Two-Player Pebbling on Diameter 2 Graphs

Garth Isaak[1] and Matthew Prudente[2]

[1] *Department of Mathematics, Lehigh University*
[2] *Department of Mathematics, Saint Vincent College*

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Abstract

A *pebbling move* refers to the act of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. The goal of graph pebbling is given an initial distribution of pebbles, use pebbling moves to reach a specified goal vertex called the root. The *pebbling number* of a graph $\pi(G)$ is the minimum number of pebbles needed so every distribution of $\pi(G)$ pebbles can reach every choice of the root. We introduce a new variant of graph pebbling, a game between two players. We aim to show configurations of various classes of graphs for which each player has a winning strategy. We will characterize the winning player for a specific class of diameter two graphs.

1 Introduction

Graph pebbling can be thought of as an optimization problem where a utility such as gas, electricity, or computing power travels across a network. While traveling through the network, some amount of the utility may be lost. A natural question that arises is what is the minimum amount of the utility that is needed to travel the network and arrive at a destination.

Graph pebbling was originally developed to solve a number theory conjecture posed by Erdös [5]. The goal is to use pebbling moves to place one pebble to a specified vertex r called the *root*. A *pebbling move* removes two pebbles from one vertex and places one pebble on an adjacent vertex. The basic question is, given an initial arrangement of the pebbles called a configuration C, can we set one pebble on r through a sequence of pebbling moves. If so, then C is *r-solvable* [4]. We say $C(v)$ is the number of pebbles at vertex v and the *size* of a configuration C is $\sum_{v \in G} C(v)$. We can see in Figure 1 that if the root is the leftmost vertex, then C is r-solvable. Define $\pi(G, r)$ as the minimum number m such that every configuration of m pebbles is r-solvable for a given root r. The *pebbling number* $\pi(G) = \max_{r \in V(G)} \pi(G, r)$ is the minimum number t such that every configuration of size t is r-solvable for every choice of r in G. Graph pebbling is well studied and has numerous variations [3, 4].

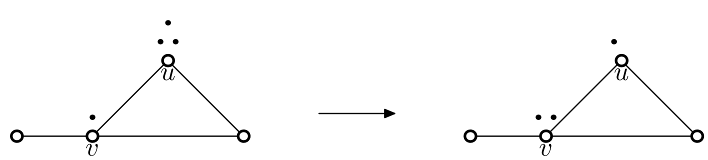

Figure 1: An example of a pebbling move on G from u to v.

From this point, all graphs will be finite and simple (no loops or multiedges). Let $V(G)$ be the set of vertices of G and $|V(G)|$ be the number of vertices in G. The *diameter* of a graph, *diam*(G), is the longest of all shortest paths in G. The *open neighborhood* of v, $N(v)$, is the set of vertices adjacent to but not including v. Likewise, the *closed neighborhood* of v, $N[v]$, is the set of vertices adjacent to and including v. Given $S \subseteq V(G)$ and $v \in V(G)$, we say the *S-restricted neighborhood of*

v $N_S(v) = N(v) - (G - S)$ is the set of neighbors of v contained only in S. Given a graph G, the complement G' is the graph such that $V(G) = V(G')$ and $uv \in E(G') \iff uv \notin E(G)$.

We introduce a new variation that extends pebbling to a two-person game. The first player, called the *mover*, uses pebbling moves to obtain a configuration C' such that $C'(r) = 1$. The second player, called the *defender*, uses pebbling moves to obtain a configuration C' that admits no pebbling moves and $C'(r) = 0$. The winner is which ever player attains their target configuration. We say a *round* consists of two pebbling moves; the initial move made by the mover and the final move made by the defender. A *turn* will be an individual player's pebbling move. Given an initial configuration C on a graph G, we begin playing round 1 with the following rules:

1. Each player must take their turn.
2. If the mover pebbles from u to v, then the defender can not pebble from v to u in the same round.
3. If $C'(r) > 0$ at any time, then the mover wins.
4. If $C'(r) = 0$ and there are no more pebbling moves, then the defender wins.

We did consider a variant where rule 1 was relaxed to allow the defender to forfeit their turn. However, this significantly limited the graphs and configurations for which the mover could win.

We examine conditions which ensure a win for the mover or a win for the defender. To do this, we must study how each player will play the game.

Definition 1.1. A *strategy* for either player is a choice function $\mathcal{S} : \mathcal{C} \to \mathcal{P}$ from the set of all possible configurations $\mathcal{C}$ to the list of all possible legal pebbling moves $\mathcal{P}$.

By this, of course, we mean a strategy is a method of playing the game based on the possible outcomes of any move. The defender also needs to be aware of the mover's previous move so the defender does not make a pebbling move that violates the rules.

Definition 1.2. A strategy $\mathcal{S}$ is *winning* for the mover (or defender) on a configuration C provided the mover (or defender) wins playing $\mathcal{S}$ no matter what the defender (or mover) does.

Now we can introduce the values for two-player pebbling.

Definition 1.3. The *rooted-two-player pebbling number*, $\eta(G, r)$, is the minimum number m such that given any configuration of m pebbles and a given root vertex, r, the mover has a winning strategy. From this, we say the *two-player pebbling number* is $\eta(G) = \max_{r \in V} \eta(G, r)$, the minimum number t such that for every configuration of size t and every choice of r, the mover has a winning strategy. However, if for a graph G, a root r, and arbitrarily large m, there exists a configuration of size at least m', for $m' > m$, for which the defender has a winning strategy, then $\eta(G, r) = \infty$.

2 Preliminary Results

We begin with some basic statements about $\eta(G)$.

Proposition 2.1. *$|V(G)| \leq \pi(G) \leq \eta(G)$.*

Proof. The mover cannot win with less than the traditional pebbling number. □

Notice if the defender is not forced to pebble in a winning pebbling move sequence for classical pebbling, then equality fails.

Proposition 2.2. *If $deg(r) = |V(G)| - 1$, then $\eta(G, r) = |V(G)|$.*

Proof. Let r be a vertex with degree $|V(G)|-1$. Suppose we have $|V(G)|-1$ pebbles. If every non-root vertex has 1 pebble, then the defender wins. So suppose we have $|V(G)|$ pebbles. If we have a configuration with 1 pebble on r, then the mover wins. Suppose we have a configuration with no pebbles on the root. Then there must exist at least one vertex with at least 2 pebbles on it. Since the mover begins the game, they will pebble to the root. □

From this, we get a corollary about the complete graph on n vertices, K_n.

Corollary 2.3. $\eta(K_n) = n$.

The proof for Proposition 2.2 os the same as the standard proof of Corollary 2.3 [1].

Proposition 2.4. *If $\eta(G,r)=\infty$ for any $r \in V(G)$, then $\eta(G)=\infty$.*

Proof. Let $\eta(G,r')=\infty$ for some $r'r' \in V(G)$. Since $\eta(G)=\max\limits_{r\in V}\eta(G,r)$, we have $\eta(G)=\eta(G,r')=\infty$. □

2.1 Sufficient Condition for Infinite η

In this section, we show there exists a graph structure for which the defender always has a winning strategy. In fact, the condition below will show that "most" configurations with arbitrarily large number of pebbles on graphs have a winning strategy for the defender. Later, we will show more structured classes of graphs that have winning strategies for the mover.

Theorem 2.5. *For a graph G, let S be a cut set of G and let $G_0, G_1, \ldots G_k$ be the components of $G-S$ with $r \in G_0$. If for every $v \in S$, $|N(v)-V(G_0)-S| \geq 2$ and for every $x \in N(v)-V(G_0)-S$, $|N(x)-S| \geq 2$, then $\eta(G)=\infty$.*

Proof. Let G be described as above. Let m be an arbitrary natural number and $\mathcal{C}$ be the family of configurations with m pebbles on the vertices of $N(x)-S$. The only way the mover can win is if the defender is forced to place a second pebble on a vertex in S. To see this, suppose the mover puts a second pebble on a vertex $v \in S$. Because $|N(v)-V(G_0)-S| \geq 2$, the defender can pebble to another vertex in $N(v)-V(G_0)-S$. Let $y \in N(v)-V(G_0)-S$ and suppose the defender must pebble from y. Because $|N(y)-S| \geq 2$, the defender can pebble to a vertex in $N(y)-S$. Therefore, the defender is never forced to place a second pebble on a vertex in S and can exhaust the use of all m pebbles. □

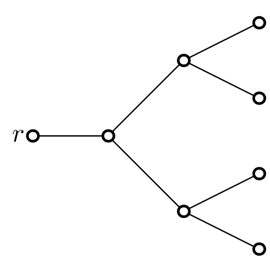

Figure 2: A small example for Theorem 2.5.

Note that Figure 2 satisfies the conditions for Theorem 2.5. We see that Figure 2 is a tree, and thus bipartite. Therefore, trees and bipartite graphs can have an infinite two-player pebbling number in general. The graph in Figure 3 has diameter 2. Thus, a graph G having diameter 2 is not a sufficient condition for a finite value of $\eta(G)$, whereas diameter-2 graphs have classical pebbling number of at most $|V(G)|+1$ [7]. In fact, we are finding that the defender has a winning strategy on the configurations for many classes of graphs. So, we must have more restrictions on graphs to find $\eta(G)<\infty$.

We have also found that grids, $P_n \square P_m$ for $m,n \geq 4$ have infinite η because they satisfy the conditions for Theorem 2.5. Consider Figure 4.

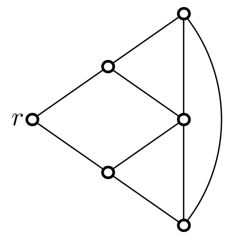

Figure 3: A graph with diameter 2 for Theorem 2.5.

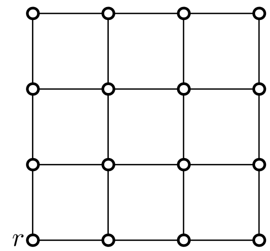

Figure 4: $P_4 \Box P_4$.

It is easy to verify that $\eta(P_4)$ is finite, and hence $\eta(P_4)\eta(P_4)$ is finite, but $\eta(P_4 \Box P_4) = \infty$. This is in direct contrast to a two-player pebbling version of Graham's Conjecture [1], a well studied problem in classical pebbling which states $\pi(G \Box H) \leq \pi(G)\pi(H)$ for any choice of G and H. So even for a simple cartesian product of graphs, a two-player pebbling analog of Graham's Conjecture will not hold.

2.2 Removal of Edges

Next, we give some examples where adding or removing edges can completely change the outcome of the game. Consider the graph in Figure 3 for example. Thereom 2.5 says that $\eta(G) = \infty$. However, if we remove one of the edges, then the conditions of Theorem 2.5 are no longer satisfied, as seen in the graph in Figure 5. It is easy to check that the mover has a winning strategy.

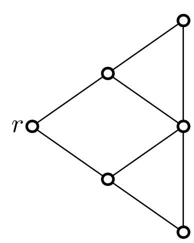

Figure 5: Removal of an Edge from Figure 3

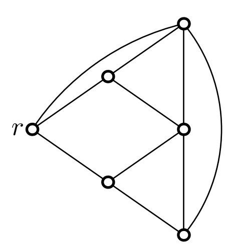

Figure 6: Adding an Edge from Figure 3

Also, if we add an edge from the root to a nonadjacent vertex, as in Figure 6, then it can be verified that the mover has a winning strategy.

The removal of edges does not just benefit the mover. Consider the graph in Figure 7.

It is straightforward to check that the graph in Figure 7 has a finite value for $\eta(G, r)$. But if we remove an edge, the game shifts. The graph in Figure 8 now satisfies the conditions of Theorem 2.5.

The removal of an edge changed the outcome of the game for either player. The edge removed can determine who is helped. The removal of an edge adjacent to the root will only help the defender.

Proposition 2.6. *Let G be a graph and e be an edge adjacent to the root. If the defender has a winning strategy on G for a given configuration C, then the defender has a winning strategy on $G - e$ with configuration C.*

Proof. Given a configuration C on the graph G, the defender will never pebble on an edge adjacent to the root unless forced to. So let the defender have a winning strategy on G. Then the defender will play the same strategy on $G - e$ and win. □

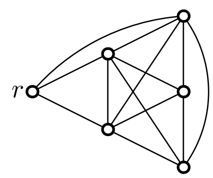

Figure 7: Graph for Which $\eta(G, r)$ is Finite

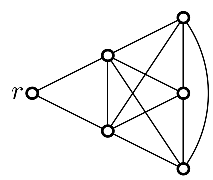

Figure 8: Figure 7 Minus One Edge

3 Certain Diameter 2 Graphs

We move on to the study of two-player pebbling on certain graphs of diameter 2. Specifically, we characterize the winning player for nearly every configuration for a specific class of diameter 2 graph, characterize the winning player for every configuration on complete bipartite and complete multipartite graphs, and find exact η values for complete bipartite and complete multipartite graphs.

First, we need a definition.

Definition 3.1. For any two graphs H and G, the *join* of H and G, $H \vee G$, is the graph such that $V(H \vee G) = V(H) \cup V(G)$ and $E(H \vee G)$ contains all edges in H, all edges in G, and edges connecting every vertex in H with every vertex with G.

Now, we define a subset of diameter 2 graphs.

Definition 3.2. We say $\mathcal{G}_{s,t} = \big((K_1 \cup K_t') \vee H\big)$ is a subset of diameter 2 graphs where H is arbitrary. We define $S = V(H)$ and $T = V(K_t')$ with $|S| = s$ and $|T| = t$. We let the root be K_1 and $s \geq 1, t \geq 2$.

We will save the case when $t = 1$ for later, as it is unique. Figure 9 gives us an example of a graph in $\mathcal{G}_{s,t}$.

If a starting configuration has two pebbles on any vertex in S, then the mover can pebble to the root and win. So, we say a *non-trivial configuration* on the vertices of G will have 0 or 1 pebbles on

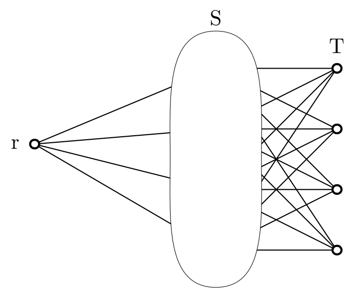

Figure 9: The class $\mathcal{G}_{s,t}$.

vertices in S. Let k be the number of vertices in S that are pebble-free. We say a vertex is *even* if there are an even number of pebbles distributed on it and a vertex *odd* if there are an odd number of pebbles distributed on it. A vertex is *pebbled* provided it has at least 1 pebble on it and is *unpebbled* or *pebble-free* otherwise.

We develop a condition on the distribution of pebbles on T based on the pebble-free vertices in S. Informally, it appears that we can compare how many pebbling moves are in T to the number of pebble-free vertices in S. If there are many more pebbling moves in T than pebble-free vertices in S, then the mover wins. Both players are pebbling to S. Eventually, S will have no pebble-free vertices and it will be the defender's turn. They will pebble to S; the mover will pebble to r on their next turn. On the other hand, if there are many more pebble-free vertices than pebbling moves in T, the defender wins. The defender will always have a pebble-free vertex in S to pebble to. We would like a way to count the number of pebbling moves in T. Notice for any vertex $v \in T$ that $\left\lfloor \dfrac{C(v)}{2} \right\rfloor$ will tell us the number of pebbling moves on v. We have the following definition.

Definition 3.3. We say $C_T = \displaystyle\sum_{v \in T} \left\lfloor \frac{C(v)}{2} \right\rfloor$ is the number pebbling moves in T with configuration C.

In fact, if there are k pebble-free vertices in S and $C_T \geq k+3$, then the mover has a winning strategy. If $C_T \leq k$, then the defender has a winning strategy. If $C_T = k+2$ or $k+1$, then it depends on the parity of k and the structure of S to find the winning player.

We can see that C_T will change from configuration to configuration. When a pebbling move is made from T, we can say that the number of pebbling moves in T for the new configuration C' is $C'_T = C_T - 1$ with original configuration C.

We want to see a configuration where the mover has a winning strategy and define such a strategy. The winning strategy for the mover is to force the defender to place a second pebble on a vertex in S.

3.1 When k is odd

Lemma 3.4 is the base case for induction when k is odd.

Lemma 3.4. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with 1 pebble-free vertex in S. The mover has a winning strategy if and only if $C_T \geq 2$.*

Proof. Suppose $C_T \geq 2$. The mover will pebble to the unpebbled vertex. Now there is one more move in T and all vertices in S have a pebble on them. The defender must pebble to a vertex in S, placing a second pebble on a vertex. The mover pebbles to r and wins.

Conversely, suppose $C_T \leq 1$. If $C_T = 0$, then there are no pebbling moves in T and the defender wins. Suppose $C_T = 1$. Since there is 1 pebbling move in T, all the vertices in T without the pebbling move have 0 or 1 pebble on them. The mover has two choices, to pebble to the unpebbled vertex or to place a second pebble on a vertex in S. If the mover pebbles to the pebble-free vertex, then for the new configuration C', $C'_T = 0$. There are no more pebbling moves and the defender wins. So suppose the mover pebbles to a pebbled a vertex in S. If they can, then the defender will pebble to the pebble-free vertex in S or T and win. If all vertices in T are pebbled, then the defender will place a second pebble on one vertex in T, yielding an extra pebbling move. The mover has the same two options as earlier. Suppose the mover places a second pebble on a vertex in S, or else they will lose. The vertex in T with the original pebbling move can now have 0 or 1 pebbles on it. The defender will pebble to it. If it is unpebbled, then the defender wins. If it is pebbled, then the defender adds a new pebbling move. The mover will pebble from that vertex to S with the same two options. Again we suppose the mover pebbles to a pebbled vertex. Now there is guaranteed to be an unpebbled vertex from the mover's last two pebbling moves for the defender to pebble to. The defender does so and wins. $\square$

Lemma 3.5. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is odd and $C_T \geq k+1$, then the mover has a winning strategy on G.*

Proof. Let k be odd and $C_T \geq k+1$. The mover will pebble to a pebble-free vertex in S. If the defender places a second pebble on a vertex in S, the mover wins. If the defender pebbles to a pebble-free vertex in S, then there are $k-2$ pebble-free vertices in S and the resulting configuration C' has $C'_T = C_T - 2$. Thus $C'_T = k-1$. Hence, by induction, the mover has a winning strategy. $\square$

Next is a result when the defender has a winning strategy.

Lemma 3.6. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is odd and $C_T \leq k$, then the defender has a winning strategy on G.*

Proof. By induction on C_t.

Base: Let $C_t = 0 \leq k$. There are no pebbling moves in T so the defender wins.

Induction: Let $C_t \leq k$ for k-pebble-free vertices in S. The mover has two choices, to pebble to a pebble-free vertex in S or to place a second pebble on a vertex in S. If the mover pebbles to a pebble-free vertex and there are no more pebble free vertices, then $k = 1$ and by Lemma 3.4 the defender wins. If the mover pebbles to a pebble-free vertex and there is another unpebbled vertex, then the defender will pebble to a pebble-free vertex. We have $C_t \leq k-2$ and by induction, the defender has a winning strategy. If the mover places a second pebble on a vertex in S, then the defender will pebble back to an even vertex in T, if one exists. Now $C_t \leq k+1$ and by induction the defender wins.

$\square$

So for k odd, we have the following:

Initital Value of C_T	Winning Player
$C_T \geq k+1$	Mover
$C_T \leq k$	Defender

Table 1: Value of C_T and its winning player for k odd

3.2 When k is even

The section when the number of pebble-free vertices on S is even is a little more difficult. We first show the number of pebbling moves needed in T for the mover to win.

Lemma 3.7. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T \geq k+3$, then the mover has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T \geq 3$. The mover will pebble to S, placing a second pebble on one of the vertices. The defender will pebble back to T or lose. The new configuration C' has $C'_T \geq 2$ and now $k = 1$. By Lemma 3.4, the mover wins.

Induction: Let $C_T \geq k+3$ for $k \geq 1$. The mover will pebble to a free vertex. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a free vertex in S, then the new configuration C' has $C'_T = C_T - 2 \geq k+3-2 = k+1$. Since S now has $k-2$ pebble-free vertices, the mover has a wining strategy by induction. $\square$

We will forgo the case when $C_T = k+2$ for now and leave it for its own section.

Lemma 3.8. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T \leq k+1$, then the defender has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T \leq 1$. If $C_T = 0$, then there are no pebbling moves in T and the defender wins. If $C_T = 1$, then all but one vertex in T as at most 1 pebble on it. The mover has no choice but to place a second pebble on a vertex in S. The defender will pebble from the vertex in S with two pebbles on it to any vertex in T. For the new configuration C', we have $C'_T \leq 1$ and $k = 1$. So by Lemma 3.4, the defender has a winning strategy.

Induction: Let k be even and $C_T \leq k+1$. If the mover pebbles to a pebble-free vertex in S, then the defender will as well. The new configuration C' has $k-2$ pebble-free vertices and $C'_T = C_T - 2 \leq k-1$. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex in S, the defender will pebble to a vertex in T. The resulting configuration C'' has $k+1$ pebble-free vertex in S and $C''_T \leq C_T \leq k+1$. Since $k+1$ is odd, the defender has a winning strategy by Lemma 3.6. □

So for k even, we have the following:

Initital Value of C_T	Winning Player
$C_T \geq k+3$	Mover
$C_T \leq k+1$	Defender

Table 2: Value of C_T and its winning player for k even

3.3 When k is even and $C_T = k+2$

When $C_T = k+2$, the difficulty increases. The number of pebbles in S and how many vertices in T have a non-zero even number of pebbles on them will determine which player has a winning strategy. Each player's strategy changes a little. The mover's goal is to force the defender to pebble to a vertex in T with an odd number of pebbles on it. This will increase the number of pebbling moves in T and yield one of the mover's winning configurations described in an early section. The defender will try to pebble to a vertex in T with an even number of pebbles on it. This adds no new pebbling moves and yields one of the defender's winning configurations.

First we consider the configuration where all the vertices in T have an odd number of pebbles on them.

Lemma 3.9. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T = k+2$ and for all $v \in T$, $C(v)$ is odd, then the mover has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T = 2$ with every vertex in T having an odd number of pebbles on it. The mover will pebble to S, placing a second pebble on one of the vertices. The defender will pebble back to T or lose. Since every vertex in T has an odd number of pebbles, the new configuration C' has $C'_T = 2$ with 1 unpebbled vertex in S. By Lemma 3.4 the mover wins.

Induction: Let k be even and $C_T \geq k+2$ for $k \geq 1$. The mover will pebble to a free vertex. If the defender places a second pebble on a vertex in S, then the mover wins. So the defender will pebble to a free vertex in S. Now for the new configuration C', $C'_T = C_T - 2 \geq k+2-2 = k$. Since S now has $k-2$ pebble-free vertices, the mover has a wining strategy by induction. □

Now, we look at the case when some vertices in T have an even number of pebbles on them. This becomes more difficult. The strategies for each player depends on how many pebbles on are the vertex with an even number of pebbles.

Lemma 3.10. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even and $C_T = k+2$ and there is either at least one $x \in T$ such that $C(x) = 0$ or at least two vertices $x, y \in T$ such that $C(x)$ and $C(y)$ are even, then the defender has a winning strategy.*

Proof. By induction on k.

Base: Let $k = 0$ and $C_T = 2$. The mover will place a second pebble on a vertex in S. The defender will pebble from that vertex in S to the pebble-free vertex in T or to an even vertex in T. For the new configuration C', we have $C'_T = 1$ and $k = 1$. Thus by Lemma 3.4, the defender wins.

Induction: Let k be even and $C_T \geq k+2$. The mover can place a second pebble on a vertex in S or pebble to a pebble-free vertex in S. If the mover places a second pebble on a vertex in S, then the defender will pebble to the unpebbled vertex in T or to an even vertex in T, not adding any pebbling moves to T. For our new configuration C', we have $C'_T = k+1$ and k is now odd. Hence, the defender wins by Lemma 3.6. If the mover pebbles to a pebble-free vertex in S, then defender will also pebble to a pebble-free vertex in S. Now for our new configuration C', we have $C'_T = k$ and there are $k-2$ pebble-free vertices in S. Since there were no pebbling moves back to T, we can see that T will still have at least one pebble-free vertex or at least two even vertices. Thus, the defender wins by induction. □

So for k even and $C_T = k+2$, we have the following:

Number of Even Vertices in T	Winning Player
None	Mover
At least one pebble-free or two even	Defender

Table 3: Number of even vertices in T and its winning player for k even

3.4 A New Game

In this section, we will characterize the winning player for specific structures on S and certain configurations on $\mathcal{G}_{s,t}$ with an even number of unpebbled vertices, one even vertex in T, and the number of pebbling moves from T is two more than the number of pebble-free vertices in S. We will partition S into two subsets.

Definition 3.11. Let S_0 be the pebble-free vertices of S and S_1 be the pebbled vertices of S.

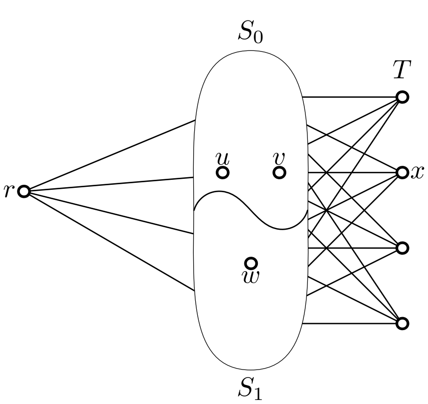

Figure 10: Partitioning S

We cannot characterize the winning player for all configurations and all structures on S. We will introduce a new game, called the Element Selecting Game (ESG), to help explain why this task is particularly difficult.

Let $N_1, N_2, \ldots N_k$ be a collection of subsets, possibly empty and intersecting, from a universal set U. There are two players, Mary and Dan. Each player will take turns, Mary beginning and Dan following, selecting one element from U. After a specified number of rounds, we say Mary wins if at least one of the subsets N_i has every one of its element selected and Dan wins if none of the N_i's has been completely selected. If there exists a subset N_j which is empty, then we say Mary wins vacuously. Which player has a winning strategy?

This game directly relates to this case of exactly one even vertex in T with $C_T = k + 2$ and k pebble-free vertices in S of two-player pebbling.

Definition 3.12. Given an instance $G \in \mathcal{G}_{s,t}$ with configuration C containing $2j$ pebble-free vertices in S and $C_T = 2j + 2$, we define $\mathcal{E}(G, C)$ as the instance of the Element Selecting Game constructed in the following way: Let $U = S_0$, the set of unpebbled vertices in S. For every vertex $v_i \in S$, let $N_i = N[v_i] \cap U$. For $k = 2j$ pebble-free vertices in S and $C_T = 2j + 2$, Mary and Dan play j rounds of the new game. Mary represents the motives of the mover and Dan represents the motives of the defender.

Here we see two lemmas to illustrate why we want $C_T = 2j + 2$ given we are playing j rounds.

Lemma 3.13. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. Suppose there exists a pebbled vertex $v \in S$ such that all its neighbors in S are pebbled. If k is even and $C_T = k + 2$ and there is one $x \in T$ such that $C(x) \geq 2$ and all other vertices in T have an odd number of pebbles, then the mover has a winning strategy.*

Proof. The mover will pebble from x to v. The defender can either pebble to a neighbor of v or pebble to an odd vertex in T. If the defender pebbles to a neighbor of v, then that vertex will have two pebbles on it and the mover wins. If the defender pebbles to an odd vertex in T, then they will add a pebbling move. Now our new configuration C' has $k + 1$ pebble-free vertices in S and $C'_T = k + 2$. By Lemma 3.5, the mover has a winning strategy. □

So, we have covered the case when the only even vertex in T has 2 pebbles on it. If S is independent, then the conditions for Lemma 3.13 will hold vacuously. Here is a configuration for the defender's winning strategy.

Lemma 3.14. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. Suppose that for every pebbled vertex $v \in S$, there exist at least one pebble-free neighbor in $u \in S$. If k is even and $C_T = k + 2$ and there is one $x \in T$ such that $C(x) = 2$ and all other vertices in T have an odd number of pebbles, then the defender has a winning strategy.*

Proof. The mover can pebble to a pebbled vertex or an unpebbled vertex. If the mover pebbles to a pebbled vertex v, then the defender will pebble from v to its pebble-free neighbor, which exists by our hypothesis. Now k is unchanged and our new configuration C' is such that $C'_T = k + 1$. By Lemma 3.8, the defender has a winning strategy. If the mover pebbles to an unpebbled vertex, then the defender will pebble from x to another vertex in S which is pebble-free, which exists because k is even and at least 2. By Lemma 3.10, the defender has a winning strategy. □

By the time the j rounds are completed, the mover wants to have a pebbled closed neighborhood for some vertex in S and still have at least 2 pebbles on the one even vertex in T.

Now, we can show that the two games are equivalent when we restrict Two-Player Pebbling to this current case.

Theorem 3.15. *Let $G \in \mathcal{G}_{s,t}$ and C be a configuration containing $2j$ pebble-free vertices in S and $C_T = 2j+2$ and $\mathcal{E}(G, C)$ be the instance of the Element Selecting Game constructed from G. Mary has a winning strategy for $\mathcal{E}(G, C)$ if and only if the mover has a winning strategy in G with configuration C.*

Proof. Given $G \in \mathcal{G}_{s,t}$, let C be a non-trivial configuration with $2j$ pebble-free vertices in S, exactly one even vertex in T and $C_T = 2j + 2$. We construct the $\mathcal{E}(G, C)$ as in Definition 3.12. Suppose Mary has a winning strategy for the $\mathcal{E}(G, C)$. Then, Mary and Dan have a sequence of elements that they each selected such that at least one of the N_i's has been selected. Every element in U that Mary selects, the mover will pebble from an odd vertex in T to the corresponding vertex in S_0. If the defender ever places a 2nd pebble on a vertex in S, then the mover wins. If the defender places a pebble on a pebble-free vertex, then the mover will pebble to the vertex that corresponds to the next element that Mary selected. Since Mary was able to select every element in one of the N_i's, the mover will be able to have a pebbled closed neighborhood with a new configuration C' such that $C'_T \geq 2$. Thus the mover has a winning strategy.

Conversely, suppose the mover has a winning strategy on G with configuration C. If the mover can not pebble a closed neighborhood after j rounds, then for the new configuration C' every vertex in S will have an unpebbled neighbor and $C'_T = 2$. So the defender wins by Lemma 3.14. Thus the mover must be able to pebble a closed neighborhood in S. Mary can select an element in U that corresponds to a pebble-free vertex in S_0 that the mover selects. Because a closed neighborhood is pebbled for some $v_i \in S$, then N_i must be able to have its elements selected. Thus Mary has a winning strategy. □

It will be easier to show cases of $\mathcal{E}(G, C)$ for which Mary has a winning strategy and then show how a case for pebbling can apply.

Lemma 3.16. *If there exists an i such that $|N_i| = j$ while playing at least j rounds, then Mary wins the Element Selecting Game.*

Proof. Suppose there exists a set N_i with j elements in it. Suppose Mary and Dan play at least j rounds. Mary can select every element in N_i with her turn and win in at most j rounds. □

Corollary 3.17. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. If k is even, $C_T = k + 2$ and there is one even vertex $x \in T$ such that $C(x) \geq k + 2$ and all other vertices in T have an odd number of pebbles, then the mover has a winning strategy.*

Proof. Let $k = 2j$. Having k pebble-free vertices in S with $C(x) \geq k + 2$ is equivalent to some $|N_i| = j$ and playing j rounds. □

Unfortunately, Lemma 3.16 and Corollary 3.17 are not necessary conditions in general. There are 'boundary' cases which can violate the conditions of converse Corollary 3.17 and the mover still has a winning strategy (Lemma 3.13 for example). Specifically, we can have many more pebble-free vertices in S than pebbles on x and the mover has a winning strategy. We see why having exactly one vertex in T with a non-zero even number of pebbles on it is so difficult. It depends on how S is structured. The informal strategy for the mover is to pebble from the even vertex in T to a vertex in S whose neighbors all have pebbles on them. Then the defender must pebble to an odd vertex in T, yielding the odd configuration in Lemma 3.5. If the defender can pebble in S, then the mover will lose.

We begin to characterize the winning strategy for each player for the case where $C(x) = 4$ with x as the only even vertex in T. Notice that for the mover to have a winning strategy in the $C(x) = 2$ case we needed a vertex $v \in S_1$ to be such that $N_S(v) \subseteq S_1$. The mover will make a pebbling move from an odd vertex in T to try and force the defender to pebble in such a way that for the next round, the conditions for Lemma 3.13 are satisfied.

Lemma 3.18. *Suppose Mary and Dan play only 1 round. Then Mary wins the Element Selecting Game if and only if there is an i such that N_i is empty, $N_i = \{y\}$ or there exists an $y \in U$ such that for every $z \in U$, $N_i = \{y, z\}$.*

Proof. Let Mary and Dan play only 1 round.

Suppose that there is some $y \in U$ such that for each $z \in U$, there is a subset such that N_i is empty, $N_i = \{y\}$ or $N_i = \{y, z\}$. If N_i is empty, then Mary wins vacuously. If all N_i's are nonempty, then Mary will select element y. Then Dan will select any other element. By our hypothesis, there must exist a subset of U that is equal to y or equal to y and the element Dan chose. Thus there will be a subset that is selected. Thus Mary wins.

Conversely, suppose for every $y \in U$ there exists a $z \in U$ so that for every N_i is nonempty, $N_i \neq \{y\}$, and $N_i \neq \{y, z\}$. Mary will chose any element y'. By our assumption, there must exist another element z' in U so that for every subset N_i, we have $\{y', z'\}$ is a proper subset of N_i. Thus after 1 round, no subset has been completely selected. Hence, Dan wins. □

Corollary 3.19. *Let $G \in \mathcal{G}_{s,t}$ and C be a non-trivial configuration with k pebble-free vertices in S. Suppose k is even and $C_T = k+2$ and there is only one even vertex $x \in T$. The mover has a winning strategy if $C(x) \geq 4$ and there exists a vertex v in S_0 that for every vertex $u \in S_0$ that either:*

a) there is some vertex $w \in S_1$ such that $N_{S_0}(w) = \{v\}$ or $\{u, v\}$, or

b) $N_{S_0}(u) = \{v\}$.

The defender has a winning strategy if $C(x) \leq 4$ and for every vertex v in S_0 there exists a vertex $u \in S_0$ such that there is no vertex $w \in S_1$ with $N_{S_0}(w) = \{v\}$ or $\{u, v\}$ and (b) $N_{S_0}(u) \neq \{v\}$.

Proof. We can consider $C(x) \geq 2(1) + 2$. Thus having $C(x) \geq 4$ is equivalent to playing 1 round in ESG. Let the vertex v in Two-Player Pebbling represent the element y in ESG. Suppose there is some vertex $w \in S_1$ such that $N_{S_0}(w) = \{v\}$ or $\{u, v\}$. Then for the ESG, $N_w = \{v\}$ or $\{u, v\}$. The mover wins by Lemmas 3.15 and 3.18. Suppose $N_{S_0}(u) = \{v\}$. Then for the ESG, $N_u = \{u, v\}$. The mover wins by Lemmas 3.15 and 3.18. □

So for k even with $C_T = k+2$ and one even vertex $x \in T$, we have the following:

Structure of S	$C(x)$	Winning Player
Any structure	$C(x) \geq k+2$	Mover
Some pebbled vertex with all pebbled neighbors	$C(x) \geq 2$	Mover
All pebbled vertices have an unpebbled neighbor	$C(x) = 2$	Defender
$\exists v \in S_0, \forall u \in S_0$ either $\exists w \in S_1$ such that $N_{S_0}(w) = \{v\}, \{u, v\}$ or $N_{S_0}(u) = \{v\}$	$C(x) \geq 4$	Mover
$\forall v \in S_0$, $\exists u \in S_0$ such that $\forall w \in S_1$, $N_{S_0}(w) \neq \{v\}, \{u, v\}$ and $N_{S_0}(u) \neq \{v\}$	$C(x) = 4$	Defender

Table 4: Structure of S and its Winning Player

3.5 Configurations on Complete Multipartite Graphs

We attempted to find a nice necessary condition for Mary to have a winning strategy in the Element Selecting Game while playing 2, 3, etc. rounds. We believe it would be easier to find the winning player for different scenarios in the Element Selecting Game and then translate them to Two-Player Pebbling. However, characterizing scenarios for which Mary has a winning strategy turns out to be very difficult and based on the structure of the subsets $N_1, N_2, \ldots, N_m$. So, we narrow our focus from any $G \in \mathcal{G}_{s,t}$ to G being a complete multipartite graph, and we can characterize the winning player without the aid of the Element Selecting Game.

The goal is to determine the winning player for all configurations on complete bipartite and complete multipartite graphs. Sections 3.2, 3.3, and 3.4 cover all cases except when the number of unpebbled vertices in S, k, is even, $C_T = k+2$ and there is one even vertex $x \in T$. Notice that for complete bipartite graphs, S is independent so Lemma 3.13 and Lemma 3.14 finish the argument for complete bipartite graphs. To finish the task for complete multipartite graphs, we need to complete the above argument. If S is a clique, then Corollary 3.17 shows when the mover has a winning strategy.

Lemma 3.20. *Let G be a complete multipartite graph with partite sets $A_1, A_2, \ldots, A_m$, $r \in A_1$, $|A_1| \geq 3$ and C be a non-trivial configuration with k pebble-free vertices in $G - A_1$. Let A_ℓ have the maximum number of unpebbled vertices in $G - A_1$ and k_ℓ denote the number of unpebbled vertices in A_ℓ. Let k be even, the number of pebbling moves in A_1 be $k+2$, and one even vertex $x \in A_1$. The mover has a winning strategy if and only if $C(x) \geq 2(k - k_\ell) + 2$.*

Proof. By induction on $k - k_\ell$.

Base: Let $k - k_\ell = 0$. Suppose $C(x) \geq 2$. If $k = 0$, then by Lemma 3.13 the mover has a winning strategy. So, suppose $k > 0$. The mover will pebble from a vertex in A_1 other than x, if one exists, to a pebble-free vertex in A_ℓ. If the defender pebbles to a pebbled vertex, then the mover can pebble to r

and win. If the defender pebbles to an unpebbled vertex in A_ℓ, then there is at least one pebbled vertex in A_ℓ with all neighbors $G - A_1$ pebbled. Then by Lemma 3.13 the mover has a winning strategy.

Conversely, suppose $C(x) = 0$. Then by Lemma 3.10, the defender has a winning strategy

Induction: Assume this is true for all $i < k - k_\ell$. First, suppose $C(x) \geq 2(k - k_\ell) + 2$. The mover will pebble from a vertex in A_1 not x, if one exists, to one of the pebble-free vertices in $G - A_1 - A_\ell$. The defender will pebble to any pebble-free vertex in $G - A_1$ (or lose). The resulting configuration C' is such that $C'(x) \geq 2(k - k_\ell)$, $C'_{A_1} = k$ and A_ℓ has at least $k_\ell - 1$ pebble-free vertices. So by induction, the mover has a winning strategy.

Conversely, suppose $C(x) \leq 2(k - k_\ell)$. The mover can either pebble to an unpebbled vertex or to a pebbled vertex of $G - A_1$. If the mover pebbles to an unpebbled vertex of $G - A_1$, then the defender will pebble from x to an unpebbled vertex in A_ℓ. The new configuration C' has $C'(x) \leq 2(k - k_\ell) - 2$ and there are at most $k_\ell - 1$ pebble-free vertices in A_ℓ. By induction, the defender has a winning strategy. If the mover pebbles to a pebbled vertex of $G - A_1$, then the defender will pebble to an unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover chooses, after two rounds the new configuration C'' has $C''(x) \leq 2(k - k_\ell) - 2$ and there are at most $k_\ell - 2$ unpebbled vertices in A_ℓ. By induction, the defender has a winning strategy

□

While exploring the case of complete multipartite graphs, we found a result for a related graph of diameter 2, where S is a disjoint union of cliques.

Lemma 3.21. *Let $G \in \mathcal{G}_{s,t}$ and $S = K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_\ell}$ and C be a non-trivial configuration with k pebble-free vertices in S. Let k be even, $C_T = k + 2$, and one even vertex $x \in T$. Let k^* be the number of pebble-free vertices in K_{m_j}, where K_{m_j} has the least number of unpebbled vertices in S The mover has a winning strategy if and only if $C(x) \geq k^* + 2$.*

Proof. By induction on k^*.

Base: The case when $k^* = 0$ is proven in a more general case by Lemma 3.13 and Lemma 3.14.

Induction: Assume this is true for all $i < k^*$. Let C be a configuration with k^* pebble-free vertices in K_{m_j}, where K_{m_j} has the minimum number of unpebbled vertices in S. First, suppose $C(x) \geq k^* + 2$. The mover will pebble from a vertex in T not x to one of the pebble-free vertices in K_{m_j}. The defender will pebble to any pebble-free vertex in S (or lose). The resulting configuration C' is such that $C'(x) \geq k^*$, $C_T = k$ and K_{m_j} has at least $k^* - 1$ pebble-free vertices. So by induction, the mover has a winning strategy.

Conversely, suppose $C(x) \leq k^*$. The mover can either pebble to an unpebbled vertex or to a pebbled vertex of S. If the mover pebbles to an unpebbled vertex of S, then the defender will pebble from x to an unpebbled vertex not in K_{m_j}. The new configuration C' has $C'(x) \leq k^* - 2$ and there are at most k^* pebble-free vertices in K_{m_j}. By induction, the defender has a winning strategy. If the mover pebbles to a pebbled vertex of S, then the defender will pebble to an unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C'' has $C''(x) \leq k^* - 2$ and there are at most k^* unpebbled vertices in K_{m_j}. By induction, the defender has a winning strategy

□

Lemma 3.20, along with Lemma 3.17, characterize the winning player for complete multipartite graphs. So, we have the following:

G is Complete Multipartite	$C(X)$	Winning Player
k_ℓ Pebble-Free Vertices in A_ℓ, Where A_ℓ has Minimum Number of Unpebbled Vertices in $G - A_1$	$C(x) \geq 2(k - k_\ell) + 2$	Mover
k_ℓ Pebble-Free Vertices in A_ℓ, Where A_ℓ has Minimum Number of Unpebbled Vertices in $G - A_1$	$C(x) \leq 2(k - k_\ell)$	Defender

Table 5: G Multipartite and its Winning Player

3.6 Determining $\eta(\mathcal{G}_{s,t}, r)$

Now we have the main result of the section which follows from the previous lemmas.

Theorem 3.22. *Let G in $\mathcal{G}_{s,t}$ and C be a configuration with k pebble-free vertices in S. Then we have the following:*

The mover has a winning strategy on G provided	The defender has a winning strategy on G provided
k is odd and $C_T \geq k+1$	k is odd and $C_T \leq k$
k is even and $C_T \geq k+3$	k is even and $C_T \leq k+1$
k is even and $C_T = k+2$ and all vertices in T are odd	k is even and $C_T = k+2$ and T has atleast one unpebbled vertex or two even vertices

And if k is even and $C_T = k+2$ and exactly one vertex in T is even, then the game is equivalent to the Element Selecting Game.

There is still one case we have not discussed yet: the case when T is a single vertex, because previous results allowed for a move back to T by the defender. Lemmas 3.23 and 3.24 are the base case of induction for Lemma 3.25, Lemma 3.26 and Lemma 3.27

Lemma 3.23. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$ If there exists a pebbled vertex $v \in S$ such that all of its neighbors in S are pebbled and $C(x) \geq 2$, then the mover has a winning strategy.*

Proof. The mover will pebble to v. The defender can not pebble back to x. So the defender can either pebble to a neighbor of v, which all have pebbles, or to r. In either case, the mover wins. □

Lemma 3.24. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. For every $v \in S$, suppose there exists at least one $u \in N_S[v]$ such that u is not pebbled. If $C(x) \leq 2$, then the defender has a winning strategy.*

Proof. If $C(x) < 2$, then there are no pebbling moves in T and the defender wins. If $C(x) = 2$, then the mover will pebble to some vertex $v \in S$. If v is unpebbled, then the defender wins. If v is pebbled, then there must exist an unpebbled neighbor by assumption. The defender will pebble to this vertex and win. □

Lemma 3.25. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. For every $v \in S$, suppose there exists at least one $u \in N_S[v]$ such that u is not pebbled and $S \neq N[v]$ for some v. Let k^* be the number of pebble-free vertices in $N[v^*]$ where $N[v^*] \in S$ has the minimum number of unpebbled vertices and $k \geq 2k^*$. Then the mover has a winning strategy if and only if $C(x) \geq 4k^* + 2$.*

Proof. By induction on k^*.

Base: Let $k^* = 0$. This is done by Lemmas 3.23 and 3.24.

Induction: Let k^* be even. First, suppose $C(x) \geq 4k^* + 2$. The mover will a pebble-free vertex of $N[v^*]$. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a pebble-free vertex in S, then for the new configuration C' we have $C'(x) \geq 4k^* - 2 = 4(k^* - 1) + 2$ and there are $k^* - 2$ unpebbled vertices in $N[v^*]$. By induction, the mover has a winning strategy.

Conversely, suppose $C(x) < 4k^* + 2$. The mover can pebble to any pebble-free vertex or place a second pebble on a vertex in S. If the mover pebbles to an unpebbled vertex in S, then the defender will pebble to an unpebbled vertex not in $N[v^*]$. The new configuration C' has $C'(x) < 4k^* - 2 = 4(k^* - 1) + 2$ and there are at most k^* unpebbled vertices in $N[v^*]$. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex, then the defender will pebble to its unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C'' has $C''(x) < 4k^* - 2 = 4(k^* - 1) + 2$ and there are at most k^* unpebbled vertices in $N[v^*]$. The defender wins by induction. □

Lemma 3.26. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. For every $v \in S$, suppose there exists at least one $u \in N_S[v]$ such that u is not pebbled and $S \neq N[v]$ for some v. Let k^* be the number of pebble-free vertices in $N[v^*]$ where $N[v^*] \in S$ has the minimum number of unpebbled vertices and $k < 2k^*$. Then the mover has a winning strategy if and only if $C(x) \geq 2k + 2$.*

Proof. By induction on k.

Base: Let $k = 0$. This is done by Lemmas 3.23 and 3.24.

Induction: Let k be even. First, suppose $C(x) \geq 2k + 2$. The mover will a pebble-free vertex of S. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a pebble-free vertex in S, then for the new configuration C' we have $C'(x) \geq 2k - 2$ and there are $k - 2$ unpebbled vertices in S. By induction, the mover has a winning strategy.

Conversely, suppose $C(x) < 2k + 2$. The mover can pebble to any pebble-free vertex or place a second pebble on a vertex in S. If the mover pebbles to an unpebbled vertex in S, then the defender will pebble to an unpebbled vertex not in S. The new configuration C' has $C'(x) < 2k - 2$ and there are at most $|S_0|$ unpebbled vertices in S. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex, then the defender will pebble to its unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C' has $C'(x) < 2k - 2$ and there are at most k unpebbled vertices in S. The defender wins by induction. □

Lemma 3.27. *Let $G \in \mathcal{G}_{s,t}$ and C be a nontrivial configuration with k pebble-free vertices in S and $T = \{x\}$. Suppose $S = N[v]$ for some v. Then the mover has a winning strategy if and only if $C(x) \geq 2k + 2$.*

Proof. By induction on k.

Base: Let $k = 0$. This is done by Lemmas 3.23 and 3.24.

Induction: Let k be even. First, suppose $C(x) \geq 2k + 2$. The mover will a pebble-free vertex of S. If the defender places a second pebble on a vertex in S, then the mover wins. If the defender pebbles to a pebble-free vertex in S, then for the new configuration C' we have $C'(x) \geq 2k - 2 = 2(k - 2) + 2$ and there are $k - 2$ unpebbled vertices in S. By induction, the mover has a winning strategy.

Conversely, suppose $C(x) < 2k + 2$. The mover can pebble to any pebble-free vertex or place a second pebble on a vertex in S. If the mover pebbles to an unpebbled vertex in S, then the defender will pebble to an unpebbled vertex in S. The new configuration C' has $C'(x) < 2k - 2 = 2(k - 2) + 2$ and there are $k - 2$ unpebbled vertices in $N[v]$. By induction, the defender has a winning strategy. If the mover places a second pebble on a vertex, then the defender will pebble to its unpebbled neighbor. Now the mover has the same two options and the defender has the same two responses. No matter which one the mover choose, after two rounds the new configuration C'' has $C''(x) < 2k - 2 = 4(k - 2) + 2$ and there are at most k unpebbled vertices in S. The defender wins by induction. □

Obtaining the winning configurations for the mover allow us to get $\eta(G, r)$ for $G \in \mathcal{G}_{s,t}$.

Theorem 3.28. *If $G \in \mathcal{G}_{s,t}$, then* $\eta(G, r) = \begin{cases} t + 2s + 4, & s \text{ is even} \\ t + 2s + 3, & s \text{ is odd.} \end{cases}$

Proof. Case 1: Let s be even. A configuration of $t + 2s + 3$ pebbles on the vertices of G which gives the defender a winning strategy is the following: in T, leave one vertex pebble-free, put one pebble on $t - 2$ vertices and the remaining $2s + 5$ pebbles on one vertex and keep S pebble-free. With this configuration, $C_T = s+2$ with one vertex in T having no pebbles on it. By Lemma 3.10, the defender wins.

Now suppose there are $m \geq t+2s+4$ pebbles on the vertices in G. Let k of the vertices in S be pebble-free. Thus there are $(s-k)$ pebbles in S. Now there are $m-(s-k) \geq t+2s+4-s+k = t+s+k+4$ pebbles on the vertices in T. To show the mover has a winning strategy, we show any configuration of the remaining pebbles on T, C_T and the configuration satisfies the condition of one of the previous lemmas.

If all of the vertices in T are pebbled, then at most t pebbles can be placed on the vertices and $C_T = 0$. There are $s+k+4$ pebbles left to arrange. First, let k be even. Then no matter how the rest are arranged, $C_T \geq \frac{s+k}{2} + 2 \geq k+2$. If there are all distributed evenly, then all vertices have an odd number of pebbles on them. So the mover wins. If they are not distributed evenly, then $C_T \geq k+3$. So the mover has a winning strategy by Lemma 3.22. Now let k be odd. No matter how the $s+k+4$ pebbles are broken up, $C_T \geq \frac{s+k}{2} + 2 \geq k+2$. Since k is odd, the mover has a winning strategy by Theorem 3.22.

Now suppose not all of the vertices of T have pebbles on them. Let ℓ of the vertices in T be pebble-free. Then at most $t-\ell$ pebbles can be placed on T so $C_T = 0$. There are $s+k+\ell+4$ pebbles left. Let k be even. If the pebbles are broken up in piles of even numbers, then $C_T = \frac{s+k}{2} + \frac{\ell}{2} + 2 \geq k+2$. The mover wins. If the pebbles are broken up with some odd piles, then $C_T \geq k+3$ and the mover wins. Now let k be odd. No matter how the pebbles are arranged, $C_T \geq \frac{s+k}{2}\frac{\ell}{2} + 2 \geq k+2$. Since k is odd, the mover has a winning strategy.

Case 2: Let s be odd. The configuration of $t+2s+2$ pebbles on the vertices of G which give the defender a winning strategy is the following: place 1 pebble on any vertex in S, place 1 pebble on $t-1$ vertices and the remaining $2s+1$ pebbles on one vertex. With this configuration, $C_T = s$ and there are $s-1$ pebble-free vertices in S, with $s-1$ even. By Lemma 3.8, the defender has a winning strategy.

A similar argument holds from above for $m \geq t+2s+3$ pebbles on the vertices of G. □

3.7 Complete Bipartite & Complete Multipartite Graphs

Now we get η for complete bipartite and multipartite graphs. We notice that complete bipartite graphs and complete multipartite graphs fall into this class of graphs, with the root in one partite set begin equivalent to $T \cup r$. Since $K_{u,v} \in \mathcal{G}_{s,t}$ with partite sets U and V, we have $u = s$ and $v = t+1$ if $r \in V$ or $u = t+1$ and $v = s$ if $r \in U$

Corollary 3.29. *Let $3 \leq u \leq v$. Then* $\eta(K_{u,v}) = \begin{cases} v+2u+3, & u \textit{ is even} \\ v+2u+2, & u \textit{ is odd.} \end{cases}$

Proof. We need to check which placement of the root yields a larger configuration to be r-solvable.

Let $u = v+i$.

If $r \in V$, then by Theorem 3.28, $\eta(K_{v+i,v}, r) = \begin{cases} v+2v+2i+3, & v+i \text{ is even} \\ v+2v+2i+2, & v+i \text{ is odd.} \end{cases}$

If $r \in U$, then by Theorem 3.28, $\eta(K_{v,v+i}, r) = \begin{cases} v+i+2v+3, & v \text{ is even} \\ v+i+2v+2, & v \text{ is odd.} \end{cases}$

We can see for every value of $i \geq 0$, the maximum configurations will be when $r \in V$.

□

The following result is for when at least one of the partite sets contains exactly two vertices.

Theorem 3.30. *If $u = 2$, then $\eta(K_{2,v}) = v+7$*

Proof. If $r \in U$, then Lemma 3.25 says we need at least 6 pebbles in U with no pebble in V so the mover has a winning strategy. By the Pigeonhole Principle, we need $v+1$ pebbles in V and no pebbles in U for the mover to have a winning strategy. So we need a total of $\max v+1, 6$ pebbles for the mover to have a winning strategy. If $r \in v$, then Theorem 3.28 says we need $v-1+2u+4 = v-1+4+4 = v+7$ pebbles for the mover to have a winning strategy. □

The final result for complete bipartite graphs is when one partite is a single vertex.

Corollary 3.31. *Let $v \geq 3$. If $u = 1$, then $\eta(K_{1,v}) = v + 4$*

Proof. If $U = \{r\}$, then by the Pigeonhole Principle the mover has a winning strategy with $v + 1$ pebbles. If $r \in V$, then Theorem 3.28 says $v - 1 + 2(1) + 3 = v + 4$ pebbles gives the mover a winning strategy. □

Here we find the Two-Player Pebbling Numbers for complete multipartite graphs.

Corollary 3.32. *If $3 \leq a_1 \leq a_2 \leq \cdots \leq a_m < n$ and $\sum_{i=1}^{m} a_i = n$, then*

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_1 + 3, & \sum_{i=2}^{m} a_i \text{ is even} \\ 2n - a_1 + 2, & \sum_{i=2}^{m} a_i \text{ is odd.} \end{cases}$$

Proof. If $r \in A_k$ for $k \neq 1$, then by Theorem 3.28,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = \begin{cases} a_k + 2\sum_{i\neq k} a_i + 3, & \sum_{i\neq k} a_i \text{ is even} \\ a_k + 2\sum_{i\neq k} a_i + 2, & \sum_{i\neq k} a_i \text{ is odd.} \end{cases}$$

Hence, in this case we have the following:

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_k + 3, & \sum_{i\neq k} a_i \text{ is even} \\ 2n - a_k + 2, & \sum_{i\neq k} a_i \text{ is odd.} \end{cases}$$

If $r \in A_1$, then by Theorem 3.28,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = \begin{cases} a_1 + 2\sum_{i=2}^{n} a_i + 3, & \sum_{i=2}^{n} a_i \text{ is even} \\ a_k + 2\sum_{i=2}^{n} a_i + 2, & \sum_{i=2}^{n} a_i \text{ is odd.} \end{cases}$$

So, in this case we have

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_1 + 3, & \sum_{i=2}^{m} a_i \text{ is even} \\ 2n - a_1 + 2, & \sum_{i=2}^{m} a_i \text{ is odd.} \end{cases}$$

Since $a_1 \leq a_k$ for all $k \geq 2$,

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 2n - a_1 + 3, & \sum_{i=2}^{m} a_i \text{ is even} \\ 2n - a_1 + 2, & \sum_{i=2}^{m} a_i \text{ is odd.} \end{cases}$$

□

Corollary 3.33. *If $2 = a_1 \leq a_2 \leq \cdots \leq a_m < n$ and $\sum_{i=1}^{m} a_i = n$, then*

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 4n - 4a_m - 3a_1, & a_m \geq \sum_{i=2}^{m-1} a_i \\ 2n - a_1, & a_m < \sum_{i=2}^{m-1} a_i. \end{cases}$$

Proof. If $r \in A_k$ for $a_k \geq 3$, then by Theorem 3.28,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = \begin{cases} a_k + 2\sum_{i \neq k} a_i + 3, & \sum_{i \neq k} a_i \text{ is even} \\ a_k + 2\sum_{i \neq k} a_i + 2, & \sum_{i \neq k} a_i \text{ is odd.} \end{cases}$$

If $r \in A_1$ and $a_m \geq \sum_{i=2}^{m-1} a_i$, then by Lemma 3.25,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = 4\sum_{i=2}^{m-1} a_i + 2$$

.

If $r \in A_1$ and $a_m < \sum_{i=2}^{m-1} a_i$, then by Lemma 3.26,

$$\eta(K_{a1,a_2,\ldots,a_m}, r) = 2\sum_{i=2}^{m} a_i + 2$$

□

Corollary 3.34. *If $1 = a_1 \leq a_2 \leq \cdots \leq a_m < n$ with a_k the size of the smallest partite set not equal to 1 and $\sum_{i=1}^{m} a_i = n$, then*

$$\eta(K_{a_1,a_2,\ldots,a_m}) = \begin{cases} 4n - 4a_m - 3a_1, & a_k = 2 \textit{ and } a_m \geq \sum_{i=2}^{m-1} a_i \\ 2n - a_1, & a_k = 2 \textit{ and } a_m < \sum_{i=2}^{m-1} a_i \\ 2n - a_k + 3, & a_k > 2 \textit{ and } \sum_{i \neq k} a_i \textit{ is even} \\ 2n - a_k + 2, & a_k > 2 \textit{ and } \sum_{i \neq k} a_i \textit{ is odd.} \end{cases}$$

Proof. If $r \in A_1$, then by the Pigeonhole Principle the mover has a winging strategy with $\sum_{i \neq 1} a_i + 1$ pebbles.

If $r \in A_k$ where a_k the size of the smallest partite set not equal to 1, then see Corollary 3.32 and 3.33.

□

4 Conclusion

We saw that we could find η for complete bipartite and complete multipartite graphs. We are currently working on finding other classes of graphs for which the mover has a winning strategy. One class of graphs we have worked on extensively is paths and variations of paths. We know that the mover has a winning strategy on paths and only have partial results to this point. We would also like to try and completely characterize the winning configurations for the special cases of diameter 2 graphs, specifically those cases relating to the Element Selecting Game. We believe it will be easier to obtain results for the Element Selecting Game and then translate them to Two-Player Pebbling. More results will follow [8].

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