Contents

1 Sets

Abbreviation 1. $A \ni a$ iff $a \in A$.

1.1 Extensionality

The axiom of set extensionality says that sets are determined by their *extension*, that is, two sets are equal iff they have the same elements.

Axiom 2. **(Set extensionality)** Suppose for all a we have $a \in A$ iff $a \in B$. Then $A = B$.

This axiom is also available as the justification "... by set extensionality", which applies it to goals of the form "$A = B$" and "$A \neq B$".

Proposition 3. **(Witness for disequality)** Suppose $A \neq B$. Then there exists c such that either $c \in A$ and $c \notin B$ or $c \notin A$ and $c \in B$.

Proof. Suppose not. Then $A = B$ by set extensionality. Contradiction. □

1.2 Subsets

Definition 4. $A \subseteq B$ iff for all $a \in A$ we have $a \in B$.

Abbreviation 5. A is a subset of B iff $A \subseteq B$.

Abbreviation 6. $B \supseteq A$ iff $A \subseteq B$.

Proposition 7. $A \subseteq A$.

Proposition 8. Suppose $A \subseteq B \subseteq A$. Then $A = B$.

Proof. Follows by set extensionality. □

Proposition 9. Suppose $a \in A \subseteq B$. Then $a \in B$.

Proposition 10. Suppose $A \subseteq B$ and $c \notin B$. Then $c \notin A$.

Proposition 11. Suppose $A \subseteq B \subseteq C$. Then $A \subseteq C$.

Definition 12. $A \subset B$ iff $A \subseteq B$ and $A \neq B$.

Proposition 13. $A \not\subset A$.

Proposition 14. Suppose $A \subseteq B \subseteq C$. Then $A \subseteq C$.

Proposition 15. Suppose $A \subset B$. Then there exists $b \in B$ such that $b \notin A$.

Proof. $A \subseteq B$ and $A \neq B$. □

Abbreviation 16. F is a family of subsets of X iff for all $A \in F$ we have $A \subseteq X$.

1.3 The empty set

Axiom 17. For all a we have $a \notin \emptyset$.

Definition 18. A is inhabited iff there exists a such that $a \in A$.

Abbreviation 19. A is empty iff A is not inhabited.

Proposition 20. If x and y are empty, then $x = y$.

Proposition 21. For all a we have $\emptyset \subseteq a$.

Proposition 22. $A \subseteq \emptyset$ iff $A = \emptyset$.

1.4 Disjointness of sets

Definition 23. A is disjoint from B iff there exists no a such that $a \in A, B$.

Abbreviation 24. $A \not\between B$ iff A is disjoint from B.

Abbreviation 25. $A \between B$ iff A is not disjoint from B.

Proposition 26. If A is disjoint from B, then B is disjoint from A.

1.5 Unordered pairing and set adjunction

Finite set expressions are desugared to iterated application of cons to $\emptyset$. Thus $\{x, y, z\}$ is an abbreviaton of $\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \emptyset)))$. The cons operation is determined by the following axiom:

Axiom 27. $x \in \mathsf{cons}(y, X)$ iff $x = y$ or $x \in X$.

Proposition 28. $x \in \mathsf{cons}(x, X)$.

Proposition 29. If $y \in X$, then $y \in \mathsf{cons}(x, X)$.

Proposition 30. $a \in \{a, b\}$.

Proposition 31. $b \in \{a, b\}$.

Proposition 32. Suppose $c \in \{a, b\}$. Then $a = c$ or $b = c$.

Proposition 33. $c \in \{a, b\}$ iff $a = c$ or $b = c$.

Proposition 34. $a \in \{a\}$.

Proposition 35. If $a \in \{b\}$, then $a = b$.

Proposition 36. $a \in \{b\}$ iff $a = b$.

Abbreviation 37. A is a subsingleton iff for all $a, b \in A$ we have $a = b$.

Proposition 38. $\{a\}$ is inhabited.

Proposition 39. Let A be a subsingleton. Let $a \in A$. Then $A = \{a\}$.

Proof. Follows by set extensionality. □

Proposition 40. Suppose $a \in C$. Then $\{a\} \subseteq C$.

Proposition 41. Suppose $\{a\} \subseteq C$. Then $a \in C$.

1.6 Union and intersection

1.6.1 Union of a set

Axiom 42. $z \in \bigcup X$ iff there exists $Y \in X$ such that $z \in Y$.

Proposition 43. Suppose $A \in B \in C$. Then $A \in \bigcup C$.

Proof. There exists $B \in C$ such that $A \in B$. □

Proposition 44. $\bigcup \emptyset = \emptyset$.

Proposition 45. Let F be a family of subsets of X. Then $\bigcup F \subseteq X$.

Abbreviation 46. T is closed under arbitrary unions iff for every subset M of T we have $\bigcup M \in T$.

1.6.2 Intersection of a set

Definition 47. $\bigcap A = \{x \in \bigcup A \mid \text{for all } a \in A \text{ we have } x \in a\}$.

Proposition 48. $z \in \bigcap X$ iff X is inhabited and for all $Y \in X$ we have $z \in Y$.

Proposition 49. Suppose C is inhabited. Suppose for all $B \in C$ we have $A \in B$. Then $A \in \bigcap C$.

Proposition 50. Suppose $A \in \bigcap C$. Suppose $B \in C$. Then $A \in B$.

Proposition 51. Suppose A is inhabited. Suppose for all $a \in A$ we have $C \subseteq a$. Then $C \subseteq \bigcap A$.

Proposition 52. Suppose A is inhabited. Then $C \subseteq \bigcap A$ iff for all $a \in A$ we have $C \subseteq a$.

Proposition 53. Let $B \in A$. Then $\bigcap A \subseteq B$.

Proposition 54. $\bigcap \{a\} = a$.

Proof. Every element of a is an element of $\bigcap \{a\}$ by propositions [36], [38] and [48]. Follows by set extensionality. □

Proposition 55. $\bigcap \{\emptyset\} = \emptyset$.

Proof. Follows by set extensionality. □

1.6.3 Binary union

Axiom 56. Let A, B be sets. $a \in A \cup B$ iff $a \in A$ or $a \in B$.

Proposition 57. If $c \in A$, then $c \in A \cup B$.

Proposition 58. If $c \in B$, then $c \in A \cup B$.

Proposition 59. $\bigcup \{x, y\} = x \cup y$.

Proof. Follows by set extensionality. □

Proposition 60. **(Commutativity of union)** $A \cup B = B \cup A$.

Proof. Follows by set extensionality. $\square$

Proposition 61. **(Associativity of union)** $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof. Follows by set extensionality. $\square$

Proposition 62. **(Idempotence of union)** $A \cup A = A$.

Proof. Follows by set extensionality. $\square$

Proposition 63. $A \cup B \subseteq C$ iff $A \subseteq C$ and $B \subseteq C$.

Proposition 64. $A \subseteq A \cup B$.

Proposition 65. $B \subseteq A \cup B$.

Proposition 66. Suppose $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

Proposition 67. $A \cup \emptyset = A$.

Proof. Follows by set extensionality. $\square$

Proposition 68. Suppose $A = \emptyset$ and $B = \emptyset$. Then $A \cup B = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 69. Suppose $A \cup B = \emptyset$. Then $A = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 70. Suppose $A \cup B = \emptyset$. Then $B = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 71. Suppose $A \subseteq B$. Then $A \cup B = B$.

Proof. Follows by set extensionality. $\square$

Proposition 72. Suppose $A \subseteq B$. Then $B \cup A = B$.

Proof. Follows by set extensionality. $\square$

Proposition 73. If $A \cup B = B$, then $A \subseteq B$.

Proposition 74. $\bigcup \mathsf{cons}(b, A) = b \cup \bigcup A$.

Proof. Follows by set extensionality. $\square$

Proposition 75. $\mathsf{cons}(b, A) \cup C = \mathsf{cons}(b, A \cup C)$.

Proof. Follows by set extensionality. $\square$

Proposition 76. $A \cup (A \cup B) = A \cup B$.

Proof. Follows by set extensionality. □

Proposition 77. $(A \cup B) \cup B = A \cup B$.

Proof. Follows by set extensionality. □

Proposition 78. $A \cup (B \cup C) = B \cup (A \cup C)$.

Proof. Follows by set extensionality. □

Abbreviation 79. T is closed under binary unions iff for every $U, V \in T$ we have $U \cup V \in T$.

1.6.4 Binary intersection

Definition 80. $A \cap B = \{a \in A \mid a \in B\}$.

Proposition 81. If $c \in A, B$, then $c \in A \cap B$.

Proposition 82. If $c \in A \cap B$, then $c \in A$.

Proposition 83. If $c \in A \cap B$, then $c \in B$.

Proposition 84. $\bigcap\{A, B\} = A \cap B$.

Proof. $\{A, B\}$ is inhabited. Thus for all c we have $c \in \bigcap\{A, B\}$ iff $c \in A \cap B$ by propositions [33] and [48] and definition [80]. Follows by extensionality. □

Proposition 85. (Commutativity of intersection) $A \cap B = B \cap A$.

Proof. Follows by set extensionality. □

Proposition 86. (Associativity of intersection) $(A \cap B) \cap C = A \cap (B \cap C)$.

Proof. Follows by set extensionality. □

Proposition 87. (Idempotence of intersection) $A \cap A = A$.

Proof. Follows by set extensionality. □

Proposition 88. $A \cap B \subseteq A$.

Proposition 89. $A \cap \emptyset = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 90. Suppose $A \subseteq B$. Then $A \cap B = A$.

Proof. Follows by set extensionality. □

Proposition 91. Suppose $A \subseteq B$. Then $B \cap A = A$.

Proof. Follows by set extensionality. □

Proposition 92. Suppose $A \cap B = A$. Then $A \subseteq B$.

Proposition 93. $C \subseteq A \cap B$ iff $C \subseteq A$ and $C \subseteq B$.

Proposition 94. $A \cap B \subseteq A$.

Proposition 95. $A \cap B \subseteq B$.

Proposition 96. $A \cap (A \cap B) = A \cap B$.

Proof. Follows by set extensionality. □

Proposition 97. $(A \cap B) \cap B = A \cap B$.

Proof. Follows by set extensionality. □

Proposition 98. $A \cap (B \cap C) = B \cap (A \cap C)$.

Proof. Follows by set extensionality. □

Abbreviation 99. T is closed under binary intersections iff for every $U, V \in T$ we have $U \cap V \in T$.

1.6.5 Interaction of union and intersection

Proposition 100. (Binary intersection over binary union) $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

Proof. Follows by set extensionality. □

Proposition 101. (Binary union over binary intersection) $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

Proof. Follows by set extensionality. □

Proposition 102. Suppose $C \subseteq A$. Then $(A \cap B) \cup C = A \cap (B \cup C)$.

Proof. Follows by set extensionality. □

Proposition 103. Suppose $(A \cap B) \cup C = A \cap (B \cup C)$. Then $C \subseteq A$.

Proposition 104. $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$.

Proof. Follows by set extensionality. □

Proposition 105. (Intersection over binary union) Suppose A and B are inhabited. Then $\bigcap A \cup B = (\bigcap A) \cap \bigcap B$.

Proof. $A \cup B$ is inhabited. Thus for all c we have $c \in \bigcap A \cup B$ iff $c \in (\bigcap A) \cap \bigcap B$ by definition [80], axiom [56], and proposition [48]. Follows by set extensionality. □

1.7 Set difference

Definition 106. $A \setminus B = \{a \in A \mid a \notin B\}$.

Proposition 107. If $a \in A$ and $a \notin B$, then $a \in A \setminus B$.

Proposition 108. If $a \in A \setminus B$, then $a \in A$.

Proposition 109. If $a \in A \setminus B$, then $a \notin B$.

Proposition 110. $x \setminus \emptyset = x$.

Proof. Follows by set extensionality. □

Proposition 111. $\emptyset \setminus x = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 112. $x \setminus x = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 113. $x \setminus (x \setminus y) = x \cap y$.

Proof. Follows by set extensionality. □

Proposition 114. Suppose $y \subseteq x$. $x \setminus (x \setminus y) = y$.

Proof. Follows by propositions [91] and [113]. □

Proposition 115. $x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z)$.

Proof. Follows by set extensionality. □

Proposition 116. $x \setminus (y \cup z) = (x \setminus y) \cap (x \setminus z)$.

Proof. Follows by set extensionality. □

Proposition 117. $x \cap (y \setminus z) = (x \cap y) \setminus (x \cap z)$.

Proof. Follows by set extensionality. □

Proposition 118. Let A, B be sets. Suppose $A \subset B$. Then $B \setminus A$ is inhabited.

Proof. Take b such that $b \in B$ and $b \notin A$. Then $b \in B \setminus A$. □

Proposition 119. $B \setminus A \subseteq B$.

Proposition 120. Suppose $C \subseteq A$. Suppose $C \cap B = \emptyset$. Then $C \subseteq A \setminus B$.

Proposition 121. Suppose $A \subseteq B$. Then $C \setminus A \supseteq C \setminus B$.

Proposition 122. Suppose $A \cap B = \emptyset$. Then $A \setminus B = A$.

Proposition 123. $A \setminus B = \emptyset$ iff $A \subseteq B$.

Proposition 124. Suppose $B \subseteq A \setminus C$ and $c \notin B$. Then $B \subseteq A \setminus \mathsf{cons}(c, C)$.

Proposition 125. Suppose $B \subseteq A \setminus \mathsf{cons}(c, C)$. Then $B \subseteq A \setminus C$ and $c \notin B$.

Proposition 126. $A \setminus \mathsf{cons}(a, B) = (A \setminus \{a\}) \setminus B$.

Proof. Follows by set extensionality. □

Proposition 127. $A \setminus \mathsf{cons}(a, B) = (A \setminus B) \setminus \{a\}$.

Proof. Follows by set extensionality. □

Proposition 128. $A \cap (B \setminus A) = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 129. Suppose $A \subseteq B$. $A \cup (B \setminus A) = B$.

Proof. Follows by set extensionality. □

Proposition 130. $A \subseteq B \cup (A \setminus B)$.

Proposition 131. Suppose $A \subseteq B \subseteq C$. Then $B \setminus (C \setminus A) = A$.

Proof. Follows by set extensionality. □

Proposition 132. Then $(A \cup B) \setminus (B \setminus A) = A$.

Proof. Follows by set extensionality. □

Proposition 133. Suppose $A, B \subseteq C$. Then $A \setminus B = A \cap (C \setminus B)$.

Proof. Follows by set extensionality. □

1.8 Tuples

As with unordered pairs, orderd pairs are a primitive construct and n-tuples desugar to iterated applications of the primitive operator $(-,-)$. For example (a, b, c, d) equals $(a, (b, (c, d)))$ by definition. While ordered pairs could be encoded set-theoretically, we simply postulate the defining property to prevent misguiding proof automation.

Axiom 134. $(a, b) = (a', b')$ iff $a = a' \wedge b = b'$.

Axiom 135. $(a, b) \neq \emptyset$.

Axiom 136. $(a, b) \neq a$.

Axiom 137. $(a, b) \neq b$.

Repeated application of the defining property of pairs yields the defining property of all tuples.

Proposition 138. $(a, b, c) = (a', b', c')$ iff $a = a' \wedge b = b' \wedge c = c'$.

There are primitive projections fst and snd that satisfy the following axioms.

Axiom 139. $\mathsf{fst}(a, b) = a$.

Axiom 140. $\mathsf{snd}(a, b) = b$.

Proposition 141. $(a, b) = (\mathsf{fst}(a, b), \mathsf{snd}(a, b))$.

Definition 142. $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Proposition 143. Suppose $(x, y) \in X \times Y$. Then $x \in X$ and $y \in Y$.

Proof. Take x', y' such that $x' \in X \wedge y' \in Y \wedge (x, y) = (x', y')$ by definition [142]. Then $x = x'$ and $y = y'$ by axiom [134]. ☐

Proposition 144. Suppose $x \in X$ and $y \in Y$. Then $(x, y) \in X \times Y$.

Proposition 145. $\emptyset \times Y = \emptyset$.

Proposition 146. $X \times \emptyset = \emptyset$.

Proposition 147. $X \times Y$ is empty iff X is empty or Y is empty.

Proof. Follows by definitions [18] and [142]. ☐

Proposition 148. Suppose $c \in A \times B$. Then $\mathsf{fst}\, c \in A$.

Proof. Take a, b such that $c = (a, b)$ and $a \in A$ by definition [142]. $a = \mathsf{fst}\, c$ by axiom [139]. ☐

Proposition 149. Suppose $c \in A \times B$. Then $\mathsf{snd}\, c \in B$.

Proof. Take a, b such that $c = (a, b)$ and $b \in B$ by definition [142]. $b = \mathsf{snd}\, c$ by axiom [140]. ☐

Proposition 150. Suppose $p \in X \times Y$. Then there exist x, y such that $x \in X$ and $y \in Y$ and $p = (x, y)$.

Proposition 151. Suppose $p \in X \times Y$. Then $\mathsf{fst}\, p \in X$ and $\mathsf{snd}\, p \in Y$.

1.9 Additional results about cons

Proposition 152. Suppose $x \in X$. Suppose $Y \subseteq X$. Then $\mathsf{cons}(x, Y) \subseteq X$.

Proposition 153. Suppose $\mathsf{cons}(x, Y) \subseteq X$. Then $x \in X$ and $Y \subseteq X$.

Proposition 154. $\mathsf{cons}(x, Y) \subseteq X$ iff $x \in X$ and $Y \subseteq X$.

Proposition 155. If $C \subseteq B$, then $C \subseteq \mathsf{cons}(a, B)$.

Corollary 156. $X \subseteq \mathsf{cons}(y, X)$.

Abbreviation 157. $B \setminus \{a\} = B \setminus \{a\}$.

Proposition 158. Suppose $a \in C \wedge C \setminus \{a\} \subseteq B$. Then $C \subseteq \mathsf{cons}(a, B)$.

Proof. Follows by propositions [123] and [126]. ☐

Proposition 159. Suppose $C \subseteq B$. Then $C \subseteq \mathsf{cons}(a, B)$.

Proposition 160. Suppose $C \subseteq \mathsf{cons}(a, B)$. Then $C \subseteq B \vee (a \in C \wedge C \setminus \{a\} \subseteq B)$.

Proof. Follows by propositions [123] and [126], definition [4], and axiom [27]. ☐

Proposition 161. $C \subseteq \mathsf{cons}(a, B)$ iff $C \subseteq B \vee (a \in C \wedge C \setminus \{a\} \subseteq B)$.

Proposition 162. $B \setminus \{a\} = B \setminus \{a\}$.

Proof. Follows by set extensionality. ☐

Proposition 163. $\{a\} \cup B = \mathsf{cons}(a, B)$.

Proof. Follows by set extensionality. ☐

Proposition 164. $\mathsf{cons}(a, \mathsf{cons}(b, C)) = \mathsf{cons}(b, \mathsf{cons}(a, C))$.

Proof. Follows by set extensionality. ☐

Proposition 165. Suppose $a \in A$. Then $\mathsf{cons}(a, A) = A$.

Proof. Follows by set extensionality. ☐

Proposition 166. Suppose $a \in A$. Then $\mathsf{cons}(a, A \setminus \{a\}) = A$.

Proof. Follows by set extensionality. ☐

Proposition 167. Then $\mathsf{cons}(a, \mathsf{cons}(a, B)) = \mathsf{cons}(a, B)$.

Proof. Follows by set extensionality. ☐

Proposition 168. Suppose B is inhabited. Then $\bigcap \mathsf{cons}(a, B) = a \cap \bigcap B$.

Proof. $\mathsf{cons}(a, B)$ is inhabited. Thus for all c we have $c \in \bigcap \mathsf{cons}(a, B)$ iff $c \in a \cap \bigcap B$ by proposition [48], axiom [27], and definition [80]. Follows by extensionality. ☐

1.10 Successor

Definition 169. $x^+ = \mathsf{cons}(x, x)$.

Proposition 170. $x \in x^+$.

Proposition 171. Suppose $x \in y$. Then $x \in y^+$.

Proposition 172. Suppose $x \in y^+$. Then $x = y$ or $x \in y$.

Proposition 173. $x \in y^+$ iff $x = y$ or $x \in y$.

Proposition 174. $x^+ \neq \emptyset$.

Proposition 175. Suppose $x^+ \subseteq y$. Then $x \in y$.

Proposition 176. $x^+ \neq x$.

Proposition 177. Suppose $x^+ = y^+$. Then $x = y$.

Proof. Suppose not. $x^+ \subseteq y^+$. Hence $x \in y^+$. Then $x \in y$. $y^+ \subseteq x^+$. Hence $y \in x^+$. Then $y \in x$. Contradiction. ☐

Proposition 178. $x \subseteq x^+$.

Proposition 179. Suppose $x \in y$ and $x \subseteq y$. Then $x^+ \subseteq y$.

Proposition 180. Suppose $x^+ \subseteq y$. Then $x \in y$ and $x \subseteq y$.

Proposition 181. There exists no z such that $x \subset z \subset x^+$.

Proof. Follows by definitions [4], [12] and [169] and propositions [15] and [172]. ☐

1.11 Symmetric difference

Definition 182. $x \triangle y = (x \setminus y) \cup (y \setminus x)$.

Proposition 183. $x \triangle y = (x \cup y) \setminus (y \cap x)$.

Proof. Follows by set extensionality. ☐

Proposition 184. If $z \in x \triangle y$, then either $z \in x$ or $z \in y$.

Proposition 185. If either $z \in x$ or $z \in y$, then $z \in x \triangle y$.

Proof. If $z \in x$ and $z \notin y$, then $z \in x \setminus y$. If $z \notin x$ and $z \in y$, then $z \in y \setminus x$. ☐

Proposition 186. $x \triangle (y \triangle z) = (x \triangle y) \triangle z$.

Proof. Follows by set extensionality. ☐

Proposition 187. $x \triangle y = y \triangle x$.

Proof. Follows by set extensionality. ☐

Proposition 188. Suppose $A \subseteq C$. Then $A \times B \subseteq C \times B$.

Proof. It suffices to show that for all $w \in A \times B$ we have $w \in C \times B$. ☐

Proposition 189. Suppose $B \subseteq D$. Then $A \times B \subseteq A \times D$.

Proof. It suffices to show that for all $w \in A \times B$ we have $w \in A \times D$. ☐

Proposition 190. Suppose $w \in (A \cap B) \times (C \cap D)$. Then $w \in (A \times C) \cap (B \times D)$.

Proof. Take a, c such that $w = (a, c)$ by proposition [150]. Then $a \in A, B$ and $c \in C, D$ by proposition [143] and definition [80]. Thus $w \in (A \times C), (B \times D)$. ☐

Proposition 191. Suppose $w \in (A \times C) \cap (B \times D)$. Then $w \in (A \cap B) \times (C \cap D)$.

Proof. $w \in A \times C$. Take a, c such that $w = (a, c)$. $a \in A, B$ by definition [80] and proposition [143]. $c \in C, D$ by definition [80] and proposition [143]. Thus $(a, c) \in (A \cap B) \times (C \cap D)$ by definition [142] and proposition [81]. ☐

Proposition 192. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Proof. Follows by set extensionality. ☐

Proposition 193. $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$.

Proof. Follows by set extensionality. $\square$

Proposition 194. $X \times (Y \cap Z) = (X \times Y) \cap (X \times Z)$.

Proof. Follows by set extensionality. $\square$

Proposition 195. Suppose $w \in (A \cup B) \times (C \cup D)$. Then $w \in (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$.

Proof. Take a, c such that $w = (a, c)$. $a \in A$ or $a \in B$ by axiom [56] and proposition [143]. $c \in C$ or $c \in D$ by axiom [56] and proposition [143]. Thus $(a, c) \in (A \times C)$ or $(a, c) \in (B \times D)$ or $(a, c) \in (A \times D)$ or $(a, c) \in (B \times C)$. Thus $(a, c) \in (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$. $\square$

Proposition 196. Suppose $w \in (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$. Then $w \in (A \cup B) \times (C \cup D)$.

Proof. Case: $w \in (A \times C)$. Take a, c such that $w = (a, c) \wedge a \in A \wedge c \in C$ by definition [142]. Then $a \in A \cup B$ and $c \in C \cup D$. Follows by proposition [144].

Case: $w \in (B \times D)$. Take b, d such that $w = (b, d) \wedge b \in B \wedge d \in D$ by definition [142]. Then $b \in A \cup B$ and $d \in C \cup D$. Follows by proposition [144].

Case: $w \in (A \times D)$. Take a, d such that $w = (a, d) \wedge a \in A \wedge d \in D$ by definition [142]. Then $a \in A \cup B$ and $d \in C \cup D$. Follows by proposition [144].

Case: $w \in (B \times C)$. Take b, c such that $w = (b, c) \wedge b \in B \wedge c \in C$ by definition [142]. Then $b \in A \cup B$ and $c \in C \cup D$. Follows by proposition [144]. $\square$

Proposition 197. $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D) \cup (A \times D) \cup (B \times C)$.

Proof. Follows by set extensionality. $\square$

Proposition 198. $(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z)$.

Proof. Follows by set extensionality. $\square$

Proposition 199. $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

Proof. Follows by set extensionality. $\square$

1.12 Powerset

Abbreviation 200. The powerset of X denotes $\mathsf{Pow}(X)$.

Axiom 201. $B \in \mathsf{Pow}(A)$ iff $B \subseteq A$.

Proposition 202. Suppose $A \subseteq B$. Then $A \in \mathsf{Pow}(B)$.

Proposition 203. Let $A \in \mathsf{Pow}(B)$. Then $A \subseteq B$.

Proposition 204. $\emptyset \in \mathsf{Pow}(A)$.

Proposition 205. $A \in \mathsf{Pow}(A)$.

Proposition 206. Let A be a set. Let B be a subset of $\mathsf{Pow}(A)$. Then $\bigcup B \subseteq A$.

Proof. Follows by definition [4], proposition [203], and axiom [42]. □

Corollary 207. Let A be a set. Let B be a subset of $\mathsf{Pow}(A)$. Then $\bigcup B \in \mathsf{Pow}(A)$.

Proof. Follows by axiom [201] and proposition [206]. □

Proposition 208. $\bigcup \mathsf{Pow}(A) = A$.

Proof. Follows by set extensionality. □

Proposition 209. $\bigcap \mathsf{Pow}(A) = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 210. $\mathsf{Pow}(A) \cup \mathsf{Pow}(B) \subseteq \mathsf{Pow}(A \cup B)$.

Proof. $\mathsf{Pow}(A) \subseteq \mathsf{Pow}(A) \cup \mathsf{Pow}(B)$ by proposition [64]. $\mathsf{Pow}(B) \subseteq \mathsf{Pow}(A) \cup \mathsf{Pow}(B)$ by proposition [65]. Follows by definition [4], axioms [56] and [201], and propositions [14] and [203]. □

Proposition 211. $\mathsf{Pow}(\emptyset) = \{\emptyset\}$.

Proposition 212. $\mathsf{Pow}(A) \cup \mathsf{Pow}(B) \subseteq \mathsf{Pow}(A \cup B)$.

Proposition 213. $A \subseteq \mathsf{Pow}(\bigcup A)$.

Proof. Follows by definition [4], axiom [201], and proposition [43]. □

Proposition 214. $\bigcup \mathsf{Pow}(A) = A$.

Proposition 215. $\bigcup A \in \mathsf{Pow}(B)$ iff $A \in \mathsf{Pow}(\mathsf{Pow}(B))$.

Proposition 216. $\mathsf{Pow}(A \cap B) = \mathsf{Pow}(A) \cap \mathsf{Pow}(B)$.

Proof. Follows by axioms [2] and [201], definition [80], and proposition [93]. □

1.13 Bipartitions

Abbreviation 217. C is partitioned by A and B iff $A, B \neq \emptyset$ and A is disjoint from B and $A \cup B = C$.

Definition 218. $\mathsf{Bipartitions}\, X = \{p \in \mathsf{Pow}(X) \times \mathsf{Pow}(X) \mid X$ is partitioned by $\mathsf{fst}\, p$ and $\mathsf{snd}\, p\}$.

Abbreviation 219. P is a bipartition of X iff $P \in \mathsf{Bipartitions}\, X$.

Proposition 220. Suppose C is partitioned by A and B. Then (A, B) is a bipartition of C.

Proof. $(A, B) \in \mathsf{Pow}(C) \times \mathsf{Pow}(C)$. C is partitioned by $\mathsf{fst}(A, B)$ and $\mathsf{snd}(A, B)$. Thus (A, B) is a bipartition of C by definition [218]. □

Proposition 221. Suppose (A, B) is a bipartition of C. Then C is partitioned by A and B.

Proof. $\mathsf{fst}(A, B) = A$. $\mathsf{snd}(A, B) = B$. □

Proposition 222. $\mathsf{Bipartitions}\, \emptyset$ is empty.

Proposition 223. Suppose $d \notin C$. Suppose $A \cup B = \mathsf{cons}(d, C)$. Suppose $A, B \neq \{d\}$. Then $A \setminus \{d\} \cup B \setminus \{d\} = C$.

Proof. Follows by set extensionality. □

Proposition 224. Suppose $d \notin C$. Suppose $\mathsf{cons}(d, C)$ is partitioned by A and B. Suppose $A, B \neq \{d\}$. Then C is partitioned by $A \setminus \{d\}$ and $B \setminus \{d\}$.

Proof. $A \setminus \{d\}, B \setminus \{d\} \neq \emptyset$. $A \setminus \{d\} \cup B \setminus \{d\} = C$ by proposition [223]. □

1.14 Partitions

Definition 225. P is a partition iff $\emptyset \notin P$ and for all $B, C \in P$ such that $B \neq C$ we have B is disjoint from C.

Abbreviation 226. P is a partition of A iff P is a partition and $\bigcup P = A$.

Proposition 227. $\emptyset$ is a partition of $\emptyset$.

Definition 228. P' is a refinement of P iff for every $A' \in P'$ there exists $A \in P$ such that $A' \subseteq A$.

Abbreviation 229. $P' \leq P$ iff P' is a refinement of P.

Proposition 230. Suppose $P'' \leq P' \leq P$. Then $P'' \leq P$.

Proof. It suffices to show that for all $A'' \in P''$ there exists $A \in P$ such that $A'' \subseteq A$. Fix $A'' \in P''$. Take $A' \in P'$ such that $A'' \subseteq A'$ by definition [228]. Take $A \in P$ such that $A' \subseteq A$. Then $A'' \subseteq A$. Follows by definition. □

1.15 Cantor's theorem

Theorem 231. **(Cantor)** There exists no surjection from A to $\mathsf{Pow}(A)$.

Proof. Suppose not. Consider a surjection f from A to $\mathsf{Pow}(A)$. Let $B = \{a \in A \mid a \notin f(a)\}$. Then $B \in \mathsf{Pow}(A)$. There exists $a' \in A$ such that $f(a') = B$ by the definition of surjectivity. Now $a' \in B$ iff $a' \notin f(a') = B$. Contradiction. □

2 Filters

Abbreviation 232. F is upward-closed in S iff for all A, B such that $A \subseteq B \subseteq S$ and $A \in F$ we have $B \in F$.

Definition 233. F is a filter on S iff F is a family of subsets of S and S is inhabited and $S \in F$ and $\emptyset \notin F$ and F is closed under binary intersections and F is upward-closed in S.

Definition 234. $\uparrow_S A = \{X \in \mathsf{Pow}(S) \mid A \subseteq X\}$.

Proposition 235. Suppose $A \subseteq S$. Suppose A is inhabited. Then $\uparrow_S A$ is a filter on S.

Proof. S is inhabited. $\uparrow_S A$ is a family of subsets of S. $S \in \uparrow_S A$. $\emptyset \notin \uparrow_S A$. $\uparrow_S A$ is closed under binary intersections. $\uparrow_S A$ is upward-closed in S. Follows by definition [233]. □

Proposition 236. Suppose $A \subseteq S$. $A \in \uparrow_S A$.

Proof. $A \in \mathsf{Pow}(S)$. □

Proposition 237. Let $X \in \mathsf{Pow}(S)$. Suppose $X \notin \uparrow_S A$. Then $A \nsubseteq X$.

Proof. □

Definition 238. F is a maximal filter on S iff F is a filter on S and there exists no filter F' on S such that $F \subset F'$.

Proposition 239. Suppose $a \in S$. Then $\uparrow_S \{a\}$ is a filter on S.

Proof. $\{a\} \subseteq S$. $\{a\}$ is inhabited. Follows by proposition [235]. □

Proposition 240. Suppose $a \in S$. Then $\uparrow_S \{a\}$ is a maximal filter on S.

Proof. $\{a\} \subseteq S$. $\{a\}$ is inhabited. Thus $\uparrow_S \{a\}$ is a filter on S by proposition [235]. It suffices to show that there exists no filter F' on S such that $\uparrow_S \{a\} \subset F'$. Suppose not. Take a filter F' on S such that $\uparrow_S \{a\} \subset F'$. Take $X \in F'$ such that $X \notin \uparrow_S \{a\}$. $X \in \mathsf{Pow}(S)$. Thus $\{a\} \nsubseteq X$ by proposition [237]. Thus $a \notin X$. $\{a\} \in F'$ by definitions [12] and [234] and propositions [7], [9], [57], [71] and [202]. Thus $\emptyset = X \cap \{a\}$. Hence $\emptyset \in F'$ by definition [233]. Follows by contradiction to the definition of a filter. □

3 Regularity

Abbreviation 241. a is an $\in$-minimal element of A iff $a \in A$ and $a \not\between A$.

Lemma 242. For all a, A such that $a \in A$ there exists $b \in A$ such that $b \not\between A$.

Proof by $\in$-induction on a. Case: $a \not\between b$. Straightforward.

Case: $a \between b$. Take a' such that $a' \in a, b$. Straightforward. ☐

Proposition 243. **(Regularity)** Let A be an inhabited set. Then there exists a $\in$-minimal element of A.

Proof. Follows by lemma [242] and definition [18]. ☐

Theorem 244. **(Foundation)** Let A be a set. Then $A = \emptyset$ or there exists $a \in A$ such that for all $x \in a$ we have $x \notin A$.

Proof. Case: $A = \emptyset$. Straightforward.

Case: A is inhabited. Take a such that a is a $\in$-minimal element of A. Then for all $x \in a$ we have $x \notin A$. ☐

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Proposition 245. For all sets A we have $A \notin A$.

Proof by $\in$-induction. Straightforward. ☐

Proposition 246. If $a \in A$, then $a \neq A$.

Proposition 247. For all sets a, b such that $a \in b$ we have $b \notin a$.

Proof by $\in$-induction on a. Straightforward. ☐

3.1 Fixpoints

Definition 248. a is a fixpoint of f iff $a \in \mathsf{dom}\, f$ and $f(a) = a$.

Definition 249. f is $\subseteq$-preserving iff for all $A, B \in \mathsf{dom}\, f$ such that $A \subseteq B$ we have $f(A) \subseteq f(B)$.

Theorem 250. **(Knaster–Tarski)** Let f be a $\subseteq$-preserving function from $\mathsf{Pow}(A)$ to $\mathsf{Pow}(A)$. Then there exists a fixpoint of f.

Proof. $\mathsf{dom}\, f = \mathsf{Pow}(A)$. Let $P = \{a \in \mathsf{Pow}(A) \mid a \subseteq f(a)\}$. $P \subseteq \mathsf{Pow}(A)$. Thus $\bigcup P \in \mathsf{Pow}(A)$. Hence $f(\bigcup P) \in \mathsf{Pow}(A)$ by proposition [501].

Show $\bigcup P \subseteq f(\bigcup P)$. *Subproof.* It suffices to show that every element of $\bigcup P$ is an element of $f(\bigcup P)$. Fix $u \in \bigcup P$. Take $p \in P$ such that $u \in p$. Then $u \in f(p)$. $p \subseteq \bigcup P$. $f(p) \subseteq f(\bigcup P)$ by definition [249]. Thus $u \in f(\bigcup P)$. ☐

Now $f(\bigcup P) \subseteq f(f(\bigcup P))$ by definition [249]. Thus $f(\bigcup P) \in P$ by definition.

Hence $f(\bigcup P) \subseteq \bigcup P$.

Thus $f(\bigcup P) = \bigcup P$ by proposition [8]. Follows by definition [248]. ☐

4 Relations

Definition 251. R is a relation iff for all $w \in R$ there exists x, y such that $w = (x, y)$.

Definition 252. a is comparable with b in R iff $a \mathrel{R} b$ or $b \mathrel{R} a$.

Proposition 253. Let R, S be relations. Suppose for all x, y we have $x \mathrel{R} y$ iff $x \mathrel{S} y$. Then $R = S$.

Proof. Follows by set extensionality. □

Abbreviation 254. F is a family of relations iff every element of F is a relation.

Proposition 255. Let F be a family of relations. Then $\bigcup F$ is a relation.

Proposition 256. Let F be a family of relations. Then $\bigcap F$ is a relation.

Proposition 257. Let R, S be relations. Then $R \cup S$ is a relation.

Proposition 258. Suppose $R \subseteq A \times B$. Suppose $S \subseteq C \times D$. Then $R \cup S \subseteq (A \cup C) \times (B \cup D)$.

Proof. Follows by definition [4], propositions [66] and [196], and axiom [56]. □

Proposition 259. Let R, S be relations. Then $R \cap S$ is a relation.

Proposition 260. Let R, S be relations. Then $R \setminus S$ is a relation.

4.1 Converse of a relation

Definition 261. $R^{\intercal} = \{z \mid \exists w \in R. \exists x, y. w = (x, y) \wedge z = (y, x)\}$.

Proposition 262. If $y \mathrel{R} x$, then $x \mathrel{R^{\intercal}} y$.

Proposition 263. If $x \mathrel{R^{\intercal}} y$, then $y \mathrel{R} x$.

Proposition 264. $x \mathrel{R^{\intercal}} y$ iff $y \mathrel{R} x$.

Proposition 265. $R^{\intercal}$ is a relation.

Proposition 266. $x \mathrel{R^{\intercal^{\intercal}}} y$ iff $x \mathrel{R} y$.

Proposition 267. Let R be a relation. Then $R^{\intercal^{\intercal}} = R$.

Proof. Follows by set extensionality. □

Proposition 268. Suppose $R \subseteq A \times B$. Then $R^{\intercal} \subseteq B \times A$.

Proof. It suffices to show that every element of $R^{\intercal}$ is an element of $B \times A$ by definition [4]. Fix $w \in R^{\intercal}$. Take x, y such that $w = (y, x)$ and $x \mathrel{R} y$ by definition [261]. Now $(x, y) \in A \times B$ by definition [4]. Thus $x \in A$ and $y \in B$ by proposition [143]. Hence $(y, x) \in B \times A$ by proposition [144]. □

Proposition 269. Then $B \times A^{\intercal} = A \times B$.

Proof. For all w we have $w \in B \times A^{\mathsf{T}}$ iff $w \in A \times B$ by definitions [142] and [261] and propositions [143] and [150]. Follows by extensionality. ☐

Proposition 270. Then $\emptyset^{\mathsf{T}} = \emptyset$.

Proof. Follows by set extensionality. ☐

Proposition 271. Let R be a relation. If $R \subseteq S$, then $R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$.

Proof. Follows by definitions [4], [251] and [261]. ☐

Proposition 272. Let R be a relation. If $R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$, then $R \subseteq S$.

Proof. Follows by definitions [4], [251] and [261] and propositions [266] and [271]. ☐

Proposition 273. Let R be a relation. $R^{\mathsf{T}} \subseteq S^{\mathsf{T}}$ iff $R \subseteq S$.

Proof. Follows by propositions [271] and [272]. ☐

Proposition 274. $(R \cup S)^{\mathsf{T}} = R^{\mathsf{T}} \cup S^{\mathsf{T}}$.

Proof. $(R \cup S)^{\mathsf{T}}$ is a relation by proposition [265]. $R^{\mathsf{T}} \cup S^{\mathsf{T}}$ is a relation by propositions [257] and [265]. For all a, b we have $(a, b) \in (R \cup S)^{\mathsf{T}}$ iff $(a, b) \in R^{\mathsf{T}} \cup S^{\mathsf{T}}$ by axiom [56] and proposition [264]. Follows by extensionality. ☐

Proposition 275. $(R \cap S)^{\mathsf{T}} = R^{\mathsf{T}} \cap S^{\mathsf{T}}$.

Proof. $(R \cap S)^{\mathsf{T}}$ is a relation by proposition [265]. $R^{\mathsf{T}} \cap S^{\mathsf{T}}$ is a relation by propositions [259] and [265]. For all a, b we have $(a, b) \in (R \cap S)^{\mathsf{T}}$ iff $(a, b) \in R^{\mathsf{T}} \cap S^{\mathsf{T}}$ by definition [80] and proposition [264]. Follows by extensionality. ☐

Proposition 276. $(R \setminus S)^{\mathsf{T}} = R^{\mathsf{T}} \setminus S^{\mathsf{T}}$.

Proof. $(R \setminus S)^{\mathsf{T}}$ is a relation by proposition [265]. $R^{\mathsf{T}} \setminus S^{\mathsf{T}}$ is a relation by propositions [260] and [265]. For all a, b we have $(a, b) \in (R \setminus S)^{\mathsf{T}}$ iff $(a, b) \in R^{\mathsf{T}} \setminus S^{\mathsf{T}}$. Follows by extensionality. ☐

4.1.1 Domain of a relation

Definition 277. $\mathsf{dom}\, R = \{x \mid \exists w \in R. \exists y. w = (x, y)\}$.

Proposition 278. $a \in \mathsf{dom}\, R$ iff there exists b such that $a \mathrel{R} b$.

Proposition 279. Suppose $a \mathrel{R} b$. Then $a \in \mathsf{dom}\, R$.

Proof. Follows by proposition [278]. ☐

Proposition 280. $\mathsf{dom}\, \emptyset = \emptyset$.

Proof. Follows by set extensionality. ☐

Proposition 281. $\mathsf{dom}(A \times B) \subseteq A$.

Proposition 282. Suppose $b \in B$. $\mathsf{dom}(A \times B) = A$.

Proof. Follows by set extensionality. $\square$

Proposition 283. $\mathsf{dom}\,\mathsf{cons}((a,b),R) = \mathsf{cons}(a, \mathsf{dom}\,R)$.

Proof. Follows by set extensionality. $\square$

Proposition 284. $\mathsf{dom}(A \cup B) = \mathsf{dom}\,A \cup \mathsf{dom}\,B$.

Proof. Follows by set extensionality. $\square$

Proposition 285. $\mathsf{dom}(A \cap B) \subseteq \mathsf{dom}\,A \cap \mathsf{dom}\,B$.

Proof. Follows by definitions [4] and [80] and proposition [278]. $\square$

Proposition 286. $\mathsf{dom}(A \setminus B) \supseteq \mathsf{dom}\,A \setminus \mathsf{dom}\,B$.

4.1.2 Range of a relation

Definition 287. $\mathsf{ran}\,R = \{y \mid \exists w \in R. \exists x. w = (x,y)\}$.

Proposition 288. $b \in \mathsf{ran}\,R$ iff there exists a such that $a\ R\ b$.

Proposition 289. Suppose $a\ R\ b$. Then $b \in \mathsf{ran}\,R$.

Proof. Follows by proposition [288]. $\square$

Proposition 290. $\mathsf{ran}\,\emptyset = \emptyset$.

Proof. Follows by set extensionality. $\square$

Proposition 291. $\mathsf{ran}(A \times B) \subseteq B$.

Proposition 292. Suppose $a \in A$. $\mathsf{ran}(A \times B) = B$.

Proof. Follows by set extensionality. $\square$

Proposition 293. $\mathsf{ran}(\mathsf{cons}((a,b),R)) = \mathsf{cons}(b, \mathsf{ran}\,R)$.

Proof. Follows by set extensionality. $\square$

Proposition 294. $\mathsf{ran}(A \cup B) = \mathsf{ran}\,A \cup \mathsf{ran}\,B$.

Proof. Follows by set extensionality. $\square$

Proposition 295. $\mathsf{ran}(A \cap B) \subseteq \mathsf{ran}\,A \cap \mathsf{ran}\,B$.

Proof. Follows by definitions [4] and [80] and proposition [288]. $\square$

Proposition 296. $\mathsf{ran}(A \setminus B) \supseteq \mathsf{ran}\,A \setminus \mathsf{ran}\,B$.

Proof. Follows by definitions [4] and [106] and proposition [288]. $\square$

4.1.3 Domain and range of converse

Proposition 297. $\mathsf{dom}\, R^{\mathsf{T}} = \mathsf{ran}\, R$.

Proof. Follows by set extensionality. ☐

Proposition 298. $\mathsf{ran}\, R^{\mathsf{T}} = \mathsf{dom}\, R$.

Proof. Follows by set extensionality. ☐

4.1.4 Field of a relation

Definition 299. $\mathsf{field}\, R = \mathsf{dom}\, R \cup \mathsf{ran}\, R$.

Proposition 300. $c \in \mathsf{field}\, R$ iff there exists d such that $c\ R\ d$ or $d\ R\ c$.

Proof. Follows by definition [299], propositions [278] and [288], and axiom [56]. ☐

Proposition 301. Suppose $(a, b) \in R$. Then $a \in \mathsf{field}\, R$.

Proof. Follows by definitions [277] and [299] and axiom [56]. ☐

Proposition 302. Suppose $(a, b) \in R$. Then $b \in \mathsf{field}\, R$.

Proof. Follows by definitions [287] and [299] and axiom [56]. ☐

Proposition 303. Then $\mathsf{dom}\, R \subseteq \mathsf{field}\, R$.

Proof. Follows by definition [299] and proposition [64]. ☐

Proposition 304. Then $\mathsf{ran}\, R \subseteq \mathsf{field}\, R$.

Proof. Follows by definition [299] and proposition [65]. ☐

Proposition 305. $\mathsf{field}(A \times B) \subseteq A \cup B$.

Proof. Follows by definition [299] and propositions [66], [281] and [291]. ☐

Proposition 306. Let R be a relation. Suppose $w \in R$. Then $w \in \mathsf{field}\, R \times \mathsf{field}\, R$.

Proof. Take a, b such that $w = (a, b)$ by definition [251]. Then $a, b \in \mathsf{field}\, R$ by proposition [300]. Thus $(a, b) \in \mathsf{field}\, R \times \mathsf{field}\, R$ by proposition [144]. ☐

Proposition 307. Let R be a relation. Then $R \subseteq \mathsf{field}\, R \times \mathsf{field}\, R$.

Proof. Follows by proposition [306] and definition [4]. ☐

Proposition 308. $\mathsf{field}(A \times A) = A$.

Proposition 309. $\mathsf{field}\, \emptyset = \emptyset$.

Proposition 310. $\mathsf{field}(\mathsf{cons}((a, b), R)) = \mathsf{cons}(a, \mathsf{cons}(b, \mathsf{field}\, R))$.

Proposition 311. $\mathsf{field}(A \cup B) = \mathsf{field}\, A \cup \mathsf{field}\, B$.

Proof.

$$\begin{aligned}
\mathsf{field}(A \cup B) &= \mathsf{dom}(A \cup B) \cup \mathsf{ran}(A \cup B) \quad \text{[by definition [299]]} \\
&= (\mathsf{dom}\, A \cup \mathsf{dom}\, B) \cup (\mathsf{ran}\, A \cup \mathsf{ran}\, B) \quad \text{[by propositions [284] and [294]]} \\
&= (\mathsf{dom}\, A \cup \mathsf{ran}\, A) \cup (\mathsf{dom}\, B \cup \mathsf{ran}\, B) \quad \text{[by propositions [60] and [61]]} \\
&= \mathsf{field}\, A \cup \mathsf{field}\, B \quad \text{[by definition [299]]}
\end{aligned}$$

□

Proposition 312. $\mathsf{field}(A \cap B) \subseteq \mathsf{field}\, A \cap \mathsf{field}\, B$.

Proof. Follows by definition [4] and propositions [93] and [300]. □

Proposition 313. $\mathsf{field}(A \setminus B) \supseteq \mathsf{field}\, A \setminus \mathsf{field}\, B$.

Proof. Follows by propositions [119] and [300] and definitions [4] and [106]. □

Proposition 314. $\mathsf{field}\, R^{\mathsf{T}} = \mathsf{field}\, R$.

Proof. Follows by definition [299] and propositions [60], [297] and [298]. □

4.2 Image

Definition 315. $R^{\to}(A) = \{b \in \mathsf{ran}\, R \mid \exists a \in A. a\ R\ b\}$.

Proposition 316. Suppose $a \in A$ and $a\ R\ b$. Then $b \in R^{\to}(A)$.

Proof. Follows by definitions [287] and [315]. □

Proposition 317. $b \in R^{\to}(A)$ iff there exists $a \in A$ such that $a\ R\ b$.

Proposition 318. Suppose $A \subseteq B$. Then $R^{\to}(A) \subseteq R^{\to}(B)$.

Proof. Follows by definition [4] and proposition [317]. □

Proposition 319. Then $R^{\to}(A) \subseteq \mathsf{ran}\, R$.

Proposition 320. Then $R^{\to}(\mathsf{dom}\, R) = \mathsf{ran}\, R$.

Proposition 321. $R^{\to}(A \cup B) = R^{\to}(A) \cup R^{\to}(B)$.

Proof. Follows by axioms [2] and [56] and proposition [317]. □

Proposition 322. $R^{\to}(A \cap B) \subseteq R^{\to}(A) \cap R^{\to}(B)$.

Proof. Follows by proposition [317] and definitions [4] and [80]. □

Proposition 323. $R^{\to}(A \setminus B) \supseteq R^{\to}(A) \setminus R^{\to}(B)$.

Proof. Follows by proposition [317] and definitions [4] and [106]. □

Proposition 324. $b \in R^{\to}(\{a\})$ iff $a\ R\ b$.

Proposition 325. Suppose $b \in R^{\rightarrow}(\{a\})$. Then $b \in \mathsf{ran}\, R$ and $(a, b) \in R$.

Proof. Follows by propositions [9], [36], [317] and [319]. $\square$

Proposition 326. $R^{\rightarrow}(\{a\}) = \{b \in \mathsf{ran}\, R \mid (a, b) \in R\}$.

Proposition 327. $R^{\rightarrow}(\emptyset) = \emptyset$.

Proof. Follows by set extensionality. $\square$

4.3 Preimage

Definition 328. $R^{\leftarrow}(B) = \{a \in \mathsf{dom}\, R \mid \exists b \in B. a\ R\ b\}$.

Proposition 329. $a \in R^{\leftarrow}(B)$ iff there exists $b \in B$ such that $a\ R\ b$.

Proposition 330. $R^{\leftarrow}(B) = R^{\mathsf{T}\rightarrow}(B)$.

Proof. Follows by set extensionality. $\square$

Proposition 331. Suppose $A \subseteq B$. Then $R^{\leftarrow}(A) \subseteq R^{\leftarrow}(B)$.

Proposition 332. Then $R^{\leftarrow}(A) \subseteq \mathsf{dom}\, R$.

Proposition 333. $R^{\leftarrow}(A \cup B) = R^{\leftarrow}(A) \cup R^{\leftarrow}(B)$.

Proof. Follows by set extensionality. $\square$

Proposition 334. $R^{\leftarrow}(A \cap B) \subseteq R^{\leftarrow}(A) \cap R^{\leftarrow}(B)$.

Proposition 335. $R^{\leftarrow}(A \setminus B) \supseteq R^{\leftarrow}(A) \setminus R^{\leftarrow}(B)$.

4.4 Upward and downward closure

Definition 336. $a^{\uparrow R} = \{b \in \mathsf{ran}\, R \mid a\ R\ b\}$.

Definition 337. $b^{\downarrow R} = \{a \in \mathsf{dom}\, R \mid a\ R\ b\}$.

Proposition 338. $a \in b^{\downarrow R}$ iff $a\ R\ b$.

4.5 Relation (and later also function) composition

Composition ignores the non-relational parts of sets. Note that the order is flipped from usual relation composition. This lets us use the same symbol for composition of functions.

Definition 339. $S \circ R = \{(x, z) \mid x \in \mathsf{dom}\, R, z \in \mathsf{ran}\, S \mid \exists y.\ x\ R\ y\ S\ z\}$.

Proposition 340. $S \circ R$ is a relation.

Proposition 341. Suppose $x\ R\ y\ S\ z$. Then $x\ (S \circ R)\ z$.

Proof. $x \in \mathsf{dom}\, R$ and $z \in \mathsf{ran}\, S$. Then $(x, z) \in S \circ R$ by definition [339]. $\square$

Proposition 342. Suppose $x\ (S \circ R)\ z$. Then there exists y such that $x\ R\ y\ S\ z$.

Proof. There exists y such that $x \; R \; y \; S \; z$ by definition [339] and axiom [134]. $\square$

Proposition 343. $x \; (S \circ R) \; z$ iff there exists y such that $x \; R \; y \; S \; z$.

Proposition 344. $(T \circ S) \circ R = T \circ (S \circ R)$.

Proof. For all a, b we have $(a, b) \in (T \circ S) \circ R$ iff $(a, b) \in T \circ (S \circ R)$ by proposition [343]. Now $(T \circ S) \circ R$ is a relation and $T \circ (S \circ R)$ is a relation by proposition [340]. Follows by relation extensionality. $\square$

Proposition 345. Suppose $(a, c) \in R^{\mathsf{T}} \circ S^{\mathsf{T}}$. Then $(a, c) \in (S \circ R)^{\mathsf{T}}$.

Proof. Take b such that $a \; S^{\mathsf{T}} \; b \; R^{\mathsf{T}} \; c$. Now $c \; R \; b \; S \; a$ by proposition [264]. Hence $c \; (S \circ R) \; a$. Thus $a \; (S \circ R)^{\mathsf{T}} \; c$. $\square$

Proposition 346. Suppose $(a, c) \in (S \circ R)^{\mathsf{T}}$. Then $(a, c) \in R^{\mathsf{T}} \circ S^{\mathsf{T}}$.

Proof. $c \; (S \circ R) \; a$. Take b such that $c \; R \; b \; S \; a$. Now $a \; S^{\mathsf{T}} \; b \; R^{\mathsf{T}} \; c$. $\square$

Proposition 347. $(S \circ R)^{\mathsf{T}} = R^{\mathsf{T}} \circ S^{\mathsf{T}}$.

Proof. $(S \circ R)^{\mathsf{T}}$ is a relation. $R^{\mathsf{T}} \circ S^{\mathsf{T}}$ is a relation. For all x, y we have $(x, y) \in (S \circ R)^{\mathsf{T}}$ iff $(x, y) \in R^{\mathsf{T}} \circ S^{\mathsf{T}}$. Thus $(S \circ R)^{\mathsf{T}} = R^{\mathsf{T}} \circ S^{\mathsf{T}}$ by proposition [253]. $\square$

4.6 Restriction

Definition 348. $R\big|_X = \{w \in R \mid \exists x, y. x \in X \land w = (x, y)\}$.

Proposition 349. $a \; R\big|_X \; b$ iff $a \; R \; b$ and $a \in X$.

Proposition 350. $R\big|_X \subseteq R$.

Proposition 351. Suppose $x \in \mathsf{dom}\, R\big|_X$. Then $x \in \mathsf{dom}\, R, X$.

Proof. Take y such that $x \in X$ and $(x, y) \in R\big|_X$. Then $(x, y) \in R$. Thus $x \in \mathsf{dom}\, R$. $\square$

Proposition 352. Suppose $x \in \mathsf{dom}\, R, X$. Then $x \in \mathsf{dom}\, R\big|_X$.

Proof. Take y such that $(x, y) \in R$ by proposition [278]. Then $(x, y) \in R\big|_X$. Thus $x \in \mathsf{dom}\, R\big|_X$. $\square$

Proposition 353. Suppose R is a relation. $R\big|_X = R \cap (X \times \mathsf{ran}\, R)$.

Proof. For all a we have $a \in R \cap (X \times \mathsf{ran}\, R)$ iff $a \in R\big|_X$ by definitions [80] and [348] and propositions [144], [150] and [288]. Follows by extensionality. $\square$

Corollary 354. Suppose R is a relation. $\mathsf{dom}\, R\big|_X = \mathsf{dom}\, R \cap X$.

Proof. Follows by set extensionality. $\square$

Proposition 355. Suppose $V \subseteq U$. Then $R\big|_U\big|_V = R\big|_V$.

Proof. For all w we have $w \in R|_U|_V$ iff $w \in R|_V$ by definitions [4] and [348]. Follows by extensionality. ☐

Proposition 356. Let R be a relation. Then $R|_{\mathsf{dom}\, R} = R$.

Proof. For all w we have $w \in R|_{\mathsf{dom}\, R}$ iff $w \in R$ by definitions [251], [277] and [348]. Follows by extensionality. ☐

Proposition 357. Then $\mathsf{dom}\, R|_X \subseteq X$.

Proposition 358. Suppose $X \subseteq \mathsf{dom}\, R$. Let $b \in \mathsf{ran}\, R|_X$. Then $b \in R^{\rightarrow}(X)$.

Proof. Take $a \in X$ such that $(a, b) \in R|_X$ by definitions [4], [277] and [287] and proposition [357]. Then $a \mathrel{R} b$ and $b \in \mathsf{ran}\, R$. Thus $b \in R^{\rightarrow}(X)$ by definition [315]. ☐

Proposition 359. Suppose $X \subseteq \mathsf{dom}\, R$. Let $b \in R^{\rightarrow}(X)$. Then $b \in \mathsf{ran}\, R|_X$.

Proof. Follows by definition [315] and propositions [289] and [349]. ☐

Proposition 360. Suppose $X \subseteq \mathsf{dom}\, R$. Then $\mathsf{ran}\, R|_X = R^{\rightarrow}(X)$.

Proof. Follows by set extensionality. ☐

Proposition 361. Suppose $X \subseteq \mathsf{dom}\, R$. Then $R|_X{}^{\rightarrow}(A) = R^{\rightarrow}(X \cap A)$.

Proof. For all b we have $b \in R|_X{}^{\rightarrow}(A)$ iff $b \in R^{\rightarrow}(X \cap A)$ by propositions [317] and [349] and definition [80]. Follows by extensionality. ☐

4.7 Set of relations

Abbreviation 362. R is a binary relation on X iff $R \subseteq X \times X$.

Proposition 363. Let R be a relation. Suppose $\mathsf{ran}\, R \subseteq B$. Suppose $\mathsf{dom}\, R \subseteq A$. Suppose $w \in R$. Then $w \in A \times B$.

Proof. Take a, b such that $(a, b) = w$. Then $a \in \mathsf{dom}\, R$ and $b \in \mathsf{ran}\, R$. Thus $a \in A$ and $b \in B$. Thus $(a, b) \in A \times B$. ☐

Proposition 364. Let R be a relation. Suppose $\mathsf{ran}\, R \subseteq B$. Suppose $\mathsf{dom}\, R \subseteq A$. Then $R \subseteq A \times B$.

Proposition 365. Suppose $R \subseteq A \times B$. Suppose $a \in \mathsf{dom}\, R$. Then $a \in A$.

Proof. Take w, b such that $w \in R$ and $w = (a, b)$. Follows by definition [277] and propositions [9] and [143]. ☐

Proposition 366. Suppose $R \subseteq A \times B$. Then $\mathsf{dom}\, R \subseteq A$.

Proof. Follows by definition [4] and proposition [365]. ☐

Proposition 367. Suppose $R \subseteq A \times B$. Suppose $b \in \mathsf{ran}\, R$. Then $b \in B$.

Proof. Take w, a such that $w \in R$ and $w = (a, b)$. Follows by definition [287] and propositions [9] and [143]. ☐

Proposition 368. Suppose $R \subseteq A \times B$. Then $\operatorname{ran} R \subseteq B$.

Proof. Follows by definition [4] and proposition [367]. ☐

Definition 369. $\mathsf{Rel}(A, B) = \mathsf{Pow}(A \times B)$.

Proposition 370. Suppose $R \subseteq A \times B$. Then $R \in \mathsf{Rel}(A, B)$.

Proposition 371. Let R be a relation. Suppose $\operatorname{dom} R \subseteq A$. Suppose $\operatorname{ran} R \subseteq B$. Then $R \in \mathsf{Rel}(A, B)$.

Proof. $R \subseteq A \times B$. ☐

Proposition 372. Suppose $R \in \mathsf{Rel}(A, B)$. Then $R \subseteq A \times B$.

Proposition 373. Suppose $R \in \mathsf{Rel}(A, B)$. Then $\operatorname{dom} R \subseteq A$.

Proof. Follows by propositions [366] and [372]. ☐

Proposition 374. Suppose $R \in \mathsf{Rel}(A, B)$. Then $\operatorname{ran} R \subseteq B$.

Proof. Follows by propositions [368] and [372]. ☐

Proposition 375. Let $R \in \mathsf{Rel}(A, B)$. Then R is a relation.

Proof. It suffices to show that for all $w \in R$ there exists x, y such that $w = (x, y)$. Fix $w \in R$. Now $R \subseteq A \times B$ by proposition [372]. Thus $w \in A \times B$. ☐

Proposition 376. Let $R \in \mathsf{Rel}(A, B)$. Suppose $A \subseteq C$. Then $R \in \mathsf{Rel}(C, B)$.

Proof. $R \subseteq A \times B \subseteq C \times B$. Thus $R \subseteq C \times B$. ☐

Proposition 377. Let $R \in \mathsf{Rel}(A, B)$. Suppose $B \subseteq D$. Then $R \in \mathsf{Rel}(A, D)$.

Proof. $R \subseteq A \times B \subseteq A \times D$. Thus $R \subseteq A \times D$. ☐

Proposition 378. Let $R \in \mathsf{Rel}(A, B)$. Suppose $(a, b) \in R$. Then $(a, b) \in A \times B$.

Proof. $R \subseteq A \times B$ by proposition [372]. ☐

Proposition 379. Let $R \in \mathsf{Rel}(A, B)$. Suppose $(a, b) \in R$. Then $a \in A$.

Proof. $(a, b) \in A \times B$ by proposition [378]. ☐

Proposition 380. Let $R \in \mathsf{Rel}(A, B)$. Suppose $(a, b) \in R$. Then $b \in B$.

Proof. $(a, b) \in A \times B$ by proposition [378]. ☐

Proposition 381. Let $R \in \mathsf{Rel}(A, B)$. Then $R \in \mathsf{Rel}(\operatorname{dom} R, B)$.

Proof. R is a relation by proposition [375]. $\operatorname{dom} R \subseteq \operatorname{dom} R$ by proposition [7]. $\operatorname{ran} R \subseteq B$. Follows by proposition [371]. □

Proposition 382. Let $R \in \mathsf{Rel}(A, B)$. Then $R \in \mathsf{Rel}(A, \operatorname{ran} R)$.

Proof. R is a relation by proposition [375]. $\operatorname{dom} R \subseteq A$. $\operatorname{ran} R \subseteq \operatorname{ran} R$ by proposition [7]. Follows by proposition [371]. □

4.8 Identity relation

Definition 383. $\mathsf{id}_A = \{(a, a) \mid a \in A\}$.

Proposition 384. $a \mathbin{\mathsf{id}_A} b$ iff $a = b \in A$.

Proof. Follows by definition [383] and axiom [134]. □

Proposition 385. Suppose $a \in A$. Then $(a, a) \in \mathsf{id}_A$.

Proof. Follows by definition [383]. □

Proposition 386. Suppose $w \in \mathsf{id}_A$. Then there exists $a \in A$ such that $w = (a, a)$.

Proof. Follows by definition [383]. □

Proposition 387. id_A is a relation.

Proposition 388. $\operatorname{dom} \mathsf{id}_A = A$.

Proof. For every $a \in A$ we have $(a, a) \in \mathsf{id}_A$. $\operatorname{dom} \mathsf{id}_A = A$ by set extensionality. □

Proposition 389. $\operatorname{ran} \mathsf{id}_A = A$.

Proof. For every a we have $a \in \operatorname{ran} \mathsf{id}_A$ iff $a \in A$ by propositions [288] and [384]. For every $a \in A$ we have $(a, a) \in \mathsf{id}_A$. $\operatorname{ran} \mathsf{id}_A = A$ by set extensionality. □

Proposition 390. $\mathsf{id}_A^{\rightarrow}(B) = A \cap B$.

Proof. Follows by set extensionality. □

Proposition 391. $\mathsf{id}_A \in \mathsf{Rel}(A, A)$.

4.9 Membership relation

Definition 392. $\in_A = \{(a, b) \mid a \in A, b \in A \mid a \in b\}$.

Proposition 393. Suppose $a, b \in A$. Suppose $a \in b$. Then $(a, b) \in \in_A$.

Proposition 394. Suppose $w \in \in_A$. Then there exists $a, b \in A$ such that $w = (a, b)$ and $a \in b$.

Proof. Follows by definition [392]. □

Proposition 395. $\in_A$ is a relation.

4.10 Subset relation

Definition 396. $\subseteq_A = \{(a, b) \mid a \in A, b \in A \mid a \subseteq b\}$.

Proposition 397. $\subseteq_A$ is a relation.

4.11 Properties of relations

Definition 398. R is left quasireflexive iff for all x, y such that $x \mathrel{R} y$ we have $x \mathrel{R} x$.

Definition 399. R is right quasireflexive iff for all x, y such that $x \mathrel{R} y$ we have $y \mathrel{R} y$.

Definition 400. R is quasireflexive iff for all x, y such that $x \mathrel{R} y$ we have $x \mathrel{R} x$ and $y \mathrel{R} y$.

Definition 401. R is coreflexive iff for all x, y such that $x \mathrel{R} y$ we have $x = y$.

Definition 402. R is reflexive on X iff for all $x \in X$ we have $x \mathrel{R} x$.

Definition 403. R is irreflexive iff for all x we have $(x, x) \notin R$.

Proposition 404. Suppose R is quasireflexive. Then R is reflexive on $\mathsf{field}\, R$.

Proposition 405. Suppose R is reflexive on $\mathsf{field}\, R$. Then R is quasireflexive.

Proposition 406. Let F be an inhabited family of relations. Suppose every element of F is reflexive on A. Then $\bigcap F$ is reflexive on A.

Proof. For all $a \in A$ we have for all $R \in F$ we have $a \mathrel{R} a$. Thus for all $a \in A$ we have $a \mathrel{(\bigcap F)} a$. □

Definition 407. R is antisymmetric iff for all x, y such that $x \mathrel{R} y \mathrel{R} x$ we have $x = y$.

Definition 408. **(Symmetry)** R is symmetric iff for all x, y we have $x \mathrel{R} y \iff y \mathrel{R} x$.

Definition 409. R is asymmetric iff for all x, y such that $x \mathrel{R} y$ we have $y \not\mathrel{R} x$.

Proposition 410. Suppose R is asymmetric. Then R is irreflexive.

Proposition 411. Suppose R is asymmetric. Then R is antisymmetric.

Proposition 412. Suppose R is antisymmetric. Suppose R is irreflexive. Then R is asymmetric.

Definition 413. **(Transitivity)** R is transitive iff for all x, y, z such that $x \mathrel{R} y \mathrel{R} z$ we have $x \mathrel{R} z$.

Proposition 414. Suppose R is transitive. Suppose $a \in b^{\downarrow R}$. Suppose $c \in a^{\downarrow R}$. Then $c \in b^{\downarrow R}$.

Proof. $c \mathrel{R} a \mathrel{R} b$. Thus $c \mathrel{R} b$ by transitivity. □

Proposition 415. Suppose R is transitive. Suppose $a \in b^{\downarrow R}$. Then $a^{\downarrow R} \subseteq b^{\downarrow R}$.

Definition 416. R is dense iff for all x, z such that $x \mathrel{R} z$ there exists y such that $x \mathrel{R} y \mathrel{R} z$.

Definition 417. R is quasiconnex iff for all $x, y \in \mathsf{field}\, R$ such that $x \neq y$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Definition 418. R is connex on X iff for all $x, y \in X$ such that $x \neq y$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Definition 419. R is strongly quasiconnex iff for all $x, y \in \mathsf{field}\, R$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Definition 420. R is strongly connex on X iff for all $x, y \in X$ we have $x \mathrel{R} y$ or $y \mathrel{R} x$.

Proposition 421. R is strongly quasiconnex iff R is quasiconnex and quasireflexive.

Proof. Follows by definitions [299], [402], [417] and [419] and propositions [404] and [405]. □

Proposition 422. Suppose R is connex on A. Let $a, b \in A \setminus \mathsf{ran}\, R$. Then $a = b$.

Proof. Suppose not. $a, b \in A$. Then $(a, b) \in R$ or $(b, a) \in R$ by definition [418]. $(a, b) \notin R$. $(b, a) \notin R$. Thus $a = b$. □

Definition 423. R is right Euclidean iff for all a, b, c such that $a \mathrel{R} b, c$ we have $b \mathrel{R} c$.

Definition 424. R is left Euclidean iff for all a, b, c such that $a, b \mathrel{R} c$ we have $a \mathrel{R} b$.

4.12 Quasiorders

Abbreviation 425. R is a quasiorder iff R is quasireflexive and transitive.

Abbreviation 426. R is a quasiorder on A iff R is a binary relation on A and R is reflexive on A and transitive.

Struct 427. A quasiordered set X is a onesorted structure equipped with

1. $\leq$

such that

1. $\leq_X$ is a binary relation on X.
2. $\leq_X$ is reflexive on X.
3. $\leq_X$ is transitive.

Lemma 428. Let X be a quasiordered set. Let $a, b, c, d \in X$. Suppose $a \leq_X b \leq_X c \leq_X d$. Then $a \leq_X d$.

Proof. $\leq_X$ is transitive. Thus $a \leq_X c \leq_X d$ by transitivity. Hence $a \leq_X d$ by transitivity. □

Proposition 429. $\subseteq_A$ is a quasiorder on A.

Proof. $\subseteq_A$ is reflexive on A. $\subseteq_A$ is transitive. □

4.13 Equivalences

Abbreviation 430. E is a partial equivalence iff E is transitive and symmetric.

Proposition 431. Let E be a partial equivalence. Then E is quasireflexive.

Abbreviation 432. E is an equivalence iff E is a symmetric quasiorder.

Abbreviation 433. E is an equivalence on A iff E is a symmetric quasiorder on A.

Proposition 434. Let F be a family of relations. Suppose every element of F is an equivalence. Then $\bigcap F$ is an equivalence.

Proof. $\bigcap F$ is quasireflexive by definition [400] and propositions [48] and [50]. $\bigcap F$ is symmetric by definition [408] and propositions [48] and [50]. $\bigcap F$ is transitive by definition [413] and propositions [48] and [50]. □

Proposition 435. Let F be an inhabited family of relations. Suppose every element of F is an equivalence on A. Then $\bigcap F$ is an equivalence on A.

Proof. $\bigcap F$ is reflexive on A by proposition [406]. $\bigcap F$ is symmetric. $\bigcap F$ is transitive. □

4.13.1 Equivalence classes

Abbreviation 436. $[a]_E = a^{\downarrow E}$.

Abbreviation 437. The E-equivalence class of a is $[a]_E$.

Proposition 438. Let E be an equivalence. Let $a \in \mathsf{field}\, E$. Then $a \in [a]_E$.

Proof. $a\ E\ a$ by definition [402] and proposition [404]. □

Proposition 439. Let E be an equivalence on A. Let $a \in A$. Then $a \in [a]_E$.

Proof. $a\ E\ a$ by definition [402]. □

Proposition 440. Let E be an equivalence on A. Let $a, b \in A$. Suppose $a\ E\ b$. Then $[a]_E = [b]_E$.

Proof. Follows by set extensionality. □

Proposition 441. Let E be an equivalence on A. Let $a, b \in A$. Suppose $[a]_E = [b]_E$. Then $a\ E\ b$.

Proposition 442. Let E be an equivalence on A. Let $a, b \in A$. Then $a\ E\ b$ iff $[a]_E = [b]_E$.

Proposition 443. Let E be a partial equivalence. Suppose $[a]_E \neq [b]_E$. Then $[a]_E$ is disjoint from $[b]_E$.

Proof. Suppose not. Take c such that $c \in [a]_E, [b]_E$. Then $c\ E\ a$ and $c\ E\ b$. E is symmetric. Thus $a\ E\ c$ by symmetry. E is transitive. Thus $a\ E\ b$ by transitivity. Then $b\ E\ a$ by symmetry. Thus $a \in [b]_E$ and $b \in [a]_E$ by proposition [338]. Hence $[a]_E \subseteq [b]_E \subseteq [a]_E$ by proposition [415]. Contradiction by proposition [8]. □

Corollary 444. Let E be an equivalence. Suppose $[a]_E \neq [b]_E$. Then $[a]_E$ is disjoint from $[b]_E$.

Proof. Follows by proposition [443]. □

Corollary 445. Let E be an equivalence on A. Suppose $[a]_E \neq [b]_E$. Then $[a]_E$ is disjoint from $[b]_E$.

Proof. Follows by proposition [443]. □

4.13.2 Quotients

Definition 446. $A/E = \{[a]_E \mid a \in A\}$.

Proposition 447. $\emptyset/\emptyset = \emptyset$.

Proposition 448. Let E be an equivalence on A. Suppose $B, C \in A/E$ and $B \neq C$. Then B is disjoint from C.

Proof. Take b such that $B = [b]_E$. Take c such that $C = [c]_E$. Then B is disjoint from C by corollary [445]. □

Proposition 449. Let E be an equivalence on A. Suppose $C \in A/E$. Then C is inhabited.

Proof. Take $a \in A$ such that $C = [a]_E$. Then $a \in [a]_E$. C is inhabited by definitions [18] and [446] and proposition [438]. □

Proposition 450. Let E be an equivalence on A. Suppose $a \in C \in A/E$. Then $a \in A$.

Proof. Take $b \in A$ such that $C = [b]_E$ by definition [446]. Then $a\ E\ b$. Thus $a \in A$ by proposition [143] and definition [4]. □

Corollary 451. Let E be an equivalence on A. $\emptyset \notin A/E$.

Proposition 452. Let E be an equivalence on A. A/E is a partition.

Proof. $\emptyset \notin A/E$. For all $B, C \in A/E$ such that $B \neq C$ we have B is disjoint from C. □

Proposition 453. Let E be an equivalence on A. A/E is a partition of A.

Proof. $\bigcup(A/E) = A$ by set extensionality. □

Definition 454. $E_P = \{(a, b) \mid a \in A, b \in A \mid \exists C \in P.\ a, b \in C\}$.

Proposition 455. Let P be a partition of A. Let $a, b \in A$. Suppose $a, b \in C \in P$. Then $a\ E_P\ b$.

Proposition 456. Let P be a partition of A. E_P is reflexive on A.

Proposition 457. Let P be a partition. E_P is symmetric.

Proof. Follows by definitions [408] and [454] and axiom [17]. □

Proposition 458. Let P be a partition. E_P is transitive.

Proposition 459. Let P be a partition of A. E_P is an equivalence on A.

Proposition 460. Let E be an equivalence on A. Then $E_{A/E} = E$.

Proof. Follows by set extensionality. □

Proposition 461. Let P be a partition of A. Then $A/E_P = P$.

Proof. Follows by set extensionality. □

4.14 Closure operations on relations

Definition 462. $\mathsf{ReflCl}_X(R) = R \cup \mathsf{id}_X$.

Proposition 463. $\mathsf{ReflCl}_X(R)$ is reflexive on X.

Definition 464. $\mathsf{ReflReduc}_X(R) = R \setminus \mathsf{id}_X$.

Definition 465. $\mathsf{SymCl}(R) = R \cup R^\mathsf{T}$.

4.15 Injective relations

Definition 466. R is injective iff for all a, a', b such that $a, a' \mathrel{R} b$ we have $a = a'$.

Abbreviation 467. R is left-unique iff R is injective.

Proposition 468. Suppose $S \subseteq R$. Suppose R is injective. Then S is injective.

Proposition 469. Suppose R is injective. Then $R\big|_A$ is injective.

Proof. $R\big|_A \subseteq R$. □

Proposition 470. Suppose R and S are injective. Then $S \circ R$ is injective.

Proposition 471. Then id_A is injective.

4.16 Right-unique relations

Definition 472. R is right-unique iff for all a, b, b' such that $a \mathrel{R} b, b'$ we have $b = b'$.

Abbreviation 473. R is one-to-one iff R is right-unique and injective.

Proposition 474. Suppose $S \subseteq R$. Suppose R is right-unique. Then S is right-unique.

Proposition 475. Suppose R and S are right-unique. Then $S \circ R$ is right-unique.

4.17 Left-total relations

Definition 476. R is left-total on A iff for all $a \in A$ there exists b such that $a \mathrel{R} b$.

4.18 Right-total relations

Definition 477. R is right-total on B iff for all $b \in B$ there exists a such that $a\ R\ b$.

Abbreviation 478. R is surjective on B iff R is right-total on B.

5 Functions

Abbreviation 479. f is a function iff f is right-unique and f is a relation.

Definition 480. $f(x) = \bigcup f^{\rightarrow}(\{x\})$.

Proposition 481. Let f be a function. Suppose $(a, b), (a, b') \in f$. Then $b = b'$.

Proof. Follows by right-uniqueness. □

Proposition 482. Let f be a function. Suppose $(a, b) \in f$. Then $f(a) = b$.

Proof. Let $B = f^{\rightarrow}(\{a\})$. $B = \{b' \in \mathsf{ran}\, f \mid (a, b') \in f\}$ by proposition [326]. $b \in \mathsf{ran}\, f$. For all $b' \in B$ we have $(a, b') \in f$. For all $b', b'' \in B$ we have $b' = b''$ by right-uniqueness. Then $B = \{b\}$ by proposition [39]. Then $\bigcup B = b$. Thus $f(a) = b$ by definition [480]. □

Proposition 483. Let f be a function. Suppose $w \in f$. Then there exists $x \in \mathsf{dom}\, f$ such that $w = (x, f(x))$.

Proof. Follows by definitions [251], [277] and [480] and proposition [482]. □

Proposition 484. Let f be a function. Suppose $x \in \mathsf{dom}\, f$. Then $(x, f(x)) \in f$.

Proof. Follows by propositions [278] and [482]. □

Proposition 485. Let f be a function. $(a, b) \in f$ iff $a \in \mathsf{dom}\, f$ and $f(a) = b$.

Proposition 486. Let f, g be functions. Suppose $\mathsf{dom}\, f \subseteq \mathsf{dom}\, g$. Suppose for all $x \in \mathsf{dom}\, f$ we have $f(x) = g(x)$. Then $f \subseteq g$.

Proof. For all x, y such that $(x, y) \in f$ we have $(x, y) \in g$. Follows by definitions [4] and [251]. □

Proposition 487. **(Function extensionality)** Let f, g be functions. Suppose $\mathsf{dom}\, f = \mathsf{dom}\, g$. Suppose for all x we have $f(x) = g(x)$. Then $f = g$.

Proof. $\mathsf{dom}\, f \subseteq \mathsf{dom}\, g \subseteq \mathsf{dom}\, f$. For all $x \in \mathsf{dom}\, f$ we have $f(x) = g(x)$. Thus $f \subseteq g$. For all $x \in \mathsf{dom}\, g$ we have $f(x) = g(x)$. Thus $g \subseteq f$. □

Proposition 488. Let f be a function. $b \in \mathsf{ran}\, f$ iff there exists $a \in \mathsf{dom}\, f$ such that $f(a) = b$.

Proof. Follows by definition [287] and proposition [485]. □

Abbreviation 489. f is a function on X iff f is a function and $X = \mathsf{dom}\, f$.

Abbreviation 490. f is a function to Y iff f is a function and for all $x \in \mathsf{dom}\, f$ we have $f(x) \in Y$.

Proposition 491. Let f be a function to B. Suppose $B \subseteq C$. Then f is a function to C.

Proposition 492. Let f be a function to B. Then $\mathsf{ran}\, f \subseteq B$.

Proof. Follows by definitions [4], [277], [287] and [480], proposition [483], and axiom [134]. □

Definition 493. $\mathsf{Fun}(A, B) = \{f \in \mathsf{Rel}(A, B) \mid A = \mathsf{dom}\, f \text{ and } f \text{ is right-unique}\}$.

Abbreviation 494. f is a function from X to Y iff $f \in \mathsf{Fun}(X, Y)$.

Proposition 495. Let $f \in \mathsf{Fun}(A, B)$. Then f is a relation.

Proof. Follows by definition [493] and proposition [375]. □

Proposition 496. Let $f \in \mathsf{Fun}(A, B)$. Then f is a function.

Proposition 497. $\mathsf{Fun}(A, B) \subseteq \mathsf{Rel}(A, B)$.

Proof. Follows by definitions [4] and [493]. □

Proposition 498. Let f be a function to B such that $A = \mathsf{dom}\, f$. Then $f \in \mathsf{Fun}(A, B)$.

Proof. $\mathsf{dom}\, f \subseteq A$ by proposition [7]. $\mathsf{ran}\, f \subseteq B$ by proposition [492]. Thus $f \in \mathsf{Rel}(A, B)$ by proposition [371]. Thus $f \in \mathsf{Fun}(A, B)$ by definition [493]. □

Proposition 499. Let $f \in \mathsf{Fun}(A, B)$. Then f is a function to B such that $A = \mathsf{dom}\, f$.

Proof. f is a function by proposition [496]. It suffices to show that for all $a \in \mathsf{dom}\, f$ we have $f(a) \in B$ by definition [493]. Fix $a \in \mathsf{dom}\, f$. Take b such that $f(a) = b$. Thus $(a, b) \in f$ by proposition [484]. Now $b \in \mathsf{ran}\, f$ by proposition [288]. Finally $\mathsf{ran}\, f \subseteq B$ by definition [493] and proposition [374]. □

Proposition 500. Let $f \in \mathsf{Fun}(A, B)$. Suppose $B \subseteq D$. Then $f \in \mathsf{Fun}(A, D)$.

Proof. $f \in \mathsf{Rel}(A, D)$ by definition [493] and proposition [377]. Follows by definition [493]. □

Proposition 501. Let $f \in \mathsf{Fun}(A, B)$. Let $a \in A$. Then $f(a) \in B$.

Proof. $(a, f(a)) \in f$ by propositions [485] and [499]. Thus $f(a) \in B$ by definition [493] and proposition [380]. □

Proposition 502. Let $f \in \mathsf{Fun}(A, B)$. Let $a \in A$. Then there exists $b \in B$ such that $(a, b) \in f$.

Proof. $(a, f(a)) \in f$ by propositions [485] and [499]. $f(a) \in B$ by proposition [501]. □

Proposition 503. Let $f \in \mathsf{Fun}(A, B)$. Then $\mathsf{ran}\, f \subseteq B$.

Proof. f is a function to B. $\square$

5.1 Image of a function

Proposition 504. Let f be a function. Suppose $x \in \mathsf{dom}\, f \cap X$. Then $f(x) \in f^{\rightarrow}(X)$.

Proof. $x \in X$ by proposition [83]. Thus $(x, f(x)) \in f$ by propositions [82] and [484]. $\square$

Proposition 505. Let f be a function. Suppose $y \in f^{\rightarrow}(X)$. Then there exists $x \in \mathsf{dom}\, f \cap X$ such that $y = f(x)$.

Proof. Take $x \in X$ such that $(x, y) \in f$. Then $x \in \mathsf{dom}\, f$ and $y = f(x)$ by propositions [279] and [485]. $\square$

Proposition 506. Suppose f is a function. $f^{\rightarrow}(X) = \{f(x) \mid x \in \mathsf{dom}\, f \cap X\}$.

Proof. Follows by propositions [504] and [505]. $\square$

5.2 Families of functions

Abbreviation 507. F is a family of functions iff every element of F is a function.

Proposition 508. Let F be a family of functions. Suppose that for all $f, g \in F$ we have $f \subseteq g$ or $g \subseteq f$. Then $\bigcup F$ is a function.

Proof. $\bigcup F$ is a relation by proposition [255]. For all x, y, y' such that $(x, y), (x, y') \in \bigcup F$ there exists $f \in F$ such that $(x, y), (x, y') \in f$ by axiom [42] and definition [4]. Thus $\bigcup F$ is right-unique by definition [472]. $\square$

5.3 The empty function

Proposition 509. $\emptyset$ is a function.

Proposition 510. $\emptyset$ is a function on $\emptyset$.

Proposition 511. $\emptyset$ is a function to X.

Proposition 512. $\emptyset$ is injective.

5.4 Function composition

Abbreviation 513. g is composable with f iff $\mathsf{ran}\, f \subseteq \mathsf{dom}\, g$.

Proposition 514. Suppose f and g are right-unique. Then $g \circ f$ is a function.

Proposition 515. Let f, g be functions. Suppose g is composable with f. Let $x \in \mathsf{dom}\, f$. Then $(g \circ f)(x) = g(f(x))$.

Proof. $(x, g(f(x))) \in g \circ f$ by definitions [4], [287] and [339] and proposition [484]. $g \circ f$ is a function by proposition [514]. Thus $(g \circ f)(x) = g(f(x))$ by proposition [482]. $\square$

Proposition 516. Let f, g be functions. Suppose g is composable with f. $\mathsf{dom}\, g \circ f = f^{\leftarrow}(\mathsf{dom}\, g)$.

Proof. Every element of $\mathsf{dom}\, g \circ f$ is an element of $f^{\leftarrow}(\mathsf{dom}\, g)$ by definitions [277], [328] and [339] and axiom [134]. Follows by set extensionality. ☐

Proposition 517. Let f, g be functions. Suppose $\mathsf{ran}\, f = \mathsf{dom}\, g$. $\mathsf{dom}\, g \circ f = \mathsf{dom}\, f$.

Proof. Every element of $\mathsf{dom}\, g \circ f$ is an element of $\mathsf{dom}\, f$. Follows by set extensionality. ☐

Proposition 518. Let f, g be functions. Suppose g is composable with f. Suppose $y \in g^{\rightarrow}(\mathsf{ran}\, f)$. Then $y \in \mathsf{ran}\, g \circ f$.

Proof. Take $x \in \mathsf{ran}\, f$ such that $(x, y) \in g$. Take $x' \in \mathsf{dom}\, f$ such that $(x', x) \in f$. Then $(x', y) \in g \circ f$. Follows by proposition [289]. ☐

Proposition 519. Let f, g be functions. Suppose g is composable with f. Suppose $y \in \mathsf{ran}\, g \circ f$. Then $y \in g^{\rightarrow}(\mathsf{ran}\, f)$.

Proof. Take $x \in \mathsf{dom}\, f$ such that $(x, y) \in g \circ f$ by definitions [277], [287] and [339] and proposition [343]. $f(x) \in \mathsf{ran}\, f$. $(f(x), y) \in g$ by propositions [343] and [482] and definition [480]. Follows by proposition [317]. ☐

Proposition 520. Let f, g be functions. Suppose g is composable with f. Then $\mathsf{ran}\, g \circ f = g^{\rightarrow}(\mathsf{ran}\, f)$.

Proof. Follows by set extensionality. ☐

Proposition 521. Let f, g be functions. Suppose $\mathsf{ran}\, f = \mathsf{dom}\, g$. Then $\mathsf{ran}\, g \circ f = \mathsf{ran}\, g$.

Proof.

$$\begin{aligned} \mathsf{ran}\, g \circ f &= g^{\rightarrow}(\mathsf{ran}\, f) \quad \text{[by propositions [7] and [520]]} \\ &= g^{\rightarrow}(\mathsf{dom}\, g) \\ &= \mathsf{ran}\, g \quad \text{[by proposition [320]]} \end{aligned}$$

☐

Proposition 522. Let f, g be functions. Let A be a set. Suppose $\mathsf{ran}\, f \subseteq \mathsf{dom}\, g$. Suppose $c \in g \circ f^{\rightarrow}(A)$. Then $c \in g^{\rightarrow}(f^{\rightarrow}(A))$.

Proof. Take $a \in A$ such that $(a, c) \in g \circ f$. Take b such that $(a, b) \in f$ and $(b, c) \in g$. Then $b \in f^{\rightarrow}(A)$. Follows by proposition [317]. ☐

Proposition 523. Let f, g be functions. Let A be a set. Suppose $\mathsf{ran}\, f \subseteq \mathsf{dom}\, g$. Then $g \circ f^{\rightarrow}(A) = g^{\rightarrow}(f^{\rightarrow}(A))$.

Proof. For all c we have $c \in g^{\rightarrow}(f^{\rightarrow}(A))$ iff $c \in g \circ f^{\rightarrow}(A)$ by propositions [317] and [343]. Follows by extensionality. ☐

Proposition 524. Let f be a function. Let A be a set. $f\big|_A$ is a function.

Proposition 525. Let f be a function. Suppose $A \subseteq \mathsf{dom}\, f$. Let $a \in A$. Then $(f\big|_A)(a) = f(a)$.

Proof. Then $(a, f(a)) \in f$. Then $(a, f(a)) \in f\big|_A$ by proposition [349]. Thus $(f\big|_A)(a) = f(a)$. ☐

Proposition 526. Suppose $x \notin \mathsf{dom}\, f$. Then $f(x) = \emptyset$.

Proof. $f^{\rightarrow}(\{x\}) = \emptyset$ by axioms [2] and [17] and propositions [279] and [324]. Follows by definition [480] and proposition [44]. ☐

5.5 Injections

Proposition 527. Suppose f is a function. f is injective iff for all $x, y \in \mathsf{dom}\, f$ we have $f(x) = f(y) \implies x = y$.

Proof. Follows by definition [466] and proposition [485]. ☐

Abbreviation 528. f is an injection iff f is an injective function.

Definition 529. $\mathsf{Inj}(A, B) = \{f \in \mathsf{Fun}(A, B) \mid \text{for all } x, y \in A \text{ such that } f(x) = f(y) \text{ we have } x = y\}$.

5.6 Surjections

Abbreviation 530. f is a surjection onto Y iff f is a function such that f is surjective on Y.

Definition 531. $\mathsf{Surj}(A, B) = \{f \in \mathsf{Fun}(A, B) \mid \text{for all } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b\}$.

Abbreviation 532. f is a surjection from A to B iff $f \in \mathsf{Surj}(A, B)$.

Lemma 533. Let f be a function. Then f is surjective on $\mathsf{ran}\, f$.

Proof. It suffices to show that for all $y \in \mathsf{ran}\, f$ there exists $x \in \mathsf{dom}\, f$ such that $f(x) = y$. Fix $y \in \mathsf{ran}\, f$. Take x such that $(x, y) \in f$. Then $x \in \mathsf{dom}\, f$ and $f(x) = y$ by definition [277] and proposition [485]. ☐

Lemma 534. Let $f \in \mathsf{Surj}(A, B)$. Then $f \in \mathsf{Fun}(A, B)$.

Lemma 535. Let $f \in \mathsf{Fun}(A, B)$. Then $f \in \mathsf{Surj}(A, \mathsf{ran}\, f)$.

Proof. $f \in \mathsf{Rel}(A, \mathsf{ran}\, f)$ by definition [493] and proposition [382]. Thus $f \in \mathsf{Fun}(A, \mathsf{ran}\, f)$ by definition [493]. It suffices to show that for all $b \in \mathsf{ran}\, f$ there exists $a \in A$ such that $f(a) = b$ by definition [531]. Fix $b \in \mathsf{ran}\, f$. Take a such that $(a, b) \in f$. Thus $f(a) = b$ by propositions [485] and [499]. We have $a \in \mathsf{dom}\, f = A$. Follows by assumption. ☐

Corollary 536. Let $f \in \mathsf{Surj}(A, B)$. Then $\mathsf{ran}\, f = B$.

Proof. We have f is a function by definition [531] and proposition [496]. Now $\mathsf{ran}\, f \subseteq B$ by definition [531] and proposition [503]. It suffices to show that every element of B is an element of $\mathsf{ran}\, f$. It suffices to show that for all $b \in B$ there exists $a \in \mathsf{dom}\, f$ such that $f(a) = b$ by proposition [488]. Fix $b \in B$. Take $a \in A$ such that $f(a) = b$ by definition [531]. Then $\mathsf{dom}\, f = A$ by definition [531] and proposition [499]. ☐

Corollary 537. Let $f \in \mathsf{Fun}(A, B)$. Then $f \in \mathsf{Surj}(A, B)$ iff $\mathsf{ran}\, f = B$.

Proof. Follows by definition [531], lemma [535], and corollary [536]. ☐

Proposition 538. Let $f \in \mathsf{Surj}(A, B)$. Let $g \in \mathsf{Surj}(B, C)$. Then $g \circ f \in \mathsf{Surj}(A, C)$.

Proof. $\mathsf{dom}\, f = A$ by definitions [493] and [531]. $\mathsf{dom}\, g = B = \mathsf{ran}\, f$ by definitions [493] and [531] and corollary [536]. $\mathsf{dom}\, g \circ f = A$ by lemma [534] and propositions [496] and [517]. Omitted. ☐

5.7 Bijections

Definition 539. $\mathsf{Bi}(A, B) = \{f \in \mathsf{Surj}(A, B) \mid f \text{ is injective}\}$.

Abbreviation 540. f is a bijection from A to B iff $f \in \mathsf{Bi}(A, B)$.

Proposition 541. Every element of $\mathsf{Bi}(A, B)$ is an element of $\mathsf{Fun}(A, B)$.

Proof. Follows by definitions [531] and [539]. ☐

Proposition 542. Every element of $\mathsf{Bi}(A, B)$ is a function.

Proof. Follows by propositions [496] and [541]. ☐

Proposition 543. Let $f \in \mathsf{Bi}(A, B)$. Then $\mathsf{dom}\, f = A$.

Proof. $f \in \mathsf{Fun}(A, B)$ by proposition [541]. Follows by definition [493]. ☐

Proposition 544. Let f be a bijection from A to B. Let g be a bijection from B to C. Then $g \circ f$ is a bijection from A to C.

Proof. $g \circ f \in \mathsf{Surj}(A, C)$ by definition [539] and proposition [538]. $g \circ f$ is an injection. ☐

5.8 Converse as a function

Proposition 545. Let f be a function. Then f^{T} is injective.

Proposition 546. Suppose f is injective. Then f^{T} is a function.

Proposition 547. Let f be a bijection from A to B. Then f^{T} is a function.

Proof. Follows by definition [539] and proposition [546]. ☐

Proposition 548. Let f be a bijection from A to B. Then $f^{\mathsf{T}} \in \mathsf{Fun}(B, A)$.

Proof. $\mathsf{dom}\, f^{\mathsf{T}} = \mathsf{ran}\, f = B$ by definition [539], proposition [297], and corollary [536]. Omitted. ☐

Proposition 549. Let f be a bijection from A to B. Then $f^{\mathsf{T}} \in \mathsf{Surj}(B, A)$.

Proof. We have $f^{\mathsf{T}} \in \mathsf{Fun}(B, A)$ by proposition [548]. It suffices to show that $\mathsf{ran}\, f^{\mathsf{T}} = A$ by corollary [537]. We have $\mathsf{dom}\, f = A$ by proposition [543]. Thus $\mathsf{ran}\, f^{\mathsf{T}} = A$ by proposition [298]. ☐

Proposition 550. Let f be a bijection from A to B. Then f^{T} is a bijection from B to A.

Proof. $f^{\mathsf{T}} \in \mathsf{Fun}(B, A)$ by proposition [548]. f^{T} is injective by propositions [542] and [545]. $f^{\mathsf{T}} \in \mathsf{Surj}(B, A)$ by proposition [549]. Follows by definition [539]. ☐

5.8.1 Inverses of a function

Abbreviation 551. g is a left inverse of f iff for all $x \in \mathsf{dom}\, f$ we have $g(f(x)) = x$.

Abbreviation 552. g is a right inverse of f iff $f \circ g = \mathsf{id}_{\mathsf{dom}\, g}$.

Abbreviation 553. g is a right inverse of f on B iff $f \circ g = \mathsf{id}_B$.

Proposition 554. Let f be an injection. Then f^{T} is a left inverse of f.

Proof. f^{T} is a function by proposition [546].
Omitted. ☐

5.9 Identity function

Proposition 555. id_A is right-unique.

Proof. Follows by definitions [383] and [472] and axiom [134]. ☐

Proposition 556. id_A is a function.

Proposition 557. id_A is a function on A.

Proposition 558. id_A is a function to A.

Proposition 559. id_A is a function from A to A.

Proof. $\mathsf{id}_A \in \mathsf{Rel}(A, A)$ by proposition [391]. Follows by definition [493] and propositions [557] and [558]. ☐

Proposition 560. Suppose $a \in A$. Suppose $f = \mathsf{id}_A$. Then $f(a) = a$.

Proof. $(a, a) \in \mathsf{id}_A$ by proposition [384]. Follows by propositions [482] and [556]. ☐

Proposition 561. $\mathsf{id}_A \in \mathsf{Fun}(A, A)$.

Proof. id_A is a function. $\mathsf{id}_A \in \mathsf{Rel}(A, A)$. $\mathsf{dom}\, \mathsf{id}_A \subseteq A$. ☐

Proposition 562. $\mathsf{id}_A \in \mathsf{Surj}(A, A)$.

Proof. We have $\mathsf{id}_A \in \mathsf{Fun}(A, A)$ by proposition [561]. Omitted. $\square$

Proposition 563. $\mathsf{id}_A \in \mathsf{Bi}(A, A)$.

Proof. $\mathsf{id}_A \in \mathsf{Surj}(A, A)$ by proposition [562]. id_A is injective by proposition [471]. Follows by definition [539]. $\square$

6 Transitive sets

We use the word *transitive* to talk about sets as relations, so we will explicitly talk about *$\in$-transitivity* here.

Definition 564. A set A is $\in$-transitive iff for all x, y such that $x \in y \in A$ we have $x \in A$.

Proposition 565. A is $\in$-transitive iff for all $a \in A$ we have $a \subseteq A$.

Proposition 566. A is $\in$-transitive iff $A \subseteq \mathsf{Pow}(A)$.

Proof. For all $a \in A$ we have $a \subseteq A \iff a \in \mathsf{Pow}(A)$. Follows by propositions [9] and [565], definition [4], and axiom [201]. $\square$

Proposition 567. A is $\in$-transitive iff $\bigcup A^+ = A$.

Proof. Follows by definitions [4], [169] and [564], propositions [8], [179], [205], [206] and [566], and axiom [42]. $\square$

Proposition 568. A is $\in$-transitive iff $\bigcup A \subseteq A$.

Proposition 569. Suppose A is $\in$-transitive. Suppose $\{a, b\} \in A$. Then $a, b \in A$.

6.0.1 Closure properties of $\in$-transitive sets

Proposition 570. $\emptyset$ is $\in$-transitive.

Proposition 571. Suppose A and B are $\in$-transitive. Then $A \cup B$ is $\in$-transitive.

Proposition 572. Let A, B be $\in$-transitive sets. Then $A \cap B$ is $\in$-transitive.

Proposition 573. Let A be an $\in$-transitive set. Then A^+ is $\in$-transitive.

Proposition 574. Let A be an $\in$-transitive set. Then $\bigcup A$ is $\in$-transitive.

Proposition 575. Suppose every element of A is an $\in$-transitive set. Then $\bigcup A$ is $\in$-transitive.

Proof. Follows by definition [564] and axiom [42]. $\square$

Proposition 576. Suppose every element of A is an $\in$-transitive set. Then $\bigcap A$ is $\in$-transitive.

Proof. Follows by definitions [47] and [564] and proposition [575]. $\square$

7 Ordinals

Definition 577. α is an ordinal iff α is $\in$-transitive and every element of α is $\in$-transitive.

Proposition 578. Suppose α is $\in$-transitive. Suppose every element of α is $\in$-transitive. Then α is an ordinal.

Proposition 579. Let α be an ordinal. Then α is $\in$-transitive.

Proposition 580. Let α be an ordinal. Suppose $A \in \alpha$. Then A is $\in$-transitive.

Proposition 581. Let α be an ordinal. Suppose $\beta \in \alpha$. Then β is an ordinal.

Proposition 582. Suppose α^+ is an ordinal. Then α is an ordinal.

Proposition 583. Let α be an ordinal. Suppose $\beta \subseteq \alpha$. Suppose β is $\in$-transitive. Then β is an ordinal.

Proof. Follows by definitions [4] and [577]. ☐

Proposition 584. Let α, β be ordinals. Suppose $\alpha \in \beta$. Then $\alpha \subseteq \beta$.

Proposition 585. Let α be an ordinal. Suppose $\gamma \in \beta \in \alpha$. Then $\gamma \in \alpha$.

Proof. Follows by definitions [564] and [577]. ☐

Proposition 586. Let β be an ordinal. Suppose $\alpha \in \beta$. Then $\alpha^+ \subseteq \beta$.

Abbreviation 587. $\alpha < \beta$ iff β is an ordinal and $\alpha \in \beta$.

Abbreviation 588. $\alpha \leq \beta$ iff β is an ordinal and $\alpha \subseteq \beta$.

Lemma 589. Let α, β be sets. Suppose $\alpha < \beta$. Then α is an ordinal.

Proof. Follows by proposition [581]. ☐

We already have global irreflexivity and asymmetry of $\in$. $\in$ is transitive on ordinals by definition. To show that $\in$ is a strict total order it only remains to show that $\in$ is connex.

Proposition 590. For all ordinals α, β we have $\alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta$.

Proof by $\in$-induction on α. Assume α is an ordinal. Show for all ordinals γ we have $\alpha \in \gamma \vee \gamma \in \alpha \vee \alpha = \gamma$. *Subproof.* [Proof by $\in$-induction on γ] Assume γ is an ordinal. Follows by axiom [2] and definitions [564] and [577]. ☐ ☐

Proposition 591. Let α, β be ordinals. Suppose $\alpha \subset \beta$. Then $\alpha \in \beta$.

Proof. $\beta \setminus \alpha$ is inhabited. Take γ such that γ is an $\in$-minimal element of $\beta \setminus \alpha$. Now $\gamma \in \beta$ by proposition [108]. Hence $\gamma \subseteq \beta$ by definition [577] and proposition [565]. For all $\delta \in \beta \setminus \alpha$ we have $\delta \notin \gamma$. Thus $\gamma \setminus \alpha = \emptyset$. Hence $\gamma \subseteq \alpha$. It suffices to show that for all $\delta \in \alpha$ we have $\delta \in \gamma$. Suppose not. Take $\delta \in \alpha$ such that $\delta \notin \gamma$. Now if $\delta = \gamma$ or $\gamma \in \delta$, then $\gamma \in \alpha$ by definition [577] and propositions [9], [565], [581] and [590]. ☐

Proposition 592. Let α, β be ordinals. Suppose $\alpha \in \beta$. Then $\alpha \subset \beta$.

Proof. $\alpha \subseteq \beta$. $\square$

Proposition 593. Let α, β be ordinals. Suppose $\alpha \leq \beta$. Then $\alpha \subseteq \beta$.

Proof. Case: $\alpha = \beta$. Trivial.
Case: $\alpha < \beta$. $\alpha \subset \beta$. $\square$

Proposition 594. Let α, β be ordinals. Then $\alpha \in \beta$ or $\beta \subseteq \alpha$.

Proposition 595. Let α, β be ordinals. Then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Proposition 596. Let α, β be ordinals. Suppose $\alpha \subseteq \beta$. Then $\alpha \in \beta$ or $\alpha = \beta$.

Corollary 597. Let α, β be ordinals. Then $(\alpha \subset \beta \vee \beta \subset \alpha) \vee \alpha = \beta$.

Proposition 598. Let α, β be ordinals. Suppose neither $\alpha \in \beta$ nor $\beta \in \alpha$. Then $\alpha = \beta$.

Proof. Neither $\alpha \subset \beta$ nor $\beta \subset \alpha$. $\square$

Proposition 599. Let α, β be ordinals. Then $(\alpha \in \beta \vee \beta \in \alpha) \vee \alpha = \beta$.

Proof. Suppose not. Then neither $\alpha \in \beta$ nor $\beta \in \alpha$. Thus $\alpha = \beta$ by proposition [598]. Contradiction. $\square$

Corollary 600. Let α, β be ordinals. Suppose neither $\alpha < \beta$ nor $\beta < \alpha$. Then $\alpha = \beta$.

Proof. Follows by proposition [598]. $\square$

Corollary 601. Let α, β be ordinals. Then $\alpha \in \beta$ or $\beta \subseteq \alpha$.

7.0.1 Construction of ordinals

Proposition 602. $\emptyset$ is an ordinal.

Proposition 603. Let α be an ordinal. α^+ is an ordinal.

Proof. α^+ is $\in$-transitive by definition [577] and proposition [573]. For every $\beta \in \alpha$ we have that β is $\in$-transitive. $\square$

Proposition 604. α is an ordinal iff α^+ is an ordinal.

Proposition 605. Let α be an ordinal. Then $\alpha \in \alpha^+$.

Corollary 606. Let α be an ordinal. Then $\alpha < \alpha^+$.

Proposition 607. Let α, β be ordinals. Suppose $\alpha \in \beta$. Then $\alpha \subseteq \beta^+$.

Proof. $\alpha \subset \beta$. In particular, $\alpha \subseteq \beta$. Hence $\alpha \subseteq \mathsf{cons}(\beta, \beta)$. $\square$

Proposition 608. Let α be an ordinal. Then $\bigcup \alpha$ is an ordinal.

Proof. For all x, y such that $x \in y \in \bigcup \alpha$ we have $x \in \bigcup \alpha$ by proposition [43], axiom [42], and definitions [564] and [577]. Thus $\bigcup \alpha$ is $\in$-transitive. Every element of $\bigcup \alpha$ is $\in$-transitive. $\square$

Lemma 609. Let α be an ordinal. Then $\bigcup \alpha \subseteq \alpha$.

Proof. Follows by definition [577] and proposition [568]. $\square$

Proposition 610. Let α, β be ordinals. Then $\alpha \cup \beta$ is an ordinal.

Proof. $\alpha \cup \beta$ is $\in$-transitive by proposition [571] and definition [577]. Every element of $\alpha \cup \beta$ is $\in$-transitive by definitions [564] and [577] and axiom [56]. Follows by definition [577]. $\square$

Proposition 611. For all ordinals α we have $\alpha = \emptyset$ or $\emptyset \in \alpha$.

Proof by $\in$-induction. Straightforward. $\square$

Proposition 612. Let A be a set. Suppose that for every $\alpha \in A$ we have α is an ordinal. Suppose that A is $\in$-transitive. Then A is an ordinal.

Theorem 613. (Burali-Forti antimony) There exists no set Ω such that for all α we have $\alpha \in \Omega$ iff α is an ordinal.

Proof. Suppose not. Take Ω such that for all α we have $\alpha \in \Omega$ iff α is an ordinal. For all x, y such that $x \in y \in \Omega$ we have $x \in \Omega$. Thus Ω is $\in$-transitive. Thus Ω is an ordinal. Therefore $\Omega \in \Omega$. Contradiction. $\square$

Proposition 614. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then $\bigcap A$ is an ordinal.

Proof. It suffices to show that $\bigcap A$ is $\in$-transitive. $\square$

Proposition 615. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then for all $\alpha \in A$ we have $\bigcap A \subseteq \alpha$.

Proposition 616. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then $\bigcap A \in A$.

Proof. Follows by propositions [48], [53], [246], [596] and [614]. $\square$

Proposition 617. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then $\bigcap A$ is an $\in$-minimal element of A.

Proof. For all $\alpha \in A$ we have $\bigcap A \subseteq \alpha$. $\square$

Proposition 618. Let A be an inhabited set. Suppose for every $\alpha \in A$ we have α is an ordinal. Then for all $\alpha \in A$ we have $\bigcap A = \alpha$ or $\bigcap A \in \alpha$.

Proof. For all $\alpha \in A$ we have $\bigcap A \subseteq \alpha$. $\square$

Proposition 619. Let α, β be ordinals. Then $\alpha \cap \beta$ is an ordinal.

Proof. $\alpha \cap \beta$ is $\in$-transitive by definitions [80], [564] and [577]. Every element of $\alpha \cap \beta$ is $\in$-transitive by definitions [80], [564] and [577]. Follows by definition [577]. ☐

7.0.2 Limit and successor ordinals

Definition 620. λ is a limit ordinal iff $\emptyset < \lambda$ and for all $\alpha \in \lambda$ we have $\alpha^+ \in \lambda$.

Definition 621. α is a successor ordinal iff there exists an ordinal β such that $\alpha = \beta^+$.

Lemma 622. Let α be an ordinal such that $\emptyset < \alpha$. Then α is a limit ordinal or α is a successor ordinal.

Proof. Case: α is a limit ordinal. Trivial.

Case: α is not a limit ordinal. Take β such that $\beta \in \alpha$ and $\beta^+ \notin \alpha$ by definition [620]. ☐

Lemma 623. $\emptyset$ is not a successor ordinal.

Lemma 624. $\emptyset$ is not a limit ordinal.

Proof. Suppose not. Then $\emptyset < \emptyset$ by axiom [17] and definition [620]. Thus $\emptyset \in \emptyset$. Contradiction. ☐

Lemma 625. Let λ be a limit ordinal. Let $\alpha \in \lambda$. Then $\alpha^+ \in \lambda$.

Proof. Follows by definition [620]. ☐

Lemma 626. Let λ be a limit ordinal. Then $\bigcup \lambda = \lambda$.

Proof. $\bigcup \lambda \subseteq \lambda$ by definition [620] and lemma [609]. For all $\alpha \in \lambda$ we have $\alpha \in \alpha^+ \in \lambda$ by proposition [170] and lemma [625]. Thus $\lambda \subseteq \bigcup \lambda$ by definition [4] and proposition [43]. Follows by proposition [8]. ☐

7.1 Natural numbers as ordinals

Lemma 627. Let $n \in \mathcal{N}$. Suppose $n \neq \emptyset$. Then n is a successor ordinal.

Proof. Let $M = \{m \in \mathcal{N} \mid m = \emptyset$ or m is a successor ordinal$\}$. M is an inductive set by propositions [602] and [603], axiom [633], and definition [621]. Now $M \subseteq \mathcal{N} \subseteq M$ by definition [4] and axiom [634]. Thus $M = \mathcal{N}$. Follows by definition [4]. ☐

Lemma 628. $\mathcal{N}$ is $\in$-transitive.

Proof. Let $M = \{m \in \mathcal{N} \mid$ for all $n \in m$ we have $n \in \mathcal{N}\}$. $\emptyset \in M$. For all $n \in M$ we have $n^+ \in M$ by axiom [633] and definition [169]. Thus M is an inductive set. Now $M \subseteq \mathcal{N} \subseteq M$ by definition [4] and axiom [634]. Hence $\mathcal{N} = M$. ☐

Lemma 629. Every natural number is an ordinal.

Proof. Follows by propositions [174], [582] and [603], axiom [633], lemma [627], and definition [621]. ☐

Lemma 630. $\mathcal{N}$ is an ordinal.

Proof. Follows by lemmas [628] and [629] and proposition [612]. ☐

Lemma 631. $\mathcal{N}$ is a limit ordinal.

Proof. $\emptyset < \mathcal{N}$. If $n \in \mathcal{N}$, then $n^+ \in \mathcal{N}$. ☐

8 Natural numbers

Abbreviation 632. A is an inductive set iff $\emptyset \in A$ and for all $a \in A$ we have $a^+ \in A$.

Axiom 633. $\mathcal{N}$ is an inductive set.

Axiom 634. Let A be an inductive set. Then $\mathcal{N} \subseteq A$.

Abbreviation 635. n is a natural number iff $n \in \mathcal{N}$.

9 Cardinality

Definition 636. X is finite iff there exists a natural number k such that there exists a bijection from k to X.

Abbreviation 637. X is infinite iff X is not finite.

10 Magmas

Struct 638. A magma A is a onesorted structure equipped with

1. mul

such that

1. for all $a, b \in A$ we have $\mathsf{mul}_A(a, b) \in A$.

Abbreviation 639. $a \cdot b = \mathsf{mul}(a, b)$.

Abbreviation 640. a is an idempotent element of A iff $a \in A$ and $\mathsf{mul}_A(a, a) = a$.

Definition 641. $\mathsf{Idempotent}(A) = \{a \in A \mid \mathsf{mul}_A(a, a) = a\}$.

Abbreviation 642. a commutes with b iff $a \cdot b = b \cdot a$.

Definition 643. A is a submagma of B iff A is a magma and B is a magma and $A \subseteq B$ and $\mathsf{mul}_A \subseteq \mathsf{mul}_B$.

Proposition 644. Suppose A is a submagma of B. Suppose B is a submagma of C. Then A is a submagma of C.

Proof. Follows by definition [643] and proposition [11]. □

Struct 645. A unital magma A is a magma equipped with

1. e

such that

1. $\mathsf{e}_A \in A$.
2. for all $a \in A$ we have $\mathsf{mul}_A(a, \mathsf{e}_A) = a$.
3. for all $a \in A$ we have $\mathsf{mul}_A(\mathsf{e}_A, a) = a$.

Proposition 646. Let A be a unital magma. Then $\mathsf{mul}(\mathsf{e}, \mathsf{e}) = \mathsf{e}$.

Proposition 647. Let A be a unital magma. Let e be a set such that $e \in A$ and for all $x \in A$ we have $\mathsf{mul}(x, e) = x = \mathsf{mul}(e, x)$. Then $e = \mathsf{e}$.

Proof. Follows by items [1] and [3]. □

Definition 648. **(Left orbit)** $A \cdot x = \{\mathsf{mul}_A(a, x) \mid a \in A\}$.

Proposition 649. Let A be a magma. Let $e, f \in A$. Suppose $A \cdot e = A \cdot f$. Let $x \in A$. Then there exists $y \in A$ such that $x \cdot e = y \cdot f$.

Proof. We have $x \cdot e \in A \cdot e$ by definition [648]. Thus $x \cdot e \in A \cdot f$ by assumption. Take $y \in A$ such that $x \cdot e = y \cdot f$ by definition [648]. □

11 Semigroups

Struct 650. A semigroup A is a magma such that

1. for all a, b, c we have $\mathsf{mul}_A(a, \mathsf{mul}_A(b, c)) = \mathsf{mul}_A(\mathsf{mul}_A(a, b), c)$.

12 Regular semigroups

Struct 651. A regular semigroup A is a semigroup such that

1. for all a there exists $b \in A$ such that $\mathsf{mul}_A(a, \mathsf{mul}_A(b, a)) = a$.

13 Inverse semigroups

Struct 652. An inverse semigroup A is a regular semigroup such that

1. for all $a, b \in \mathsf{Idempotent}(A)$ we have $\mathsf{mul}_A(a, b) = \mathsf{mul}_A(b, a)$.

Proposition 653. Suppose A is an inverse semigroup. Then A is a semigroup.

Proposition 654. Suppose A is an inverse semigroup. Then A is a regular semigroup.

Proposition 655. Let A be an inverse semigroup. Let $e, f \in \mathsf{Idempotent}(A)$. Suppose for all $x \in A$ there exists $y \in A$ such that $x \cdot e = y \cdot f$. Suppose for all $x \in A$ there exists $y \in A$ such that $x \cdot f = y \cdot e$. Then $e = f$.

Proof. Take $x, y \in A$ such that $e = x \cdot f$ and $f = y \cdot e$ by definition [641].

$$
\begin{aligned}
e &= x \cdot f \quad \text{[by assumption]} \\
&= x \cdot (f \cdot f) \quad \text{[by definition [641]]} \\
&= (x \cdot f) \cdot f \quad \text{[by item [1] and proposition [653]]} \\
&= e \cdot f \quad \text{[by assumption]} \\
&= f \cdot e \quad \text{[by commutativity of idempotent elements]} \\
&= (y \cdot e) \cdot e \quad \text{[by assumption]} \\
&= y \cdot (e \cdot e) \quad \text{[by item [1] and proposition [653]]} \\
&= y \cdot e \quad \text{[by definition [641]]} \\
&= f \quad \text{[by assumption]}
\end{aligned}
$$

∎

Abbreviation 656. R is an order iff R is an antisymmetric quasiorder.

Abbreviation 657. R is an order on A iff R is an antisymmetric quasiorder on A.

Abbreviation 658. R is a strict order iff R is transitive and asymmetric.

Struct 659. An ordered set X is a quasiordered set such that

1. $\leq_X$ is antisymmetric.

Definition 660. $\mathsf{StrictOrderFromOrder}(R) = \{w \in R \mid \mathsf{fst}\, w \neq \mathsf{snd}\, w\}$.

Definition 661. $\mathsf{OrderFromStrictOrder}_A(R) = R \cup \mathsf{id}_A$.

Proposition 662. $(a, b) \in \mathsf{StrictOrderFromOrder}(R)$ iff $(a, b) \in R$ and $a \neq b$.

Proof. Follows by definition [660] and axioms [139] and [140]. ∎

Proposition 663. $\mathsf{OrderFromStrictOrder}_A(R)$ is reflexive on A.

Proposition 664. Suppose $(a, b) \in R$. Then $(a, b) \in \mathsf{OrderFromStrictOrder}_A(R)$.

Proof. $R \subseteq \mathsf{OrderFromStrictOrder}_A(R)$. ∎

Proposition 665. Suppose $(a, b) \in \mathsf{OrderFromStrictOrder}_A(R)$. Then $(a, b) \in R$ or $a = b$.

Proof. Follows by definitions [383] and [661], axiom [56], and propositions [31] and [662]. ∎

Proposition 666. $(a, b) \in \mathsf{OrderFromStrictOrder}_A(R)$ iff $(a, b) \in R$ or $a = b \in A$.

Proposition 667. Suppose R is an order. Then $\mathsf{StrictOrderFromOrder}(R)$ is a strict order.

Proof. $\mathsf{StrictOrderFromOrder}(R)$ is asymmetric. $\mathsf{StrictOrderFromOrder}(R)$ is transitive. $\square$

Proposition 668. Suppose R is a strict order. Suppose R is a binary relation on A. Then $\mathsf{OrderFromStrictOrder}_A(R)$ is an order on A.

Proof. $\mathsf{OrderFromStrictOrder}_A(R)$ is antisymmetric. $\mathsf{OrderFromStrictOrder}_A(R)$ is transitive by definition [413] and proposition [666]. $\mathsf{OrderFromStrictOrder}_A(R)$ is reflexive on A. $\square$

Proposition 669. $\subseteq_A$ is antisymmetric.

Proof. Follows by definitions [396] and [407], axiom [134], and proposition [8]. $\square$

Proposition 670. $\subseteq_A$ is an order on A.

Proof. $\subseteq_A$ is a quasiorder on A by proposition [429]. $\subseteq_A$ is antisymmetric by proposition [669]. $\square$

Struct 671. A meet semilattice X is a partial order equipped with

1. $\sqcap$

such that

1. for all $x, y \in X$ we have $\sqcap_X(x, y) \in X$.
2. for all $x, y \in X$ we have $\sqcap_X(x, y) \leq_X x, y$.
3. for all $a, x, y \in X$ such that $a \leq_X x, y$ we have $a \leq_X \sqcap_X(x, y)$.

Proposition 672. Let X be a meet semilattice. Then $\sqcap(x, x) = x$.

Proof. $\sqcap(x, x) \leq x$. $x \leq_X x, x$. Thus $x \leq_X \sqcap(x, x)$. $\square$

14 Topological spaces

Struct 673. A topological space X is a onesorted structure equipped with

1. $\mathcal{O}$

such that

1. $\mathcal{O}_X$ is a family of subsets of X.
2. $\emptyset \in \mathcal{O}_X$.
3. $X \in \mathcal{O}_X$.

4. For all $A, B \in \mathcal{O}_X$ we have $A \cap B \in \mathcal{O}_X$.
5. For all $F \subseteq \mathcal{O}_X$ we have $\bigcup F \in \mathcal{O}_X$.

Abbreviation 674. U is open iff $U \in \mathcal{O}$.

Abbreviation 675. U is open in X iff $U \in \mathcal{O}_X$.

Proposition 676. Let X be a topological space. Suppose A, B are open. Then $A \cup B$ is open.

Proof. $\{A, B\} \subseteq \mathcal{O}$. $\bigcup\{A, B\}$ is open. $\bigcup\{A, B\} = A \cup B$. □

Definition 677. **(Interiors)** $\mathsf{Int}_X\, A = \{U \in \mathcal{O}_X \mid U \subseteq A\}$.

Definition 678. **(Interior)** $\mathsf{int}_X\, A = \bigcup \mathsf{Int}_X\, A$.

Proposition 679. **(Interior)** Suppose $U \in \mathcal{O}_X$ and $a \in U \subseteq A$. Then $a \in \mathsf{int}_X\, A$.

Proof. $U \in \mathsf{Int}_X\, A$. □

Proposition 680. **(Interior)** Suppose $a \in \mathsf{int}_X\, A$. Then there exists $U \in \mathcal{O}_X$ such that $a \in U \subseteq A$.

Proof. Take $U \in \mathsf{Int}_X\, A$ such that $a \in U$. □

Proposition 681. **(Interior)** $a \in \mathsf{int}_X\, A$ iff there exists $U \in \mathcal{O}_X$ such that $a \in U \subseteq A$.

Proof. Follows by propositions [679] and [680]. □

Proposition 682. Let X be a topological space. Suppose U is open in X. Then $\mathsf{int}_X\, U = U$.

Proof. $U \in \mathsf{Int}_X\, U$. Follows by definition [4] and propositions [3] and [681]. □

Proposition 683. Let X be a topological space. Then $\mathsf{int}_X\, A$ is open.

Proof. $\mathsf{Int}_X\, A \subseteq \mathcal{O}_X$. □

Proposition 684. Then $\mathsf{int}_X\, A \subseteq A$.

Proposition 685. Let X be a topological space. Suppose $U \subseteq A \subseteq X$. Suppose U is open. Then $U \subseteq \mathsf{int}_X\, A$.

Proposition 686. Let X be a topological space. Suppose $\mathsf{int}_X\, A = A$. Then A is open.

Corollary 687. Let X be a topological space. Then $\mathsf{int}_X\, A = A$ iff A is open in X.

Proposition 688. Let X be a topological space. $\mathsf{int}_X\, X = X$.

Proof. $X \in \mathcal{O}_X$. $X \subseteq X$ by proposition [7]. Thus $X \in \mathsf{Int}_X\, X$ by definition [677]. Follows by set extensionality. □

Proposition 689. Let X be a topological space. Then $\mathsf{int}_X A \in \mathsf{Pow}(X)$.

Proof. We have $\mathsf{Int}_X A \subseteq \mathsf{Pow}(X)$. Thus $\mathsf{int}_X A \subseteq X$ by definition [678] and proposition [206]. □

14.1 Closed sets

Definition 690. A is closed in X iff $X \setminus A$ is open in X.

Abbreviation 691. A is clopen in X iff A is open in X and closed in X.

Proposition 692. Let X be a topological space. Then $\emptyset$ is closed in X.

Proof. $X \setminus \emptyset = X$. □

Proposition 693. Let X be a topological space. Then $\emptyset$ is closed in X.

Proof. $X \setminus X = \emptyset$. □

Definition 694. **(Closed sets)** $\mathcal{C}_X = \{A \in \mathsf{Pow}(X) \mid A \text{ is closed in } X\}$.

Proposition 695. Let X be a topological space. Let $U \in \mathcal{O}_X$. Then $X \setminus U \in \mathcal{C}_X$.

Proof. $X \setminus U \in \mathsf{Pow}(X)$. $U \subseteq X$ by item [1]. Hence $X \setminus (X \setminus U) = U$ by proposition [114]. $X \setminus U$ is closed in X. □

Definition 696. **(Closed covers)** $\mathsf{Cl}_X A = \{D \in \mathsf{Pow}(X) \mid A \subseteq D \text{ and } D \text{ is closed in } X\}$.

Definition 697. **(Closure)** $\mathsf{cl}_X A = \bigcap \mathsf{Cl}_X A$.

Proposition 698. Let X be a topological space. Then $\mathsf{cl}_X \emptyset = \emptyset$.

Proof. $\emptyset \in \mathsf{Cl}_X \emptyset$. □

Proposition 699. Let X be a topological space. Then $\mathsf{cl}_X X = X$.

Proof. For all $D \in \mathsf{Cl}_X X$ we have $X = D$ by axiom [201], definition [696], and proposition [8]. Now $X \in \mathsf{Cl}_X X$. Thus $\mathsf{Cl}_X X = \{X\}$ by proposition [39]. Follows by proposition [54] and definition [697]. □

Proposition 700. $X \setminus \mathsf{int}_X A = \mathsf{cl}_X (X \setminus A)$.

Proof. Omitted. □

Definition 701. **(Frontier)** $\mathsf{fr}_X A = \mathsf{cl}_X A \setminus \mathsf{int}_X A$.

Proposition 702. $\mathsf{fr}_X A = \mathsf{cl}_X A \cap \mathsf{cl}_X (X \setminus A)$.

Proof. Omitted. □

Proposition 703. Let X be a topological space. Then $\mathsf{fr}_X \emptyset = \emptyset$.

Proof. Follows by set extensionality. □

Proposition 704. Let X be a topological space. Then $\mathsf{fr}_X X = \emptyset$.

Proof. $\mathsf{fr}_X X = X \setminus X$ by definition [701] and propositions [688] and [699]. Follows by proposition [112]. ☐

Definition 705. $\mathsf{N}_X x = \{U \in \mathcal{O}_X \mid x \in U\}$.

14.2 Topological basis

Abbreviation 706. C covers X iff for all $x \in X$ there exists $U \in C$ such that $x \in U$.

Proposition 707. Suppose C covers X. Then $X \subseteq \bigcup C$.

Proposition 708. Suppose $X \subseteq \bigcup C$. Then C covers X.

Abbreviation 709. B is a topological prebasis for X iff $\bigcup B = X$.

Proposition 710. B is a topological prebasis for X iff B is a family of subsets of X and B covers X.

Proof. If B is a family of subsets of X and B covers X, then $\bigcup B = X$ by propositions [8], [45] and [707]. If $\bigcup B = X$, then B is a family of subsets of X and B covers X by propositions [7], [707] and [708]. ☐

Definition 711. B is a topological basis for X iff B is a topological prebasis for X and for all U, V, x such that $U, V \in B$ and $x \in U, V$ there exists $W \in B$ such that $x \in W \subseteq U, V$.

14.3 Disconnections

Definition 712. $\mathsf{Disconnections}\, X = \{p \in \mathsf{Bipartitions}\, X \mid \mathsf{fst}\, p, \mathsf{snd}\, p \in \mathcal{O}_X\}$.

Abbreviation 713. D is a disconnection of X iff $D \in \mathsf{Disconnections}\, X$.

Definition 714. X is disconnected iff there exist $U, V \in \mathcal{O}_X$ such that X is partitioned by U and V.

Proposition 715. Let X be a topological space. Suppose X is disconnected. Then there exists a disconnection of X.

Proof. Take $U, V \in \mathcal{O}_X$ such that X is partitioned by U and V by definition [714]. Then (U, V) is a bipartition of X. Thus (U, V) is a disconnection of X by definition [712] and propositions [144] and [151]. ☐

Proposition 716. Let X be a topological space. Let D be a disconnection of X. Then X is disconnected.

Proof. $\mathsf{fst}\, D, \mathsf{snd}\, D \in \mathcal{O}_X$. X is partitioned by $\mathsf{fst}\, D$ and $\mathsf{snd}\, D$. ☐

Abbreviation 717. X is connected iff X is not disconnected.