

DECISION TREE POCKET GUIDE

Probability Theory Final Exam - December 16, 2025

LEVEL 0: START HERE

READ PROBLEM → FIND KEY TERMS

Distribution Named?

- “Gaussian” = NORMAL! → §3.3
- “Gaussian vector” = MVN → §4.5
- “Exponential” → §3.4
- “Poisson” → §2.3
- “Binomial” → §2.2
- “Lognormal” → §7.3
- “Beta” → §3.6, §7.2

Multiple Variables?

- “Joint” → §4.1
- “Bivariate Normal” → §4.5
- “Sum X+Y” → MGF §5.2
- “Max/Min” → §4.7, Template L
- “X—Y=y” → §4.2, 4.5

Approximation/Limits?

- “Large n” → CLT §6.1
- “Approximate” → CLT §6.1
- “i.i.d. + sample mean” → CLT
- “n games” → Template B

Bayesian?

- “Prior/Posterior” → §7.2
- “Update belief” → §1.3, 7.2
- “Defective rate” → Template E
- “Monty Hall” → Template J

Expectation?

- “E[X—Y]” → §7.1
- “Total Expectation” → §7.1
- “Conditional variance” → §7.1

!! CRITICAL TRAPS !!

“Mean $\theta = 3$ ” (Exp)
 $\Rightarrow \lambda = 1/3$ NOT 3!

“Independent components” (MVN)
 $\Rightarrow \rho = 0$ = INDEPENDENT!

“ $\psi(t)$ ” = MGF
(Professor’s notation)

FAST FORMULAS

Top 10 for Finals

- $P(A|B) = P(A \cap B)/P(B)$
- Bayes: $P(H|E) \propto P(E|H)P(H)$
- $E[X] = \sum xP(X = x)$
- Var: $E[X^2] - (E[X])^2$
- CLT: $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$
- BVN sum: $\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y$
- $E[e^X] = e^{\mu+\sigma^2/2}$ (lognormal)
- Max: $P(\max \leq a) = [F(a)]^n$
- Min: $P(\min > a) = [1 - F(a)]^n$
- Total Exp: $E[X] = E[E[X|Y]]$

KEY TEMPLATES

A: Gaussian Vector

$Y_1 = aX_1 + X_2, Y_2 = X_1 + bX_2$
Independence: $\text{Cov} = 0 \Rightarrow b = -a$

B: CLT Games

$S_n \approx N(n\mu, n\sigma^2)$
Find $E[X]$, $\text{Var}(X)$ for single trial

C: Exponential + CLT

$\bar{X} \approx N(1/\lambda, 1/(n\lambda^2))$
Mean $\theta \rightarrow \lambda = 1/\theta$!

D: Lognormal

$E[e^X] = e^{\mu+\sigma^2/2}$ for $X \sim N(\mu, \sigma^2)$
 $P(S > K) = 1 - \Phi\left(\frac{\ln(K/S_0) - \mu}{\sigma}\right)$

E: Bayesian Discrete

$P(\theta_i|x) = \frac{P(x|\theta_i)P(\theta_i)}{\sum_j P(x|\theta_j)P(\theta_j)}$

F: BVN Conditional

$Y|X = x \sim N(\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X), (1 - \rho^2)\sigma_Y^2)$

NORMAL TABLE SHORTCUTS

- $\Phi(0) = 0.5$
- $\Phi(1) \approx 0.841$
- $\Phi(1.645) = 0.95$
- $\Phi(1.96) = 0.975$
- $\Phi(2) \approx 0.977$
- $\Phi(2.576) = 0.995$

CHECKLIST

- “Gaussian” = Normal
- Check λ vs mean
- MVN: $\rho = 0 \Leftrightarrow$ indep
- Continuity correction
- $|J|$ absolute value
- Normalize posterior

PROBLEM TYPE → SOLUTION PATH

NORMAL/GAUSSIAN

Single Variable

$$X \sim N(\mu, \sigma^2)$$

1. Standardize: $Z = (X - \mu)/\sigma$
2. Use Φ table
3. $P(X > a) = 1 - \Phi((a - \mu)/\sigma)$

Linear Transform

$$Y = aX + b \text{ where } X \sim N(\mu, \sigma^2)$$

$$\Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$$

Sum of Normals

$$X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2) \text{ indep}$$

$$\Rightarrow X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Bivariate Normal

(X, Y) jointly normal, correlation ρ :

- $aX + bY$ is normal
- $\text{Var}(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$
- $\rho = 0 \Leftrightarrow$ independent (MVN only!)

CLT PROBLEMS

Standard Setup

X_1, \dots, X_n i.i.d., $E[X] = \mu$, $\text{Var}(X) = \sigma^2$

1. $\bar{X} \approx N(\mu, \sigma^2/n)$
2. $S_n = \sum X_i \approx N(n\mu, n\sigma^2)$
3. Standardize: $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Game/Coin Problems

1. Define X_i = single game payoff
2. List PMF: $P(X_i = x)$
3. Calculate $\mu = E[X_i]$
4. Calculate $\sigma^2 = E[X^2] - \mu^2$
5. Total: $S_n \approx N(n\mu, n\sigma^2)$

Finding n

Want $P(\bar{X} > c) \geq p$:

$$n \geq \left(\frac{z^* \sigma}{c - \mu} \right)^2$$

where z^* from $\Phi(z^*) = 1 - p$

BAYESIAN

Discrete Prior

1. List $\theta_1, \theta_2, \dots$
2. Priors: $P(\theta_i)$
3. Likelihoods: $P(\text{data}|\theta_i)$
4. Posterior: $\propto P(\text{data}|\theta_i)P(\theta_i)$
5. Normalize: sum = 1

Monty Hall

Sober Monty: Knows car location
- Switch wins 2/3

Dizzy Monty: Random choice
- No advantage

Conjugate Priors

$$\text{Beta}(\alpha, \beta) + \text{Binomial}(n, x) = \\ \text{Beta}(\alpha + x, \beta + n - x)$$

Posterior Mean

$$\text{Beta}(\alpha, \beta): E[\theta] = \frac{\alpha}{\alpha + \beta}$$

LOGNORMAL

If $Y = e^X$ where $X \sim N(\mu, \sigma^2)$:

- $E[Y] = e^{\mu + \sigma^2/2}$
- $P(Y > K) = P(X > \ln K)$
- $P(Y \leq k) = \Phi\left(\frac{\ln k - \mu}{\sigma}\right)$

Stock Price

$S = S_0 e^Z$ where $Z \sim N(r - \sigma^2/2, \sigma^2)$
 $E[e^{-r}S] = S_0$ (risk-neutral!)

Product

X, Y lognormal indep
 XY is lognormal
 $\ln(XY) = \ln X + \ln Y$

ORDER STATISTICS

X_1, \dots, X_n i.i.d. with CDF F

Maximum

$$P(\max \leq a) = [F(a)]^n$$

$$P(\max > a) = 1 - [F(a)]^n$$

Minimum

$$P(\min > a) = [1 - F(a)]^n$$

$$P(\min \leq a) = 1 - [1 - F(a)]^n$$

Uniform(0,1)

$$\begin{aligned}E[X_{(n)}] &= n/(n+1) \\E[X_{(1)}] &= 1/(n+1)\end{aligned}$$

CONDITIONAL

Conditional Expectation

$$\begin{aligned}E[X|Y=y] &= \int xf(x|y)dx \\E[X] &= E[E[X|Y]]\end{aligned}$$

Total Variance

$$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$$

BVN Conditional

$$\begin{aligned}Y|X=x &\sim N(\mu_{Y|X}, \sigma_{Y|X}^2) \\ \mu_{Y|X} &= \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (x - \mu_X) \\ \sigma_{Y|X}^2 &= \sigma_Y^2 (1 - \rho^2)\end{aligned}$$

EXAM STRATEGY

1. Read ALL problems first (5 min)
2. Start with easiest/most familiar
3. 30 min per question max
4. Write formulas even if stuck
5. Check units/reasonableness