Work on Hasegawa-Mima

Adel Saleh

February 26, 2020

Contents

The Hasegawa-Mima equation [1, 2, 3, 4] is a second order non linear differential equation which can be written as a coupled system of linear equations [5, 6] as follows

$$\begin{cases} -\Delta u + u = w, \\ w_t + \vec{V}(u) \cdot \nabla w = -k u_y, \end{cases}$$

In the above system, we seek solutions $u, v \in H^1(\Omega)$ with $\Omega = [0, w] \times [0, h] \subset \mathbb{R}^2$ for some $w, h > 0$. We also have $k \in \mathbb{R}$ is a constant and

$$\vec{V}(u) = \begin{pmatrix} -\partial u / \partial y \\ \partial u / \partial x \end{pmatrix}.$$

In this small report, we prove existence of traveling wave solutions or *Modons* as in [1] for the Hasegawa-Mima equation and compare results. Later, we simulate using FreeFrem++ some solutions on the time interval $[0, T_{\text{max}}]$ and then generate some H^∞ estimates of the solution at specific times.

1 The Modon Solution

If one looks for traveling wave solution for the Hasegawa-Mima equation of the form $u(x, y, t) = \Phi(x, y - ct)$, we obtain the following problem [3]

$$\Delta\Phi - \Phi + x = f(\Phi - cx), \quad \text{for all } (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \tag{1}$$

where f is an arbitrary function.

1.1 Determining the Function f.

Under certain continuity conditions on f and some boundary/asymptotic conditions on Φ, we can determine f.

Proposition 1. *Suppose that the functions Φ and f satisfy (1). Furthermore, suppose that $\Phi, \Delta\Phi \to 0$ as $x, y \to \infty$. Then,*

$$f(-cx) = x,$$

whenever f is continuous at the point $-cx$.

Proof. This is clear by fixing x and letting $y \to \infty$ in (1). $\square$

The above claim is still true even if we do not assume the condition $\Delta\Phi \to 0$, but the proof more involved and quite technical. It requires the usage of Green's Formula, Lebesgue Dominated Convergence and Fubini's Theorem.

Claim 1. *Suppose that the functions Φ and f satisfy (1). Furthermore, suppose that $\Phi \to 0$ as $x, y \to \infty$ and f is continuous. Then,*

$$f(-cx) = x,$$

for all $x \in \mathbb{R}$.

Proof. Let $\{y_n\}$ be any sequence that increases to infinity. Choose two arbitrary functions $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R})$ and then define

$$\psi_n(x, y) = \varphi_1(x)\varphi_2(y - y_n).$$

Multiply both sides of (1) by ψ_n and integrate

$$\int \Delta\Phi \cdot \psi_n - \int \Phi \cdot \psi_n + \int (x - f(\Phi - cx))\psi_n = 0.$$

Integration by parts once on the first term in the L.H.S yields

$$-\int \nabla\Phi \cdot \nabla\psi_n - \int \Phi \cdot \psi_n + \int (x - f(\Phi - cx))\psi_n = 0.$$

We expand the first term

$$-\int \frac{\partial\Phi}{\partial x}\frac{\partial\psi_n}{\partial x} - \int \frac{\partial\Phi}{\partial y}\frac{\partial\psi_n}{\partial y} - \int \Phi \cdot \psi_n + \int (x - f(\Phi - cx))\psi_n = 0.$$

In the above equation, we replace ψ_n by it's appropriate value and get

$$\begin{aligned} 0 = &-\int \frac{\partial\Phi}{\partial x}(x,y)\frac{d\varphi_1}{dx}(x)\varphi_2(y - y_n)\,dx\,dy \\ &-\int \frac{\partial\Phi}{\partial y}(x,y)\varphi_1(x)\frac{d\varphi_2}{dy}(y - y_n)\,dx\,dy \\ &-\int \Phi(x,y)\varphi_1(x)\varphi_2(y - y_n)\,dx\,dy \\ &+\int \big(x - f(\Phi(x,y) - cx)\big)\varphi_1(x)\varphi_2(y - y_n)\,dx\,dy \end{aligned} \tag{2}$$

Now let us apply integration by parts to the first term in the R.H.S of the above equation,

$$
\begin{aligned}
&-\int \frac{\partial \Phi}{\partial x}(x,y)\frac{d\varphi_1}{dx}(x)\varphi_2(y-y_n)\,dx\,dy \\
&= -\int \varphi_2(y-y_n)\left(\int \frac{\partial \Phi}{\partial x}(x,y)\frac{d\varphi_1}{dx}(x)\,dx\right)\,dy \\
&= -\int \varphi_2(y-y_n)\left(-\int \Phi(x,y)\frac{d^2\varphi_1}{dx^2}(x)dx\right)\,dy \\
&= \quad \int \Phi(x,y)\varphi_1''(x)\varphi_2(y-y_n)\,dx\,dy,
\end{aligned}
$$

and with similar reasoning,

$$
-\int \frac{\partial \Phi}{\partial y}(x,y)\varphi_1(x)\frac{d\varphi_2}{dy}(y-y_n)dxdy = \int \Phi(x,y)\varphi_1(x)\varphi_2''(y-y_n)dxdy.
$$

By replacing in (2) we have,

$$
\begin{aligned}
0 = &\int \Big(\varphi_1''(x)\varphi_2(y-y_n) + \varphi_1(x)\varphi_2''(y-y_n) + \varphi_1(x)\varphi_2(y-y_n)\Big)\Phi(x,y)dxdy \\
&+ \int \varphi_1(x)\varphi_2(y-y_n)\Big(x - f\big(\Phi(x,y)-cx\big)\Big)\,dx\,dy.
\end{aligned}
$$

Using the change of variables $(x,y)\to(x,y-y_n)$ we get

$$
\begin{aligned}
0 = &\int \Big(\varphi_1''(x)\varphi_2(y) + \varphi_1(x)\varphi_2''(y) + \varphi_1(x)\varphi_2(y)\Big)\Phi(x,y+y_n)\,dx\,dy \\
&+ \int \varphi_1(x)\varphi_2(y)\Big(x - f\big(\Phi(x,y+y_n)-cx\big)\Big)\,dx\,dy
\end{aligned}
$$

Let $y_n\to\infty$ and use Lebesgue Dominated Convergence. Then we get

$$
0 = \int \varphi_1(x)\varphi_2(y)\big(x-f(-cx)\big)\,dx\,dy = \int \varphi_2(y)\,dy \int \varphi_1(x)(x-f(-cx))\,dx.
$$

Since φ_1 can be chosen arbitrarily, it follows that $f(-cx)=x$ for all $x\in \text{supp}(\varphi_1)$. By varying the support of φ_1 over all of $\mathbb{R}$, we get the result is true for all $x\in\mathbb{R}$. □

A similar version to the above claim is true if Ω is a rectangle and Φ vanishes on one side of the boundary of the rectangle. Without loss of generality, let $\Omega=(-w,w)\times(-L,L)$ where $w,L>0$.

Claim 2. *Suppose that the functions Φ and f satisfy (1) in Ω. Assume that $\Phi(x,L)=0$ for all $x\in(-w,w)$ and that f is continuous. Then*

$$
f(-cx)=x,\ \textit{for all } x\in(-w,w).
$$

The proof of this fact follows a similar line of reasoning to the proof of Claim 1. The idea is to construct a sequence $\{\varphi_{2,n}\}_{n\in\mathbb{N}}$ of compactly supported smooth functions on $(-L, L)$ whose support "approaches" L as $n \to \infty$ and use the integral identities obtained in the proof above to show that $\int \varphi(x - f(\Phi - cx)) = 0$ for all $\varphi \in C_c^\infty((-w, w))$.

Proof. Let $\varphi_1 \in C_c^\infty((-w, w))$ be arbitrary. We start by choosing any $\varphi_2 \in C_c^\infty((-L, L))$ such that $\operatorname{supp}(\varphi_2) = [0, L/2]$.[1] Then for $n \in \mathbb{N}$, define first

$$\ell_n(y) = \frac{L}{2}n(n+1)\left(y - L + \frac{L}{n}\right),$$

and then define $\varphi_{2,n} : \Omega \to \mathbb{R}$ as

$$\varphi_{2,n}(y) = \frac{4}{n^2(n+1)^2 L}(\varphi_2 \circ \ell_n)(y), \qquad n = 1, 2, 3, \ldots$$

If on one hand we take

$$L - \frac{L}{n} \leq y \leq L - \frac{L}{n+1},$$

then $0 < \ell_n(y) < L/2$ and therefore $\varphi_{2,n}(y) = \varphi_2(\ell_n(y)) \neq 0$. On the other hand, for any other value of y, $\phi_{2,n}(y) = 0$ and we can conclude that

$$\operatorname{supp}(\varphi_{2,n}) = [L - L/n, L - L/(n+1)]$$

Intuitively, the support of $\varphi_{2,n}$ is "moving" towards $\{L\}$ in the sense that the distance between $\{L\}$ and $\operatorname{supp}(\varphi_{2,n})$ goes to 0, and "shrinks" by a factor of $1/n$ at each step n, in the sense that $\operatorname{diam}(\operatorname{supp}(\varphi_{2,n})) = 1/n(n+1) \to 0$.

We also have the following equalities

$$\|\varphi_{2,n}\|_\infty = \frac{4\|\varphi_2\|_\infty}{n^2(n+1)^2 L} \leq \|\varphi_2\|_\infty \text{ and } \|\varphi_{2,n}''\|_\infty = \|\varphi_2''\|_\infty.$$

Now let $\psi_n(x, y) = \varphi_1(x)\varphi_{2,n}(y)$ and proceed as in the proof of above claim to obtain

$$\begin{aligned} 0 = &\int \Big(\varphi_1''(x)\varphi_{2,n}(y) + \varphi_1(x)\varphi_{2,n}''(y) + \varphi_1(x)\varphi_{2,n}(y)\Big)\Phi(x, y)\,dx\,dy \\ &+ \int \varphi_1(x)\varphi_{2,n}(y)\Big(x - f\big(\Phi(x, y) - cx\big)\Big)\,dx\,dy. \end{aligned} \tag{3}$$

[1] One can take for instance $\varphi_2(y) = \exp(-\frac{1}{y^2}\frac{1}{(y-L/2)^2})$ if $0 < y < L/2$ and 0 otherwise.

If we look at the first term of the R.H.S of the above equation, it goes to zero as $n \to \infty$ since

$$\begin{aligned}
&\int \Big(\varphi_1''(x)\varphi_{2,n}(y) + \varphi_1(x)\varphi_{2,n}''(y) + \varphi_1(x)\varphi_{2,n}(y)\Big)\Phi(x,y)\,dx\,dy \\
&\leq \frac{L\cdot\lambda\big(\mathrm{supp}(\varphi_1)\big)}{n(n+1)}\Big(\|\varphi_1''\|_\infty\|\varphi_2\|_\infty + \|\varphi_1\|_\infty\|\varphi_2''\|_\infty + \|\varphi_1\|_\infty\|\varphi_2\|_\infty\Big)\|\Phi\|_\infty
\end{aligned}$$

where λ is the one dimensional Lebesgue measure. On the other hand, using the change to variables $(x,y) \to (x, \ell_n(y))$ on the second term of (3) we get

$$\begin{aligned}
&\int \varphi_1(x)\varphi_{2,n}(y)\Big(x - f\big(\Phi(x,y) - cx\big)\Big)\,dx\,dy \\
&= \int \varphi_1(x)\varphi_2(y)\left(x - f\left(\Phi\left(x, \frac{2y}{n(n+1)L} + L - \frac{L}{n}\right) - cx\right)\right)\,dx\,dy.
\end{aligned}$$

Taking limits in and using the Lebesgue Dominated Convergence on the second term of (3) we get

$$\begin{aligned}
0 &= \int \varphi_1(x)\varphi_2(y)\big(x - f(-cx)\big)\,dx\,dy \\
&= \left(\int \varphi_1(x)\big(x - f(-cx)\big)\,dx\right)\left(\int \varphi_2(y)\,dy\right).
\end{aligned}$$

Since φ_1 is arbitrary, we get that $f(-cx) = x$ for all $x \in \mathrm{supp}(\varphi_1)$ and hence for all $x \in (-w, w)$.

□

By replacing f in (1) with it's appropriate value, it follows immediately that

$$\|\Delta\Phi\|_\infty = |1 - 1/c|\;\|\Phi\|_\infty,$$

from which we deduce that the H^∞-norm of a traveling wave solution that decays at infinity is bounded at each time t.

1.2 The Problem as Solved in the Original Paper

In the paper [1], the problem of finding a Modon is stated as follows. Fix t, choose a radius a and a velocity c and consider the region

$$\Omega_t = \{(x,y) : x^2 + (y - ct)^2 < a\}.$$

In the unbounded region Ω_t^c, the fluid elements are not "trapped", and so the condition $\Phi \to 0$ as $x, y \to \infty$ is added. This applied to (1) implies that $f(z) = -z/c$ in that region. In Ω_t, f is arbitrary and hence was chosen to be $f(z) = -(1+s^2)z$ with $s = \gamma^2/a^2$, where γ is a parameter that will chosen later. To restate the problem, we seek $\Phi \in C^2(\mathbb{R}^2)$ such that:

$$\begin{cases} \Delta\Phi + (1/c - 1)\Phi = 0 & \text{if } (x,y) \in \Omega_t^c, \\ \Delta\Phi + s^2\Phi = ((1+s^2)c - 1)\, x & \text{if } (x,y) \in \Omega_t. \end{cases} \tag{4}$$

Using the change of coordinates,

$$r^2 = x^2 + (y - ct)^2, \qquad \cos\theta = x/r,$$

the Modon equation is given by

$$\Phi(r,\theta) = \begin{cases} AK_1(\beta r/a)\cos\theta & \text{if } r > a, \\ \dfrac{Br}{a}\cos\theta + CJ_1(\gamma r/a)\cos\theta & \text{if } r < a, \end{cases} \tag{5}$$

for some constants $a, c \in \mathbb{R}$, $\beta = a(1 - 1/c)$, γ a parameter that will be chosen later and,

$$A = \frac{ac}{K_1(\beta)}\,, \quad B = ac\left(1 + \frac{1}{\gamma^2}\right), \quad \text{and } \; C = -\frac{\beta^2}{\gamma^2}\frac{ac}{J_1(\gamma)}.$$

Concern: Let's look at the issue of compatibility between (1) and (4). By Proposition 1, the assumption that Φ decays for w.r.t the spacial variables, implies that $f(z) = -z/c$ at the points of continuity of f. Thus if Φ satisfies (4) and f is continuous at the point z then

$$-z/c = f(z) = -(1+s^2)z, \text{ and therefore } c = (1+s^2)^{-1}.$$

Thus we obtain a relation between a and c, and this may have impact on the physical interpretation of the Modon.

1.3 The Continuity Equation in the Original Paper

Suppose that we take the Modon solution as given in (5). As said there, there is a parameter γ that will be chosen to that Φ is C^2. This parameter satisfies the following equation, called the continuity equation, given in [**?**] by

$$\frac{K_2(\beta)}{\beta K_1(\beta)} = \frac{J_2(\gamma)}{\gamma J_1(\gamma)}. \tag{6}$$

This equation which gives many values γ for the fixed value β; this can be seen in Figure 1 where γ is any root of the function plotted.

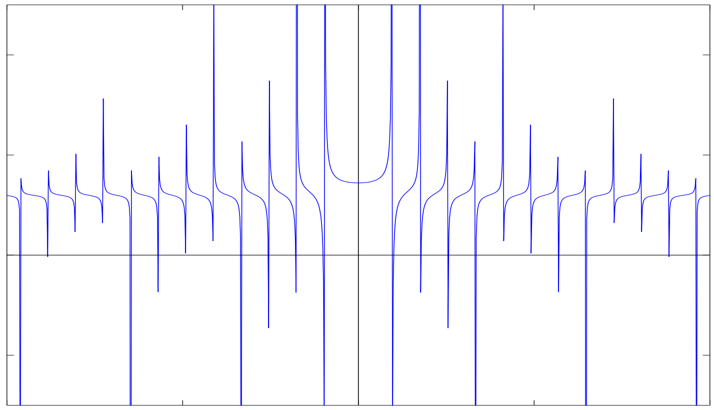

Figure 1: Possible values of γ for $a = 2$ and $c = 6$.

Concern: There is no mentioning on how the above equation is obtained, so I tried to re-derive the result. Continuity of (5) implies that by fixing $\theta \in [0, 2\pi]$ and letting $r \to a^{\pm}$, one should obtain

$$AK_1(\beta) = B + CJ_1(\gamma), \tag{7}$$

After simplification, the above equation reduces to $\beta = \pm 1$ which is definitely not related to (6) in any way. Also, this means that the variables a and c have to be related in order for proposed solution (5) to be continuous, which is also a problem.

1.4 Deriving C^2 continuity (can ignore this part)

Now assuming that the needed condition on β is satisfied, we look for higher order continuity. We need only to look at the derivatives with respect to the r variable as partial derivatives with respect to θ provide the same continuity conditions. Thus we compute

$$\frac{\partial \Phi}{\partial r}(r,\theta) = \begin{cases} A \cdot \dfrac{\beta}{a} \cdot \dfrac{K_0(\beta r/a) + K_2(\beta r/a)}{-2} \cdot \cos\theta & \text{if } r > a, \\[2ex] \dfrac{B}{a} \cdot \cos\theta + C \cdot \dfrac{\gamma}{a} \cdot \dfrac{J_0(\gamma r/a) - J_2(\gamma r/a)}{2} \cdot \cos\theta & \text{if } r < a. \end{cases}$$

Canceling $\cos(\theta)/2a$ from both sides and taking limits as $r \to a^+, a^-$ we get:

$$-A\beta(K_0(\beta) + K_2(\beta)) = 2B + C\gamma(J_0(\gamma) - J_2(\gamma)). \tag{8}$$

This equation guarantees C^1 continuity at $r = a$. To get C^2 continuity, we compute

$$\frac{\partial^2 \Phi}{\partial r^2}(r,\theta) = \begin{cases} \dfrac{A}{4}\dfrac{\beta^2}{a^2}\cos\theta\left(3K_1(\beta r/a) + K_3(\beta r/a)\right), & \text{if } r > a, \\ \\ \dfrac{C}{4}\dfrac{\gamma^2}{a^2}\cos\theta\left(-3J_1(\gamma r/a) + J_3(\gamma r/a)\right), & \text{if } r < a. \end{cases}$$

Taking limits again for all $\theta \in [0, 2\pi]$ we get:

$$A\beta^2(2K_1(\beta) + K_3(\beta)) = C\gamma^2(-3J_1(\gamma) + J_3(\gamma)). \tag{9}$$

The continuity of the partial derivatives $\partial_{\theta\theta}\Phi$ and $\partial_{\theta r}$ gives the same result as (7) and (8) respectively. So we look for β satisfying (7) and γ simultaneously satisfying (8) and (9).

2 Testing for an Upper Bound on $\|w\|_\infty$

We are interested in testing for an upper bound on the H^∞-norm of on the solutions w of the coupled system:

$$\begin{cases} -\Delta u + u = w, \\ w_t + \vec{V}(u)\cdot\nabla w = -ku_y. \end{cases}$$

The simulator is written in FreeFem++ and can be found here: `https://github.com/adelsaleh/hmSimulator`

2.1 The Algorithm

We give a description of the algorithm that approximates

$$\sup_{0\le t\le T}(\|\nabla w\|_\infty + \|w\|_\infty),$$

for some given time T. This algorithm is correct as long as $w \in \mathcal{V}_h$. It works as follows, suppose T_i is a triangle in the mesh $\mathcal{T}$. Let $w_i = w|_{T_i}$ then there are $a_i, b_i, c_i \in \mathbb{R}$ such that for all $(x, y) \in T_i$,

$$w(x,y) = w_i(x,y) = a_i x + b_i y + c_i, \quad \text{and therefore } \nabla w_i = (a_i, b_i).$$

The code recovers the values of a_i, b_i, c_i. We are given three vertices $P_j^i = (x_j^i, y_j^i) \in T_i$ for $j = 1, 2, 3$ from which we solve:

$$\begin{pmatrix} x_1^i & y_1^i & 1 \\ x_2^i & y_2^i & 1 \\ x_3^i & y_3^i & 1 \end{pmatrix}\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} w_i(x_1^i, y_1^i) \\ w_i(x_2^i, y_2^i) \\ w_i(x_3^i, y_3^i) \end{pmatrix}.$$

If $\mathcal{P}$ is the set of nodes of the mesh then what we are computing at each $0 \leq t \leq 75$:

$$\|w\|_\infty + \|\nabla w\|_\infty = \max_{p \in \mathcal{P}} |w(p)| + \max_{T_j \in \mathcal{T}} (|a_j| + |b_j|).$$

The code has been tested for 3 different initial conditions, and for endt=76.

2.2 The Tests

Case 1: $u_0(x, y) = \sin 3x$.

- for $dt = 1$

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	9.72845e+06
meshp = 32	0.00932348	0.00932348	0.0700644	9.48621e+06
meshp = 64	0.00935401	0.00935401	0.25487	3.36182e+07

- for $dt = 0.1$

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	0.00920216
meshp = 32	0.00932348	0.00932348	0.00932348	0.00932348
meshp = 64	0.00935401	0.00935401	0.00935401	0.00935401

- For dt=0.01

	t=1	t=25	t=50	t=75
meshp = 16	0.00920216	0.00920216	0.00920216	0.00920216
meshp = 32	0.00932348	0.00932348	0.00932348	0.00932348
meshp = 64	0.00935401	0.00935401	0.00935401	0.00935401

Case 2: $u_0(x, y) = \sin 3y$.

dt	meshsize	t=1	t=25	t=50	t=75
1	16	0.00819026	226093	2.35954e+13	3.38578e+20
	32	0.0174697	274395	3.46674e+13	2.54711e+18
	64	0.049547	288790	3.84457e+13	8.44555e+19
0.1	16	0.00109678	0.0228526	1.44038	92.6633
	32	0.00177724	0.0251274	1.70835	118.378
	64	0.00541686	0.0267284	1.79248	126.325
0.01	16	0.000573946	0.000709836	0.000882754	0.001284
	32	0.000786692	0.000811604	0.0010299	0.0013535
	64	0.0012995	0.00132479	0.00137715	0.00169396

Case 3: The Gauss fucntion.

- $dt = 1$.

	t=1	t=25	t=50	t=75
meshp = 16	0.271741	0.297416	0.304924	0.309987
meshp = 32	0.268623	0.291456	0.298063	0.298935
meshp = 64	0.263379	0.289162	0.294879	0.294879

- $dt = 0.1$.

	t=1	t=25	t=50	t=75
meshp = 16	0.27173	0.295808	0.303704	0.307203
meshp = 32	0.26973	0.290931	0.29701	0.29796
meshp = 64	0.262307	0.288401	0.293987	0.293987

- $dt = 0.01$.

	t=1	t=25	t=50	t=75
meshp = 16	0.271735	0.295687	0.303588	0.306932
meshp = 32	0.269837	0.29089	0.296902	0.297862
meshp = 64	0.262465	0.288328	0.293898	0.293898

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