

Report on Hasegawa-Mima

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1 Introduction

The Hasegawa-Mima equation **crotinger**, **hasegawa-mima**, **nycander**, **farah** is a second order non linear differential equation which can be written as a coupled system of linear equations **hagop**, **hagopp** as follows

$$\begin{cases} -\Delta u + u = w, \\ w_t + \vec{V}(u) \cdot \nabla w = -ku_y, \end{cases} \quad (1)$$

In the above system, we seek solutions $u, v \in H^1(\Omega)$ with $\Omega = [0, w] \times [0, h] \subset \mathbb{R}^2$ for some $w, h > 0$. We also have $k \in \mathbb{R}$ is a constant and

$$\vec{V}(u) = \begin{pmatrix} -\partial u / \partial y \\ \partial u / \partial x \end{pmatrix}.$$

In this small report, we prove existence of traveling wave solutions or *Modons* as in **crotinger** for the Hasegawa-Mima equation and compare results. Later, we simulate using FreeFem++ some solutions on the time interval $[0, T_{\max}]$ and then generate some $W^{1,\infty}$ estimates of the solution at specific times.

2 Traveling Wave Solutions

In **crotinger**, **nycander**, traveling wave solutions⁽ⁱ⁾ for the Hasegawa-Mima equation were found on \mathbb{R}^2 and under restrictive assumptions on the second derivatives of u . In this section, we prove stronger results. Let $\mathcal{R} = [0, L] \times [0, L]$. If one looks for traveling wave solution for the Hasegawa-Mima equation traveling in the y direction with constant velocity c , one would seek a solution of the form

$$u(x, y, t) = \Psi(\xi, \zeta), \quad (\xi, \zeta) = (x, y - ct), \quad \forall (x, y, t) \in \mathcal{R} \times [0, \infty).$$

Replacing in (??), the traveling wave form of the Hasegawa-Mima equation is

$$\begin{cases} \Delta \Psi - \Psi = F(\Psi - c\xi) - k\xi, & \text{on } \Omega := [0, L] \times (-\infty, L], \\ \text{B.C's inherited from } u, & \text{on } \Gamma = \partial\Omega, \\ \text{A.C's inherited from } u, & \text{as } \zeta \rightarrow -\infty \text{ (i.e as } t \rightarrow \infty). \end{cases} \quad (2)$$

⁽ⁱ⁾A traveling wave solution is called a *dipole vortex* or *modon* in the literature.

where $F : D(F) \subset \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary function and $k \in \mathbb{R}$ is a constant. It is important to note since $D(F)$ must contain the range of Ψ , then different assumptions on the solution Ψ provide different functions F . We will mainly require that $D(F)$ is an interval containing the essentially bounded values of $\Psi - c\xi$. If Ψ is not essentially bounded, then we take $D(F) = \mathbb{R}$. Note that by letting $\Phi = \Psi - c\xi$ we get

$$F(\Phi) = \Delta\Phi - \Phi + (k - c)\xi, \quad \text{for all } (\xi, \zeta) \in \Omega. \quad (3)$$

Depending on convenience we will use either (??) or (??). Our goal is to investigate the nature of the function F , which can be determined completely by the boundary/asymptotic behaviour of Ψ . We will be mainly interested in the conditions that linearize equation (??), but some interesting non linear cases might help in investigating the $L^\infty(\Omega)$ behaviour of Ψ in ζ , and hence of u in t . Equation (??) tells us that

$$\begin{cases} F(\Psi(0, \zeta) - cL) = \Delta\Psi(0, \zeta) - \Psi(0, \zeta), & \text{for all } \zeta \in (-\infty, L], & (4a) \\ F(\Psi(L, \zeta) - cL) = kL + \Delta\Psi(L, \zeta) - \Psi(L, \zeta), & \text{for all } \zeta \in (-\infty, L], & (4b) \\ F(\Psi(\xi, L) - c\xi) = k\xi + \Delta\Psi(\xi, L) - \Psi(\xi, L), & \text{for all } \xi \in [0, L]. & (4c) \end{cases}$$

and if F is continuous

$$F\left(\lim_{\zeta \rightarrow \infty} \Psi(\xi, \zeta) - c\xi\right) = k\xi + \lim_{\zeta \rightarrow \infty} (\Delta\Psi - \Psi)(\xi, \zeta), \quad \text{for all } \xi \in [0, L]. \quad (5)$$

provided that these limits exist. Indeed, these equations tell us that the boundary and asymptotic conditions of Ψ determine F and hence the nature of problem, ie whether it's linear or non-linear. Not only this, but if we assume some periodicity conditions like $\Psi(0, \zeta) = \Psi(L, \zeta)$ and $\Delta\Psi(0, \zeta) = \Delta\Psi(L, \zeta)$ for all $\zeta \in (-\infty, L]$ then using (??) and (??) we get $kL = 0$ and thus $k = 0$.

2.1 Properties of Traveling Wave Solutions

Note. Throughout this section, we assume that $F : D(F) \subset \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Consider the problem.

$$\begin{cases} \Delta\Psi - \Psi = F(\Psi - c\xi) - k\xi, & \text{on } \Omega := [0, L] \times (-\infty, L], \\ \text{B.C's inherited from } u, & \text{on } \Gamma = \partial\Omega, \\ \lim_{\zeta \rightarrow \infty} \Psi(\xi, \zeta) = 0, & \text{for all } \xi \in [0, L]. \end{cases} \quad (6)$$

Equation (??) which tells us that asymptotic behaviour of Ψ , if regular enough, can help in determining F (and vice versa of course if F is known beforehand). This has been noted in **crotinger**, **nycander**, **farah**, where the authors claimed that if Ψ goes to zero for large t then F is a linear function. This is indeed true, however no proofs were provided for that. Furthermore, for their claim to hold it seems that they implicitly assumed that $\Delta\Psi$ decays at infinity and F is continuous. Here is the obvious result as intended by the authors.

Proposition 2.1. *Suppose that Ψ and F satisfy (??) and that*

$$\lim_{\zeta \rightarrow \infty} \Psi(\xi, \zeta) = \lim_{\zeta \rightarrow \infty} \Delta\Psi(\xi, \zeta) = 0, \quad \text{for all } \xi \in [0, L]. \quad (\text{ii})$$

Then for all $z \in D(F)$, we have $F(z) = -z/c$.

This is clearly true by equation (??). However, this result is true for a much larger class of functions, namely if we assume that $\Psi \in H^1(\Omega) \cap L^\infty(\Omega)$ instead of $C^2(\overline{\Omega})$. The problem becomes however how to define the limit for such functions, as limits along lines are not well defined in H^1 . The following definition makes precise this notion.

Definition (Trace Along Lines). *Let $\Psi \in H^1(\Omega)$. For each $\xi \in [0, L]$, let $\text{Tr}_\xi : H^1(\Omega) \rightarrow L^2((-\infty, L])$ be the densely defined continuous linear operator with the property that for all $\varphi \in C_c^\infty(\Omega)$ we have*

$$\text{Tr}_\xi(\varphi)(\zeta) = \varphi(\xi, \zeta), \quad \text{for all } \zeta \in (-\infty, L].$$

For shorthand set

$$\Psi_\infty^-(\xi) := \lim_{\zeta \rightarrow -\infty} \text{Tr}_\xi(\Psi)(\zeta), \quad (\text{iii}) \quad \text{for almost all } \xi \in (-\infty, L],$$

whenever this limit exists. For continuous functions, it is clear that $\Psi_\infty^-(\xi) = \lim_{\zeta \rightarrow -\infty} \Psi(\xi, \zeta)$.

With this definition in mind, we can turn (??) into a variational equation

$$\begin{cases} \int_{\Omega} \nabla \Psi \cdot \nabla v + \int_{\Omega} \Psi \cdot v = \int_{\Omega} (k\xi - F(\Psi - c\xi))v + \int_{\Gamma} (\vec{n} \cdot \nabla \Psi)v, & \forall v \in H(\Omega), \\ \Psi|_{\Gamma} = g \in L^2(\Gamma), \quad \Psi_\infty^- = f \in L^2([0, L]). \end{cases} \quad (7)$$

⁽ⁱⁱ⁾Of course, Ψ is assumed to be a classical $C^2(\overline{\Omega})$ solution for the discussion of limits to make sense.

⁽ⁱⁱⁱ⁾This definition makes sense since if $\Psi \in H^1(\Omega)$ then $\text{Tr}_\xi(\Psi) \in C((-\infty, L])$ for almost all ξ and the limit makes sense.

where $H(\Omega)$ is an appropriate test space which we choose later. Eventually, we will require that H encodes in the asymptotic/boundary conditions of Ψ inherited from u and that $C_c^\infty(\Omega)$ be dense in $H(\Omega)$ with respect to it's norm. We can actually determine F from this equation only with the following result.

Proposition 2.2. *Suppose that the functions $\Psi \in H^1(\Omega) \cap L^\infty(\Omega)$ and F solve equation (??). Suppose that $\Psi_\infty^-(\xi) = 0$ for almost all $\xi \in [0, L]$. Then*

$$F(-c\xi) = k\xi, \quad (\text{a.e.}) \text{ on } [0, L].$$

Thus F and consequently equation (??) are linear.

Proof. Let $\{\zeta_n\}$ be any decreasing sequence such that $\zeta_n \rightarrow -\infty$. Choose an arbitrary function $\varphi \in C_c^\infty([0, L])$. Fix $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi > 0$ on the interior of it's support. For $n \in \mathbb{N}$ and for all $(\xi, \zeta) \in \Omega$ we let

$$\phi_n(\zeta) := \phi(\zeta - \zeta_n) \quad \text{and} \quad \psi_n(\xi, \zeta) = \varphi(\xi)\phi_n(\zeta).$$

Notice that ψ_n has compact support and $\text{supp}(\psi_n)$ approaches $+\infty$ as ζ_n tends to $-\infty$. Replace v by ψ_n in (??) and rearrange to get

$$\int \phi_n \left(F(\Psi - c\xi) - k\xi \right) \varphi = - \int \left(\phi_n \frac{d\varphi}{d\xi} \frac{\partial \Psi}{\partial \xi} + \varphi \frac{d\phi_n}{d\zeta} \frac{\partial \Psi}{\partial \zeta} + \varphi \phi_n \Psi \right). \quad (8)$$

We have that by Fubini and integration by parts

$$- \int \phi_n \frac{\partial \Psi}{\partial \xi} \frac{d\varphi_1}{d\xi} = - \int_{-\infty}^L \phi_n \left(\int_0^L \frac{\partial \Psi}{\partial \xi} \frac{d\varphi}{d\xi} d\xi \right) d\zeta = \int_{-\infty}^L \phi_n \left(\int_0^L \Psi \frac{d^2 \varphi}{d\xi^2} d\xi \right) d\zeta = \int \varphi'' \phi_n \Psi,$$

and with similar reasoning,

$$- \int \varphi \frac{d\phi_n}{d\zeta} \frac{\partial \Psi}{\partial \zeta} = \int \varphi \phi_n'' \Psi.$$

Note that in the above we have made constant use of the fact that the integrands have compact support and thus allowing us to cancel boundary terms. Now, replacing the obtained identities in (5) to get

$$\int \phi_n \left(F(\Psi - c\xi) - k\xi \right) \varphi = \int \left(\varphi'' \phi_n + \varphi \phi_n'' + \varphi \phi_n \right) \Psi.$$

Using the change of variables $(\xi, \zeta - \zeta_n) \rightarrow (\xi, \zeta)$ in the above equation

$$\begin{aligned} & \int \phi(\zeta) \left(F(\Psi(\xi, \zeta + \zeta_n) - c\xi) - k\xi \right) \varphi(\xi) d\xi d\zeta \\ &= \int (\varphi''(\xi)\phi(\zeta) + \varphi(\xi)\phi''(\xi) + \varphi(\xi)\phi(\zeta)) \Psi(\xi, \zeta + \zeta_n) d\xi d\zeta. \end{aligned} \quad (9)$$

If we consider the integrand of the R.H.S integral, by Cauchy-Shwarz that it is less than or equal to

$$\left(\int (\varphi''(\xi)\phi(\zeta) + \varphi(\xi)\phi''(\xi) + \varphi(\xi)\phi(\zeta))^2 d\xi d\zeta \right)^{1/2} \cdot \left(\int (\Psi(\xi, \zeta + \zeta_n))^2 d\xi d\zeta \right)^{1/2}.$$

The first term is clearly bounded, while the second term goes to 0 since $\Psi_\infty^- = 0$ and $\Psi \in L^2(\Omega)$, and thus making sure that the L.H.S of (??) goes to zero. On the other hand, the continuity of F and the fact that for almost all $(\xi, \zeta) \in \Omega$ we have

$$|\Psi(\xi, \zeta + \zeta_n) - c\xi| \leq \|\Psi\|_\infty + cL, \quad \text{for all } n \in \mathbb{N},$$

tell us that the absolute value of integrand of the L.H.S integral in (??) is bounded by the $L^1(\Omega)$ function

$$|\phi| \cdot |\varphi| \cdot (\sup F + |k|L), \quad \text{where } \sup F \text{ is taken in } [-\|\Psi\|_\infty - cL, \|\Psi\|_\infty + cL].$$

Therefore, by letting $n \rightarrow \infty$ and using dominated convergence in (??) we get

$$\begin{aligned} 0 &= \int \phi(\zeta) \left(F(\Psi_\infty^-(\xi) - c\xi) - k\xi \right) \varphi(\xi) d\xi d\zeta \\ &= \underbrace{\int \phi(\zeta) d\zeta}_{\neq 0} \cdot \int (F(-c\xi) - k\xi) \varphi(\xi) d\xi. \end{aligned} \quad (10)$$

Notice that we have used the continuity of F and the existence of the limit of Ψ at infinity in the above step. Since φ can be chosen arbitrarily, it follows that $F(-c\xi) - k\xi = 0$. \square

What is interesting in the above proposition is that the boundary conditions on Γ were not needed to determine F . Hence we can now safely look at the following problem.

Problem 1: (HM decaying traveling wave, linear)

Seek $\Psi : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} \Delta \Psi = (1 - k/c)\Psi, & \text{on } \Omega, \\ \text{BC's inherited from } u, & \text{on } \Gamma = \partial\Omega, \\ \Psi_\infty^-(\xi) = 0, & \text{on } [0, L]. \end{cases} \quad (11)$$

In variational from

$$\int_{\Omega} \nabla \Psi \cdot \nabla v + (k/c - 1) \int_{\Omega} \Psi \cdot v = \int_{\Gamma} (\vec{n} \cdot \nabla \Psi) v, \quad \forall v \in H(\Omega). \quad (12)$$

We conclude this section with an interesting result that will be helpful in later problems.

Proposition 2.3. *Suppose now $\Omega = [0, L] \times [0, L]$ and that Ψ satisfies (??) on Ω . If Ψ vanishes on any side of $\partial\Omega$, then $F(-c\xi) = k\xi$.*

Proof. Suppose that $\Psi(\xi, L) = 0$ for all $\xi \in [0, L]$. Let $\varphi_1 \in C_c^\infty([0, L])$ be arbitrary. Let φ_2 be the standard bump function whose support is $[0, L/2]$. Then for $n \in \mathbb{N}$ set

$$\rho_n(\zeta) = \frac{L}{2}n(n+1)\zeta - \frac{L}{2}n(n+1)(L - 1/n),$$

and

$$\varphi_n(\zeta) = \begin{cases} 2(n(n+1)L)^{-1}\varphi_2 \circ \rho_n(\zeta) & \text{if } \zeta \in [L - 1/n, L - 1/(n+1)] \\ 0 & \text{otherwise.} \end{cases}$$

Now multiply both sides of (??) by $\varphi(\xi)\varphi_n(\zeta)$ and proceed as in the proof of Proposition (??).

$$\begin{aligned} 0 &= \int \int_{L-1/n}^{L-1/(n+1)} \left(\varphi''(\xi)\varphi_n(\zeta) + \varphi(\xi)\varphi_n''(\zeta) + \varphi(\xi)\varphi_n(\zeta) \right) \Psi(\xi, \zeta) d\zeta d\xi \\ &+ \int \int_{L-1/n}^{L-1/(n+1)} \varphi(\xi)\varphi_n(\zeta) \left(\xi - f(\Psi(\xi, \zeta) - c\xi) \right) d\zeta d\xi. \end{aligned} \quad (13)$$

We bound the first integral in the R.H.S of the above equation. We have that

$$\left| \int \varphi''(\xi)\varphi_n(\zeta) \right| \leq |\text{supp}(\varphi_n)| \cdot \|\varphi''\|_\infty \cdot \|\phi\|_\infty \cdot \|\Psi\|_\infty \rightarrow 0,$$

and that

$$\left| \int \varphi(\xi) \varphi_n(\zeta) \right| \leq |\text{supp}(\varphi_n)| \cdot \|\varphi\|_\infty \cdot \|\phi\|_\infty \cdot \|\Psi\|_\infty \rightarrow 0.$$

As for the last integrand, we have

$$\begin{aligned} \left| \int \varphi(\xi) \varphi_n''(\zeta) \Psi(\xi, \zeta) d\xi d\zeta \right| &\leq \|\varphi \cdot \varphi_n''\|_{L^2(\Omega)} \cdot \|\Psi\|_{L^2(\Omega)} && \text{(by Cauchy-Schwarz)} \\ &= \|\varphi\|_{L^2(\mathbb{R})} \cdot \|\varphi_n''\|_{L^2(\mathbb{R})} \cdot \|\Psi\|_{L^2(\Omega)}. \end{aligned}$$

To complete the proof, we will need this term to go to zero and hence to choose ϕ appropriately to guarantee that $\|\varphi_n''\|_{L^2(\mathbb{R})} \rightarrow 0$. Such a ϕ can be shown to exist^(iv). On the other hand, using the change to variables $(\xi, \zeta) \rightarrow (\xi, \rho_n^{-1}(\zeta))$ in the second integral of (5)

$$\begin{aligned} 0 &= \int \varphi(\xi) \varphi_n(\zeta) \left(\xi - f(\Psi(\xi, \zeta) - c\xi) \right) \\ &= \int_{\text{supp}(\varphi)} \int_{\text{supp}(\varphi_n)} \varphi(\xi) \varphi_n(\zeta) \left(\xi - f(\Psi(\xi, \zeta) - c\xi) \right) d\zeta d\xi \\ &= \int \int_{1-\frac{1}{n}}^{1-\frac{1}{n+1}} \varphi(\xi) \varphi_n(\zeta) \left(\xi - f(\Psi(\xi, \zeta) - c\xi) \right) d\zeta d\xi \\ &= \int \int_0^{L/2} \varphi(\xi) \frac{\phi(\zeta)}{n+1} \left(\xi - f(\Psi(\xi, \rho_n^{-1}(\zeta)) - c\xi) \right) (n+1) d\zeta d\xi, \\ &= \int \varphi(\xi) \varphi(\zeta) \left(\xi - f(\Psi(\xi, \rho_n^{-1}(\zeta)) - c\xi) \right). \end{aligned}$$

Taking limits in the above equation, then using the Lebesgue Dominated Convergence and the continuity of f we get

$$0 = \int \varphi(\xi) \phi(\zeta) (\xi - f(-c\xi)) d\xi d\zeta = \left(\int \varphi(\xi) (\xi - f(-c\xi)) dx \right) \left(\int \phi(\zeta) d\zeta \right).$$

Since φ is arbitrary, we get that $f(-c\xi) = \xi$. □

^(iv)It turns out that the standard bump function

$$\phi(y) = \begin{cases} \exp\left(\frac{1}{y(y-1/2)}\right) & \text{if } 0 < y < 1/2, \\ 0 & \text{otherwise} \end{cases}$$

guarantees that $\|\varphi_n\|_{L^2} \rightarrow 0$,

2.2 Weighted Sobolev spaces on semi-infinite strip

So we have obtained a first instance where an asymptotic boundary conditions turns (??) into a linear equation. We can linearize equation (??) with even more general assumptions. Let us start with the classical case.

Proposition 2.4. *Suppose that*

- $\Psi \in C^2(\overline{\Omega})$ satisfies (??).
- $\Psi_{\infty}^{-}(\xi)$ exists for all $\xi \in [0, L]$.
- $\lim_{\zeta \rightarrow -\infty} \Delta(\xi, \zeta) = 0$ for all $\xi \in [0, L]$.

Then F is linear if and only if Ψ_{∞}^{-} is linear.

Proof. This is clearly true by fixing ξ and letting $\zeta \rightarrow -\infty$ in (??). □

We aim to generalize this result to a larger class of functions. But we encounter a problem: if $\Psi \in C^2(\overline{\Omega})$ and Ψ_{∞}^{-} is not zero, then $\Psi \notin L^p(\Omega)$ for all $1 \leq p < \infty$ and we cannot use standard Sobolev spaces and integration techniques. Thus we are lead to look at weighted Sobolev spaces, which are similar to standard Sobolev spaces but they are endowed with weights prescribing the functions' growth or decay at infinity. They are defined as follows.

Definition. *Fix an almost everywhere positive function $w \in L^1_{\text{loc}}(\Omega)$, a constant $\alpha \in \mathbb{R}$ and an integer p with $1 \leq p < \infty$. Define*

$$L^p_{w,\alpha}(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable and } w^{\alpha}u \in L^p(\Omega)\} \subset L^1_{\text{loc}}(\Omega),$$

endowed with the norm $\|u\|_{L^p_{w,\alpha}(\Omega)} = \|w^{\alpha}u\|_{L^p(\Omega)}$. In particular, $L^2_{w,\alpha}(\Omega)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle : L^2_{w,\alpha}(\Omega) \times L^2_{w,\alpha}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\langle u, v \rangle_{L^2_{w,\alpha}(\Omega)} := \langle w^{\alpha}u, w^{\alpha}v \rangle_{L^2(\Omega)},^{(v)} \text{ for all } u, v \in L^2_{w,\alpha}(\Omega).$$

^(v)This inner product is finite since by Cauchy-Schwarz $|\langle w^{\alpha}u, w^{\alpha}v \rangle| \leq \|w^{\alpha}u\|_{L^2(\Omega)} \cdot \|w^{\alpha}v\|_{L^2(\Omega)} < \infty$ for all $u, v \in L^2_{w,\alpha}(\Omega)$.

This turns $L_{w,\alpha}^2(\Omega)$ into a Hilbert space. If $\alpha = 1$ we write $L_w^p(\Omega)$ for shorthand. In general for $m \geq 0$, set

$$W_{w,\alpha}^{m,p}(\Omega) := \{u \in L_{w,\alpha}^p(\Omega) : D^\beta u \in L_{w,\alpha}^2(\Omega), \forall (0 \leq |\beta| \leq m)\},$$

which convention $W_{w,\alpha}^{0,p}(\Omega) := L_{w,\alpha}^p(\Omega)$. These spaces are Banach spaces with norm

$$\|u\|_{W_{w,\alpha}^{m,p}(\Omega)} = \left(\sum_{0 \leq |\beta| \leq m} \|D^\beta u\|_{L_{w,\alpha}^p(\Omega)}^p \right)^{1/p}.$$

In particular, we write $H_{w,\alpha}^m(\Omega) = W_{w,\alpha}^{m,2}(\Omega)$, which is a Hilbert space with inner product

$$\langle u, v \rangle_{H_{w,\alpha}^m(\Omega)} = \sum_{0 \leq |\beta| \leq m} \langle D^\beta u, D^\beta v \rangle_{L_{w,\alpha}^2(\Omega)}.$$

The theory of such spaces is well established **kufner, turesson** for several domain types and for weights w satisfying the A_p condition^(vi) defined as follows.

Definition (A_p condition). A weight w having the property there is a constant C such that for every ball $B \subset \Omega$,

$$\left(\int_B w \right) \left(\int_B w^{-1/(p-1)} \right)^{p-1} \leq C|B|^p, \quad (A_p\text{-condition})$$

is said to satisfy the A_p condition and is called an A_p weight if $1 < p < \infty$. If $p = 1$ the A_p condition becomes

$$\left(\int_B w \right) \|w^{-1}\|_{L^\infty(B)} \leq C|B|. \quad (A_1\text{-condition})$$

On the other hand, we say that w is a A_∞ weight if there are two constants C and δ such that

$$w(Q) \geq C \left(\frac{|Q|}{|E|} \right)^\delta w(E). \quad (A_\infty\text{-condition})$$

for every cube $Q \subset \Omega$ and any non-null measurable subset $E \subset Q$.

^(vi)The A_p condition guarantees that the Hardy-Littlewood maximal operator $M : L_w^p(\mathbb{R}^n) \rightarrow L_w^p(\mathbb{R}^n)$ defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

is continuous when $1 < p < \infty$.

If $w \in A_p$ with $1 \leq p \leq \infty$, then the infimum of all constants C is called the A_p constant of w , and is usually denoted by A . The set of all weights w satisfying the A_p condition is denoted by A_p for shorthand.

Proposition 2.5 (Properties of Weighted Spaces). *Suppose $\Omega \subset \mathbb{R}^2$ is open. Let w be a weight and $1 \leq p < \infty$. We have the following simple properties.*

(i) *From Hölder's inequality we have*

$$\int_B |f| \leq \left(\int_B w^{-1/(p-1)} \right)^{1/p'} \left(\int_B |f|^p \right)^{1/p},$$

for every ball $B \subset \Omega$.

(ii) *If $w^{-1/(p-1)} \in L^1_{loc}(\Omega)$ and $w^{-1} \in L^\infty(B)$ for every ball $B \subset \Omega$ then $L^p_w(\Omega) \subset L^1_{loc}(\Omega)$.*

(iii) *If $w \in A_p$ then $L^p_w(\Omega) \subset L^1_{loc}(\Omega)$.*

(iv) *If $1 \leq p < q < \infty$ then $A_p \subset A_q$.*

(v) *If $w \in A_p$ then $w^{-1/(p-1)} \in A'_p$.*

(vi) *The A_p condition is invariant under translations and dilations.*

(vii) *If $w \in A_p$ and A is its A_p constant then $A \geq 1$.*

(viii) *If $w \in A_1$, then $w \geq (1 + |x|)^{-N}$ for all $x \in \Omega$.*

(ix) *If there are constants $C, D \in \mathbb{R}$ such that $0 < C \leq w \leq D$ then $w \in A_p$ for all $1 \leq p < \infty$.*

(x) *If $w(x) = |x|^\alpha$ then*

$$w \in \begin{cases} A_1 & \text{if } -N < \alpha \leq 0, \\ A_p & \text{if } -N < \alpha < N(p-1). \end{cases} \quad (14)$$

(xi) *$w^\alpha \in A_2$ for some $\alpha > 0$ if and only if w has bounded mean oscillation.*

To be added: Some results on weighted Sobolev spaces to help generalize Proposition 3.4 from $\Psi \in C^2(\overline{\Omega})$ with $\Psi_\infty^- = c$ to $\Psi \in H^1_{w,\alpha}(\Omega)$ for some appropriate weight w and exponent α .

2.3 Traveling waves with periodicity in the ζ -direction

Let us introduce some periodicity in problem (??). Assume $u(0, y, t) = u(L, y, t)$ for all $(y, t) \in [0, L] \times \mathbb{R}^+$ in (??) and u becomes linear for large t . For the traveling wave function Ψ , this means that $\Psi_\infty^-(\xi) = b$, where b is some real constant. Hence if we take $\Psi \in C^2(\overline{\Omega})$, equation (??) tells us that

$$F(b - c\xi) = k\xi - b,$$

and therefore

$$F(\xi) = F(b - c(-\xi/c + b/c)) = -\frac{k}{c}\xi + \left(\frac{k}{c} - 1\right)b.$$

This leads to the following boundary value problem.

Problem 2: (HM Traveling Wave, L -periodic in ξ , Constant at Infinity)

Seek $\Psi : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} \Delta\Psi + (k/c - 1)\Psi = (k/c - 1)b, \\ \Psi(0, \zeta) = \Psi(L, \zeta), \quad \Psi_\infty^-(\xi) = b. \end{cases} \quad (15)$$

The setup of Problem 2 bears resemblance with the one in **weightSobolev** where existence and uniqueness of a 1-periodic solution to the Laplace equation $\Delta u = f$ equation on the strip $[0, 1] \times \mathbb{R}$ was shown in some weighted Sobolev spaces.

Definition. Let Ω be the strip $[0, 1] \times \mathbb{R}$. Define

$$\mathcal{D}_\#(\Omega) = \{\varphi : \Omega \rightarrow \mathbb{R} \mid y \rightarrow \varphi(x, y) \in C^\infty([0, 1]) \text{ 1-periodic, } x \rightarrow \varphi(x, y) \in C_c^\infty(\mathbb{R})\},$$

and $\mathcal{D}_\#^*(\Omega)$ be its dual. Define the weight $w = w(x, y) = (1 + y^2)^{1/2}$. For $\alpha \in \mathbb{R}$, let

$$L_\alpha^2(\Omega) = \{u \in \mathcal{D}_\#^*(\Omega) : w^\alpha u \in L^2(\Omega)\}.$$

It can be shown that $L_\alpha^2(\Omega)$ is a Banach space with norm $\|u\|_{L_\alpha^2(\Omega)} = \|w^\alpha u\|_{L^2(\Omega)}$ and that $C_c^\infty(\Omega)$ is dense in $L_\alpha^2(\Omega)$. If $m \in \mathbb{N}$ and

$$k = \begin{cases} m - 1/2 - \alpha & \text{if } \alpha \in \{1/2, \dots, m - 1/2\}, \\ -1 & \text{if } \alpha \notin \{1/2, \dots, m - 1/2\}. \end{cases}$$

We let $H_{\alpha,\#}^0(\Omega) = L_\alpha^2(\Omega)$ and in general for $m \in \mathbb{N}$, we define the weighted Sobolev spaces

$$H_{\alpha,\#}^m(\Omega) = \left\{ u \in \mathcal{D}_\#^*(\Omega) \left| \begin{array}{l} w^{-m+|\beta|}(\ln(1+w^2))^{-1} D^\beta u \in L_\alpha^2(\Omega), \quad \forall 0 \leq |\beta| \leq k, \\ w^{-m+|\beta|} D^\beta u \in L_\alpha^2(\Omega), \quad \forall k < |\beta| < m \end{array} \right. \right\}.$$

Also, the norm

$$\|u\|_{H_{\alpha,\#}^m(\Omega)} = \left(\sum_{0 \leq |\beta| \leq k} \left\| \frac{w^{-m+|\beta|}}{\ln(1+w^2)} D^\beta u \right\|_{L_\alpha^2(\Omega)}^2 + \sum_{k < |\beta| \leq m} \|w^{-m+|\beta|} D^\beta u\|_{L_\alpha^2(\Omega)}^2 \right)^{1/2},$$

turns $H_{\alpha,\#}^m(\Omega)$ into a Banach space.

Theorem 2.1 (V. Milisic, U.Razafison, **weightSobolev**). For $1/2 \leq \alpha \leq 3/2$ and for $m \in \mathbb{N}$ with $m \geq 3$, the Laplace operator Δ is an isomorphism when seen as

$$\Delta : H_{\alpha,\#}^m(\Omega)/\mathbb{P}'_{m-2} \rightarrow H_{\alpha,\#}^{m-2}(\Omega)/\mathbb{P}'_{m-4} \quad \text{and} \quad \Delta : H_{\alpha,\#}^m(\Omega)/\mathbb{P}_{m-2}^\Delta \rightarrow H_{\alpha,\#}^{m-2}(\Omega),$$

where \mathbb{P}'_j is the set of polynomials in y only of degree at most j and \mathbb{P}_j^Δ is the subset of \mathbb{P}'_j consisting of harmonic polynomials.

To be added: The technique used in the above theorem can be used to tackle Problem 2. I am currently reading **weightSobolev** in detail to see how to apply it for Problem 2 appropriately.

2.4 Traveling waves with PBC's

Of interest are traveling waves Ψ inheriting the P.B.C's of $u \in H_P^1(\mathcal{R})$. For starters, suppose $u \in H_P^1(\mathcal{R}) \cap C^2(\mathcal{R})$. We obtain the following problem.

Problem 3: (HM Traveling Wave, L -periodic in ξ , L -periodic in ζ).

Seek $\Psi : \mathcal{R} \rightarrow \mathbb{R}$ such that if β is a multi-index and $0 \leq |\beta| \leq 3$ then

$$\begin{cases} \Delta \Psi = \Psi + F(\Psi - c\xi) - k\xi, \\ D^\beta \Psi(\xi, -ct) = D^\beta \Psi(\xi, L - ct), \\ D^\beta \Psi(0, \zeta) = D^\beta \Psi(L, \zeta). \end{cases} \quad (16)$$

Periodicity allows us restrict our study to the rectangle $[0, L] \times [0, L]$ and so Problem 3 becomes

$$\begin{cases} \Delta\Psi = \Psi + F(\Psi - c\xi) - k\xi, & \forall(\xi, \zeta) \in [0, L] \times [0, L], \\ D^\beta\Psi(\xi, 0) = D^\beta\Psi(\xi, L) = f_\beta(\xi), & \forall\xi \in [0, L], \\ D^\beta\Psi(0, \zeta) = D^\beta\Psi(L, \zeta) = g_\beta(\zeta) & \forall\zeta \in [0, L]. \end{cases} \quad (17)$$

To linearize this equation without any assumption on $\Delta\Psi$, we can assume that Ψ vanishes on any side of the rectangle \mathcal{R} and Proposition 3.3 would apply. Note here that the discussion of asymptotic behaviour becomes irrelevant as periodicity automatically tells us that Ψ_∞^- doesn't exist.

To be added: Seek boundary conditions on u that linearize Problem 3 and investigate a boundary condition that does not allow for a linear choice of F .

3 The Problem as Solved in the Original Paper

In the paper **crotinger**, the problem of finding a Modon is stated as follows. Fix t , choose a radius a and a velocity c and consider the region

$$\Omega_t = \{(x, y) : x^2 + (y - ct)^2 < a\}.$$

In the unbounded region Ω_t^c , the fluid elements are not "trapped", and so the condition $\Phi \rightarrow 0$ as $x, y \rightarrow \infty$ is added. This applied to (1) implies that $f(z) = -z/c$ in that region. In Ω_t , f is arbitrary and hence was chosen to be $f(z) = -(1 + s^2)z$ with $s = \gamma^2/a^2$, where γ is a parameter that will be chosen later. To restate the problem, we seek $\Phi \in C^2(\mathbb{R}^2)$ such that:

$$\begin{cases} \Delta\Phi + (1/c - 1)\Phi = 0 & \text{if } (x, y) \in \Omega_t^c, \\ \Delta\Phi + s^2\Phi = ((1 + s^2)c - 1)x & \text{if } (x, y) \in \Omega_t. \end{cases} \quad (18)$$

Using the change of coordinates,

$$r^2 = x^2 + (y - ct)^2, \quad \cos \theta = x/r,$$

the Modon equation is given by

$$\Phi(r, \theta) = \begin{cases} AK_1(\beta r/a) \cos \theta & \text{if } r > a, \\ \frac{Br}{a} \cos \theta + CJ_1(\gamma r/a) \cos \theta & \text{if } r < a, \end{cases} \quad (19)$$

for some constants $a, c \in \mathbb{R}$, $\beta = a(1 - 1/c)$, γ a parameter that will be chosen later and,

$$A = \frac{ac}{K_1(\beta)}, \quad B = ac \left(1 + \frac{1}{\gamma^2}\right), \quad \text{and } C = -\frac{\beta^2}{\gamma^2} \frac{ac}{J_1(\gamma)}.$$

Concern: Let's look at the issue of compatibility between (1) and (4). By Proposition 1, the assumption that Φ decays for w.r.t the spacial variables, implies that $f(z) = -z/c$ at the points of continuity of f . Thus if Φ satisfies (4) and f is continuous at the point z then

$$-z/c = f(z) = -(1 + s^2)z, \text{ and therefore } c = (1 + s^2)^{-1}.$$

Thus we obtain a relation between a and c , and this may have impact on the physical interpretation of the Modon.



Figure 1: Possible values of γ for $a = 2$ and $c = 6$.

3.1 The Continuity Equation in the Original Paper

Suppose that we take the Modon solution as given in (5). As said there, there is a parameter γ that will be chosen so that Φ is C^2 . This parameter satisfies the following equation, called the continuity equation, given in **1** by

$$\frac{K_2(\beta)}{\beta K_1(\beta)} = \frac{J_2(\gamma)}{\gamma J_1(\gamma)}. \quad (20)$$

This equation gives many values γ for the fixed value β ; this can be seen in Figure 1 where γ is any root of the function plotted.

Concern: There is no mentioning on how the above equation is obtained, so I tried to re-derive the result. Continuity of (5) implies that by fixing $\theta \in [0, 2\pi]$ and letting $r \rightarrow a^\pm$, one should obtain

$$AK_1(\beta) = B + CJ_1(\gamma), \quad (21)$$

After simplification, the above equation reduces to $\beta = \pm 1$ which is definitely not related to (6) in any way. Also, this means that the variables a and c have to be related in order for proposed solution (5) to be continuous, which is also a problem.

3.2 Deriving C^2 continuity (can ignore this part)

Now assuming that the needed condition on β is satisfied, we look for higher order continuity. We need only to look at the derivatives with respect to the r variable as partial derivatives with respect to θ provide the same continuity conditions. Thus we compute

$$\frac{\partial \Phi}{\partial r}(r, \theta) = \begin{cases} A \cdot \frac{\beta}{a} \cdot \frac{K_0(\beta r/a) + K_2(\beta r/a)}{-2} \cdot \cos \theta & \text{if } r > a, \\ \frac{B}{a} \cdot \cos \theta + C \cdot \frac{\gamma}{a} \cdot \frac{J_0(\gamma r/a) - J_2(\gamma r/a)}{2} \cdot \cos \theta & \text{if } r < a. \end{cases}$$

Canceling $\cos(\theta)/2a$ from both sides and taking limits as $r \rightarrow a^+, a^-$ we get:

$$-A\beta(K_0(\beta) + K_2(\beta)) = 2B + C\gamma(J_0(\gamma) - J_2(\gamma)). \quad (22)$$

This equation guarantees C^1 continuity at $r = a$. To get C^2 continuity, we compute

$$\frac{\partial^2 \Phi}{\partial r^2}(r, \theta) = \begin{cases} \frac{A}{4} \frac{\beta^2}{a^2} \cos \theta (3K_1(\beta r/a) + K_3(\beta r/a)), & \text{if } r > a, \\ \frac{C}{4} \frac{\gamma^2}{a^2} \cos \theta (-3J_1(\gamma r/a) + J_3(\gamma r/a)), & \text{if } r < a. \end{cases}$$

Taking limits again for all $\theta \in [0, 2\pi]$ we get:

$$A\beta^2(2K_1(\beta) + K_3(\beta)) = C\gamma^2(-3J_1(\gamma) + J_3(\gamma)). \quad (23)$$

The continuity of the partial derivatives $\partial_{\theta\theta}\Phi$ and $\partial_{\theta r}$ gives the same result as (7) and (8) respectively. So we look for β satisfying (7) and γ simultaneously satisfying (8) and (9).

4 Simulations and numerical results

We are interested in testing for an upper bound on the H^∞ -norm of on the solutions w of the coupled system:

$$\begin{cases} -\Delta u + u = w, \\ w_t + \vec{V}(u) \cdot \nabla w = -ku_y. \end{cases}$$

The simulator is written in FreeFem++ and can be found here: <https://github.com/adelsaleh/hmSimulator>

4.1 The Algorithm

We give a description of the algorithm that approximates

$$\sup_{0 \leq t \leq T} (\|\nabla w\|_\infty + \|w\|_\infty),$$

for some given time T . This algorithm is correct as long as $w \in \mathcal{V}_h$. It works as follows, suppose T_i is a triangle in the mesh \mathcal{T} . Let $w_i = w|_{T_i}$ then there are $a_i, b_i, c_i \in \mathbb{R}$ such that for all $(x, y) \in T_i$,

$$w(x, y) = w_i(x, y) = a_i x + b_i y + c_i, \quad \text{and therefore} \quad \nabla w_i = (a_i, b_i).$$

The code recovers the values of a_i, b_i, c_i . We are given three vertices $P_j^i = (x_j^i, y_j^i) \in T_i$ for $j = 1, 2, 3$ from which we solve:

$$\begin{pmatrix} x_1^i & y_1^i & 1 \\ x_2^i & y_2^i & 1 \\ x_3^i & y_3^i & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} w_i(x_1^i, y_1^i) \\ w_i(x_2^i, y_2^i) \\ w_i(x_3^i, y_3^i) \end{pmatrix}.$$

If \mathcal{P} is the set of nodes of the mesh then what we are computing at each $0 \leq t \leq 75$:

$$\|w\|_\infty + \|\nabla w\|_\infty = \max_{p \in \mathcal{P}} |w(p)| + \max_{T_j \in \mathcal{T}} (|a_j| + |b_j|).$$

The code has been tested for 3 different initial conditions, and for endt=76.

Figure 2: $u_0 = \sin 3x$, $\Omega = [0, 0.5]^2$, $\Delta t = 0.1$ and $d = 32$.

4.2 The Tests

Case 1: $u_0(x, y) = \sin 3x$.

- for $dt = 1$

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	9.72845e+06
meshp = 32	0.00932348	0.00932348	0.0700644	9.48621e+06
meshp = 64	0.00935401	0.00935401	0.25487	3.36182e+07

- for $dt = 0.1$

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	0.00920216
meshp = 32	0.00932348	0.00932348	0.00932348	0.00932348
meshp = 64	0.00935401	0.00935401	0.00935401	0.00935401

- For $dt=0.01$

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	0.00920216
meshp = 32	0.00932348	0.00932348	0.00932348	0.00932348
meshp = 64	0.00935401	0.00935401	0.00935401	0.00935401

Case 2: $u_0(x, y) = \sin 3y$.

dt	meshsize	t=1	t=25	t=50	t=75
1	16	0.00819026	226093	2.35954e+13	3.38578e+20
	32	0.0174697	274395	3.46674e+13	2.54711e+18
	64	0.049547	288790	3.84457e+13	8.44555e+19
0.1	16	0.00109678	0.0228526	1.44038	92.6633
	32	0.00177724	0.0251274	1.70835	118.378
	64	0.00541686	0.0267284	1.79248	126.325
0.01	16	0.000573946	0.000709836	0.000882754	0.001284
	32	0.000786692	0.000811604	0.0010299	0.0013535
	64	0.0012995	0.00132479	0.00137715	0.00169396

(a) L^∞ -norm of $u(t)$.

(b) $W^{1,\infty}$ -norm of $u(t)$.

Figure 3: Gauss initial condition on $\Omega = [-10, 10]^2$, $\Delta t = 0.01$ and $d = 64$.

- $dt = 1$.

	t=1	t=25	t=50	t=75
meshp = 16	0.271741	0.297416	0.304924	0.309987
meshp = 32	0.268623	0.291456	0.298063	0.298935
meshp = 64	0.263379	0.289162	0.294879	0.294879

- $dt = 0.1$.

	t=1	t=25	t=50	t=75
meshp = 16	0.27173	0.295808	0.303704	0.307203
meshp = 32	0.26973	0.290931	0.29701	0.29796
meshp = 64	0.262307	0.288401	0.293987	0.293987

- $dt = 0.01$.

	t=1	t=25	t=50	t=75
meshp = 16	0.271735	0.295687	0.303588	0.306932
meshp = 32	0.269837	0.29089	0.296902	0.297862
meshp = 64	0.262465	0.288328	0.293898	0.293898