Probability
Problem Set 5

Name: Jonah Aden **Date:** November 25, 2025

Problem 1.

Let X and Y be jointly normal random variables with parameters $\mu_X = 1$, $\sigma_X^2 = 2$, $\mu_Y = -2$, $\sigma_Y^2 = 3$, and $\rho = -\frac{2}{3}$.

(a) Find $P(X + Y > 0)$.

Let $Z = X + Y$. Then Z is normal with mean μ_Z and variance σ_Z^2 where:

$$\mu_Z = -1, \quad \sigma_Z^2 = 5 - \frac{4\sqrt{6}}{3}$$

Therefore:

$$P(X + Y > 0) \approx 0.2238$$

(b) Find the constant a if we know $aX + Y$ and $X + 2Y$ are independent.

For independence, $\mathrm{Cov}(aX + Y, X + 2Y) = 0$. Solving:

$$a = 1 + \sqrt{6} \approx 3.4495$$

(c) Find $P(X + Y > 0 \mid 2X - Y = 0)$.

Let $U = X + Y$ and $V = 2X - Y$. The conditional distribution U given $V = 0$ is normal with:

$$E[U|V = 0] = -\frac{99}{47} + \frac{24\sqrt{6}}{47}, \quad \mathrm{Var}(U|V = 0) = \frac{198 - 48\sqrt{6}}{47}$$

Therefore:

$$P(X + Y > 0 \text{ given } 2X - Y = 0) \approx 0.2565$$

Problem 2.

Let X have the lognormal distribution with parameters 3 and 1.44. Find the probability that $X < 6.05$.

If X is Lognormal(3, 1.44), then $\log(X)$ is normal with mean 3 and variance 1.44.

$$P(X < 6.05) \approx 0.1587$$

Problem 3.

Let X and Y be independent random variables such that $\log(X)$ has the normal distribution with mean 1.6 and variance 4.5 and $\log(Y)$ has the normal distribution with mean 3 and variance 6. Find the mean of the product XY and the probability that $XY > 10$. Use simulation to evaluate the probability.

Since X and Y are independent, $\log(XY)$ is normal with mean 4.6 and variance 10.5.
Mean: $E[XY] = e^{9.85} \approx 18{,}958.35$
Probability (simulation, n = 10,000,000): $P(XY > 10) \approx 0.7608$

Problem 4.

Suppose that two different tests A and B are to be given to a student chosen at random from a certain population. Suppose also that the mean score on test A is 85, and the standard deviation is 10; the mean score on test B is 90, and the standard deviation is 16; the scores on the two tests have a bivariate normal distribution; and the correlation of the two scores is 0.8. If the student's score on test A is 80, what is the probability that her score on test B will be higher than 90?
The conditional distribution B given $A = 80$ is normal with:

$$E[B|A = 80] = 83.6, \quad \sigma(B|A = 80) = 9.6$$

Therefore:

$$P(B > 90 \text{ given } A = 80) \approx 0.2525$$

Problem 5.

Suppose that X_1 and X_2 have a bivariate normal distribution for which $E(X_1|X_2) = 3.7 - 0.15X_2$, $E(X_2|X_1) = 0.4 - 0.6X_1$, and $\mathrm{Var}(X_2|X_1) = 3.64$. Find the mean and the variance of X_1, the mean and the variance of X_2, and the correlation of X_1 and X_2.
From the conditional expectations, we find $\rho^2 = 0.09$, so $\rho = -0.3$.
From $\mathrm{Var}(X_2|X_1) = 3.64$: $\sigma_2^2 = 4$, $\sigma_1^2 = 1$.
From the constant terms: $\mu_1 = 4$, $\mu_2 = -2$.
Answers: $\mu_1 = 4$, $\sigma_1^2 = 1$, $\mu_2 = -2$, $\sigma_2^2 = 4$, $\rho = -0.3$

Problem 6.

Using Yahoo Finance data from November 10, 2015 to November 10, 2025 (2,513 trading days):

Parameter Estimates:

SP500 Returns: Mean = 0.0533%, Std = 1.145%
VIX Returns: Mean = 0.345%, Std = 8.637%
Correlation: $\rho = -0.7143$

Probability that SP500 returns are less than 3%:

Empirical: 0.9920
Theoretical (normal): 0.9950