Voting Systems

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1 Introduction

Voting systems are an interesting concept in game theory that were not covered in class, but we are still able to analyze them using techniques that we have learned in class. Fundamentally, the job of a voting system is to take the individual opinions or preferences of a set of voters about a group of candidates, and output a winning candidate. An obvious implementation of voting systems is electing candidates in a democracy, but they can be used in many other domains as well. For example, our candidates could movies nominated for an Oscar, or athletes nominated for an MVP award.

This paper will analyze the game theoretic concepts underpinning different types of mechanisms for determining the outcome of an election, and explain the strengths and weaknesses of each system. We will use several different criteria that will allow us to analyze the properties of a given system, define what a good social welfare outcome looks like, and examine how often a system gives us a good outcome. To help analyze systems, we will also use Python models to simulate different outcomes and see which systems perform well. Lastly, we propose some randomized mechanisms to demonstrate how we can encourage voters to truthfully report their preferences.

2 Broad Criteria for Evaluating a System

We will examine voting systems mostly based of of 2 broad criteria: 1) how happy is the group with the outcomes the system produces and 2) assurances the system gives based on what properties it has. When we investigate the properties of a system, we are looking for assurances that might tell us that a system will give us a certain type of outcome, or that voters are free to act in a certain way without being punished (i.e. getting a less preferred outcome). There is no strong consensus on how to best measure either of these criteria, which makes analyzing voting systems especially challenging. As we will see, no system can give everything that we are looking for, so there is no obvious answer to what system we will use. Our hope then is not that we will be able to offer a system that is optimal, but rather that we will be able to make clear the benefits and drawbacks of choosing a certain system.

In all of the following sections, we denote the number of voters as n and the number of candidates as m.

3 Ranking Systems

Ranking systems are the standard type of voting system. In a ranked system, each voter i's action profile consists of a ballot b_i that specifies an ordinal ranking of the candidates. The ballots are then collected, and one candidate is chosen as the winner. Note that the agents (voters) do not know how other agents act when they submit their ballot.

3.1 Gibbard-Satterthwaite Theorem

Before examining these systems further, it is important to recognize one of the most immediate constraints with ranked systems. The Gibbard-Satterthwaite Theorem[1] says that any ranked voting system with more than 2 possible outcomes (winners) that is not dictatorial is susceptible to **tactical voting**.

A dictatorial system is a system where 1 voter has the power to unilaterally determine the winning candidate. For instance, if one ballot was arbitrarily picked from all ballots, and the most preferred candidate of that ballot was the winner, then this system would be a dictatorship.

By susceptible to tactical voting, we mean that it is not a dominant strategy for voters to report their preferences truthfully. That is, a voter may be able to get a more preferred winning candidate by reporting false preferences. So this theorem establishes that no ranking system has the property of **truthfulness**. We will explore the issue of tactical voting more while examining some specific ranked systems in depth.

3.2 Condorcet Winner

As stated earlier, there is no clear way to define the best outcome of a ranked vote. One intuitive and historical method to determine who should be the winner is the **Condorcet winner**[2]. A Condorcet winner is a candidate that would beat every other candidate in a head to head matchup. That is, candidate A is a Condorcet winner if for A and for any other candidate X, A is more preferred than X by a majority[3] of the voters. We also define a **Condorcet loser** to be a candidate that loses to all other candidates in a head-to-head matchup. A ranked voting system is considered **Condorcet consistent** if for any possible

[1] Gibbard, Allan (1973), Satterthwaite, Mark Allen (April 1975)
[2] de Condorcet, Marquis (1785)
[3] if we allow for ties then the candidate is a weak Condorcet winner

set of ballots, it outputs the Condorcet winner, if one exists. Note that a Condorcet winner does not necessarily exist. Consider the following two examples, each with 3 voters and 3 candidates, A, B, and C:

Example 1:

$$b_1 : A > B > C$$
$$b_2 : C > A > B$$
$$b_3 : B > A > C$$

In this example, the head-to-head scores are: $A - B$: 2-1, $A - C$: 2-1, and $B - C$: 2-1. A wins both of its matchups 2-1, making it the Condorcet winner.

Example 2:

$$b_1 : A > B > C$$
$$b_2 : C > A > B$$
$$b_3 : B > C > A$$

In this example, the the only ballot that was changed was b_3, where the rankings of A and C were swapped. The head-to-head scores are: $A-B$: 2-1, $A-C$: 1-2, and $B-C$: 2-1. No candidate wins both of its head-to-head matchups, meaning that there is no Condorcet winner. In order for no Condorcet winner to exist, the collective preferences of all the voters must be cyclical. In this example, collectively $A > B$, $B > C$, and $C > A$. The idea that the collective preference can be cyclical despite individual preferences being non-cyclical is known as **Condorcet's Paradox**. In some simplified models, such a situation cannot occur. For example, if voters and candidates both lie on a one-dimensional line, and voters prefer candidates based on proximity on that line, then a Condorcet winner will always exist[4]. In fact, a candidate located at the "median voter" on this line is guaranteed to be a Condorcet winner.

3.3 Monotonicity and the No Show Paradox

A system is said to **monotonic** if it can never hurt a candidate to receive unilaterally more support, or help a candidate to receive unilaterally less support. We can formalize this for ranked systems as follows:

> - If candidate A is the winning candidate and a subset of voters move A up in their rankings, while leaving their preferences between all non A candidates the same, A is always still the winning candidate.
>
> - If candidate A is a non winning candidate and a subset of voters move A down in their rankings, while leaving their preferences between all non A candidates the same, A is always still a non winning candidate.

[4]Black, Duncan (1958)

A related concept is the **no show paradox**. The no show paradox is where a subset of voters would have been given a more preferred winning candidate had they not voted at all (i.e. they are ignored by the system). A voting system is susceptible to the no show paradox if it is possible that voters could be victims of the no show paradox under the given system.

3.4 Plurality Rule

The plurality rule is a simple and well established system which selects the candidate that has the plurality of first place votes. The downside of its simplicity is that the plurality rule is not expressive of rankings, as it only considers each voters top choice. This can lead to a candidate getting selected with a small plurality of voters, even if a vast majority of voters strongly dislike the candidate. More formally, the plurality rule is not Condorcet consistent. In fact, the plurality rule can even select a Condorcet loser, as the following example with 7 voters and 3 candidates shows:

Number of voters	Ballot
3	$A > B > C$
2	$B > C > A$
2	$C > B > A$

In this example, the plurality rule would select A as the winner because it has the most first-place votes. However, A loses it's head-to head matchups with B and C both by a score of 4-3, making it a Condorcet loser.

3.5 Plurality with Runoff

Plurality with runoff is a system where the two candidates that get the most first place votes advance to a head to head runoff. The winner of the runoff is then output as the winning candidate. Plurality with runoff attempts to correct the case of a plurality rule where an unpopular candidate wins with a small plurality. Note that plurality with runoff cannot select a Condorcet loser, since the winner of the runoff must beat the loser of the runoff in a head to head matchup. However, plurality with runoff is not Condorcet consistent. Consider the following example with 8 voters and 3 candidates:

Number of voters	Ballot
3	$A > C > B$
2	$B > A > C$
3	$C > B > A$

In this example, A and C tie for the most first-place votes, and advance to a runoff. The head-to-head scores are: $A - B$: 3-5, $A - C$: 5-3, and $B - C$: 5-3. Note that B is the Condorcet winner, but is eliminated before the runoff. Since

A beats C in the runoff, A is elected the winner. Since the Condorcet winner was not selected, this method is not Condorcet consistent.

In addition, plurality with runoff is not monotonic, and is susceptible to the no show paradox. The following example with 11 voters and 3 candidates demonstrates how some voters may be better off by not participating:

Number of voters	Ballot
3	$A > C > B$
4	$B > A > C$
4	$C > A > B$

In this example, B and C advance to a runoff where C is elected as the winner, since it beats B 7-4 in a head-to-head matchup. Now, consider what would happen if 2 of the voters with preferences $B > A > C$ do not participate:

Number of voters	Ballot
3	$A > C > B$
2 $\cancel{4}$	$B > A > C$
4	$C > A > B$

Now, candidates A and C advance to a runoff, where candidate A wins with a head-to-head score of 5-4. By not participating, the 2 voters with preferences $B > A > C$ receive a more preferred outcome (A) than the outcome had they participated (C).

3.6 Instant Runoff Voting

Note that plurality with runoff only takes into account first place votes when determining which 2 candidates make the final runoff. Instant runoff voting (IRV) attempts to do better by taking into account the full range of ranking to determine who makes the runoff. IRV works as follows:
The candidate who has the fewest first place votes is eliminated, and their supporters' votes are redistributed to the next highest remaining candidate on their ballot. This continues until we have 1 candidate left, who is the winner (once we reach 2 candidates this is the same as a head to head runoff). Note that if we have 3 candidates, this is the same as plurality with runoff.

This version of IRV is called the Hare rule. There is also a version called Coombs rule that eliminates the candidate with the most last place votes instead of the candidate with the fewest first place votes. Unless specified, when we say IRV, we are referring to the Hare rule.

Instant runoff voting is becoming more popular in the US and around the world. One of the reasons for this is that proponents claim that IRV allows voters to rank their preferences more truthfully, and not have to worry about voting strategically. While strategic voting in IRV is unlikely to occur in practice, as with all ranked systems, IRV is still not theoretically truthful. In practice, IRV is successful at addressing the problem known as the spoiler effect. The spoiler effect is when a spoiler candidate with little chance of winning draws votes away from a main candidate with a similar ideology. Since the candidates have similar ideologies, it is assumed that people who voted for the spoiler candidate would have otherwise voted for the main candidate, possibly allowing them to win. The following example with 4 candidates shows how IRV can successfully address the spoiler effect:

Number of voters	Ballot
21	$A > C > B > D$
10	$B > A > C > D$
10	$B > C > A > D$
19	$C > D > A > B$
3	$D > C > A > B$

In this case candidate D, is the spoiler candidate, taking votes away from candidate C. If we use plurality or plurality with runoff, C loses, however when we use IRV we get the following:

Round 1: A=21, B=20, C=19, D=3 — D is eliminated
Round 2: C=22, A=21, B=20 — B is eliminated
Round 3: C=32, A=31 — A is eliminated
So using IRV, C wins.

Even though IRV can help address the spoiler effect, as we have said, IRV is still susceptible to tactical voting. The following example using the Hare Rule shows how a voter can vote tactically in order to get a more preferred winner:

Number of voters	Ballot
4	$A > C > B$
4	$B > C > A$
3	$C > A > B$

Round 1: A=4, B=4, C=3 — C is eliminated
Round 2: A=7, B=4 — B is eliminated
A wins.

Now, if one of the voters that ranked B first decides to rank C first instead, the result is:

Number of voters	Ballot
4	$A > C > B$
3 $\cancel{4}$	$B > C > A$
1	$C > B > A$
3	$C > A > B$

Round 1: A=4, B=3, C=4	B is eliminated
Round 2: C=7, A=4	A is eliminated

C wins.

By switching the order of her preferences, the voter was able to receive a more preferred outcome (C) than the result had she reported her true preferences (A).

In addition, IRV is not Condorcet consistent. The 3 candidate example that we used to show that plurality with runoff was not Condorcet consistent works again here (since with 3 candidates the systems are the same). IRV is also not monotonic and is susceptible to the no show paradox. For intuition, suppose with 3 candidates A will easily make the final runoff, but will lose to B and beat C. If B and C are basically tied in first place votes, A might be better off losing some first place votes to C in order to ensure that C makes the runoff instead of B. Similarly, the 3 candidate example we used to show that plurality with runoff was susceptible to the no show paradox works to show that IRV is susceptible to the no show paradox. So while IRV may allow voters to be more truthful in practice, in theory it does not offer us any guarantees.

3.7 Scoring systems

As opposed to non-scoring systems, scoring systems give a numerical value to each candidate based on their ranking in ballots. The **Borda count** is a well-known scoring system that was developed around the same time as the Condorcet winner[5]. The original formula for calculating a candidate's Borda count is to give $m - k + 1$ points for each ballot that ranks that candidate at position k. The candidate with the highest Borda count is chosen as the winner. It is useful to note that strictly monotonic sum-scoring systems are not Condorcent consistent[6]. That is, if the system assigns greater scores to higher rankings, and a candidate's total score is the sum of all scores for that candidate, the system is not Condorcet consistent. The Borda count falls into this criterion, and is therefore not Condorcet consistent. Consider the following example with 16 voters:

Number of voters	Ballot
10	$A > B > C$
6	$B > C > A$

[5] de Borda, Jean Charles (1770)
[6] Fishburn, PC (1984)

The Condorcet winner in this example is A. A wins against both B and C with a score of 10-6. The Borda scores for the example above are:

$$score(A) = 10(3) + 0(2) + 6(1) = 36$$
$$score(B) = 6(3) + 10(2) + 0(1) = 38$$
$$score(C) = 0(3) + 6(2) + 10(1) = 22$$

B has the largest Borda score in this example and would be chosen as the winner, even though A is the Condorcet winner.

Exponential Scoring is another scoring system, where scores increase exponentially as rankings go up. If the score is 2^{m-r+1}, where r is a candidate's ranking, the scores in the example above are:

$$score(A) = 10(8) + 0(4) + 6(2) = 92$$
$$score(B) = 6(8) + 10(4) + 0(2) = 88$$
$$score(C) = 0(8) + 6(4) + 10(2) = 44$$

Exponential scoring is still a monotonically increasing sum-scoring function, so it is not Condorcet consistent. But, in this example, it selects A, which is the Condorcet winner, while the Borda count selects B.

3.8 Condorcet Systems

No system we have seen thus far has been Condorcet consistent. A **Condorcet system** is a voting system that always selects a Condorcet winner if once exists (the system can find this winner simply by computing head to head scores). There are many different systems for what to do if a Condorcet winner doesn't exist. A few examples are Black's, which chooses the Borda winner if there is no Condorcet winner, and Copeland's, which chooses the candidate with the best overall head to head record.

A **Condorcet component** is a group of ballots that express a cycle of preferences over candidates, with votes evenly distributed across each ballot ordering. For example, in a system with four candidates, A, B, C, and D, an example of a Condorcet component would be a group of 4 ballots, with 1 each of the forms $A > B > C > D$, $D > A > B > C$, $C > D > A > B$, and $B > C > D > A$. It is reasonable to consider a Condorcet component meaningless with respect to the collective preferences, and throw out all of those ballots, since the preferences cancel each other out. A system is said to **cancel properly** if for any set of ballots V, the system always selects the same winner if a Condorcet component is added to V. For example, a system that cancels properly should select the same winner from the two sets of ballots below, since the Condorcet component described above was added to the first set of ballots:

Number of voters	Ballot
2	$A > B > C > D$
3	$B > A > C > D$

Number of voters	Ballot
3	$A > B > C > D$
1	$D > A > B > C$
1	$C > D > A > B$
1	$B > C > D > A$
3	$B > A > C > D$

A flaw in Condorcet systems is that is had been shown that no Condorcet system cancels properly[7]. In the example above, B is the Condorcet winner in the first set of ballots. After adding in the Condorcet component, the new Condorcet winner is A, which beats B with a score of $5-4$.

Another flaw in Condorcet systems is that any Condorcet system with at least 4 candidates is susceptible to the no-show paradox[8]. For some scholars, these flaws call into question the importance of Condorcet consistency.

4 Grading Systems

Grading systems are a generalized type of voting system, where voters assign some sort of grade to each candidate. Ranked systems are a subset of grading systems where a voter's grades give us their ordinal ranking of candidates. Non-ranking grading systems often require more thought from voters, but can also potentially be more expressive of voters' preferences.

k-Approval Voting is a system where voters select a subset of candidates of size $k < m$ of which they approve. The candidate in the largest number of subsets is the winner.

Cumulative Voting is a system where voters have a fixed number of points that they can allocate between candidates any way they wish. The candidate with the most total points wins.

Range Voting (or Score Voting) is where voters give candidates a grade from a finite set of numbers, and the candidate with the largest average grade is elected.

Majority Judgement is a recently developed system, where voters assign each candidate an integer grade (on a scale from 1-4) and then the candidate

[7] Balinski, Michel and Laraki, Rida (2010)
[8] Moulin, Herve (1988)

with the highest median grade wins[9].

These systems all have advantages and disadvantages, which we will not describe in depth for the sake of brevity. What is important to understand about them is, that like ranked systems, none of these systems are immune from tactical voting if: (a) there are more than two candidates, (b) the system is deterministic, and (c) the system is not dictatorial[10]. This result was proven by Gibbard's theorem, a generalization of the Gibbard-Satterthwaite Theorem. Still, some grading systems, like the Majority Judgement, claim to significantly reduce the incentives of tactical voting.

5 Simulation

While we can see how these systems work in theory, it's important to get empirical data on how they work in practice. While we don't have a human population to test on, we can do our best to simulate humans with randomness injected into the right places.

5.1 Methodology

For our simulation, we built a simple Python framework and used Matplotlib to graph the results of our simulations. For each trial, we generate a variable length list of voters that each have preferences assigned to 4 distinct candidates. Preferences to candidates are integer scores randomly generated on the interval [0...10]. An example voter may look as follows:

```
{
  'name': 1,
  'prefs': {'a': 0, 'b': 4, 'c': 1, 'd': 7}
}
```

In our model, the higher score the better i.e. voter 1 prefers d to a in this example.
We then define functions to model the following voting systems:

- Plurality
- Borda
- Random Scoring[11]
- Exponential Scoring

[9] Balinski, Michel and Laraki, Rida (2010)
[10] Gibbard, Allan (1973)
[11] Scoring rule where scores for each ranking are randomly generated integers in the range [1...m] and sorted in decreasing order. Similar to Borda count, but adjacent rankings may have the same score.

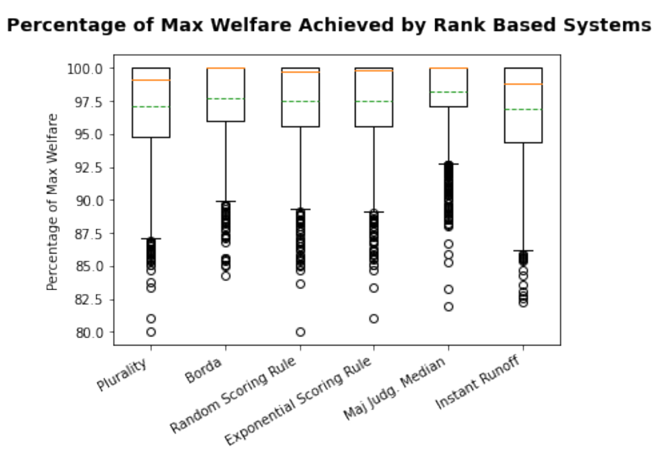
- Majority Judgement Median
- Instant Runoff (Hare's rule)

These functions take in a list of voters and candidates and run their respective algorithms on the voters' preferences over candidates. We then run each of these systems for 1000 iterations with varying population sizes to get our final data. Welfare is evaluated by taking the social welfare that a system gives (based on the winning candidate) and comparing it to the max welfare achievable over all choices of winner (i.e. the candidate c that maximizes the sum of v.prefs[c] for all v in the voter list).

We also looked at how reporting untruthful preferences might impact the outcomes generated by the various systems. We ran simulations that, given an outcome by a system and a list of voters, re-randomized the preferences of voters whose favorite candidate did *not* win. We had a parameter that tuned what percentage of the losing voters are re-randomized (as opposed to being held constant). We also tuned a parameter that decreased the rating for the winning candidate in the re-randomized untruthful preferences, so as to increase the likelihood of altering the outcome.

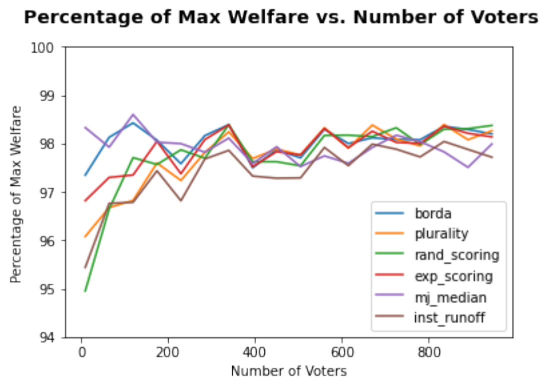
5.2 Results & Analysis

First, we plot the total welfare of our population vs the optimal total welfare of the population:

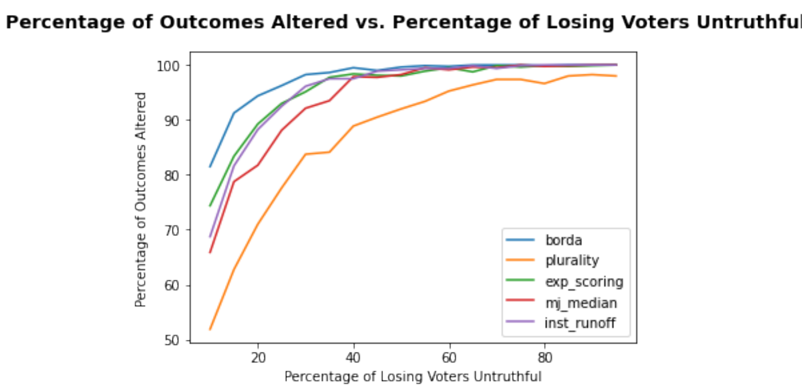


As we can see, while instant runoff is a widely lauded voting method, it doesn't perform as well as we would hope in comparison to our welfare function. In particular, it seems to have have the most consistently low results. However, we see great things from Majority Judgement Median as well as exponential scoring. These both show good clustering at close to 100% welfare and rarely dip below 85% welfare.

We were also curious as to how different systems performed at different scales of voters:



As we can see, these algorithms perform relatively consistently regardless of the input size. While there is some variation in the very low numbers, that can probably be attributed the the noise associated with small sample sizes. Once we get into large enough sample sizes, things stabilize, and, as before, we can see instant runoff consistently under-performing vs the other algorithms.
Finally, we analyzed how frequently outcomes were altered as a result of tactical untruthful voting as described above:



As we can see, all of these systems are highly susceptible to large-scale tactically untruthful voting.

We must recognize that this is all calculated as a posterior. Hence, this type of coordination among voters, especially of such scale, would be quite difficult to achieve. Also, an altered outcome does not guarantee that the new outcome is better for the majority of the voters. Yet, we still see a certain degree of susceptibility when less than 20% of losing voters are altering their preferences. An example of a scenario in which this type of untruthful voting could succeed is when a small but fervent minority who collude to harm a fairly popular candidate. This group could work to sink the popular candidate in each of their preferences and would likely end up with a different winner. This type of counterfactual are critical for voting systems to address.

6 Randomized Truthfulness

So far, we have struggled to find a system that ensures that it is a dominant strategy for voters to truthfully report their preferences. For intuition, this is largely because a voter may want to change their preferences depending on what other voters do. So we seek to find a system such that a voter does not care about the actions of other voters, and thus can act truthfully. Lets see if randomization can give us such a system. Zeckhauser showed in his paper, *Voting Systems, Honest Preferences and Pareto Optimality* (1973), how randomization can deliver truthfulness for a 1 voter, 3 candidate system, where the voter reports their utilities for given candidates to the system. We will attempt to extend his result to a system for an arbitrary number of voters and candidates.

6.1 The System

Consider a system with n candidates and m voters, where voter v receives a certain utility based on which candidate wins. Let the utility that voter v receives if their most preferred candidate wins be 1, and the utility they receive if their least preferred candidate wins be 0. If voters have utility valuations on a different scale we can just normalize them from 0 to 1. For voter v, let $u_{v,1}$ be the utility that v gets if their most preferred candidate wins, $u_{v,2}$ be the utility if their second most preferred candidate wins, $u_{v,3}$ for their third most preferred, and so on. So $u_{v,1} = 1$ and $u_{v,n} = 0$.

As an input to our system we will ask voters to report utilities for each of the candidates, with their highest reported utility being 1 and their lowest reported utility being 0. Once again if reported utilities are on a different scale we can just normalize them from 0 to 1. Also assume that all reported utilities are unique. If they aren't we can choose one of the tied reported utilities at random and add an arbitrarily small epsilon ϵ to break the tie. For voter v, let $r_{v,1}$ be the highest reported utility, $r_{v,2}$ be the second highest, $r_{v,3}$ the third highest, and so on. So $r_{v,1} = 1$ and $r_{v,n} = 0$. Let r_v represent a given set of reported utilities for voter v.

Our system will convert the reported utilities for each candidate into a total weight for that candidate, and the probability of a candidate winning will be proportional to their total weight. For candidate i, let W_i be their total weight. So then we have:

$\Pr[\text{Candidate } i \text{ wins}] = \frac{W_i}{\sum_{j=1}^{n} W_j}$

We define a class of functions $f_1,\ f_2,\ ...,\ f_n$ for voter v, that each take r_v as their input. Voter v will contribute weight of $f_1(r_v)$ to their highest reported candidate, weight of $f_2(r_v)$ to their second highest reported candidate, weight of $f_3(r_v)$ to their third highest reported candidate, and so on.

We would like for our system to have the following properties:
- the sum of the weights that each voter contributes to all the candidates is 1
- the amount that voter v contributes to each candidate to be sorted in the same order as their reported preferences
- voters cannot contribute negative weights to candidates
- the amount that voter v contributes to candidate i is never decreasing with respect to $r_{v,i}$

So we want to define this class of functions such that:

TOTAL SUM PROPERTY: for any r_v,
$\sum_{i=1}^{n} f_i(r_v) = 1$

DECREASING ORDER PROPERTY: for any r_v,
$f_1(r_v) > f_2(r_v) > f_3(r_v) > ... > f_{n-1}(r_v) > f_n(r_v)$

NON NEGATIVITY PROPERTY: for all i from 1 to n and for any r_v,
$f_i(r_v) \geq 0$

NON DECREASING PROPERTY: for all i from 1 to n, $f_i(r_v)$ is never decreasing with respect to $r_{v,i}$

Note that since we have the Total Sum Property that: $\sum_{v=1}^{m} \sum_{i=1}^{n} f_i(r_v) = m$. That is the total weight of all candidates from all voters is m. So now:

Pr[Candidate i wins] $= \frac{W_i}{m}$

Voter v will submit preferences to maximize their expected utility. Assume that v submits reported utilities in the same order as true utilities[12]. Let X_i be the total weight that v's ith highest reported candidate receives from all voters OTHER than v. Then the expected utility for voter v is:

E[Utility] $=\sum_{i=1}^{n} u_{v,i} \frac{W_i}{m} = \sum_{i=1}^{n} u_{v,i} \frac{X_i + f_i(r_v)}{m} = \frac{1}{m} \sum_{i=1}^{n} u_{v,i} \big[X_i + f_i(r_v)\big]$

$= \frac{1}{m} \big[\sum_{i=1}^{n} u_{v,i} * X_i + \sum_{i=1}^{n} u_{v,i} * f_i(r_v) \big] = \frac{1}{m} \sum_{i=1}^{n} u_{v,i} * X_i + \frac{1}{m} \sum_{i=1}^{n} u_{v,i} * f_i(r_v)$

Note that in the final expression, $\frac{1}{m}$ and the first sum do not depend on the reported utilities of voter v. Therefore voter v reports utilities to do the following:

[12] we will show later why v must do this to maximize utility

$max \quad \sum_{i=1}^{n} u_{v,i} * f_i(r_v)$

and since $u_{v,1} = 1$ and $u_{v,n} = 0$ we have:

$max \quad f_1(r_v) + \sum_{i=2}^{n-1} u_{v,i} * f_i(r_v)$ call this expression "M"

If voter v did NOT report utilities in the same order as true utilities, then $u_{v,i}$ and $f_i(r_v)$ might not match up for each term. For example one term of the sum could be: $u_{v,1} * f_3(r_v)$. Due to the Decreasing Order Property, the misaligned sum must have a maximum smaller than the aligned sum. So voter v will report utilities in order to maximize expected utility. However, we want reported utilities to be equal to true utilities, not just in the same order. To do this we must define our class of functions such that:

for all i from 1 to n,
$\frac{\partial M}{\partial r_{v,i}} = 0$ when v reports utilities truthfully

So now we must try to find a class of functions that works. Consider the following class of functions, where c's and k's are constants:

$f_1(r_v) = c_1 - \sum_{i=2}^{n-1} k_i * r_{v,i}^2$

for i from 2 to (n-1): $f_i(r_v) = c_i + 2k_i * r_{v,i}$

$f_n(r_v) = 1 - \sum_{i=1}^{n-1} f_i(r_v) = 1 - \sum_{i=1}^{n-1} c_i - \sum_{i=2}^{n-1}(2k_i * r_{v,i} - k_i * r_{v,i}^2)$

Then that gives us:

$\frac{\partial M}{\partial r_{v,1}} = 0$ (since M does not depend on $r_{v,1}$)

for i from 2 to n-1: $\frac{\partial M}{\partial r_{v,i}} = -2k_i * r_{v,i} + u_{v,i} * 2k_i$
so if $r_{v,i} = u_{v,i}$ (reported utilities are truthful), then this equals 0

$\frac{\partial M}{\partial r_{v,n}} = 0$ (since M does not depend on $r_{v,n}$)

So this class of functions will incentivize voters to report truthful utilities. (i.e. it is a Dominant Strategy to report truthfully)

However, this class of functions only works with constants that uphold our properties. We define the bounds on viable constants as follows:

for all i for which corresponding constants are defined:
RULE 1: $0 \leq c_i \leq 1$ and $0 \leq k_i \leq 1$

To uphold the Non Negativity Property:
since constants and reported utilities are positive, only $f_1(r_v)$ and $f_n(r_v)$ could possibly be negative.
For $f_1(r_v)$ to be non negative we must have that for any r_v:
$c_1 \geq \sum_{i=2}^{n-1} k_i * r_{v,i}^2$
which gives:
RULE 2: $c_1 \geq \sum_{i=2}^{n-1} k_i$ (from when $r_{v,i}$ values approach 1)

For $f_n(r_v)$ to be non negative we must have for any r_v:
$\sum_{i=1}^{n-1} c_i + \sum_{i=2}^{n-1}(2k_i * r_{v,i} - k_i * r_{v,i}^2) \leq 1$
Note that $(2k_i * r_{v,i} - k_i * r_{v,i}^2)$ is maximized when $r_{v,i} = 1$, so this gives:
RULE 3: $\sum_{i=1}^{n-1} c_i + \sum_{i=2}^{n-1} k_i \leq 1$ (sum of all constants at most 1)

To uphold the Decreasing Order Property:
To ensure that all functions from $f_1(r_v)$ to $f_{n-1}(r_v)$ are in decreasing order for any r_v we must have:
RULE 4: $c_1 > c_2 > ... > c_{n-2} > c_{n-1}$ (from when $r_{v,i}$ values approach 0)
RULE 5: $k_2 > k_3 > ... > k_{n-2} > k_{n-1}$ (from when $r_{v,i}$ values approach 1)

also to ensure that $f_1(r_v) > f_2(r_v)$ for any r_v, we have:
RULE 6: $3k_2 + \sum_{i=3}^{n-1} < c_1 - c_2$ (from when $r_{v,i}$ values approach 1)

and to ensure that $f_{n-1}(r_v) > f_n(r_v)$ for any r_v, we have:
RULE 7: $c_{n-1} > 1 - \sum_{i=1}^{n-1} c_i$ (from when $r_{v,i}$ values approach 0)
note that this implies that c_{n-1} must always be at least $\frac{1}{n}$, but can be smaller in some cases.

So we have completely defined our system. An example of constants that would work if we have 4 candidates is:

$c_1 = 0.50$, $c_2 = 0.20$, $c_3 = 0.18$, $k_2 = 0.07$, $k_3 = 0.04$
Then voter v would report utilities to maximize:
$max \quad 0.5 - 0.07 * r_{v,2}^2 - 0.04 * r_{v,3}^2 + u_{v,2}(0.20 + 0.14r_{v,2}) + u_{v,3}(0.18 + 0.08r_{v,3})$

Taking partial derivatives gives us:
$\frac{\partial M}{\partial r_{v,1}} = \frac{\partial M}{\partial rv,4} = 0$

$\frac{\partial M}{\partial r_{v,2}} = -0.14r_{v,2} + 0.14u_{v,2}$

$\frac{\partial M}{\partial r_{v,3}} = -0.08r_{v,2} + 0.08u_{v,2}$

We see that v maximizes expected utility by reporting truthfully. Note that class of functions that we found is not the only class of functions that works. There are other classes of functions that satisfy the partial derivative constraints.

6.2 Evaluation

We have already shown that this system is truthful. In addition, note that due to the Non Decreasing Property and the Decreasing Order Property, the system is monotonic (i. e. it is always better for candidates to receive higher reported utilities from voters). This also means not susceptible to no show paradox.

Another question we might ask is what kind of constants should we choose to maximize social welfare. This obviously is dependent on what the voters' utilities look like, but we can loosely generalize some ideas. If there is broad consensus on a top candidate among the voters, then we would want $f_1(v_r)$ to be as big as possible. This would mean a high c_1 value, low values for the other c constants, and low k values. If there is not broad consensus then the best way to make social welfare relatively stable is high values for $f_2(v_r)$ through $f_{n-1}(v_r)$, at the expense of $f_1(v_r)$. This would mean a low c_1 value, high values for the other c constants, and high k values.

7 Conclusion

Our research shows that all voting systems have flaws, and that there is no obvious way to determine who the winner of an election should be. Thus we suggest that in order to prevent untruthful behavior and other phenomena like the no-show paradox, while still achieving a good outcome, more unconventional methods should be considered. The randomized system we propose at the end of the paper is an example of a system that meets a lot of the criteria that we set forth at the beginning of the paper, including truthfulness. Majority Judgement is also an unconventional system that performed well in our analysis. Another unconventional idea that we think is promising and could be explored more is a randomization system where candidates have probabilities of winning based on a nonlinear function of the number of votes they receive, and voters can switch their votes to continuously maximize their expected utility. We were unable to show that such a system could be designed so that it will always reach an equilibrium, but we still think that there is a lot of promise in the idea.

The systems for electing political officials are difficult and risky to change, so it would be best to first test unconventional methods on lower-stakes situations such as internet awards and rankings. If any of these systems are proven to be robust and effective in practice, then they can be trialed in elections and other more important situations.