Probability
Problem Set 5

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Problem 1.

Let X and Y be jointly normal random variables with parameters $\mu_X = 1$, $\sigma_X^2 = 2$, $\mu_Y = -2$, $\sigma_Y^2 = 3$, and $\rho = -\frac{2}{3}$.

(a) Find $P(X+Y>0)$.

Let $Z = X+Y$. Then Z is normal with mean μ_Z and variance σ_Z^2 where:

$$\mu_Z = \mu_X + \mu_Y = -1$$

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y = 5 - \frac{4\sqrt{6}}{3} \approx 1.7340$$

Therefore:

$$P(X+Y>0) = P(Z>0) \approx 0.2238$$

(b) Find the constant a if we know $aX+Y$ and $X+2Y$ are independent.

For independence, $\text{Cov}(aX+Y, X+2Y) = 0$:

$$a\sigma_X^2 + (2a+1)\rho\sigma_X\sigma_Y + 2\sigma_Y^2 = 0$$

Solving for a:

$$a = \frac{-(\rho\sigma_X\sigma_Y + 2\sigma_Y^2)}{\sigma_X^2 + 2\rho\sigma_X\sigma_Y} = 1 + \sqrt{6} \approx 3.4495$$

(c) Find $P(X+Y>0 \mid 2X-Y=0)$.

Let $U = X+Y$ and $V = 2X-Y$. The conditional distribution U given $V=0$ is normal with:

$$E[U|V=0] = -\frac{99}{47} + \frac{24\sqrt{6}}{47} \approx -0.8556$$

$$\text{Var}(U|V=0) = \frac{198 - 48\sqrt{6}}{47} \approx 1.7112$$

Therefore:

$$P(X+Y>0 \text{ given } 2X-Y=0) \approx 0.2565$$

Problem 2.

Let X have the lognormal distribution with parameters 3 and 1.44. Find the probability that $X < 6.05$.

If X is Lognormal(3, 1.44), then $\log(X)$ is normal with mean 3, variance 1.44, and $\sigma = 1.2$.

$$P(X<6.05) = P(\log(X) < \log(6.05)) \approx 0.1587$$

Problem 3.

Let X and Y be independent random variables such that $\log(X)$ has the normal distribution with mean 1.6 and variance 4.5 and $\log(Y)$ has the normal distribution with mean 3 and variance 6. Find the mean of the product XY and the probability that $XY > 10$. Use simulation to evaluate the probability.

Since X and Y are independent, $\log(XY) = \log(X) + \log(Y)$ is normal with mean 4.6 and variance 10.5.

Mean:

$$E[XY] = \exp\left(4.6 + \frac{10.5}{2}\right) = e^{9.85} \approx 18{,}958.35$$

Probability (simulation with n = 10,000,000):

$$P(XY > 10) \approx 0.7608$$

Problem 4.

Suppose that two different tests A and B are to be given to a student chosen at random from a certain population. Suppose also that the mean score on test A is 85, and the standard deviation is 10; the mean score on test B is 90, and the standard deviation is 16; the scores on the two tests have a bivariate normal distribution; and the correlation of the two scores is 0.8. If the student's score on test A is 80, what is the probability that her score on test B will be higher than 90?

For bivariate normal (A, B), the conditional distribution B given $A = 80$ is normal with:

$$E[B|A = 80] = 90 + 0.8 \cdot \frac{16}{10}(80 - 85) = 83.6$$

$$\mathrm{Var}(B|A = 80) = 16^2(1 - 0.8^2) = 92.16, \quad \sigma = 9.6$$

Therefore:

$$P(B > 90 \text{ given } A = 80) \approx 0.2525$$

Problem 5.

Suppose that X_1 and X_2 have a bivariate normal distribution for which $E(X_1|X_2) = 3.7 - 0.15X_2$, $E(X_2|X_1) = 0.4 - 0.6X_1$, and $\mathrm{Var}(X_2|X_1) = 3.64$. Find the mean and the variance of X_1, the mean and the variance of X_2, and the correlation of X_1 and X_2.

From the conditional expectations:

$$\rho\frac{\sigma_1}{\sigma_2} = -0.15, \quad \rho\frac{\sigma_2}{\sigma_1} = -0.6$$

Multiplying: $\rho^2 = 0.09$, so $\rho = -0.3$ (negative since both coefficients are negative).

From $\mathrm{Var}(X_2|X_1) = \sigma_2^2(1 - \rho^2) = 3.64$:

$$\sigma_2^2 = 4, \quad \sigma_2 = 2$$

From $\rho(\sigma_1/\sigma_2) = -0.15$:

$$\sigma_1 = 1, \quad \sigma_1^2 = 1$$

For the means:

$$\mu_1 + 0.15\mu_2 = 3.7, \quad \mu_2 + 0.6\mu_1 = 0.4$$

Solving: $\mu_1 = 4$ and $\mu_2 = -2$.

Answers: $\mu_1 = 4$, $\sigma_1^2 = 1$, $\mu_2 = -2$, $\sigma_2^2 = 4$, $\rho = -0.3$

Problem 6.

Using Yahoo Finance data via the `quantmod` package in R:

1. Download SP500 (^GSPC) and VIX (^VIX) data from November 10, 2015 to November 10, 2025
2. Calculate daily returns for both series
3. Estimate the following parameters:

   SP500 Returns:

   - Mean: 0.0533% per day
   - Standard Deviation: 1.145% per day

   VIX Returns:

   - Mean: 0.345% per day
   - Standard Deviation: 8.637% per day

   Correlation: $\rho = -0.7143$

4. Calculate:

   - Empirical probability that SP500 returns are less than 3%: **0.9920** (99.20%)
   - Theoretical probability from the fitted normal distribution that SP500 returns are less than 3%: **0.9950** (99.50%)