

# TOP 20 PROBLEM PATTERNS

Probability Theory Final Exam — December 16, 2025 — With Section References

## 1. CLT Game/Total

“n games”, “total winnings”

**Steps:**

1. Find  $E[X_i]$ ,  $\text{Var}(X_i)$

2.  $S_n \approx N(n\mu, n\sigma^2)$

3. Standardize

**Ref:** §6.1, Template B

## 2. Gaussian Vector

“Gaussian vector”, “indep. components”

**Key:**  $\rho = 0 \Leftrightarrow$  independent (MVN)

**Solve:** Cov = 0

**Ref:** §4.5, Template A, G

## 3. Lognormal Stock

“ $S = S_0 e^Z$ ”, “stock price”

**Key:**  $E[e^X] = e^{\mu + \sigma^2/2}$

**Ref:** §7.3, Template D

## 4. Bayesian Discrete

“prior”, “posterior”, “defective”

**Steps:**

1. List  $\theta_i$ , priors  $P(\theta_i)$

2. Likelihoods  $P(\text{data}|\theta_i)$

3. Bayes: posterior  $\propto$  prior  $\times$  likelihood

4. Normalize

**Ref:** §7.2, Template E

## 5. BVN Conditional

“ $Y|X = x$ ”, “bivariate normal”

$\mu_{Y|X} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$

$\sigma_{Y|X}^2 = \sigma_Y^2(1 - \rho^2)$

**Ref:** §4.5, Template F

## 6. Exponential + CLT

“i.i.d. Exp”, “mean  $\theta$ ”

Mean  $\theta \Rightarrow \lambda = 1/\theta!$

$\bar{X} \approx N(1/\lambda, 1/(n\lambda^2))$

**Ref:** §3.4, §6.1, Template C

## 7. Monty Hall

“Monty Hall”, “contestant picks”

**Sober:** Switch wins 2/3

**Dizzy:** No advantage

**Ref:** §8.1, Template J

## 8. Max/Min i.i.d.

“maximum of n”, “minimum”

$P(\max \leq a) = [F(a)]^n$

$P(\min > a) = [1 - F(a)]^n$

**Ref:** §4.7, Template L

## 9. Linear Combo BVN

“ $P(X + Y > c)$ ”, “sum of jointly normal”

$X + Y \sim N(\mu_X + \mu_Y, \sigma_{X+Y}^2)$

$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y$

**Ref:** §4.5, §8.3

## 10. Predictive Dist.

“predict next”, “posterior predictive”

$P(X_{n+1}|\text{data}) = \sum_{\theta} P(X|\theta)P(\theta|\text{data})$

**Ref:** Template H

## 11. Product Lognormal

“ $XY$  where lognormal”

$\ln(XY) = \ln X + \ln Y$

Still lognormal!

**Ref:** Template I

## 12. BVN Parameters

“find  $\mu_Y, \sigma_Y, \rho$ ”

Match  $E[Y|X]$ ,  $\text{Var}(Y|X)$  formulas

**Ref:** Template N

## 13. Ratio + CLT

“ $\bar{X}/(\bar{X} + c) < p$ ”

**Key:** Transform inequality first!

Then apply CLT

**Ref:** Template O

## 14. Find n for CLT

“smallest n such that”

$n \geq (z^*\sigma/(c - \mu))^2$

**Ref:** Template K

## 15. Confidence Int.

“95% CI”, “confidence interval”

$\bar{X} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$

$z_{0.025} = 1.96$

**Ref:** §6.4

## 16. Conjugate Prior

“Beta-Binomial”

Beta( $\alpha, \beta$ ) +  $x$  successes in  $n$

$\Rightarrow$  Beta( $\alpha + x, \beta + n - x$ )

**Ref:** §7.2

## 17. Total Expectation

“ $E[X] = ?$ ”, condition on Y

$E[X] = E[E[X|Y]]$

**Ref:** §7.1, §8.9

## 18. Total Variance

“ $\text{Var}(X)$ ”, condition on Y

$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$

**Ref:** §7.1

## 19. Sum of Poisson

“sum of Poisson”

$X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$

**Ref:** Template P

## 20. Transformation

“find dist. of  $Y = g(X)$ ”

CDF:  $F_Y(y) = P(g(X) \leq y)$

Jacobian:  $f_Y = f_X|dx/dy|$

**Ref:** §4.6

<b>CLT:</b> $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$	<b>Lognormal:</b> $E[e^X] = e^{\mu+\sigma^2/2}$	<b>Cov:</b> $E[XY] - E[X]E[Y]$	<b>Total Exp:</b> $E[X] = E[E[X Y]]$
<b>Bayes:</b> $P(H E) = \frac{P(E H)P(H)}{P(E)}$	<b>BVN Cond:</b> $\mu_{Y X} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$	<b>Var Sum:</b> $\sigma_X^2 + \sigma_Y^2 + 2\text{Cov}$	<b>Max:</b> $P(\max \leq a) = [F(a)]^n$

**CRITICAL TRAP:** “Mean  $\theta = 3$ ” (Exponential)  $\Rightarrow \lambda = 1/3$  NOT 3!    —    “Gaussian” = Normal    —    “Independent components” =  $\rho = 0$  for MVN