \\
&= f \cdot e \quad \text{[by commutativity of idempotent elements]} \\
&= (y \cdot e) \cdot e \\
&= y \cdot (e \cdot e) \quad \text{[by item [1] and proposition [644]]} \\
&= y \cdot e \quad \text{[by definition [632]]} \\
&= f
\end{aligned}$$

□

Abbreviation 647. R is an order iff R is an antisymmetric quasiorder.

Abbreviation 648. R is an order on A iff R is an antisymmetric quasiorder on A.

Abbreviation 649. R is a strict order iff R is transitive and asymmetric.

Struct 650. An ordered set X is a quasiordered set such that

1. $\leq_X$ is antisymmetric.

Definition 651. $\mathsf{StrictOrderFromOrder}(R) = \{w \in R \mid \mathsf{fst}\, w \neq \mathsf{snd}\, w\}$.

Definition 652. $\mathsf{OrderFromStrictOrder}_A(R) = R \cup \mathsf{id}_A$.

Proposition 653. $(a, b) \in \mathsf{StrictOrderFromOrder}(R)$ iff $(a, b) \in R$ and $a \neq b$.

Proof. Follows by definition [651] and axioms [138] and [139]. ☐

Proposition 654. $\mathsf{OrderFromStrictOrder}_A(R)$ is reflexive on A.

Proposition 655. Suppose $(a, b) \in R$. Then $(a, b) \in \mathsf{OrderFromStrictOrder}_A(R)$.

Proof. $R \subseteq \mathsf{OrderFromStrictOrder}_A(R)$. ☐

Proposition 656. Suppose $(a, b) \in \mathsf{OrderFromStrictOrder}_A(R)$. Then $(a, b) \in R$ or $a = b$.

Proof. Follows by definitions [381] and [652], axiom [56], and propositions [31] and [653]. ☐

Proposition 657. $(a, b) \in \mathsf{OrderFromStrictOrder}_A(R)$ iff $(a, b) \in R$ or $a = b \in A$.

Proposition 658. Suppose R is an order. Then $\mathsf{StrictOrderFromOrder}(R)$ is a strict order.

Proof. $\mathsf{StrictOrderFromOrder}(R)$ is asymmetric. $\mathsf{StrictOrderFromOrder}(R)$ is transitive. ☐

Proposition 659. Suppose R is a strict order. Suppose R is a binary relation on A. Then $\mathsf{OrderFromStrictOrder}_A(R)$ is an order on A.

Proof. $\mathsf{OrderFromStrictOrder}_A(R)$ is antisymmetric. $\mathsf{OrderFromStrictOrder}_A(R)$ is transitive by definition [411] and proposition [657]. $\mathsf{OrderFromStrictOrder}_A(R)$ is reflexive on A. ☐

Proposition 660. $\subseteq_A$ is antisymmetric.

Proof. Follows by definitions [394] and [405], axiom [133], and proposition [8]. ☐

Proposition 661. $\subseteq_A$ is an order on A.

Proof. $\subseteq_A$ is a quasiorder on A by proposition [427]. $\subseteq_A$ is antisymmetric by proposition [660]. ☐

Struct 662. A meet semilattice X is a partial order equipped with

1. $\sqcap$

such that

1. for all $x, y \in X$ we have $\sqcap_X(x, y) \in X$.

2. for all $x, y \in X$ we have $\sqcap_X(x, y) \leq_X x, y$.
3. for all $a, x, y \in X$ such that $a \leq_X x, y$ we have $a \leq_X \sqcap_X(x, y)$.

Proposition 663. Let X be a meet semilattice. Then $\sqcap(x, x) = x$.

Proof. $\sqcap(x, x) \leq x$. $x \leq_X x, x$. Thus $x \leq_X \sqcap(x, x)$. □

14 Topological spaces

Struct 664. A topological space X is a onesorted structure equipped with

1. $\mathcal{O}$

such that

1. $\mathcal{O}_X$ is a family of subsets of X.
2. $\emptyset \in \mathcal{O}_X$.
3. $X \in \mathcal{O}_X$.
4. For all $A, B \in \mathcal{O}_X$ we have $A \cap B \in \mathcal{O}_X$.
5. For all $F \subseteq \mathcal{O}_X$ we have $\bigcup F \in \mathcal{O}_X$.

Axiom 665. For all A, B we have $\bigcup\{A, B\} = A \cup B$.

Abbreviation 666. U is open iff $U \in \mathcal{O}$.

Abbreviation 667. U is open in X iff $U \in \mathcal{O}_X$.

Proposition 668. Let X be a topological space. Suppose A, B are open. Then $A \cup B$ is open.

Proof. $\{A, B\} \subseteq \mathcal{O}$. $\bigcup\{A, B\}$ is open. $\bigcup\{A, B\} = A \cup B$. □

Definition 669. **(Interiors)** $\mathsf{Int}_X\, A = \{U \in \mathcal{O}_X \mid U \subseteq A\}$.

Definition 670. **(Interior)** $\mathsf{int}_X\, A = \bigcup \mathsf{Int}_X\, A$.

Proposition 671. **(Interior)** Suppose $U \in \mathcal{O}_X$ and $a \in U \subseteq A$. Then $a \in \mathsf{int}_X\, A$.

Proof. $U \in \mathsf{Int}_X\, A$. □

Proposition 672. **(Interior)** Suppose $a \in \mathsf{int}_X\, A$. Then there exists $U \in \mathcal{O}_X$ such that $a \in U \subseteq A$.

Proof. Take $U \in \mathsf{Int}_X\, A$ such that $a \in U$. □

Proposition 673. **(Interior)** $a \in \mathsf{int}_X\, A$ iff there exists $U \in \mathcal{O}_X$ such that $a \in U \subseteq A$.

Proof. Follows by propositions [671] and [672]. ☐

Proposition 674. Let X be a topological space. Suppose U is open in X. Then $\mathsf{int}_X U = U$.

Proof. $U \in \mathsf{Int}_X U$. Follows by definition [4] and propositions [3] and [673]. ☐

Proposition 675. Let X be a topological space. Then $\mathsf{int}_X A$ is open.

Proof. $\mathsf{Int}_X A \subseteq \mathcal{O}_X$. ☐

Proposition 676. Then $\mathsf{int}_X A \subseteq A$.

Proposition 677. Let X be a topological space. Suppose $U \subseteq A \subseteq X$. Suppose U is open. Then $U \subseteq \mathsf{int}_X A$.

Proposition 678. Let X be a topological space. Suppose $\mathsf{int}_X A = A$. Then A is open.

Corollary 679. Let X be a topological space. Then $\mathsf{int}_X A = A$ iff A is open in X.

Proposition 680. Let X be a topological space. $\mathsf{int}_X X = X$.

Proof. $X \in \mathcal{O}_X$. $X \subseteq X$ by proposition [7]. Thus $X \in \mathsf{Int}_X X$ by definition [669]. Follows by set extensionality. ☐

14.1 Closed sets

Definition 681. A is closed in X iff $X \setminus A$ is open in X.

Abbreviation 682. A is clopen in X iff A is open in X and closed in X.

Proposition 683. Let X be a topological space. Then $\emptyset$ is closed in X.

Proof. $X \setminus \emptyset = X$. ☐

Proposition 684. Let X be a topological space. Then $\emptyset$ is closed in X.

Proof. $X \setminus X = \emptyset$. ☐

Definition 685. **(Closed sets)** $\mathcal{C}_X = \{A \in \mathsf{Pow}(X) \mid A \text{ is closed in } X\}$.

Proposition 686. Let X be a topological space. Let $U \in \mathcal{O}_X$. Then $X \setminus U \in \mathcal{C}_X$.

Proof. $X \setminus U \in \mathsf{Pow}(X)$. $U \subseteq X$ by item [1]. Hence $X \setminus (X \setminus U) = U$ by proposition [113]. $X \setminus U$ is closed in X. ☐

Definition 687. **(Closed covers)** $\mathsf{Cl}_X A = \{D \in \mathsf{Pow}(X) \mid A \subseteq D \text{ and } D \text{ is closed in } X\}$.

Definition 688. **(Closure)** $\mathsf{cl}_X A = \bigcap \mathsf{Cl}_X A$.

Proposition 689. Let X be a topological space. Then $\mathsf{cl}_X \emptyset = \emptyset$.

Proof. $\emptyset \in \mathsf{Cl}_X \emptyset$. ☐

Proposition 690. Let X be a topological space. Then $\mathsf{cl}_X\, X = X$.

Proof. For all $D \in \mathsf{Cl}_X\, X$ we have $X = D$ by axiom [200], definition [687], and proposition [8]. Now $X \in \mathsf{Cl}_X\, X$. Thus $\mathsf{Cl}_X\, X = \{X\}$ by proposition [39]. Follows by proposition [54] and definition [688]. ☐

Proposition 691. $\mathsf{cl}_X\, A \cap (X \setminus \mathsf{int}_X\, A) = \mathsf{cl}_X\, (X \setminus A)$.

Proof. Omitted. ☐

Definition 692. (Frontier) $\mathsf{fr}_X\, A = \mathsf{cl}_X\, A \setminus \mathsf{int}_X\, A$.

Proposition 693. $\mathsf{fr}_X\, A = \mathsf{cl}_X\, A \cap \mathsf{cl}_X\, (X \setminus A)$.

Proof. Omitted. ☐

Proposition 694. Let X be a topological space. Then $\mathsf{fr}_X\, \emptyset = \emptyset$.

Proof. Follows by set extensionality. ☐

Proposition 695. Let X be a topological space. Then $\mathsf{fr}_X\, X = \emptyset$.

Proof. $\mathsf{fr}_X\, X = X \setminus X$ by definition [692] and propositions [680] and [690]. Follows by proposition [111]. ☐

Definition 696. $\mathsf{N}_X x = \{U \in \mathcal{O}_X \mid x \in U\}$.

14.2 Topological basis

Abbreviation 697. C covers X iff for all $x \in X$ there exists $U \in C$ such that $x \in U$.

Proposition 698. Suppose C covers X. Then $X \subseteq \bigcup C$.

Proposition 699. Suppose $X \subseteq \bigcup C$. Then C covers X.

Abbreviation 700. B is a topological prebasis for X iff $\bigcup B = X$.

Proposition 701. B is a topological prebasis for X iff B is a family of subsets of X and B covers X.

Proof. If B is a family of subsets of X and B covers X, then $\bigcup B = X$ by propositions [8], [45] and [698]. If $\bigcup B = X$, then B is a family of subsets of X and B covers X by propositions [7], [698] and [699]. ☐

Definition 702. B is a topological basis for X iff B is a topological prebasis for X and for all U, V, x such that $U, V \in B$ and $x \in U, V$ there exists $W \in B$ such that $x \in W \subseteq U, V$.

14.3 Disconnections

Definition 703. $\mathsf{Disconnections}\,X = \{p \in \mathsf{Bipartitions}\,X \mid \mathsf{fst}\,p, \mathsf{snd}\,p \in \mathcal{O}_X\}$.

Abbreviation 704. D is a disconnection of X iff $D \in \mathsf{Disconnections}\,X$.

Definition 705. X is disconnected iff there exist $U, V \in \mathcal{O}_X$ such that X is partitioned by U and V.

Proposition 706. Let X be a topological space. Suppose X is disconnected. Then there exists a disconnection of X.

Proof. Take $U, V \in \mathcal{O}_X$ such that X is partitioned by U and V by definition [705]. Then (U, V) is a bipartition of X. Thus (U, V) is a disconnection of X by definition [703] and propositions [143] and [150]. ☐

Proposition 707. Let X be a topological space. Let D be a disconnection of X. Then X is disconnected.

Proof. $\mathsf{fst}\,D, \mathsf{snd}\,D \in \mathcal{O}_X$. X is partitioned by $\mathsf{fst}\,D$ and $\mathsf{snd}\,D$. ☐

Abbreviation 708. X is connected iff X is not disconnected.