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Crew pairing at Air France

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Abstract

In the airline industry, crew schedules consist of a number of pairings. These are round trips originating and terminating at the same crew home base composed of legal work days, called duties, separated by rest periods. The purpose of the airline crew pairing problem is to generate a set of minimal cost crew pairings covering all flight legs. The set of pairings must satisfy all the rules in the work convention and all the appropriate air traffic regulations. The resulting constraints can affect duty construction, may restrict each pairing, or be imposed on the overall crew schedule.

The pairing problem is formulated as an integer, nonlinear multi-commodity network flow problem with additional resource variables. Nonlinearities occur in the objective function as well as in a large subset of constraints. A branch-and-bound algorithm based on an extension of the Dantzig-Wolfe decomposition principle is used to solve this model. The master problem becomes a Set Partitioning type model, as in the classical formulation, while pairings are generated using resource constrained shortest path subproblems. This primal approach implicitly considers all feasible pairings and also provides the optimality gap value on a feasible solution. A nice feature of this decomposition process is that it isolates all nonlinear aspects of the proposed multi-commodity model in the subproblems which are solved by means of a specialized dynamic programming algorithm.

We present the application and implementation of this approach at Air France. It is one of the first implementations of an optimal approach for a large airline carrier. We have chosen a subproblem network representation where the duties rather than the legs are on the arcs. This ensures feasibility relative to duty restrictions by definition. As opposed to Lavoie, Minoux and Odier (1988), the nonlinear cost function is modeled without approximations. The computational experiments were conducted using actual Air France medium haul data. Even if the branch-and-bound trees were not fully explored in all cases, the gaps certify that the computed solutions are within a fraction of one percentage point of the optimality. Our results illustrate that our approach produced substantial improvements over solutions derived by the expert system in use at Air France. Their magnitude led to the eventual implementation of the approach.

Keywords: Crew pairing; Integer nonlinear programming; Multi-commodity flow model; Dantzig-Wolfe decomposition; Set partitioning

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1. Introduction

The objective of the airline crew pairing problem is to minimize the costs associated with assigning crews to flight legs. Exactly one operational crew must be assigned to each leg of a given flight schedule. Legs must be combined to form legal work days or duties separated by rest periods to yield pairings. These are round trips originating and terminating at the same crew home base. To be legal, the set of pairings must satisfy all the rules in the work convention and all the appropriate air traffic regulations. The resulting constraints can affect duty construction, may restrict each pairing, or be imposed on the overall crew schedule. Generally, pairing construction has a time horizon of several days (e.g., a week for medium haul carriers). These pairings are then used as input for the monthly (or longer) crew workload assignment problem, also referred to as the block assignment problem.

Crew pairing is a very important problem in today's highly competitive global market. Given that crew costs are only exceeded by fixed aircraft costs and the cost of the fuel consumed, significant reductions can be generated by solving the crew pairing problem optimally. The magnitude of these savings for major airlines is exemplified by the fact that a one percent decrease in the total crew costs often amounts to tens of million of dollars per year in additional profits.

A variety of solution methodologies have been suggested for this problem over its thirty year history. Yet, they all have relied in one form or another on a common strategy consisting of two distinct phases: (1) pairing enumeration, and (2) pairing selection. Pairing enumeration, though relatively straightforward, requires careful analysis of complex rules and regulations. Unfortunately, even for small to medium size instances, an extremely large number of legal pairings must be enumerated, making it extremely difficult to identify the minimum cost set. To alleviate this, current systems employ reduction strategies such as: selecting only the best pairings in the enumeration process, splitting the problem into partial problems, or extending the solution of a daily problem to a weekly one. Generally, these have a significant negative impact on solution quality. The economics of the global commercial aviation market have led most companies to assess the capability of their current systems. Computational tests conducted using actual data suggest that solutions provided by systems presently in use can be significantly improved by novel optimization approaches.

One such algorithm is described in this paper. It is based on an integer, nonlinear multi-commodity flow model with resource variables. This is solved using an extension of the Dantzig–Wolfe decomposition principle (Dantzig and Wolfe, 1960) embedded in a branch-and-bound framework. A nice feature of this decomposition process is that it isolates all nonlinear aspects of the proposed multi-commodity model in the subproblems which are solved by means of a specialized dynamic programming algorithm.

The contribution of this paper is to apply the proposed model and solution approach to the airline crew pairing problem. The model permits to consider nonlinear cost and resource extension functions. Certainly, the complexity of these functions influences the algorithmic capability. The solution approach implicitly considers all pairings, it allows the derivation of a valid lower bound and hence a true estimation of the integrality gap. It also establishes a correspondence with solution approaches based on the classical Set Partitioning model. It is superior to such methods when these restrict a priori the number of pairings to be considered. The generic solution methodology presented here is not completely new as it has been successfully used for bus driver scheduling (Desrochers and Soumis, 1989). Yet, the airline environment contains many additional complexities not encountered in urban transportation. This approach also generalizes previous column generation schemes used to solve this problem as it considers the utilization of resource variables in the pairing generation process. This allows for the removal of the cost function approximations used in previous solution methods such as in Lavoie, Minoux and Odier (1988) and Barnhart et al. (1994). This paper describes one of the first implementations of an optimal approach for a large airline carrier.

The paper is organized as follows. The next section examines the existing literature. Then, Section 3 presents the proposed integer, nonlinear multi-commodity flow model with resource variables and the solution process used to solve it. Next, Section 4 describes their application in the context of Air France, while Section 5 discusses the computational results for medium haul data. Finally, we present our conclusions and perspectives in Section 6.

2. Literature review

2.1. Comprehensive surveys

The initial review by Arabeyre et al. (1969), several later papers by Bornemann (1982), Etschmaier and Mathaisel (1985), Gershkoff (1989), and Barutt and Hull (1990), and a chapter of Teodorović's book (1988) illustrate the variety of approaches used before 1990. Recently, Desrosiers et al. (1994) and Desaulniers et al. (1994) have proposed unified formulations for time constrained vehicle routing and crew scheduling problems. They also survey current advances in the airline crew pairing problem.

2.2. Relevant papers

Previous research has formulated the airline crew pairing problem as a Set Partitioning problem, where each variable corresponds to a feasible pairing. A Set Covering formulation is used when overcovering of flight legs is allowed at the same cost as usual covering by an active crew. A number of researchers have tackled relatively small problems, involving a manageable number of feasible pairings that can all be generated a priori. These include Gerbracht (1978), Marsten, Muller and Killion (1979), Marsten and Shepardson (1981), and Ryan and Garner (1985). Generally however, the number of feasible pairings in most real-world problems gives rise to a huge number of variables. To cope with the computational burden, many approaches have restricted either the number of pairings generated or the size of the Set Partitioning problem to be solved or both. By limiting a priori the pairings generated to the most promising ones, Baker, Bodin and Fisher (1985) solve instances of a large problem faced by Federal Express. Local improvement procedures are then applied to improve the solution.

Several approaches consist of alternating between a heuristic pairing generator and a Set Partitioning problem optimizer restricted to the variables associated with the pairings generated. These include the algorithms of Rubin (1973), Ball and Roberts (1985), Gershkoff (1989), Anbil et al. (1991). Recently, Graves et al. (1993) have enhanced this algorithmic framework to allow over- and undercovering of flight legs. Cutting planes and block echelon enumeration are utilized to solve the Set Partitioning problem. Wedelin (1993) has presented an approximation algorithm consisting of an ascent algorithm for the dual of the linear relaxation of the Set Partitioning problem, along with an approximation scheme which manipulates the reduced costs to produce integer solutions. This method is used in conjunction with a heuristic pairing generator. Hoffman and Padberg (1993) have proposed an exact branch-and-cut approach. It is based on a heuristic to obtain good integer-feasible solutions quickly and a cut generation procedure to tighten the linear relaxation.

Several column generation based approaches have also been suggested. Crainic and Rousseau (1987) use a partial enumeration method to generate new negative cost pairings and a heuristic single branch enumeration method to obtain an integer solution. Lavoie, Minoux and Odier (1988) generate pairings by solving a classical shortest path problem in an expanded network which includes all single-pairing constraints. The reduced cost of a pairing is evaluated by a linear approximation. A similar approach is also used by Barnhart et al. (1994).

2.3. Commercial systems

Over the years, several commercial systems for constructing crew pairings have been utilized by large airline companies. These include: TPACS (Rubin, 1973) at United Airlines, TRIP (Gershkoff, 1989; Anbil et al., 1991) at American and Continental Airlines, CARMEN (Wedelin, 1993) at SAS, Lufthansa, Alitalia and KLM, ALPPS at Northwest and USAir, and KORBX (developed by AT&T) at Delta, Canadian and Quantas Airlines. Furthermore, Federal Express and Air France used systems based on the approaches developed by Baker, Bodin and Fisher (1985) and Lavoie, Minoux and Odier (1988), respectively. Recently, Delta Airlines has started to use the software package BANG based on the method proposed by Carter, Marsten and Subramanian (1995).

The solution approach described in this paper has led to the development of the GENCOL optimizer which is used in various areas of the vehicle routing and crew scheduling field. It is incorporated within ALTITUDE (Desrosiers et al., 1995), an airline operations management system which solves aircraft routing problems at Air Transat, crew pairing problems at Air Transat and a large U.S. carrier and block assignment problems at Air Transat and Air Canada. This optimizer is also one of the core modules of HASTUS (Hamer and Séguin, 1992; Rousseau and Desrosiers 1995), a system designed for bus driver scheduling and currently in use in Helsinki, Lyon, Singapore, Toulouse, Tokyo and Vienna. These applications are only part of GENCOL's broad application domain. Its potential applicability in other areas has been discussed in Desrosiers et al. (1994) and Desaulniers et al. (1994).

3. An integer, nonlinear multi-commodity network flow model with resource variables

In this section, we present an integer, nonlinear multi-commodity flow formulation with resource variables for the airline crew pairing problem. Nonlinearities occur in the objective function as well as in a large subset of constraints. By applying an extension of the decomposition principle of Dantzig and Wolfe (1960) on this model, the resulting master problem to be solved is a Set Partitioning type model while the subproblem is a constrained shortest path problem with resource variables. Column generation is a classical concept first applied by Ford and Fulkerson (1958) to a multi-commodity network flow model and then by Gilmore and Gomory (1961, 1963) to the classical cutting-stock problem. The reader is also referred to Lasdon (1970) for a comprehensive presentation of this concept and other applications.

3.1. Notation

Let F, indexed by f, represent the set of operational flight legs to be covered exactly once by a set of pairings. Let K, indexed by k, be the set of commodities or crews. In the proposed model, a specific commodity is associated with each crew. Hence, the model is very general and flexible as it can make use of distinct crews, even those whose initial conditions such as current location and worked time are different. In the proposed model, a specific commodity is associated with each crew. Hence, the model is very general and flexible as it can make use of distinct crews, even those whose initial conditions such as current location and worked time are different. However, commodities can also be aggregated when they represent identical crews.

With each crew k, associate a graph $G^k = (V^k, A^k)$, where V^k is the set of nodes and A^k is the set of arcs. This graph is a time-space oriented graph whose nodes represent station locations at different times and arcs may represent various crew activities such as briefing, debriefing, operational or deadhead flight legs, ground transportation, rests, connections, duties, and even complete pairings. A single arc can represent a single or multiple crew activities. Define $A^k_f \subseteq A^k$ as the subset of arcs in G^k containing the operational flight leg f. The set V^k contains two special nodes: the source node $o(k)$ and the sink node $d(k)$. All feasible paths in G^k corresponding to feasible pairings for crew k originate at $o(k)$ and end at $d(k)$. Let $N^k = V^k \setminus \{o(k), d(k)\}$. There also exists an origin–destination arc $(o(k), d(k))$ which represents, depending on its parameter values, an unused crew or a complete pairing.

Let R^k, indexed by r, be the set of resources used on graph G^k. These resources are used to model the restrictions on a pairing such as the maximum allowed time away from base. Denote by $[a_i^{kr}, b_i^{kr}]$ the interval restricting the amount of resource r used to reach node i on a path in G^k. The initial conditions at the source node are given by $a^{kr}_{o(k)} = b^{kr}_{o(k)}$, for each resource r and for each crew k. Resource variables are further discussed in the next section and Section 4.3.

Let M, indexed by m, be a set of global constraints on a solution set of pairings. These constraints can not be checked during the path construction of a single pairing since they involve several or even all pairings. Yet, they are controlled at a higher level. For a given global constraint m, let $B^k_m \subseteq A^k$ be the subset of arcs in G^k involved in the constraint definition for crew k. Specific coefficients $b^k_{m,ij}$ are assigned to the corresponding arcs $(i,j) \in B^k_m$. Let also $\underline{b}_m$ and $\overline{b}_m$ be the lower and upper bounds of global constraint m, respectively.

Three types of decision variables are used. The *network flow variables* are defined on the arcs of G^k; they are given by $\boldsymbol{X}^k = (X^k_{ij})$, $k \in K$, $(i,j) \in A^k$. For a given crew k, the flow variable X^k_{ij} takes only binary values, i.e., the value 1 if node j directly follows node i, and the value 0, otherwise. The *resource variables* are defined on the nodes of G^k; they are given by $\boldsymbol{T}^k_i = (T^{kr}_i)$, $k \in K$, $r \in R^k$, $i \in V^k$. Each resource variable T_i^{kr} is restricted to be within the resource interval $[a_i^{kr}, b_i^{kr}]$ defined above. The model also comprises *supplementary variables* Y_s in I_s, $s \in F \cup M$, used to relax the flight covering and the global constraints.

The cost function is composed of three parts. The first involves functions g_{ij}^k, $k \in K$ and $(i,j) \in A^k$ which represent the cost of using the arcs while the second uses functions g_i^k, $k \in K$ and $i \in V^k \setminus \{o(k)\}$ which represent the cost of visiting the nodes. These functions are dependent of the resource vector $\boldsymbol{T}_i^k$ and are usually nonlinear. In the Air France case study described in Section 4, only the node costs g_i^k are complex nonlinear functions. The last part concerns the cost of the supplementary variables which is computed using the cost functions $c_s(Y_s)$, $s \in F \cup M$.

3.2. A mathematical formulation

Formally, the integer, nonlinear multi-commodity network flow model with resource variables is:

Minimize

$$\sum_{k\in K} \sum_{(i,j)\in A^k} g_{ij}^k(\boldsymbol{T}_i^k) X_{ij}^k + \sum_{k\in K} \sum_{i\in V^k\setminus\{o(k)\}} g_i^k(\boldsymbol{T}_i^k) \sum_{j:(j,i)\in A^k} X_{ij}^k + \sum_{s\in F\cup M} c_s(Y_s), \tag{3.1}$$

subject to:

$$\sum_{k\in K} \sum_{(i,j)\in A_f^k} X_{ij}^k + Y_f = 1, \quad \forall f \in F, \tag{3.2}$$

$$\underline{b}_m \leqslant \sum_{k\in K} \sum_{(i,j)\in B_m^k} b_{m,ij}^k X_{ij}^k + Y_m \leqslant \overline{b}_m, \quad \forall m \in M, \tag{3.3}$$

$$Y_s \in I_s, \quad \forall s \in F \cup M, \tag{3.4}$$

$$\sum_{j:(o(k),j)\in A^k} X_{o(k),j}^k = 1, \quad \forall k \in K, \tag{3.5}$$

$$\sum_{i:(i,j)\in A^k} X_{ij}^k - \sum_{i:(j,i)\in A^k} X_{ji}^k = 0, \quad \forall k \in K,\ \forall j \in N^k, \tag{3.6}$$

$$\sum_{i:(i,d(k))\in A^k} X_{i,d(k)}^k = 1, \quad \forall k \in K, \tag{3.7}$$

$$X_{ij}^k (f_{ij}^{kr}(\boldsymbol{T}_i^k) - T_j^{kr}) \leqslant 0, \quad \forall k \in K,\ \forall r \in R^k,\ \forall (i,j) \in A^k, \tag{3.8}$$

$$a_i^{kr} \leqslant T_i^{kr} \leqslant b_i^{kr}, \quad \forall k \in K,\ \forall r \in R^k,\ \forall i \in V^k, \tag{3.9}$$

$$X_{ij}^k \text{ binary}, \quad \forall k \in K,\ \forall (i,j) \in A^k. \tag{3.10}$$

The cost function (3.1) depends on the flow, the resource and the supplementary variables. We assume that it is separable by crew except for the total cost of the supplementary variables. We also assume that each cost function $c_s(Y_s)$ can be approximated by a piecewise linear function. Relations (3.2) require that each operational flight leg in F be covered exactly once while relations (3.3) impose a set of global constraints. The set of constraints (3.5)–(3.7) represents a path structure for each crew k: the availability of a unit crew at the origin node $o(k)$, a demand of the same unit crew at the destination node $d(k)$, and the flow conservation equations at the nodes of N^k. Bounded resource variables defined in (3.9) and flow variables defined in (3.10) are linked in (3.8), on each arc (i,j), and for each crew k. The function $f_{ij}^{kr}(\boldsymbol{T}_i^k)$ is used for the resource extension from node i to node j, for resource r and crew k. This function may be linear or nonlinear and it may depend on several resources. This is a generalization of the commonly used linear single resource extension function $f_{ij}^{kr}(\boldsymbol{T}_i^k) = T_i^{kr} + t_{ij}^{kr}$, for all $r \in R^k$.

Interval restrictions on resource variables are hard constraints. For example, if resource r involves time consumption, then waiting until the earliest admissible time is required. Other resource types are interpreted in a similar manner. In the solution process, if $X_{ij}^k = 1$, and if the cost functions g_{ij}^k and g_i^k are non-decreasing in terms of the resource variables, then the resource variables can be computed as:

$$T_j^{kr} = \max\{f_{ij}^{kr}(\boldsymbol{T}_i^k), a_j^{kr}\}, \tag{3.11}$$

provided that the extension is feasible at j, i.e., $f_{ij}^{kr}(\boldsymbol{T}_i^k) \leqslant b_j^{kr}$, for all resources $r \in R^k$. As shown in Desaulniers et al. (1994), the solution process to be described shortly is valid for any type of cost and resource extension function. However, the modeling and computational aspects are streamlined when all the functions g_{ij}^k, g_i^k and f_{ij}^{kr} are non-decreasing in terms of the resource variables as in the Air France application. When it is not the case, given integer data, one can discretize the resource intervals over the decreasing part of the functions.

Supplementary variables have been introduced by Graves et al. (1993). First, if $f \in F$, variables Y_f allow for undercovering of a flight leg when the corresponding supplementary variable takes the value 1. These also permit overcovering, i.e., deadheading on operational flight legs at a certain cost. That is, crews travel as passengers and are repositioned according to the schedule needs. Potential deadhead flights may also be chosen from a set of legs not required to be covered. An efficient methodology for dynamically selecting deadhead flights in the long haul context can be found in Barnhart, Hatay and Johnson (1995). Second, if $m \in M$, the supplementary variables Y_m allow to violate global constraints.

The resource variables are used to model the *local constraints* (3.7)-(3.9). These constraints only restrict the construction of a single pairing. Examples include certain government and collective agreement rules. Some operating rules can easily be imposed on the underlying network by adequately defining all feasible arcs and accordingly forbidding all illegal ones. If the flight legs are represented by arcs, an example of such a rule is the connection time between two legs. Alternatively when the duties are represented by arcs, the rest period between duties can be modeled in this fashion. In the latter representation, other constraints such as the time away from base, the time flown and the number of landings per pairing are modeled using resource variables. The feasibility of the corresponding restrictions for duties is ensured by definition. Therefore a duty-based representation needs fewer resources than a leg-based one.

Finally, the model incorporates the *global constraints* (3.3). These are defined on several or all pairings. Since the set of crews can be partitioned according to the home base location where manpower is assigned, examples of global constraints are those defined on the number of available crews or the number of time credits per home base. Others may consider the distribution of the number of duties per pairing and of the number of national and international pairings. Additional examples are given later in the Air France case study.

3.3. Solution process

Model (3.1)-(3.10) consists of a block angular structure (3.5)-(3.10) separable per crew, a cost function (3.1) also separable per crew and the linking constraints, i.e., the covering of the operational flight legs (3.2) and a set of global constraints (3.3). This integer, nonlinear formulation can be solved using an extension of the Dantzig-Wolfe decomposition principle embedded in a branch-and-bound framework. The master problem is comprised of relations (3.1)-(3.3), i.e., the cost function and the linking constraints. Each subproblem k is defined using the corresponding constraint sets in (3.5)-(3.10) as well as the corresponding nonlinear crew cost function in (3.1) updated according to the dual information transferred from the master problem solution. Next, we first give the master problem formulation, then the subproblem structure and finally the basic branch-and-bound strategies to obtain integer solutions.

3.3.1. Master problem

Let Ω^k, indexed by p, be the set of extreme points of subproblem $k \in K$. In fact, these extreme points correspond to feasible crew paths in G^k or pairings. Define a_{fp}^k to be equal to 1 if pairing p for crew k covers operational flight leg $f \in F$, and 0 otherwise; b_{mp}^k as the contribution to global constraint $m \in M$ of pairing p for crew k; and c_p^k as the cost of pairing p for crew k. Define also a binary variable θ_p^k for each path p in G^k, which takes the value 1 if pairing p is assigned to crew k, and 0 otherwise. These new decision variables are called *path* variables.

Using this notation, the master problem (3.1)-(3.3) can be written as:

Minimize

$$\sum_{k\in K} \sum_{p\in\Omega^k} c_p^k \theta_p^k + \sum_{s\in F\cup M} c_s(Y_s), \tag{3.12}$$

subject to:

$$\sum_{k\in K} \sum_{p\in\Omega^k} a_{fp}^k \theta_p^k + Y_f = 1, \quad \forall f \in F, \tag{3.13}$$

$$\underline{b}_m \leqslant \sum_{k\in K} \sum_{p\in\Omega^k} b_{mp}^k \theta_p^k + Y_m \leqslant \overline{b}_m, \quad \forall m \in M, \tag{3.14}$$

$$Y_s \in I_s, \quad \forall s \in F \cup M, \tag{3.15}$$

$$\sum_{p \in \Omega^k} \theta_p^k = 1, \quad \forall k \in K, \forall p \in \Omega^k, \tag{3.16}$$

$$\theta_p^k \geqslant 0, \quad \forall k \in K, \forall p \in \Omega^k, \tag{3.17}$$

$$\theta_p^k \text{ binary}, \quad \forall k \in K, \forall p \in \Omega^k. \tag{3.18}$$

As shown in Desaulniers et al. (1994), the problem formulation (3.12)–(3.18) is equivalent to the integer, nonlinear multi-commodity network flow model (3.1)–(3.10). Its linear relaxation (3.12)–(3.17) is solved at each node of the search tree to obtain a lower bound. However, since it usually contains a huge number of variables (one for each feasible crew path in each graph G^k), we resort to a column generation scheme to solve it.

Therefore at each iteration we solve a restricted master problem which only considers a relatively small subset of path variables using linear programming. Starting with just the supplementary variables (and the path variables corresponding to an initial solution if one is provided) in the restricted master problem, at each iteration new path variables and their coefficient columns are generated from the subproblems. One or several new path variables with a negative reduced cost are then added to the current restricted master problem. Barring degeneracy, the addition of these new path variables ensures an improvement over the current solution at the next iteration. This iterative process continues until the subproblems cannot further generate negative reduced cost path variables. While the pairing costs appear as constant coefficients in the master problem, the objective function of each subproblem remains nonlinear.

3.3.2. Subproblems

The modified objective function together with constraint sets (3.5)–(3.10) define $|K|$ subproblems, each separable by crew k. Let $\boldsymbol{\alpha} = \{\alpha_f | f \in F\}$, $\boldsymbol{\beta} = \{\beta_m | m \in M\}$ and $\boldsymbol{\gamma} = \{\gamma^k | k \in K\}$ be the vectors of the dual variables associated with constraint sets (3.13), (3.14) and (3.16), respectively. Therefore the subproblem for crew k defined over the graph G^k, is given by:

Minimize

$$\sum_{(i,j)\in A^k} g_{ij}^k(\boldsymbol{T}_i^k) X_{ij}^k + \sum_{i \in V^k \setminus \{o(k)\}} g_i^k(\boldsymbol{T}_i^k) \sum_{j:(j,i)\in A^k} X_{ij}^k - \sum_{(i,j)\in A_f^k} \alpha_f X_{ij}^k - \sum_{(i,j)\in B_m^k} \beta_m b_{m,ij}^k X_{ij}^k - \gamma^k, \tag{3.19}$$

subject to:

$$\sum_{j:(o(k),j)\in A^k} X_{o(k),j}^k = 1, \tag{3.20}$$

$$\sum_{i:(i,j)\in A^k} X_{ij}^k - \sum_{i:(j,i)\in A^k} X_{ji}^k = 0, \quad \forall j \in N^k, \tag{3.21}$$

$$\sum_{i:(i,d(k))\in A^k} X_{i,d(k)}^k = 1, \tag{3.22}$$

$$X_{ij}^k \left(f_{ij}^{kr}(\boldsymbol{T}_i^k) - T_j^{kr} \right) \leqslant 0, \quad \forall r \in R^k, \forall (i,j) \in A^k, \tag{3.23}$$

$$a_i^{kr} \leqslant T_i^{kr} \leqslant b_i^{kr}, \quad \forall r \in R^k, \forall i \in V^k, \tag{3.24}$$

$$X_{ij}^k \text{ binary}, \quad \forall (i,j) \in A^k. \tag{3.25}$$

This subproblem is a shortest path problem with resource variables known to be NP-hard. Since graph G^k is acyclic, it can however be solved by pseudo-polynomial time algorithms if g_{ij}^k, g_i^k and f_{ij}^{kr} are non-decreasing functions (Desaulniers et al., 1994). Otherwise, one can discretize the resource space on the intervals where each function is decreasing.

It should be noted that when several commodities are identical, the path variables corresponding to these commodities can be aggregated to reduce the number of subproblems as well as the size of the master problem. When all commodities are assumed to be identical, such an aggregation process is presented in Desaulniers et al. (1994). In this case, the commodity indices can be removed from the Set Partitioning type formulation (3.12)–(3.18), as in the classical formulation of the crew pairing problem.

3.3.3. Integer solutions

Integer solutions to the nonlinear multi-commodity flow model (3.1)–(3.10) can be obtained through branch-and-bound. For the present application, branching decisions are based on the evaluation of the current flow between pairs of legs. For a selected pair with a fractional flow value, decisions are taken by imposing a flow of zero on one branch and a flow of one on the other. When any such decision is taken, all pairings in the master problem and all duties in the subproblem networks that do not satisfy it are removed. At each node of the search tree, a lower bound is provided by the Dantzig-Wolfe decomposition process described above, i.e., new columns are generated until the linear relaxation is solved to optimality.

The crew pairing problem is a particular, yet complex case of the unified model for time constrained vehicle routing and crew scheduling problems developed by Desaulniers et al. (1994). The reader is referred to this paper for a complete description of the solution process in the general case. In particular, this paper discusses branching decisions that may be appropriate for other crew pairing applications. These include branching on resource variables, on the supplementary variables or on linear combinations of these.

4. Air France case study

At Air France, as at most airlines, the crew pairing problem is divided in two independent problems to be solved separately: the pilot pairing problem and the flight attendant pairing problem. Their mathematical models are identical except that the former contains an additional resource variable. Next, we describe the global constraints of these problems, their objective function, their subproblem network as well as the resources used in both problems.

4.1. Global constraints

Global constraints (3.14) encountered in crew pairing models usually describe the availability of personal at each crew base. In this case study, even though Air France operates two bases (Orly and Charles-de-Gaulle), such base-related constraints are not present because of the geographical proximity of the bases. Instead, the two bases are considered together as the Paris crew base. However, the work convention of the Air France medium haul flying personnel comprises constraints dealing with several pairings. These constraints aim at limiting the total number of reduced night rests, of deadheads and of aircraft changes, rather than forbidding these events. The corresponding nonnegative variables are TOTRNR, TOTDH and TOTAC. If rnr_p^k, dh_p^k and ac_p^k denote the respective contributions of pairing $p \in \Omega^k$, $k \in K$ to these three global constraints, they can be written as:

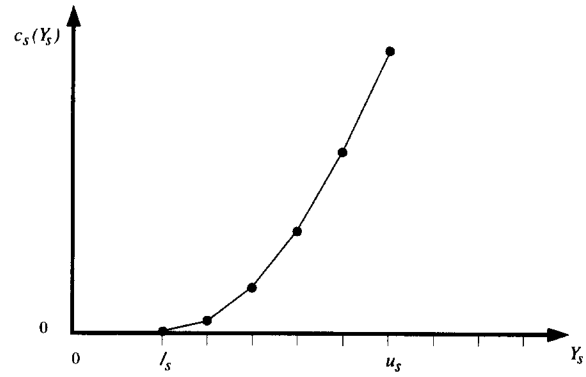

Fig. 1. Cost functions for supplementary variables.

$$\sum_{k\in K}\sum_{p\in\Omega^k} \mathrm{rnr}_p^k\theta_p^k - \mathrm{TOTRNR} = 0, \tag{4.1}$$

$$\sum_{k\in K}\sum_{p\in\Omega^k} \mathrm{dh}_p^k\theta_p^k - \mathrm{TOTDH} = 0, \tag{4.2}$$

$$\sum_{k\in K}\sum_{p\in\Omega^k} \mathrm{ac}_p^k\theta_p^k - \mathrm{TOTAC} = 0. \tag{4.3}$$

Note that the variables TOTRNR, TOTDH and TOTAC correspond to supplementary variables in constraints (3.14). We associate with these variables piecewise linear and nondecreasing cost functions. Fig. 1 illustrates such a cost function $c_s(Y_s)$ defined on the interval $[0, u_s]$. For $0 \leqslant l_s \leqslant Y_s \leqslant u_s$, $c_s(Y_s)$ is a linearization of a quadratic function:

$$Q(Y_s) = \frac{Q(u_s)\,(Y_s - l_s)^2}{(u_s - l_s)^2}.$$

Note that $c_s(Y_s)$ is equal to zero on the interval $[0, l_s]$, while it is linear on each of the five equal length intervals between l_s and u_s. Therefore, Y_s can be written as the sum of bounded variables, $Y_s = Y_s^0 + Y_s^1 + \cdots + Y_s^5$, with $Y_s^0 \in [0, l_s]$ and $Y_s^1, \ldots, Y_s^5 \in [0, (u_s - l_s)/5]$. The corresponding part of the objective function is $c_s(Y_s) = c_s^0 Y_s^0 + c_s^1 Y_s^1 + \cdots + c_s^5 Y_s^5$ where $c_s^0 = 0$ and $c_s^1, \ldots, c_s^5$ represent the slopes of this function in the corresponding intervals.

4.2. Subproblem network

4.2.1. Duty based network

Pairing feasibility is characterized by several rules mostly related to duties. Following in the footsteps of the preceding implementation at Air France (Lavoie, Minoux and Odier, 1988), a sequence of duties is used to construct pairings. As mentioned in Section 3.2, by using a duty based network, all rules related to duty feasibility do not need to be explicitly modeled. Since the construction of duties starting from a set of flight legs is a highly constrained problem, duties seldom contain more than four flight legs for medium haul problems and it is rather simple to enumerate all feasible ones, even for large-scale instances.

4.2.2. Possible states at the end of a duty

In order to generate feasible pairings, additional rules need to be satisfied. During the subproblem solution process, the set of partial paths ending by a given duty can be partitioned depending on the values of certain parameters. These parameters are the flying time of the last duty, the minimum rest time following it and the amount of insufficient residual rest time (this latter parameter is considered only for the pilot pairing problem). The partial paths belonging to a given element of the partition can be followed by duties in the same subset which is defined according to the parameter values. These paths are said to be in the same state. For the medium haul problem at Air France, the partition results in eight states for the pilot pairing problem and four for the flight attendant pairing problem. A copy of each duty is generated for each of its possible states. A detailed example of such a representation at Air France is provided in Lavoie, Minoux and Odier (1988). In the following, we will use the notation *duty/state* to designate a duty and its associated state.

4.2.3. Nodes and arcs

There are three types of nodes in the network which correspond to *station/time* combinations. We define *start_base/day* and *end_base/day* nodes for each base and day of the week. For each *duty/state* couple, we also define a *duty/state/station/time* node. Such a node represents the duty itself, its associated state, the station and the time of departure of the duty.

There are also three types of arcs in the network. The *start_of_pairing* arcs leave from a *start_base/day* node and enter in a *duty/state/station/time* node where the station is the starting base and the state is zero. The second and third types of arcs represent duties. The arcs of the second type, denoted *duty/state/rest*, connect two *duty/state/station/time* nodes. In addition to the duration of the duty, denoted *dutyduration*, such an arc includes the *rest* period prior to the next duty. The arcs of the third type start at a *duty/state/station/time* node and terminate at an *end_base/day* node. Therefore such an arc represents the end of a pairing and its type is denoted by *duty/state/end_of_pairing*.

The particular time-space network structure that we have created can be decomposed into subnetworks. Once specific *start_base/day* and *end_base/day* nodes have been selected, such a decomposition allows the use of only the corresponding subnetworks to efficiently generate pairings lasting at most from the start day to the end day. At Air France the maximum pairing duration is 6 days. Over a periodic weekly horizon, the number of subnetworks is therefore equal to seven times the number of bases.

4.3. Resource variables

4.3.1. Resource/constraint aspects

The pairing generation problem is a shortest path problem with resource variables. Resources are associated with certain constraints during the pairing generation phase. The quantity of such a resource is computed to verify if the associated constraint is met. Pairing generation at Air France involves two types of such constraints: the time away from base and the insufficiency of the residual rest time (used solely in the pilot pairing problem).

The work convention stipulates that in a pairing, the time away from base must be less than or equal to a given value. This value depends on the day and the start time of a pairing and is computed by using the function *maxtab*(*time*) depicted in Fig. 2. The maximum value of this function is 144 hours, that is 6 days. To adequately model this aspect, one non-negative resource variable is introduced: DIFF&TAB. For this resource, each *start_of_pairing* arc consumes the value $144 - maxtab(time)$ corresponding to the day of the *start_base/day* node and the start time of the *duty/state/station/time* node. This value represents the dif-

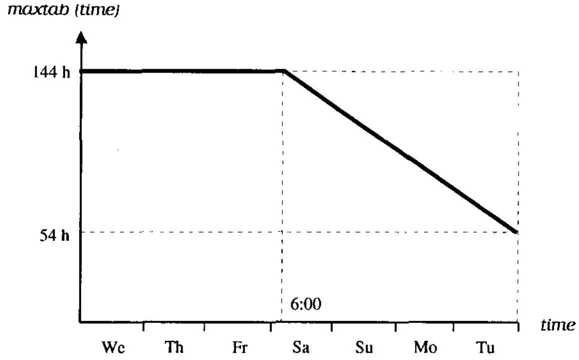

Fig. 2. The maximum time away from base function.

ference between 144 hours and the maximum permitted time away from base for this departure time. All other arcs consume the associated elapsed time. This resource variable DIFF&TAB is initialized at zero for any *start_base/day* node and is bounded from above by 144 for all other nodes.

For the pilot pairing problem, we must also take into account the insufficiency of the rest time before carrying out a duty. Given the state of a *duty/state/rest* arc, we determine a normal rest time, *normalrest*(*state*), as well as a minimum rest time, *minrest*(*state*), as authorized by the convention. The rest time before a subsequent duty, *rest*, must always be greater than or equal to *minrest*(*state*). This aspect is satisfied during the network construction since only compatible connections between duties are allowed. Nevertheless, if *minrest*(*state*) $\leqslant$ *rest* $\leqslant$ *normalrest*(*state*), there are *normalrest*(*state*) − *rest* units of additional insufficient rest time. On the other hand, if *rest* > *normalrest*(*state*), there is a partial or total recovery of the insufficient residual rest time. It is however not possible to recover the insufficient rest time before such an insufficiency occurs.

The rest time aspect of the convention is modeled by defining a non-negative resource variable INSUFF which represents the total insufficient residual rest time. To adequately treat this resource, we keep track of the number of units consumed on each arc type as follows: *duty/state/rest* arcs may increase or decrease the value of this resource variable by *normalrest*(*state*) − *rest* units, while the starting and ending arcs of any pairing do not provide any contribution. Initially, this resource variable is set at zero for any *start_base/day* node and is bounded from above by *maxinsuff*(*state*) at the *duty/state/station/time* nodes and by the largest value *maxinsuff* of the *maxinsuff*(*state*) function at the *end_base/day* nodes.

4.3.2. Resource/cost aspects

The cost of a given pairing is evaluated in hours and calculated at an *end_base/day* node as the sum of the time away from base and the home base rest subsequent to that pairing. The former value is computed using a resource variable denoted by TAB. The consumption of this resource is equal to the elapsed time on each arc in the network. The latter value is computed using a nonlinear, nondecreasing function g of the time away from base TAB, the total flying time (TOTFT) and the total number of landings (TOTLAND). The resource variables TAB, TOTFT and TOTLAND are initialized at zero and consume the elapsed time, the flying time (*flyingtime*) and the number of landings (*landings*) on each arc, respectively. As opposed to the resource variables used until now, no bounds are imposed on these three resource variables.

4.3.3. Extension and cost functions

For the Air France case study, the five resource variables DIFF&TAB, INSUFF, TAB, TOTFT and TOTLAND are extended on each arc using the usual linear single resource function $f_{ij}^{kr}(T_i^k) = T_i^{kr} + t_{ij}^{kr}$ introduced earlier in model (3.1)-(3.10). The cost functions g_{ij}^k and g_i^k of the same model are represented by:

$$g_{ij}^k(T_i^k) = t_{ij}^{k,\mathrm{TAB}}, \quad \forall k \in K,\ \forall (i,j) \in A^k,$$

and, for all $k \in K$,

$$g_i^k(T_i^k) = \begin{cases} g(\mathrm{TAB}_i^k, \mathrm{TOTFT}_i^k, \mathrm{TOTLAND}_i^k), \\ \quad i = d(k), \\ 0, \quad \text{otherwise}, \end{cases}$$

where g is the nonlinear, nondecreasing function mentioned above. Note that the function g_{ij}^k takes a constant value associated only with the resource TAB and is simply given by the elapsed time $t_{ij}^{k,\mathrm{TAB}}$ on the arc (i,j). Note also that the function g_i^k is computed solely at the end node $d(k)$ and represents the home base cost subsequent to a pairing.

4.3.4. Network reduction

In order to limit the computation time of the shortest path with resource variables algorithm, we reduce

Table 1
Resource consumption on arcs

Resource variables	Arc types			
	start_of_pairing	*ground/rest*	*duty/state/rest*	*duty/state/end_of_pairing*
DIFF&TAB	144 − *maxtab(time)*	*rest*	*dutyduration* + *rest*	*dutyduration*
INSUFF	0	−*rest*	*normalrest(state)* − *rest*	0
TAB	0	*rest*	*dutyduration* + *rest*	*dutyduration*
TOTFT	0	0	*flyingtime*	*flyingtime*
TOTLAND	0	0	*landings*	*landings*

the size of the network by using the following three strategies. The first consists of partially transforming the network into a time-line network (for example, see Barnhart et al., 1994). To do so, we need to define ground waiting arcs linking certain duties starting at the same station. As the rest time has an impact on the state of a duty unless the state is the most permissible one, we create such *ground/rest* arcs to connect, in increasing order of departure times, the *duty/state/station/time* nodes which are in the most permissible state. These *ground/rest* arcs are used at all stations except at the base stations since for these, the *start_of_pairing/day* arcs are necessary to impose the maximum time away from base restrictions. The insertion of the *ground/rest* arcs permits to remove a number of *duty/state/rest* arcs in the most permissible state.

The second strategy consists of recursively eliminating all nodes without predecessor or successor as well as their incident arcs. In the third, we aggregate the *duty/state/station/time* nodes with the same station, time and state. The reduced network permits to generate exactly the same pairings as the initial network.

Tables 1 and 2 summarize the network construction process. The first provides the resource consumption for each arc type while the second gives the resource restrictions for each node type.

Table 2
Resource restrictions at nodes

Resource variables	Node types		
	start_base/day	*duty/state/station/time*	*end_base/day*
DIFF&TAB	$[0,0]$	$[0,144]$	$[0,144]$
INSUFF	$[0,0]$	$[0, maxinsuff(state)]$	$[0, maxinsuff]$
TAB	$[0,0]$	$[0,\infty]$	$[0,\infty]$
TOTFT	$[0,0]$	$[0,\infty]$	$[0,\infty]$
TOTLAND	$[0,0]$	$[0,\infty]$	$[0,\infty]$

5. Computational results

The first phase of the implementation process of GENCOL, the commercial version of the algorithm described in Section 3, at Air France has been to test it on several medium haul problems provided by the airline. The network utilized by GENCOL was provided by the Air France software package ICARE. This network is defined on a one-week horizon and its structure is presented in detail in Section 4.2. To construct this network, ICARE utilizes the data generated by the information system CESAR in which it is integrated. At the end of the solution process, ICARE is utilized again as an interface between the optimizer and the information system as it transmits to CESAR the pairing solution obtained by GENCOL. For a given problem, the potential deadhead legs have been restricted to the active legs of the problem and those used in the Air France solution.

Table 3 highlights the attributes of each problem tested. The name of each problem indicates the date, whether it is a pilot (PP) or a flight attendant pairing problem (FAP), and the type of aircraft. For example, the name of the problem denoted by 9209 FAP A320 signifies the 1992, September flight attendant pairing problem involving aircraft of type A320. The size of each problem is characterized by the number of flight legs to cover, the number of bases and stations, and the number of nodes and arcs representing the underlying network. To complete the description of each problem, we finally indicate the number of subnetworks and resources used.

Table 3
Problem characteristics

Problems	9004 FAP A300	9004 FAP A310	9004 FAP A320	9004 FAP B727	9004 FAP B737	9201 FAP A320	9201 FAP B737	9209 FAP A320	9209 PP A320	9209 FAP B737	9209 PP B737
Flight legs	280	154	392	342	477	701	566	739	743	1157	570
Bases	2	2	2	2	2	1	2	1	1	2	2
Stations	40	20	26	69	63	30	45	33	33	63	41
Nodes	2868	675	5399	1184	3829	983	1246	2253	1860	8829	2051
Arcs	40074	4390	115089	7258	41238	3601	6356	13052	10421	95504	16148
Subnetworks	14	14	14	14	14	7	14	7	7	14	14
Resources	4	4	4	4	4	4	4	4	5	4	5

Table 4
Computational results

Computer	SUN Sparc Station 2							HP 9000/730			
Optimizer	GENCOL 2.0/XMP							GENCOL 3.0/CPLEX			
Problems	9004 FAP A300	9004 FAP A310	9004 FAP A320	9004 FAP B727	9004 FAP B737	9201 FAP A320	9201 FAP B737	9209 FAP A320	9209 PP A320	9209 FAP B737	9209 PP B737
Flight legs	280	154	392	342	477	701	566	739	743	1157	570
Linear relaxation											
Z_{LP}	2532.5	1721.8	3061.7	3312.2	4450.6	6070.3	4946.7	6547.6	6830.3	10241.1	5042.6
CPU (sec.)	769	19	1692	78	1033	44	390	369	215	8608	854
Integer solution											
Z_{IP}	2539.0	1721.8	3061.7	3328.3	4469.6	6070.3	4951.9	6549.5	6836.5	10245.6	5059.8
CPU (sec.)	1861	12	808	130	2757	42	944	376	249	13084	2140
BB	16	0	0	14	20	0	26	1	2	5	25
Absolute gap	6.5	0	0	16.1	19.0	0	5.2	1.9	6.2	4.5	17.2
% Gap	0.26	0	0	0.49	0.43	0	0.11	0.03	0.09	0.05	0.34
General information											
Number of columns	3667	960	4402	2344	7365	1844	5950	5163	6848	20127	20150
CPU SP (sec.)	2390	6	926	123	2583	7	361	268	63	11089	804
CPU MP (sec.)	43	1	24	12	133	14	106	67	83	1818	192
CPU TOT (sec.)	3004	19	1692	227	3589	44	1001	419	263	13983	2185

As can be seen from Table 3, the problems involve between 154 and almost 1200 flight legs, one or two bases, and from 20 to 69 stations. In addition, the number of nodes varies between 675 and 8829, while the number of arcs ranges from 4390 to over 115 000.

Table 4 describes the computational results. No initial solutions where used for any test problems. The top part of this table focuses on the solution of the linear relaxation (3.12)-(3.17) of the master problem. For each problem, we report the value of the objective function in credited hours (Z_{LP}) and the computation time in seconds (CPU) needed to obtain this solution. The middle part of the table depicts the best integer solution. It gives the objective function value (Z_{IP}) and the computation time in seconds (CPU) required to arrive at this solution. We then furnish the number of nodes explored in the branch and bound tree to obtain this solution (BB). Finally, we provide the gap between the integer solution Z_{IP} and the linear relaxation solution Z_{LP} in credited hours (Absolute Gap) and in percentage points (% Gap). The bottom part of the table presents additional solution summaries,

Table 5
GENCOL's improvement of the Air France solutions

Problems	Flight legs	Air France solution		GENCOL solution		
		Number of pairings	Cost	Number of pairings	Cost	Improvement (%)
9004 FAP A300	280	95	2768.4	82	2539.0	8.29
9004 FAP A310	154	60	1801.5	55	1721.8	4.43
9004 FAP A320	392	95	3238.1	99	3061.7	5.30
9004 FAP B727	342	111	3392.5	108	3328.2	1.89
9004 FAP B737	477	130	4813.6	123	4469.6	7.15
9201 FAP A320	701	218	6341.6	208	6070.3	4.28
9201 FAP B737	566	148	5343.9	152	4951.9	7.34
9209 FAP A320	739	226	7190.4	204	6549.5	8.91
9209 PP A320	743	214	7083.1	209	6836.5	3.48
9209 FAP B737	1157	270	11196.1	288	10245.6	8.49
9209 PP B737	570	133	5084.9	136	5059.8	0.49
All problems			58254.1		54833.9	6.24

such as the number of columns generated (Number of Columns), the computational time required for the subproblem (CPU SP) and for the master problem (CPU MP), and finally, the total computation time utilized by the software package (CPU TOT). It finally separates the test problems according to the machine type used which also corresponds to the software package's transition from the Fortran version GENCOL 2.0/XMP to the C version GENCOL 3.0/CPLEX. In both versions, the primal simplex method is used to solve the linear relaxation (3.12)–(3.17) of the master problem.

As can be seen from Table 4, for three problems out of eleven, the linear relaxation solution was already integer and hence did not require any branching. In the other cases, the number of branching nodes needed to arrive at the integer solution was no more than 26, while the absolute gap on the solution cost ranged from 0 to 19 hours. The maximum percentage gap was below 0.5%. Depending on the problem, GENCOL had to generate between 960 and 20 150 pairing columns to obtain an integer solution. This is incomparably less than the number of theoretically possible pairings. The time necessary for the solution of the master problem is ranged between 1 and 1819 s while the solution of the subproblem required from 6 to nearly 11 100 s. The total computation time (comprising the time to solve the master problem and the subproblem, the reading of the data, etc.) was between 19 and 14 000 s. With the exception of one problem where the solution time was 4 hours, this time was of the order of minutes. It is worth noting that the internal management includes the generation of the network, which explains the large CPU time associated with it. In addition, the solution time varies with the speed of the different computers utilized.

Table 5 illustrates the comparisons between the solution obtained by Air France using an expert system and that derived by utilizing GENCOL. We first describe each of the two solutions by indicating the number of pairings they involve (Number of pairings) and the value of the objective function to be minimized. We next indicate the percentage improvement (Improvement (%)) obtained by the GENCOL solution which is calculated as:

$$\text{Improvement} = 100 \times \frac{\text{Cost}_{\text{Air France}} - \text{Cost}_{\text{GENCOL}}}{\text{Cost}_{\text{Air France}}}.$$

As can be seen from Table 5, the GENCOL method always provides a lower cost solution and realizes an average improvement of 6.24% with a minimum and a maximum amelioration of 0.49% and 8.91%, respectively. Note that GENCOL sometimes derives more pairings than Air France since the optimization is not directed at minimizing the number of pairings but rather their cost.

6. Conclusions

This paper has described the implementation of a crew pairing problem optimizer at Air France. The problem has been modeled as an integer, nonlinear multi-commodity network flow problem with additional resource variables. It has been solved by a branch-and-bound algorithm where the lower bounds are derived from an extension of the Dantzig–Wolfe decomposition principle.

This approach generated significant improvements over the expert system in use on test problems provided by the airline. This led to its eventual implementation. The success of this methodology is not restricted to this specific application. The model and algorithm are flexible enough to be customized to many other environments, including virtually any airline crew pairing problem.

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