Report on Hasegawa-Mima

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Contents

1 Introduction **2**

2 Convergence of fully discrete system **5**

2.1 The result 5

2.2 Estimates on the elements 7

3 Traveling Wave Solutions **11**

3.1 Properties of Traveling Wave Solutions 12

3.2 Weighted Sobolev spaces on semi-infinite strip 17

3.3 Traveling waves with periodicity in the ζ-direction 20

3.4 Traveling waves with PBC's 22

4 The Problem as Solved in the Original Paper **23**

4.1 The Continuity Equation in the Original Paper 24

4.2 Deriving C^2 continuity (can ignore this part) 25

5 Simulations and numerical results **26**

5.1 The Algorithm 26

5.2 Testing for $W^{1,\infty}$ norm 26

1 Introduction

The Hasegawa-Mima equations arises in the context of plasma physics, and in particular in the Magnetic Plasma Confinement method used to generate fusion power. This relatively novel approach uses very powerful magnetic fields to heat a plasma to temperatures of up to 150 million °C while keeping it confined in space [1]. This allows fusion to take place and thus generating immense amount of thermonuclear power with relatively low hydrogen fuel consumption. This process is however difficult to control as plasma can become easily turbulent/unstable at such high energy levels and therefore disrupt the confinement process. In order to better control the plasma, one then needs a model to describe turbulence. There are many models available for describing *bi-dimensional turbulence* and the one in which the Hasegawa-Mima equation appears can be found in [2]–[4]. In it's original form, the equation can be written as

$$(I-\Delta)\frac{\partial u}{\partial t}=\{u,\Delta u\}+k\frac{\partial u}{\partial y},\quad u(0)=u_0. \tag{1}$$

where

- $u:\Omega\times[0,T]\to\mathbb{R}$ describes the electrostatic potential of the plasma,
- Ω is an open domain in $\mathbb{R}^2$,
- Δ is the two dimensional spacial Laplacian,
- $\{\cdot,\cdot\}$ is the Poisson bracket(i),
- $k\in\mathbb{R}$ is a constant (ii).

There are numerous results investigating existence and uniqueness of both strong (classical) and weak solutions to the non-linear equation (1) in various the initial data and domains. Strong solutions for the Hasegawa-Mima equation were obtained in the case $\Omega=\mathbb{R}^2$ using the classical *Lie symmetry reduction method*, where it was shown that (1) has a family of solutions that can be expressed in terms of Bessel functions and trigonometric polynomials. Of interest in particular are solutions exhibiting

(i)The Poisson bracket $\{\cdot,\cdot\}:W^{1,p}(\Omega)\times W^{1,p}(\Omega)\to L^p(\Omega)$ is defined as $\{u,v\}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$.

(ii)It is defined as $k=\partial_x\ln(n_0/\omega_{ci})$ where n_0 is the background particle density that depends only on the x-direction and ω_{ci} is the ion cyclotron frequency that depends only on the initial magnetic field.

periodic behaviour in time. More specifically, we will be interested in solutions obtained in [5] such as

$$u(x,y,t) = k_1 + k_2 \sin\left((\alpha\beta t + \alpha x + \beta y)\sqrt{(\alpha^2\beta + \beta^3)^{-1}(1-\beta)}\right) \\ + k_3 \cos\left((\alpha\beta t + \alpha x + \beta y)\sqrt{(\alpha^2\beta + \beta^3)^{-1}(1-\beta)}\right).$$

where $k_1, k_2, k_3 \in \mathbb{R}$ and $(\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R} \setminus \{0, 1\}$ are arbitrary constants. Such solutions are helpful in comparing simulated solutions to theoretical smooth solutions.

However, this thesis will deal mainly with proving existence and simulating weak solutions of (1). First we list the results in the literature on weak solutions. But to do so, we will need some preliminary definitions and notation.

Notation. In this thesis Ω will denote an open domain in $\mathbb{R}^2$. For $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, we will denote $W^{m,p}(\Omega)$ the standard Sobolev spaces of weakly-differentiable functions and $W^{m,2} := H^m(\Omega)$ with their standard norms and inner products when available. For shorthand, if Ω is specified we omit it and write $H^m := H^m(\Omega)$, $L^p := L^p(\Omega)$ and so on. Also, if X is any Banach space then we use the shorthand for the Bochner space

$$L^p(0,T;X) := L^p([0,T];X) \quad \text{with norm} \quad \|\cdot\|_{L^p,X} = \left(\int_0^T \|\cdot\|_X^p\right)^{1/p} \quad \text{for all } 1 \leq p < \infty.$$

We also define

$$C^1([0,T];\, X) := C^1\big((0,T);X\big) \cap C\big([0,T];X\big).$$

Definition 1.1. *Let $\mathcal{R}$ be a bounded rectangular domain in $\mathbb{R}^2$. Write $\Gamma := \partial\Omega = \bigcup_{i=1}^4 \Gamma_j$ where Γ_j is one of the four sides of $\mathcal{R}$ and $\Gamma_j = \Gamma_{j+2}$ are parallel. Let $H_P^1(\Omega)$ be the closed linear subspace of the Hilbert space $H^1(\Omega)$ defined as*

$$H_P^1(\mathcal{R}) := \left\{ u \in H^1(\Omega) \,\middle|\, u_{|\Gamma_j} = u_{|\Gamma_{j+2}},\ j = 1, 2 \right\},$$

For $m \geq 2$, we define $H_P^1(\Omega)$ as the closed linear subspace of $H^m(\Omega)$ as

$$H_P^m(\mathcal{R}) := \left\{ u \in H_P^1(\Omega) \,\middle|\, u_x, u_y \in H_P^{m-1}(\Omega) \right\}.$$

The spaces $H_P^m(\Omega)$ are called a periodic Sobolev space.

Note. The functions $u_{|\Gamma_j} := \mathrm{Tr}_j(u)$ where $\mathrm{Tr}_j : H^1(\mathcal{R}) \to L^2(\Gamma_j)$ is the usual densely defined trace operator.

With the above in mind, we can summarize the results obtained so far in the literature in the Table 1. In [6], the results have been obtained when $\Omega = \mathbb{R}^2$ by using analytic semi-groups. Using a

Ref	Ω	u_0	u	u_t	Unique?
[6]	$\mathbb{R}^2$	H^m $(m \geq 4)$	$L^\infty(0,T;\, H^m)$ $\cap\ C(0,T;\, H^1)$	$L^\infty(0,T;\, H^{m-1})$ $\cap\ C(0,T;\, L^2)$	Yes
		H^2	$L^\infty(0,T,H^2), \forall T$	$L^\infty(0,T;\, L^2), \forall T$	?
[7]	$[0,L]^2$	H^m_P $(m \geq 4)$	$L^2(0,T;\, H^m_P)$ $\cap\, C(0,T;\, L^2)$	$L^2(0,T;\, L^2)$	Yes
[8]		H^3_P	$C(0,T;\, H^2_P)$	$L^2(0,T;\, L^2)$	?

Table 1: Regularity results on the solution u given various settings for u_0.

similar approach in [7], the strongly elliptic operator $\epsilon(\Delta(\Delta - I) + 2I)$ was introduced to obtain the existence and uniqueness of a solution for (1) in H^1_P. In the context of [8] however, the equation was rewritten as a coupled system of equations by setting $w := u - \Delta u$ and then replacing accordingly in (1) to get the Hasegawa-Mima coupled system

$$\begin{cases} -\Delta u + u = w, & \text{on } \Omega \times [0,T) \quad (2a) \\ w_t + \nabla_{\perp} u \cdot \nabla w = k u_y, & \text{on } \Omega \times [0,T), \quad (2b) \\ u(0) = u_0, \quad w(0) = (I - \Delta) u_0. & \quad (2c) \end{cases}$$

where $\nabla_{\perp} \colon H^1_P(\mathcal{R}) \to L^2(\mathcal{R}) \times L^2(\mathcal{R})$ is the skew gradient given by

$$\nabla_{\perp} u = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x} \right).$$

Notice that the first equation is elliptic in u and the second is hyperbolic in w.

2 Convergence of fully discrete system

2.1 The result

Here is the list of notations used in the current section.

- $\Omega = [a, b] \times [c, d]$ is the domain.
- T_h is the conforming triangulation corresponding to h (uniform mesh).
- m is the number of nodes on the segement $\{c\} \times [a, b]$ and n is the number of nodes on the segment $\{b\} \times [c, d]$. In case $m = n$ we use just use the symbol n.
- V_P^h is the space of periodic piecewise $\mathbb{P}_1$ elements with basis B_h^P.
- $d = (m-1)(n-1)$ is the dimension of $V_h^P(\Omega)$.
- The semi-discrete solutions

$$u_h(x,y,t) = \sum_{j=1}^{d} u_h^{(j)}(t)\varphi_j(x,y), \quad w_h(x,y,t) = \sum_{j=1}^{d} w_h^{(j)}(t)\varphi_j(x,y)$$

  of the system

$$\begin{cases} \left\langle \dfrac{dw_h}{dt}(t), v \right\rangle = -\left\langle \{u_h(t), w_h(t)\}, v\right\rangle + k\left\langle \dfrac{\partial u_h}{\partial y}(t), v\right\rangle, & \text{for all } v \in V_P^h. \\ \langle \nabla u_h(t), \nabla v\rangle + \langle u_h(t), v\rangle = \langle w_h(t), v\rangle, \quad u(0) = u_0. \end{cases} \tag{3}$$

- In vector form we have

$$U_h(t) = (u_h^{(1)}, \ldots, u_h^{(d)}), \quad W_h(t) = (w_h^{(1)}, \ldots, w_h^{(d)}).$$

- The matrices

$$M = \Big[\langle \varphi_i, \varphi_j\rangle\Big]_{i,j=1}^{d}, \quad S = \Big[\langle \nabla\varphi_i, \nabla\varphi_j\rangle\Big]_{i,j=1}^{d}, \quad B = (S+M)^{-1}M,$$

$$R = \left[\left\langle \frac{\partial \varphi_i}{\partial y}, \varphi_j\right\rangle\right]_{i,j=1}^{d}, \quad P(z) = \left[\sum_{k=1}^{d} z_k \Big\langle \{\varphi_k, \varphi_j\}, \varphi_i\Big\rangle\right]_{i,j=1}^{d}, \quad z = (z_1, \ldots, z_d) \in \mathbb{R}^d.$$

- (System evolution functions in time). Define $L, F : \mathbb{R}^d \to \mathbb{R}^d$ as

$$L(x) = kM^{-1}RBx, \quad F(x) = -M^{-1}P(Bx)x. \tag{4}$$

- (Semi-discrete matrix form). System (3) in matrix form is written as

$$\begin{cases} M\dfrac{dW_h}{dt}(t) = -P(U_h(t))W_h(t) + kRU_h(t), \\ (S+M)U_h(t) = MW_h(t), \quad U(0) = \pi_h(u_0). \end{cases} \tag{5}$$

- (Semi discrete matrix in W_h only). In terms of W_h only, we have that

$$M\frac{dW_h}{dt}(t) = -P\big(BW_h(t)\big)W_h(t) + kRBW_h(t), \tag{6}$$

- (Approximations in time). Integrating the hyperbolic equation in (3) from t to $t+\tau$ for some $\tau > 0$ we get

$$\begin{aligned} \int_t^{t+\tau} \big\langle \big\{w_h(s), u_h(s)\big\}, v\big\rangle\, ds &= \big\langle \big\{w_h(t), u_h(t)\big\}, v\big\rangle + \epsilon_F(t,v), \\ \int_t^{t+\tau} k\left\langle \frac{\partial u_h}{\partial y}(s), v\right\rangle ds &= k\left\langle \frac{\partial u_h}{\partial y}(t), v\right\rangle + \epsilon_L(t,v), \end{aligned} \tag{7}$$

- Letting $\epsilon = \epsilon_F + \epsilon_L$, we have that (3) becomes

$$\langle w_h(t+\tau), v\rangle = \langle w_h(t), v\rangle - \big\langle \big\{u_h(s), w_h(s)\big\}, v\big\rangle + k\left\langle \frac{\partial u_h}{\partial y}(s), v\right\rangle ds + \epsilon_\tau(t,v), \tag{8}$$

for all $v \in V_h^P$. In matrix form and in terms of W_h only the above equation becomes

$$W_h(t+\tau) = W_h(t) - M^{-1}P\big(BW_h(t)\big)W_h(t) + kM^{-1}RBW_h(t) + \vec{\epsilon}_\tau(t), \tag{9}$$

where $\vec{\epsilon}_\tau(t) = \big(\vec{\epsilon}_\tau(t,\varphi_1), \ldots, \big(\vec{\epsilon}_\tau(t,\varphi_d)\big)$. Using L and F we can write in more compact form

$$W_h(t+\tau) = W_h(t) - F(W_h(t)) + L(W_h(t)) + \vec{\epsilon}_\tau(t), \quad \text{for all } t \in [0, T_{\text{thresh}} - \tau).$$

where L is linear and F is non-linear.

- (The interpolant). Given fixed $\tau > 0$, we define the interpolant $W_{h,\tau}$ as

$$\boxed{\begin{cases} \text{Constructing the interpolant:} \\ W_{h,\tau}(t+\tau) = W_{h,\tau}(t) - \tau F(W_{h,\tau}(t)) + \tau L(W_{h,\tau}(t)), & \text{for all } t \in [0, T_{\text{thresh}} - \tau), \\ W_{h,\tau}(t) = W_h(0) = BU_h(0), & \text{for all } t \in [0, \tau). \end{cases}} \tag{10}$$

Therefore, what is being simulated in the code is actually $W_{h,\tau}(k\tau)$, for $k = 1, 2, 3, \ldots$

Now we are in good shape to show the convergence of the numerical scheme. In fact, we are going to show that for an appropriate sequence of times $\{\tau_n\}$, each depending on h, that the sequence of interpolants $\{W_{h,\tau_n}(t)\}$ converges to the semi-discrete solution $W_h(t)$ for every $t \in [0, T_{\text{thresh}})$. To do so, we will use the following estimates on F and L, which are derived in the next section.

$$|L(x)| \leq |k| \|R\| \|M^{-1}\| \|B\| |x| \leq |k| (8 \cdot 3\sqrt{2}) O(h) O(h^{-2}) O(h^{-2}) O(h^2) \in O(h)|x|.$$

and

$$|F(x)| \leq \|M^{-1}\|_2 \|P(Bx)\|_2 |x| \leq \|M^{-1}\|_2 |x| \tag{11}$$

2.2 Estimates on the elements

In this part, I have computed explicitly the L^2 norm of basis elements in $\mathcal{B}_{h,P}$. In particular, I have chose the basis element $\varphi_0 \in \mathcal{B}_{h,P}$ such that $\varphi_0(L/n, L/n) = 1$. It's domain is plotted in Figure 1 and is the union of six triangles $T_1, \ldots, T_6 \in \mathcal{T}_h$.

Lemma 2.1. *The element φ_0 is given by the following formula*

$$\varphi_0(x,y) = \begin{cases} ny/L & \textit{if } (x,y) \in T_1 \\ nx/L & \textit{if } (x,y) \in T_2 \\ nx/L - ny/L + 1 & \textit{if } (x,y) \in T_3 \\ 2 - ny/L & \textit{if } (x,y) \in T_4 \\ 2 - nx/L & \textit{if } (x,y) \in T_5 \\ -nx/L + ny/L + 1 & \textit{if } (x,y) \in T_6 \end{cases}$$

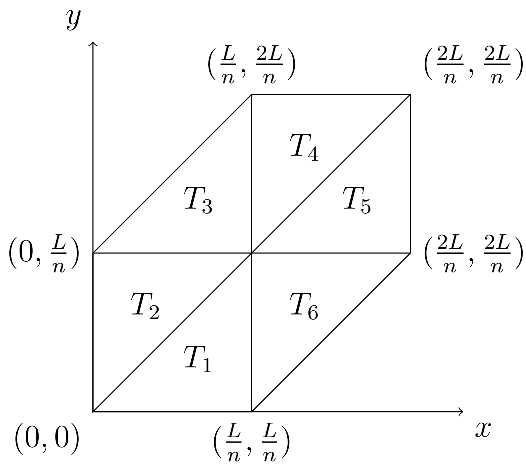


Figure 1: The domain of the element φ_0.

and therefore for any basis element $\varphi \in \mathcal{B}_{h,P}$ with $\varphi_{|\partial\Omega} = 0$ we have

$$\frac{\partial \varphi}{\partial x}(x,y) = \begin{cases} n/L & \textit{if } (x,y) \in T_2 \cup T_3 \\ -n/L & \textit{if } (x,y) \in T_5 \cup T_6 \\ 0 & \textit{if } (x,y) \in T_1 \cup T_4 \end{cases} \quad \textit{and} \quad \frac{\partial \varphi}{\partial y}(x,y) = \begin{cases} n/L & \textit{if } (x,y) \in T_1 \cup T_6 \\ -n/L & \textit{if } (x,y) \in T_3 \cup T_4 \\ 0 & \textit{if } (x,y) \in T_2 \cup T_5 \end{cases}.$$

First of all, it is clear from the above proposition and Figure 1 that

$$\|\varphi_0\|_{L^2} = \left(4\int_{T_1} \varphi_0^2 d\lambda + 2\int_{T_3} \varphi_0^2 d\lambda\right)^{\frac{1}{2}} = \frac{L}{\sqrt{2}n}.$$

This means that we have

$$\boxed{\|\varphi\|_{L^2} \le \frac{L}{\sqrt{2}n}, \quad \forall \varphi \in \mathcal{B}_{h,P} \text{ with } \varphi_{|\partial\Omega} = 0.} \tag{12}$$

Second, for any basis element with $\varphi|_{\partial\Omega} = 0$, the above proposition tells us that

$$\begin{aligned} \left\|\frac{\partial \varphi}{\partial x}\right\|_{L^2} &= \left[\int_\Omega \left(\frac{\partial \varphi}{\partial x}\right)^2 d\lambda\right]^{\frac{1}{2}} = \left[\sum_{k=1}^{6} \int_{T_k} \left(\frac{\partial \varphi}{\partial x}\right)^2 d\lambda\right]^{\frac{1}{2}} \le \left[\sum_{k=1}^{6} \int_{T_k} \frac{n^2}{L^2} d\lambda\right]^{\frac{1}{2}} \\ &\le \frac{n}{L} \cdot \sqrt{6} \cdot \max \lambda(T_k)^{\frac{1}{2}} = 6\frac{n}{L}\frac{L}{\sqrt{2}n} = \frac{6}{\sqrt{2}} = 3\sqrt{2}. \end{aligned}$$

The same estimate also holds for $\|\partial_y \varphi\|_2$. In summay, if φ is *any* basis element (also could be a boundary element) then it satisfies the following estimate

$$\boxed{\left\|\frac{\partial \varphi}{\partial x}\right\|_{L^2} \leq 3\sqrt{2}, \qquad \left\|\frac{\partial \varphi}{\partial x}\right\|_{L^2} \leq 3\sqrt{2}, \qquad \|\varphi\|_{L^2} \leq \frac{1}{\sqrt{2}}\frac{L}{n}}.$$

From this we can deduce that and Cauchy-Shwarz that

$$\begin{gathered} |\langle \varphi_i, \varphi_j \rangle| \leq \frac{1}{2}\frac{L^2}{n^2}, \quad |\langle \nabla\varphi_i, \nabla\varphi_j \rangle| \leq 6\sqrt{2}, \\ \left|\left\langle \frac{\partial \varphi_i}{\partial y}, \varphi_j \right\rangle\right| \leq \frac{3L}{n}, \quad |\langle \{\varphi_k, \varphi_j\}, \varphi_i \rangle| \leq \frac{L}{\sqrt{2}}. \end{gathered} \tag{13}$$

We now look at the Poisson bracket of basis elements. Let $\varphi_1, \dots, \varphi_6 \in \mathcal{B}_{h,P}$ be the elements such that $\varphi_i(P_j) = \delta_{ij}$ ($i, j = 1, \dots, 6$) where $P_1, \dots, P_6$ are the points in Figure 2. We have the following equalities

$$\begin{aligned}
&\nabla_{\perp}(\varphi_1) \cdot \nabla\varphi_0 = n^2/L^2, && \text{on } T_1, \\
&\nabla_{\perp}(\varphi_2) \cdot \nabla\varphi_0 = (0, -n/L) \cdot (0, n/L) = -n^2/L^2, && \text{on } T_1, \\
&\nabla_{\perp}(\varphi_2) \cdot \nabla\varphi_0 = (-(-n/L), 0) \cdot (n/L, 0) = 0, && \text{on } T_2, \\
&\nabla_{\perp}(\varphi_3) \cdot \nabla\varphi_0 = (-n/L, -n/L) \cdot (n/L, 0) = -n^2/L^2, && \text{on } T_2, \\
&\nabla_{\perp}(\varphi_3) \cdot \nabla\varphi_0 = (0, -n/L) \cdot (n/L, -n/L) = n^2/L^2, && \text{on } T_3, \\
&\nabla_{\perp}(\varphi_4) \cdot \nabla\varphi_0 = (-n/L, 0) \cdot (n/L, -n/L) = -n^2/L^2, && \text{on } T_3, \\
&\nabla_{\perp}(\varphi_4) \cdot \nabla\varphi_0 = (-n/L, -n/L) \cdot (0, -n/L) = n^2/L^2, && \text{on } T_4, \\
&\nabla_{\perp}(\varphi_5) \cdot \nabla\varphi_0 = (0, n/L) \cdot (0, -n/L) = 0, && \text{on } T_4, \\
&\nabla_{\perp}(\varphi_5) \cdot \nabla\varphi_0 = (-n/L, 0) \cdot (-n/L, 0) = n^2/L^2, && \text{on } T_5, \\
&\nabla_{\perp}(\varphi_6) \cdot \nabla\varphi_0 = (n/L, n/L) \cdot (-n/L, 0) = -n^2/L^2, && \text{on } T_5, \\
&\nabla_{\perp}(\varphi_6) \cdot \nabla\varphi_0 = (0, n/L) \cdot (0, n/L) = n^2/L^2, && \text{on } T_6, \\
&\nabla_{\perp}(\varphi_1) \cdot \nabla\varphi_0 = (n/L, 0) \cdot (-n/L, n/L) = -n^2/L^2, && \text{on } T_6.
\end{aligned}$$

If we define $I : \{1, \dots, 12\} \to \{-1, 0, 1\}$ by

$$I(\kappa) = \begin{cases} 0 & \text{if } \kappa = 3, 8, \\ -1 & \text{if } \kappa \text{ is odd}, \\ 1 & \text{if } \kappa \text{ is even}. \end{cases}$$

All of the matrices in the previous section are at most 8 diagonal. Therefore, we have by the Schur test that

$$\|A\|_2 \leq \|A\|_1\|A\|_\infty \leq \sqrt{8}|\max_{i,j} a_{ij}| = 2\sqrt{2}|\max_{i,j} a_{ij}|$$

We also have the following estimates on the condition number of the mass and stiffness matrices.

Proposition 2.1. *The condition numbers $C_2(M)$ and $C_2(K)$ of the mass and stiffness matrices satisfy the following estimates*

$$1 \leq C_2(M) \leq 4p_{max}, \qquad \frac{9}{\lambda_1 h^2 p_{max}} \leq C_2(K) \leq \frac{36p_{max}}{\lambda_1 h^2}.$$

For a proof see I. Fried [9]. As a consequence we obtain the following proposition.

Proposition 2.2. *For the matrices $P(x)$, S, M, B, R defined in* (**??**) *we have the following inequalities.*

$$\|M\|_2 \leq \sqrt{2}\,\frac{L^2}{n^2}, \quad \|K\|_2 \leq 24. \tag{14}$$

Proof. We have that

$$|P_{ij}(x)| \leq \sum_{k=1}^{d} |x_k|\big\langle\{\varphi_k, \varphi_j\}, \varphi_i\big\rangle \leq \frac{n^2}{L^2}\sum_{k=1}^{d}|x_k| \leq \frac{n^2}{L^2}\sqrt{d}|x| \leq \frac{n^2(n-1)}{L^2}|x|.$$

so that

$$\|P(x)\|_2 \leq 2\sqrt{2}\frac{n^2(n-1)}{L^2}|x| = 2\sqrt{2}\frac{n-1}{h}$$

□

3 Traveling Wave Solutions

In [2], [3], traveling wave solutions[iii] for the Hasegawa-Mima equation were found on $\mathbb{R}^2$ and under restrictive assumptions on the second derivatives of u. In this section, we prove stronger results. Let $\mathcal{R} = [0, L] \times [0, L]$. If one looks for traveling wave solution for the Hasegawa-Mima equation traveling in the y direction with constant velocity c, one would seek a solution of the form

$$u(x, y, t) = \Psi(\xi, \zeta), \qquad (\xi, \zeta) = (x, y - ct), \qquad \forall (x, y, t) \in \mathcal{R} \times [0, \infty).$$

Replacing in (1), the traveling wave form of the Hasegawa-Mima equation is

$$\begin{cases} \Delta\Psi - \Psi = F(\Psi - c\xi) - k\xi, & \text{on } \Omega := [0, L] \times (-\infty, L], \\ \text{B.C's inherited from } u, & \text{on } \Gamma = \partial\Omega, \\ \text{A.C's inherited from } u, & \text{as } \zeta \to -\infty \text{ (i.e as } t \to \infty). \end{cases} \tag{15}$$

where $F : D(F) \subset \mathbb{R} \to \mathbb{R}$ is an arbitrary function and $k \in \mathbb{R}$ is a constant. It is important to note since $D(F)$ must contain the range of Ψ, then different assumptions on the solution Ψ provide different functions F. We will mainly require that $D(F)$ is an interval containing the essentially bounded values of $\Psi - c\xi$. If Ψ is not essentially bounded, then we take $D(F) = \mathbb{R}$. Note that by letting $\Phi = \Psi - c\xi$ we get

$$F(\Phi) = \Delta\Phi - \Phi + (k - c)\xi, \quad \text{for all } (\xi, \zeta) \in \Omega. \tag{16}$$

Depending on convenience we will use either (15) or (16). Our goal is to investigate the nature of the function F, which can be determined completely by the boundary/asymptotic behaviour of Ψ. We will be mainly interested in the conditions that linearize equation (15), but some interesting non linear cases might help in investigating the $L^\infty(\Omega)$ behaviour of Ψ in ζ, and hence of u in t. Equation (15) tells us that

$$\begin{cases} F(\Psi(0, \zeta) - cL) = \Delta\Psi(0, \zeta) - \Psi(0, \zeta), & \text{for all } \zeta \in (-\infty, L], \quad \text{(17a)} \\ F(\Psi(L, \zeta) - cL) = kL + \Delta\Psi(L, \zeta) - \Psi(L, \zeta), & \text{for all } \zeta \in (-\infty, L], \quad \text{(17b)} \\ F(\Psi(\xi, L) - c\xi) = k\xi + \Delta\Psi(\xi, L) - \Psi(\xi, L), & \text{for all } \xi \in [0, L]. \quad \text{(17c)} \end{cases}$$

[iii] A traveling wave solution is called a *dipole vortex* or *modon* in the literature.

and if F is continuous

$$F\big(\lim_{\zeta\to\infty}\Psi(\xi,\zeta)-c\xi\big)=k\xi+\lim_{\zeta\to\infty}(\Delta\Psi-\Psi)(\xi,\zeta),\qquad \text{for all } \xi\in[0,L]. \tag{18}$$

provided that these limits exist. Indeed, these equations tell us that the boundary and asymptotic conditions of Ψ determine F and hence the nature of problem, ie whether it's linear or non-linear. Not only this, but if we assume some periodicity conditions like $\Psi(0,\zeta)=\Psi(L,\zeta)$ and $\Delta\Psi(0,\zeta)=\Delta\Psi(L,\zeta)$ for all $\zeta\in(\infty,L]$ then using (17a) and (17b) we get $kL=0$ and thus $k=0$.

3.1 Properties of Traveling Wave Solutions

Note. Throughout this section, we assume that $F: D(F)\subset\mathbb{R}\to\mathbb{R}$ is continuous.

Consider the problem.

$$\begin{cases}\Delta\Psi-\Psi=F(\Psi-c\xi)-k\xi, & \text{on } \Omega:=[0,L]\times(-\infty,L],\\ \text{B.C's inherited from } u, & \text{on } \Gamma=\partial\Omega,\\ \lim_{\zeta\to\infty}\Psi(\xi,\zeta)=0, & \text{for all } \xi\in[0,L].\end{cases} \tag{19}$$

Equation (18) which tells us that asymptotic behaviour of Ψ, if regular enough, can help in determining F (and vice versa of course if F is known beforehand). This has been noted in [2]–[4], where the authors claimed that if Ψ goes to zero for large t then F is a linear function. This is indeed true, however no proofs were provided for that. Furthermore, for their claim to hold it seems that they implicitly assumed that $\Delta\Psi$ decays at infinity and F is continuous. Here is the obvious result as intended by the authors.

Proposition 3.1. *Suppose that Ψ and F satisfy (15) and that*

$$\lim_{\zeta\to\infty}\Psi(\xi,\zeta)=\lim_{\zeta\to\infty}\Delta\Psi(\xi,\zeta)=0,\quad \textit{for all } \xi\in[0,L].\ ^{\text{(iv)}}$$

Then for all $z\in D(F)$, we have $F(z)=-z/c$.

(iv) Of course, Ψ is assumed to be a classical $C^2(\overline{\Omega})$ solution for the discussion of limits to make sense.

This is clearly true by equation (18). However, this result is true for a much larger class of functions, namely if we asumme that $\Psi \in H^1(\Omega) \cap L^\infty(\Omega)$ instead of $C^2(\overline{\Omega})$. The problem becomes however how to define the limit for such functions, as limits along lines are not well defined in H^1. The following definition makes precise this notion.

Definition 3.1 (Trace Along Lines)**.** *Let $\Psi \in H^1(\Omega)$. For each $\xi \in [0, L]$, let $\mathrm{Tr}_\xi : H^1(\Omega) \to L^2\big((-\infty, L]\big)$ be the densely defined continuous linear operator with the property that for all $\varphi \in C_c^\infty(\Omega)$ we have*

$$\mathrm{Tr}_\xi(\varphi)(\zeta) = \varphi(\xi, \zeta), \quad \text{for all } \zeta \in (-\infty, L].$$

For shorthand set

$$\Psi_\infty^-(\xi) := \lim_{\zeta \to -\infty} \mathrm{Tr}_\xi(\Psi)(\zeta),^{\text{(v)}} \quad \text{for almost all } \xi \in (-\infty, L],$$

whenever this limit exists. For continuous functions, it is clear that $\Psi_\infty^-(\xi) = \lim_{\zeta \to -\infty} \Psi(\xi, \zeta)$.

With this definition in mind, we can turn (15) into a variational equation

$$\begin{cases} \displaystyle\int_\Omega \nabla\Psi \cdot \nabla v + \int_\Omega \Psi \cdot v = \int_\Omega \big(k\xi - F(\Psi - c\xi)\big)v + \int_\Gamma (\vec{n} \cdot \nabla\Psi)v, \quad \forall v \in H(\Omega), \\ \Psi|_\Gamma = g \in L^2(\Gamma), \quad \Psi_\infty^- = f \in L^2([0, L]). \end{cases} \tag{20}$$

where $H(\Omega)$ is an appropriate test space which we choose later. Eventually, we will require that H encodes in the asymptotic/boundary conditions of Ψ inherited from u and that $C_c^\infty(\Omega)$ be dense in $H(\Omega)$ with respect to it's norm. We can actually determine F from this equation only with the following result.

Proposition 3.2. *Suppose that the functions $\Psi \in H^1(\Omega) \cap L^\infty(\Omega)$ and F solve equation (20). Suppose that $\Psi_\infty^-(\xi) = 0$ for almost all $\xi \in [0, L]$. Then*

$$F(-c\xi) = k\xi, \quad \text{(a.e) on } [0, L].$$

Thus F and consequently equation (15) are linear.

(v) This definition makes sense since if $\Psi \in H^1(\Omega)$ then $\mathrm{Tr}_\xi(\Psi) \in C((-\infty, L])$ for almost all ξ and the limit makes sense.

Proof. Let $\{\zeta_n\}$ be any decreasing sequence such that to $\zeta_n \to -\infty$. Choose an arbitrary function $\varphi \in C_c^\infty([0, L])$. Fix $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi > 0$ on the interior of it's support. For $n \in \mathbb{N}$ and for all $(\xi, \zeta) \in \Omega$ we let

$$\phi_n(\zeta) := \phi(\zeta - \zeta_n) \text{ and } \psi_n(\xi, \zeta) = \varphi(\xi)\phi_n(\zeta).$$

Notice that ψ_n has compact support and $\text{supp}(\psi_n)$ approaches $+\infty$ as ζ_n tends to $-\infty$. Replace v by ψ_n in (20) and rearrange to get

$$\int \phi_n \Big(F(\Psi - c\xi) - k\xi \Big) \varphi = - \int \left(\phi_n \frac{d\varphi}{d\xi} \frac{\partial \Psi}{\partial \xi} + \varphi \frac{d\phi_n}{d\zeta} \frac{\partial \Psi}{\partial \zeta} + \varphi \phi_n \Psi \right). \tag{21}$$

We have that by Fubini and integration by parts

$$- \int \phi_n \frac{\partial \Psi}{\partial \xi} \frac{d\varphi_1}{d\xi} = - \int_{-\infty}^{L} \phi_n \left(\int_0^L \frac{\partial \Psi}{\partial \xi} \frac{d\varphi}{d\xi} d\xi \right) d\zeta = \int_{-\infty}^{L} \phi_n \left(\int_0^L \Psi \frac{d^2\varphi}{d\xi^2} d\xi \right) d\zeta = \int \varphi'' \phi_n \Psi,$$

and with similar reasoning,

$$- \int \varphi \frac{d\phi_n}{d\zeta} \frac{\partial \Psi}{\partial \zeta} = \int \varphi \phi_n'' \Psi.$$

Note that in the above we have made constant use of the fact that the integrands have compact support and thus allowing us to cancel boundary terms. Now, replacing the obtained identities in (5) to get

$$\int \phi_n \Big(F\big(\Psi - c\xi\big) - k\xi \Big) \varphi = \int \Big(\varphi'' \phi_n + \varphi \phi_n'' + \varphi \phi_n \Big) \Psi.$$

Using the change of variables $(\xi, \zeta - \zeta_n) \to (\xi, \zeta)$ in the above equation

$$\begin{aligned} &\int \phi(\zeta) \Big(F\big(\Psi(\xi, \zeta + \zeta_n) - c\xi\big) - k\xi \Big) \varphi(\xi) \, d\xi d\zeta \\ &\qquad = \int \big(\varphi''(\xi)\phi(\zeta) + \varphi(\xi)\phi''(\xi) + \varphi(\xi)\phi(\zeta) \big) \Psi(\xi, \zeta + \zeta_n) \, d\xi d\zeta. \end{aligned} \tag{22}$$

If we consider the integrand of the R.H.S integral, by Cauchy-Shwarz that it is less than or equal to

$$\left(\int \big(\varphi''(\xi)\phi(\zeta) + \varphi(\xi)\phi''(\xi) + \varphi(\xi)\phi(\zeta) \big)^2 d\xi d\zeta \right)^{1/2} \cdot \left(\int \left(\Psi(\xi, \zeta + \zeta_n) \right)^2 d\xi d\zeta \right)^{1/2}.$$

The first term is clearly bounded, while the second term goes to 0 since $\Psi_\infty^- = 0$ and $\Psi \in L^2(\Omega)$, and thus making sure that the L.H.S of (22) goes to zero. On the other hand, the continuity of F and the fact that for almost all $(\xi, \zeta) \in \Omega$ we have

$$|\Psi(\xi, \zeta + \zeta_n) - c\xi| \leq \|\Psi\|_\infty + cL, \text{ for all } n \in \mathbb{N},$$

tell us that the absolute value of integrand of the L.H.S integral in (22) is bounded by the $L^1(\Omega)$ function

$$|\phi| \cdot |\varphi| \cdot (\sup F + |k|L), \quad \text{where } \sup F \text{ is taken in } [-\|\Psi\|_\infty - cL,\ \|\Psi\|_\infty + cL].$$

Therefore, by letting $n \to \infty$ and using dominated convergence in (22) we get

$$\begin{aligned} 0 &= \int \phi(\zeta)\Big(F\big(\Psi^-_\infty(\xi) - c\xi\big) - k\xi\Big)\varphi(\xi)d\xi d\zeta \\ &= \underbrace{\int \phi(\zeta)d\zeta}_{\neq 0} \cdot \int \big(F\big(- c\xi\big) - k\xi\big)\varphi(\xi)d\xi. \end{aligned} \tag{23}$$

Notice that we have used the continuity of F and the existence of the limit of Ψ at infinity in the above step. Since φ can be chosen arbitrarily, it follows that $F(-c\xi) - k\xi = 0$. $\square$

What is interesting in the above proposition is that the boundary conditions on Γ where not needed to determine F. Hence we can now safely look at the following problem.

Problem 1: (HM decaying traveling wave, linear)

Seek $\Psi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \Delta\Psi = (1 - k/c)\Psi, & \text{on } \Omega, \\ \text{BC's inherited from } u, & \text{on } \Gamma = \partial\Omega, \\ \Psi^-_\infty(\xi) = 0, & \text{on } [0, L]. \end{cases} \tag{24}$$

In variational from

$$\int_\Omega \nabla\Psi \cdot \nabla v + (k/c - 1)\int_\Omega \Psi \cdot v = \int_\Gamma (\vec{n} \cdot \nabla\Psi)v, \quad \forall v \in H(\Omega). \tag{25}$$

We conclude this section with an interesting result that will be helpful in later problems.

Proposition 3.3. *Suppose now $\Omega = [0, L] \times [0, L]$ and that Ψ satisfies (15) on Ω. If Ψ vanishes on any side of $\partial\Omega$, then $F(-c\xi) = k\xi$.*

Proof. Suppose that $\Psi(\xi, L) = 0$ for all $\xi \in [0, L]$. Let $\varphi_1 \in C_c^\infty([0, L])$ be arbitrary. Let φ_2 be the standard bump function whose support is $[0, L/2]$. Then for $n \in \mathbb{N}$ set

$$\rho_n(\zeta) = \frac{L}{2}n(n+1)\zeta - \frac{L}{2}n(n+1)(L - 1/n),$$

and

$$\varphi_n(\zeta) = \begin{cases} 2(n(n+1)L)^{-1}\varphi_2 \circ \rho_n(\zeta) & \text{if } \zeta \in [L - 1/n, L - 1/(n+1)] \\ 0 & \text{otherwise.} \end{cases}$$

Now multiply both sides of (15) by $\varphi(\xi)\varphi_n(\zeta)$ and proceed as in the proof of Proposition (3.2).

$$\begin{aligned} 0 = &\int \int_{L-1/n}^{L-1/(n+1)} \Big(\varphi''(\xi)\varphi_n(\zeta) + \varphi(\xi)\varphi_n''(\zeta) + \varphi(\xi)\varphi_n(\zeta)\Big)\Psi(\xi,\zeta)d\zeta d\xi \\ &+ \int \int_{L-1/n}^{L-1/(n+1)} \varphi(\xi)\varphi_n(\zeta)\Big(\xi - f\big(\Psi(\xi,\zeta) - c\xi\big)\Big)d\zeta d\xi. \end{aligned} \tag{26}$$

We bound the first integral in the R.H.S of the above equation. We have that

$$\left|\int \varphi''(\xi)\varphi_n(\zeta)\right| \le |\text{supp}(\varphi_n)| \cdot \|\varphi''\|_\infty \cdot \|\phi\|_\infty \cdot \|\Psi\|_\infty \to 0,$$

and that

$$\left|\int \varphi(\xi)\varphi_n(\zeta)\right| \le |\text{supp}(\varphi_n)| \cdot \|\varphi\|_\infty \cdot \|\phi\|_\infty \cdot \|\Psi\|_\infty \to 0.$$

As for the last integrand, we have

$$\begin{aligned} \left|\int \varphi(\xi)\varphi_n''(\zeta)\Psi(\xi,\zeta)d\xi d\zeta\right| &\le \|\varphi \cdot \varphi_n''\|_{L^2(\Omega)} \cdot \|\Psi\|_{L^2(\Omega)} \qquad \text{(by Cauchy-Shwarz)} \\ &= \|\varphi\|_{L^2(\mathbb{R})} \cdot \|\varphi_n''\|_{L^2(\mathbb{R})} \cdot \|\Psi\|_{L^2(\Omega)}. \end{aligned}$$

To complete the proof, we will need this term to go to zero and hence to choose ϕ appropriately to guarantee that $\|\varphi_n''\|_{L^2(\mathbb{R})} \to 0$. Such a ϕ can be shown to exist[vi]. On the other hand, using the

[vi] It turns out that the standard bump function

$$\phi(y) = \begin{cases} \exp\left(\frac{1}{y(y-1/2)}\right) & \text{if } 0 < y < 1/2, \\ 0 & \text{otherwise} \end{cases}$$

guarantees that $\|\varphi_n\|_{L^2} \to 0$, change to variables $(\xi,\zeta)\to(\xi,\rho_n^{-1}(\zeta))$ in the second integral of (5)

$$\begin{aligned}
0 &= \int \varphi(\xi)\varphi_n(\zeta)\Big(\xi - f\big(\Psi(\xi,\zeta)-c\xi\big)\Big)\\
&= \int_{\mathrm{supp}(\varphi)}\int_{\mathrm{supp}(\varphi_n)} \varphi(\xi)\varphi_n(\zeta)\Big(\xi - f\big(\Psi(\xi,\zeta)-cx\big)\Big)d\zeta d\xi\\
&= \int\int_{1-\frac{1}{n}}^{1-\frac{1}{n+1}} \varphi(\xi)\varphi_n(\zeta)\Big(\xi - f\big(\Psi(\xi,\zeta)-c\xi\big)\Big)d\zeta d\xi\\
&= \int\int_0^{L/2} \varphi(\xi)\,\frac{\phi(\zeta)}{n+1}\Big(\xi - f\left(\Psi(\xi,\rho_n^{-1}(\zeta))-c\xi\right)\Big)(n+1)d\zeta d\xi,\\
&= \int \varphi(\xi)\varphi(\zeta)\Big(\xi - f\left(\Psi(\xi,\rho_n^{-1}(\zeta))-c\xi\right)\Big).
\end{aligned}$$

Taking limits in in the above equation, then using the Lebesgue Dominated Convergence and the continuity of f we get

$$0=\int \varphi(\xi)\phi(\zeta)\big(\xi - f(-c\xi)\big)\,d\xi\,d\zeta = \left(\int \varphi(\xi)\big(\xi-f(-c\xi)\big)\,dx\right)\left(\int \phi(\zeta)\,d\zeta\right).$$

Since φ is arbitrary, we get that $f(-c\xi)=\xi$. □

3.2 Weighted Sobolev spaces on semi-infinite strip

So we have obtained a first instance where an asymptotic boundary conditions turns (15) into a linear equation. We can linearize equation (15) with even more general assumptions. Let us start with the classical case.

Proposition 3.4. *Suppose that*

- $\Psi\in C^2(\overline{\Omega})$ *satisfies (15).*
- $\Psi_\infty^-(\xi)$ *exists for all* $\xi\in[0,L]$.
- $\lim_{\zeta\to-\infty}\Delta(\xi,\zeta)=0$ *for all* $\xi\in[0,L]$.

Then F is linear if and only if Ψ_∞^- is linear.

Proof. This is clearly true by fixing ξ and letting $\zeta\to-\infty$ in (15). □

We aim to generalize this result to a larger class of functions. But we encounter a problem: if $\Psi \in C^2(\overline{\Omega})$ and Ψ_∞^- is not zero, then $\Psi \notin L^p(\Omega)$ for all $1 \leq p < \infty$ and we cannot use standard Sobolev spaces and integration techniques. Thus we are lead to look at weighted Sobolev spaces, which are similar to standard Sobolev spaces but they are endowed with weights prescribing the functions' growth or decay at infinity. They are defined as follows.

Definition 3.2. *Fix an almost everywhere positive function $w \in L^1_{\text{loc}}(\Omega)$, a constant $\alpha \in \mathbb{R}$ and an integer p with $1 \leq p < \infty$. Define*

$$L^p_{w,\alpha}(\Omega) := \left\{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } w^\alpha u \in L^p(\Omega)\right\} \subset L^1_{\text{loc}}(\Omega),$$

endowed with the norm $\|u\|_{L^p_{w,\alpha}(\Omega)} = \|w^\alpha u\|_{L^p(\Omega)}$. In particular, $L^2_{w,\alpha}(\Omega)$ is a Hilbert space with inner product $\langle \cdot\,, \cdot\rangle : L^2_{w,\alpha}(\Omega) \times L^2_{w,\alpha}(\Omega) \to \mathbb{R}$ defined by

$$\langle u, v\rangle_{L^2_{w,\alpha}(\Omega)} := \langle w^\alpha u, w^\alpha v\rangle_{L^2(\Omega)}\,,^{\text{(vii)}} \quad \text{for all } u, v \in L^2_{w,\alpha}(\Omega).$$

This turns $L^2_{w,\alpha}(\Omega)$ into a Hilbert space. If $\alpha = 1$ we write $L^p_w(\Omega)$ for shorthand. In general for $m \geq 0$, set

$$W^{m,p}_{w,\alpha}(\Omega) := \{u \in L^p_{w,\alpha}(\Omega) : D^\beta u \in L^2_{w,\alpha}(\Omega),\ \forall (0 \leq |\beta| \leq m)\},$$

which convention $W^{0,p}_{w,\alpha}(\Omega) := L^p_{w,\alpha}(\Omega)$. These spaces are Banach spaces with norm

$$\|u\|_{W^{m,p}_{w,\alpha}}(\Omega) = \left(\sum_{0\leq|\beta|\leq m} \left\|D^\beta u\right\|^p_{L^p_{w,\alpha}(\Omega)}\right)^{1/p}.$$

In particular, we write $H^m_{w,\alpha}(\Omega) = W^{m,2}_{w,\alpha}(\Omega)$, which is a Hilbert space with inner product

$$\langle u, v\rangle_{H^m_{w,\alpha}(\Omega)} = \sum_{0\leq|\beta|\leq m} \langle D^\beta u, D^\beta v\rangle_{L^2_{w,\alpha}(\Omega)}\,.$$

The theory of such spaces is well established [10], [11] for several domain types and for for weights w satisfying the A_p *condition*[(viii)] defined as follows.

(vii) This inner product is finite since by Cauchy-Shwarz $|\langle w^\alpha u, w^\alpha v\rangle| \leq \|w^\alpha u\|_{L^2(\Omega)} \cdot \|w^\alpha v\|_{L^2(\Omega)} < \infty$ for all $u, v \in L^2_{w,\alpha}(\Omega)$.

(viii) The A_p condition guarantees that the Hardy-Littlewood maximal operator $M : L^p_w(\mathbb{R}^n) \to L^p_w(\mathbb{R}^n)$ defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

is continuous when $1 < p < \infty$.

Definition 3.3 (A_p condition)**.** *A weight w having the property there is a constant C such that for every ball $B \subset \Omega$,*

$$\left(\int_B w\right)\left(\int_B w^{-1/(p-1)}\right)^{p-1} \leq C|B|^p, \qquad (A_p\text{-condition})$$

is said to satisfy the A_p condition and is called an A_p weight if $1 < p < \infty$. If $p = 1$ the A_p condition becomes

$$\left(\int_B w\right)\|w^{-1}\|_{L^\infty(B)} \leq C|B|. \qquad (A_1\text{-condition})$$

On the other hand, we say that w is a A_∞ weight if there are two constants C and δ such that

$$w(Q) \geq C\left(\frac{|Q|}{|E|}\right)^\delta w(E). \qquad (A_\infty\text{-condition})$$

for every cube $Q \subset \Omega$ and any non-null measurable subset $E \subset Q$.

If $w \in A_p$ with $1 \leq p \leq \infty$, then the infimum of all constants C is called the A_p $\hat{e}$xtitconstant of w, and is usually denoted by A. The set of all weights w satisfying the A_p condition is denoted by A_p for shorthand.

Proposition 3.5 (Properties of Weighted Spaces)**.** *Suppose $\Omega \subset \mathbb{R}^2$ is open. Let w be a weight and $1 \leq p < \infty$. We have the following simple properties.*

(i) *From Hölder's inequality we have*

$$\int_B |f| \leq \left(\int_B w^{-1/(p-1)}\right)^{1/p'}\left(\int_B |f|^p\right)^{1/p},$$

for every ball $B \subset \Omega$.

(ii) *If $w^{-1/(p-1)} \in L^1_{loc}(\Omega)$ and $w^{-1} \in L^\infty(B)$ for every ball $B \subset \Omega$ then $L^p_w(\Omega) \subset L^1_{loc}(\Omega)$.*

(iii) *If $w \in A_p$ then $L^p_w(\Omega) \subset L^1_{loc}(\Omega)$.*

(iv) *If $1 \leq p < q < \infty$ then $A_p \subset A_q$.*

(v) *If $w \in A_p$ then $w^{-1/(p-1)} \in A'_p$.*

(vi) *The A_p condition is invariant under translations and dialations.*

(vii) *If $w \in A_p$ and A is it's A_p constant then $A \geq 1$.*

(viii) *If $w \in A_1$, then $w \geq (1+|x|)^{-N}$ for all $x \in \Omega$.*

(ix) *If there are constants $C, D \in \mathbb{R}$ such that $0 < C \leq w \leq D$ then $w \in A_p$ for all $1 \leq p < \infty$.*

(x) *If $w(x) = |x|^\alpha$ then*

$$w \in \begin{cases} A_1 & \text{if } -N < \alpha \leq 0, \\ A_p & \text{if } -N < \alpha < N(p-1). \end{cases} \tag{27}$$

(xi) *$w^\alpha \in A_2$ for some $\alpha > 0$ if and only if w has bounded mean oscillation.*

To be added: Some results on weigthet Sobolev spaces to help generalize Proposition 3.4 from $\Psi \in C^2(\overline{\Omega})$ with $\Psi_\infty^- = c$ to $\Psi \in H^1_{w,\alpha}(\Omega)$ for some appropriate weight w and exponent α.

3.3 Traveling waves with periodicity in the ζ-direction

Let us introduce some periodicity in problem (15). Assume $u(0,y,t) = u(L,y,t)$ for all $(y,t) \in [0,L] \times \mathbb{R}^+$ in (1) and u becomes linear for large t. For the traveling wave function Ψ, this means that $\Psi_\infty^-(\xi) = b$, where b is some real constant. Hence if we take $\Psi \in C^2(\overline{\Omega})$, equation (18) tells us that

$$F(b - c\xi) = k\xi - b,$$

and therefore

$$F(\xi) = F(b - c(-\xi/c + b/c)) = -\frac{k}{c}\xi + \left(\frac{k}{c} - 1\right) b.$$

This leads to the following boundary value problem.

Problem 2: (HM Traveling Wave, L-periodic in ξ, Constant at Infinity)
Seek $\Psi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \Delta\Psi + (k/c - 1)\Psi = (k/c - 1)b, \\ \Psi(0,\zeta) = \Psi(L,\zeta), \ \Psi_\infty^-(\xi) = b. \end{cases} \tag{28}$$

The setup of Problem 2 bears resemblance with the one in [12] where existence and uniqueness of a 1-periodic solution to the Laplace equation $\Delta u = f$ equation on the strip $[0,1] \times \mathbb{R}$ was shown in some weighted Sobolev spaces.

Definition 3.4. *Let Ω be the strip $[0,1]\times\mathbb{R}$. Define*

$$\mathcal{D}_{\#}(\Omega)=\{\varphi:\Omega\to\mathbb{R}\mid y\to\varphi(x,y)\in C^{\infty}([0,1])\ 1\text{-periodic},\ x\to\varphi(x,y)\in C_c^{\infty}(\mathbb{R})\},$$

and $\mathcal{D}_{\#}^{*}(\Omega)$ be its dual. Define the weight $w=w(x,y)=(1+y^2)^{1/2}$. For $\alpha\in\mathbb{R}$, let

$$L_{\alpha}^{2}(\Omega)=\{u\in\mathcal{D}_{\#}^{*}(\Omega): w^{\alpha}u\in L^{2}(\Omega)\}.$$

It can be shown that $L_{\alpha}^{2}(\Omega)$ is a Banach space with norm $\|u\|_{L_{\alpha}^{2}(\Omega)}=\|w^{\alpha}u\|_{L^{2}(\Omega)}$ and that $C_c^{\infty}(\Omega)$ is dense in $L_{\alpha}^{2}(\Omega)$. If $m\in\mathbb{N}$ and

$$k=\begin{cases} m-1/2-\alpha & \text{if } \alpha\in\{1/2,\dots,m-1/2\},\\ -1 & \text{if } \alpha\notin\{1/2,\dots,m-1/2\}.\end{cases}$$

We let $H_{\alpha,\#}^{0}(\Omega)=L_{\alpha}^{2}(\Omega)$ and in general for $m\in\mathbb{N}$, we define the weighted Sobolev spaces

$$H_{\alpha,\#}^{m}(\Omega)=\left\{u\in\mathcal{D}_{\#}^{*}(\Omega)\ \middle|\ \begin{array}{r} w^{-m+|\beta|}(\ln(1+w^2))^{-1}D^{\beta}u\in L_{\alpha}^{2}(\Omega),\quad \forall\, 0\le|\beta|\le k,\\ w^{-m+|\beta|}D^{\beta}u\in L_{\alpha}^{2}(\Omega),\quad \forall\, k<|\beta|<m\end{array}\right\}.$$

Also, the norm

$$\|u\|_{H_{\alpha,\#}^{m}(\Omega)}=\left(\sum_{0\le|\beta|\le k}\left\|\frac{w^{-m+|\beta|}}{\ln(1+w^2)}D^{\beta}u\right\|_{L_{\alpha}^{2}(\Omega)}^{2}+\sum_{k<|\beta|\le m}\left\|w^{-m+|\beta|}D^{\beta}u\right\|_{L_{\alpha}^{2}(\Omega)}^{2}\right)^{1/2},$$

turns $H_{\alpha,\#}^{m}(\Omega)$ into a Banach space.

Theorem 3.1 (V. Milisic, U.Razafison, [12])**.** *For $1/2\le\alpha\le 3/2$ and for $m\in\mathbb{N}$ with $m\ge 3$, the Laplace operator Δ is an isomorphism when seen as*

$$\Delta: H_{\alpha,\#}^{m}(\Omega)/\mathbb{P}'_{m-2}\to H_{\alpha,\#}^{m-2}(\Omega)/\mathbb{P}'_{m-4}\quad\text{and}\quad \Delta: H_{\alpha,\#}^{m}(\Omega)/\mathbb{P}_{m-2}^{\Delta}\to H_{\alpha,\#}^{m-2}(\Omega),$$

where $\mathbb{P}'_{j}$ is the set of polynomials in y only of degree at most j and $\mathbb{P}_{j}^{\Delta}$ is the subset of $\mathbb{P}'_{j}$ consisting of harmonic polynomials.

To be added: The technique used in the above theorem can be used to tackle Problem 2. I am currently reading [12] in detail to see how to apply it for Problem 2 appropriately.

3.4 Traveling waves with PBC's

Of interest are traveling waves Ψ inheriting the P.B.C's of $u \in H^1_P(\mathcal{R})$. For starters, suppose $u \in H^1_P(\mathcal{R}) \cap C^2(\mathcal{R})$. We obtain the following problem.

__Problem 3__**:** (HM Traveling Wave, L-periodic in ξ, L-periodic in ζ).

Seek $\Psi : \mathcal{R} \to \mathbb{R}$ such that if β is a multi-index and $0 \leq |\beta| \leq 3$ then

$$\begin{cases} \Delta\Psi = \Psi + F(\Psi - c\xi) - k\xi, \\ D^\beta\Psi(\xi, -ct) = D^\beta\Psi(\xi, L - ct), \\ D^\beta\Psi(0, \zeta) = D^\beta\Psi(L, \zeta). \end{cases} \tag{29}$$

Periodicity allows us restrict our study'to the rectangle $[0, L] \times [0, L]$ and so Problem 3 becomes

$$\begin{cases} \Delta\Psi = \Psi + F(\Psi - c\xi) - k\xi, & \forall(\xi, \zeta) \in [0, L] \times [0, L], \\ D^\beta\Psi(\xi, 0) = D^\beta\Psi(\xi, L) = f_\beta(\xi), & \forall\xi \in [0, L], \\ D^\beta\Psi(0, \zeta) = D^\beta\Psi(L, \zeta) = g_\beta(\zeta) & \forall\zeta \in [0, L]. \end{cases} \tag{30}$$

To linearize this equation without any assumption on $\Delta\Psi$, we can assume that Ψ vanishes on any side of the rectangle $\mathcal{R}$ and Proposition 3.3 would apply. Note here that the discussion of asymptotic behaviour becomes irrelevant as periodicity automatically tells us that Ψ^-_∞ doesn't exist.

__**To be added:**__ Seek boundary conditions on u that linearize Problem 3 and investigate a boundary condition that does not allow for a linear choice of F.

4 The Problem as Solved in the Original Paper

In the paper [2], the problem of finding a Modon is stated as follows. Fix t, choose a radius a and a velocity c and consider the region

$$\Omega_t = \{(x, y) : x^2 + (y - ct)^2 < a\}.$$

In the unbounded region Ω_t^c, the fluid elements are not "trapped", and so the condition $\Phi \to 0$ as $x, y \to \infty$ is added. This applied to (1) implies that $f(z) = -z/c$ in that region. In Ω_t, f is arbitrary and hence was chosen to be $f(z) = -(1 + s^2)z$ with $s = \gamma^2/a^2$, where γ is a parameter that will chosen later. To restate the problem, we seek $\Phi \in C^2(\mathbb{R}^2)$ such that:

$$\begin{cases} \Delta\Phi + (1/c - 1)\Phi = 0 & \text{if } (x, y) \in \Omega_t^c, \\ \Delta\Phi + s^2\Phi = ((1 + s^2)c - 1)\, x & \text{if } (x, y) \in \Omega_t. \end{cases} \tag{31}$$

Using the change of coordinates,

$$r^2 = x^2 + (y - ct)^2, \qquad \cos\theta = x/r,$$

the Modon equation is given by

$$\Phi(r, \theta) = \begin{cases} AK_1(\beta r/a)\cos\theta & \text{if } r > a, \\ \dfrac{Br}{a}\cos\theta + CJ_1(\gamma r/a)\cos\theta & \text{if } r < a, \end{cases} \tag{32}$$

for some constants $a, c \in \mathbb{R}$, $\beta = a(1 - 1/c)$, γ a parameter that will be chosen later and,

$$A = \frac{ac}{K_1(\beta)}\,, \quad B = ac\left(1 + \frac{1}{\gamma^2}\right), \quad \text{and } \; C = -\frac{\beta^2}{\gamma^2}\frac{ac}{J_1(\gamma)}.$$

Concern: Let's look at the issue of compatibility between (1) and (4). By Proposition 1, the assumption that Φ decays for w.r.t the spacial variables, implies that $f(z) = -z/c$ at the points of continuity of f. Thus if Φ satisfies (4) and f is continuous at the point z then

$$-z/c = f(z) = -(1 + s^2)z, \text{ and therefore } \; c = (1 + s^2)^{-1}.$$

Thus we obtain a relation between a and c, and this may have impact on the physical interpretation of the Modon.

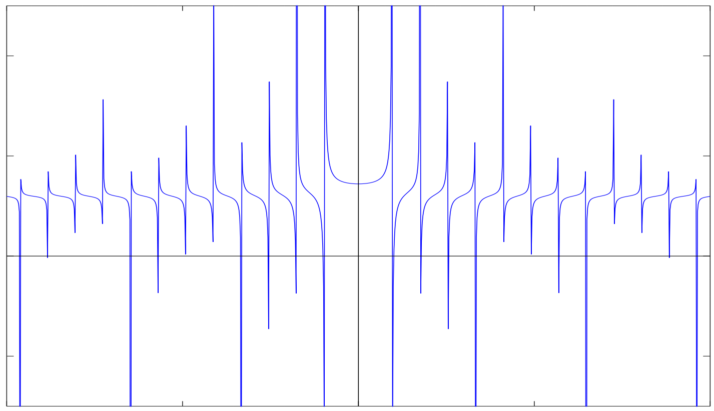


Figure 2: Possible values of γ for $a = 2$ and $c = 6$.

4.1 The Continuity Equation in the Original Paper

Suppose that we take the Modon solution as given in (5). As said there, there is a parameter γ that will be chosen to that Φ is C^2. This parameter satisfies the following equation, called the continuity equation, given in **1** by

$$\frac{K_2(\beta)}{\beta K_1(\beta)} = \frac{J_2(\gamma)}{\gamma J_1(\gamma)}. \tag{33}$$

This equation which gives many values γ for the fixed value β; this can be seen in Figure 1 where γ is any root of the function plotted.

Concern: There is no mentioning on how the above equation is obtained, so I tried to re-derive the result. Continuity of (5) implies that by fixing $\theta \in [0, 2\pi]$ and letting $r \to a^{\pm}$, one should obtain

$$AK_1(\beta) = B + CJ_1(\gamma), \tag{34}$$

After simplification, the above equation reduces to $\beta = \pm 1$ which is definitely not related to (6) in any way. Also, this means that the variables a and c have to be related in order for proposed solution (5) to be continuous, which is also a problem.

4.2 Deriving C^2 continuity (can ignore this part)

Now assuming that the needed condition on β is satisfied, we look for higher order continuity. We need only to look at the derivatives with respect to the r variable as partial derivatives with respect to θ provide the same continuity conditions. Thus we compute

$$\frac{\partial \Phi}{\partial r}(r,\theta) = \begin{cases} A \cdot \dfrac{\beta}{a} \cdot \dfrac{K_0(\beta r/a) + K_2(\beta r/a)}{-2} \cdot \cos\theta & \text{if } r > a, \\ \\ \dfrac{B}{a} \cdot \cos\theta + C \cdot \dfrac{\gamma}{a} \cdot \dfrac{J_0(\gamma r/a) - J_2(\gamma r/a)}{2} \cdot \cos\theta & \text{if } r < a. \end{cases}$$

Canceling $\cos(\theta)/2a$ from both sides and taking limits as $r \to a^+, a^-$ we get:

$$-A\beta(K_0(\beta) + K_2(\beta)) = 2B + C\gamma(J_0(\gamma) - J_2(\gamma)). \tag{35}$$

This equation guarantees C^1 continuity at $r = a$. To get C^2 continuity, we compute

$$\frac{\partial^2 \Phi}{\partial r^2}(r,\theta) = \begin{cases} \dfrac{A}{4}\dfrac{\beta^2}{a^2} \cos\theta \left(3K_1(\beta r/a) + K_3(\beta r/a)\right), & \text{if } r > a, \\ \\ \dfrac{C}{4}\dfrac{\gamma^2}{a^2} \cos\theta \left(-3J_1(\gamma r/a) + J_3(\gamma r/a)\right), & \text{if } r < a. \end{cases}$$

Taking limits again for all $\theta \in [0, 2\pi]$ we get:

$$A\beta^2(2K_1(\beta) + K_3(\beta)) = C\gamma^2(-3J_1(\gamma) + J_3(\gamma)). \tag{36}$$

The continuity of the partial derivatives $\partial_{\theta\theta}\Phi$ and $\partial_{\theta r}$ gives the same result as (7) and (8) respectively. So we look for β satisfying (7) and γ simultaneously satisfying (8) and (9).

5 Simulations and numerical results

We are interested in testing for an upper bound on the H^∞-norm of on the solutions w of the coupled system:

$$\begin{cases} -\Delta u + u = w, \\ w_t + \vec{V}(u) \cdot \nabla w = -k u_y. \end{cases}$$

The simulator is written in FreeFem++ and can be found here: `https://github.com/adelsaleh/hmSimulator`

5.1 The Algorithm

We give a description of the algorithm that approximates

$$\sup_{0 \le t \le T} (\|\nabla w\|_\infty + \|w\|_\infty),$$

for some given time T. This algorithm is correct as long as $w \in \mathcal{V}_h$. It works as follows, suppose T_i is a triangle in the mesh $\mathcal{T}$. Let $w_i = w|_{T_i}$ then there are $a_i, b_i, c_i \in \mathbb{R}$ such that for all $(x, y) \in T_i$,

$$w(x, y) = w_i(x, y) = a_i x + b_i y + c_i, \quad \text{and therefore } \; \nabla w_i = (a_i, b_i).$$

The code recovers the values of a_i, b_i, c_i. We are given three vertices $P_j^i = (x_j^i, y_j^i) \in T_i$ for $j = 1, 2, 3$ from which we solve:

$$\begin{pmatrix} x_1^i & y_1^i & 1 \\ x_2^i & y_2^i & 1 \\ x_3^i & y_3^i & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} w_i(x_1^i, y_1^i) \\ w_i(x_2^i, y_2^i) \\ w_i(x_3^i, y_3^i) \end{pmatrix}.$$

If $\mathcal{P}$ is the set of nodes of the mesh then what we are computing at each $0 \le t \le 75$:

$$\|w\|_\infty + \|\nabla w\|_\infty = \max_{p \in \mathcal{P}} |w(p)| + \max_{T_j \in \mathcal{T}} (|a_j| + |b_j|).$$

5.2 Testing for $W^{1,\infty}$ norm

The code has been tested for 3 different initial conditions, and for endt=76.

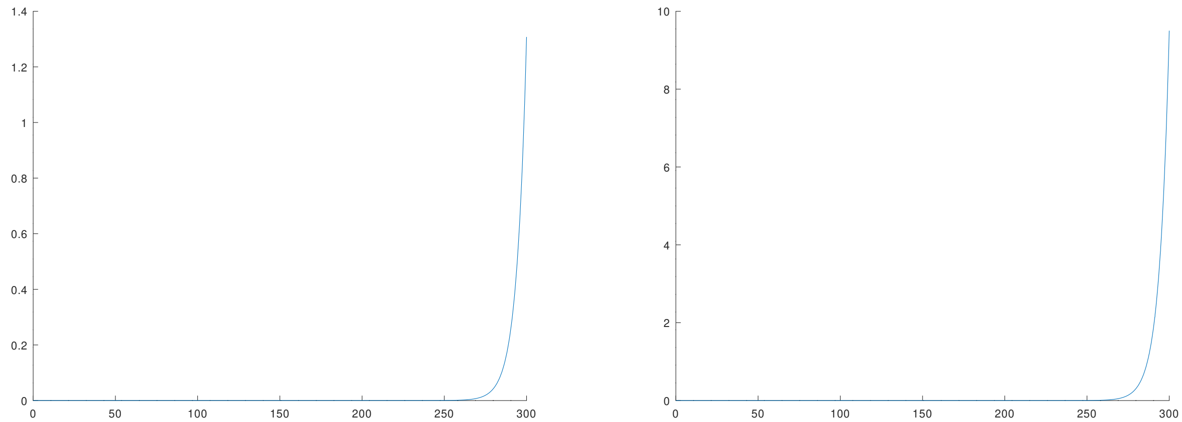


Figure 3: $u_0 = \sin 3x$, $\Omega = [0, 0.5]^2$, $\Delta t = 0.1$ and $d = 32$.

<u>**Case 1:**</u> $u_0(x, y) = \sin 3x.$

- for $dt = 1$

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	9.72845e+06
meshp = 32	0.00932348	0.00932348	0.0700644	9.48621e+06
meshp = 64	0.00935401	0.00935401	0.25487	3.36182e+07

- for $dt = 0.1$

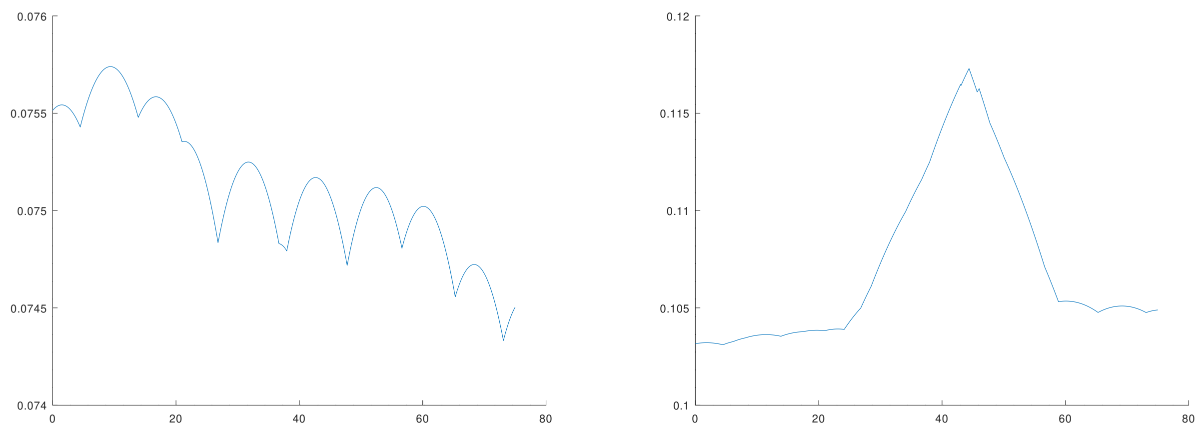
	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	0.00920216
meshp = 32	0.00932348	0.00932348	0.00932348	0.00932348
meshp = 64	0.00935401	0.00935401	0.00935401	0.00935401

- For dt=0.01

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	0.00920216
meshp = 32	0.00932348	0.00932348	0.00932348	0.00932348
meshp = 64	0.00935401	0.00935401	0.00935401	0.00935401

Case 2: $u_0(x, y) = \sin 3y$.

dt	meshsize	t=1	t=25	t=50	t=75
1	16	0.00819026	226093	2.35954e+13	3.38578e+20
	32	0.0174697	274395	3.46674e+13	2.54711e+18
	64	0.049547	288790	3.84457e+13	8.44555e+19
0.1	16	0.00109678	0.0228526	1.44038	92.6633
	32	0.00177724	0.0251274	1.70835	118.378
	64	0.00541686	0.0267284	1.79248	126.325
0.01	16	0.000573946	0.000709836	0.000882754	0.001284
	32	0.000786692	0.000811604	0.0010299	0.0013535
	64	0.0012995	0.00132479	0.00137715	0.00169396



(a) L^∞-norm of $u(t)$.

(b) $W^{1,\infty}$-norm of $u(t)$.

Figure 4: Gauss initial condition on $\Omega = [-10, 10]^2$, $\Delta t = 0.01$ and $d = 64$.

- $dt = 1$.

	t=1	t=25	t=50	t=75
meshp = 16	0.271741	0.297416	0.304924	0.309987
meshp = 32	0.268623	0.291456	0.298063	0.298935
meshp = 64	0.263379	0.289162	0.294879	0.294879

- $dt = 0.1$.

	t=1	t=25	t=50	t=75
meshp = 16	0.27173	0.295808	0.303704	0.307203
meshp = 32	0.26973	0.290931	0.29701	0.29796
meshp = 64	0.262307	0.288401	0.293987	0.293987

- $dt = 0.01$.

	t=1	t=25	t=50	t=75
meshp = 16	0.271735	0.295687	0.303588	0.306932
meshp = 32	0.269837	0.29089	0.296902	0.297862
meshp = 64	0.262465	0.288328	0.293898	0.293898

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