DECISION TREE POCKET GUIDE

Probability Theory Final Exam - December 16, 2025

LEVEL 0: START HERE

READ PROBLEM → FIND KEY TERMS

Distribution Named?

- **"Gaussian"** = NORMAL! → §3.3
- **"Gaussian vector"** = MVN → §4.5
- "Exponential" → §3.4
- "Poisson" → §2.3
- "Binomial" → §2.2
- "Lognormal" → §7.3
- "Beta" → §3.6, §7.2

Multiple Variables?

- "Joint" → §4.1
- "Bivariate Normal" → §4.5
- "Sum X+Y" → MGF §5.2
- "Max/Min" → §4.7, Template L
- "X|Y=y" → §4.2, 4.5

Approximation/Limits?

- "Large n" → CLT §6.1
- "Approximate" → CLT §6.1
- "i.i.d. + sample mean" → CLT
- "n games" → Template B

Bayesian?

- "Prior/Posterior" → §7.2
- "Update belief" → §1.3, 7.2
- "Defective rate" → Template E
- "Monty Hall" → Template J

Expectation?

- "E[X|Y]" → §7.1
- "Total Expectation" → §7.1
- "Conditional variance" → §7.1

!! CRITICAL TRAPS !!

"Mean $\theta = 3$" (Exp)
$\Rightarrow \lambda = 1/3$ NOT 3!

"Independent components" (MVN)
$\Rightarrow \rho = 0$ = INDEPENDENT!

"$\psi(t)$" = MGF
(Professor's notation)

KEY TEMPLATES

A: Gaussian Vector

$Y_1 = aX_1 + X_2$, $Y_2 = X_1 + bX_2$
Independence: Cov $= 0 \Rightarrow b = -a$

B: CLT Games

$S_n \approx N(n\mu, n\sigma^2)$
Find E[X], Var(X) for single trial

C: Exponential + CLT

$\bar{X} \approx N(1/\lambda, 1/(n\lambda^2))$
Mean $\theta \to \lambda = 1/\theta$!

D: Lognormal

$E[e^X] = e^{\mu+\sigma^2/2}$ for $X \sim N(\mu, \sigma^2)$
$P(S > K) = 1 - \Phi\left(\frac{\ln(K/S_0)-\mu}{\sigma}\right)$

E: Bayesian Discrete

$P(\theta_i|x) = \frac{P(x|\theta_i)P(\theta_i)}{\sum_j P(x|\theta_j)P(\theta_j)}$

F: BVN Conditional

$Y|X = x \sim N(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X), (1-\rho^2)\sigma_Y^2)$

FAST FORMULAS

Top 10 for Finals

1. $P(A|B) = P(A \cap B)/P(B)$
2. Bayes: $P(H|E) \propto P(E|H)P(H)$
3. $E[X] = \sum xP(X = x)$
4. Var: $E[X^2] - (E[X])^2$
5. CLT: $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$
6. BVN: $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y$
7. Lognormal: $E[e^X] = e^{\mu+\sigma^2/2}$
8. Max: $P(\max \leq a) = [F(a)]^n$
9. Min: $P(\min > a) = [1 - F(a)]^n$
10. Tower: $E[X] = E[E[X|Y]]$

NORMAL TABLE

- $\Phi(0) = 0.5$
- $\Phi(1) \approx 0.841$
- $\Phi(1.645) = 0.95$
- $\Phi(1.96) = 0.975$
- $\Phi(2) \approx 0.977$
- $\Phi(2.576) = 0.995$

CHECKLIST

- ☐ "Gaussian" = Normal
- ☐ Check λ vs mean
- ☐ MVN: $\rho = 0 \Leftrightarrow$ indep
- ☐ Continuity correction
- ☐ $|J|$ absolute value
- ☐ Normalize posterior

PROBLEM TYPE → SOLUTION PATH

NORMAL/GAUSSIAN

Single Variable

$X \sim N(\mu, \sigma^2)$

1. Standardize: $Z = (X - \mu)/\sigma$
2. Use Φ table
3. $P(X > a) = 1 - \Phi((a - \mu)/\sigma)$

Linear Transform

$Y = aX + b$ where $X \sim N(\mu, \sigma^2)$
$\Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$

Sum of Normals

$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ indep
$\Rightarrow X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Bivariate Normal

(X, Y) jointly normal, correlation ρ:

- $aX + bY$ is normal
- $\text{Var}(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$
- $\rho = 0 \Leftrightarrow$ independent (MVN only!)

CLT PROBLEMS

Standard Setup

$X_1, ..., X_n$ i.i.d., $E[X] = \mu$, $\text{Var}(X) = \sigma^2$

1. $\bar{X} \approx N(\mu, \sigma^2/n)$
2. $S_n = \sum X_i \approx N(n\mu, n\sigma^2)$
3. Standardize: $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Game/Coin Problems

1. Define X_i = single game payoff
2. List PMF: $P(X_i = x)$
3. Calculate $\mu = E[X_i]$
4. Calculate $\sigma^2 = E[X^2] - \mu^2$
5. Total: $S_n \approx N(n\mu, n\sigma^2)$

Finding n

Want $P(\bar{X} > c) \geq p$:

$$n \geq \left(\frac{z^*\sigma}{c - \mu}\right)^2$$

where z^* from $\Phi(z^*) = 1 - p$

BAYESIAN

Discrete Prior

1. List $\theta_1, \theta_2, ...$
2. Priors: $P(\theta_i)$
3. Likelihoods: $P(\text{data}|\theta_i)$
4. Posterior: $\propto P(\text{data}|\theta_i)P(\theta_i)$
5. Normalize: sum = 1

Monty Hall

Sober Monty: Knows car location
- Switch wins 2/3

Dizzy Monty: Random choice
- No advantage

Conjugate Priors

Beta(α, β) + Binomial(n,x) =
Beta($\alpha + x, \beta + n - x$)

Posterior Mean

Beta(α, β): $E[\theta] = \frac{\alpha}{\alpha+\beta}$

LOGNORMAL

If $Y = e^X$ where $X \sim N(\mu, \sigma^2)$:

- $E[Y] = e^{\mu + \sigma^2/2}$
- $P(Y > K) = P(X > \ln K)$
- $P(Y \leq k) = \Phi\left(\frac{\ln k - \mu}{\sigma}\right)$

Stock Price

$S = S_0 e^Z$ where $Z \sim N(r - \sigma^2/2, \sigma^2)$
$E[e^{-r}S] = S_0$ (risk-neutral!)

Product

X, Y lognormal indep
XY is lognormal
$\ln(XY) = \ln X + \ln Y$

ORDER STATISTICS

$X_1, ..., X_n$ i.i.d. with CDF F

Maximum

$$P(\max \leq a) = [F(a)]^n$$
$$P(\max > a) = 1 - [F(a)]^n$$

Minimum

$$P(\min > a) = [1 - F(a)]^n$$
$$P(\min \leq a) = 1 - [1 - F(a)]^n$$

Uniform(0,1)

$E[X_{(n)}] = n/(n+1)$
$E[X_{(1)}] = 1/(n+1)$

CONDITIONAL

Conditional Expectation

$E[X|Y = y] = \int x f(x|y) dx$
$E[X] = E[E[X|Y]]$

Total Variance

$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$

BVN Conditional

$Y|X = x \sim N(\mu_{Y|X}, \sigma^2_{Y|X})$
$\mu_{Y|X} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$
$\sigma^2_{Y|X} = \sigma^2_Y(1 - \rho^2)$

EXAM STRATEGY

1. Read ALL problems first (5 min)

2. Start with easiest/most familiar

3. 30 min per question max

4. Write formulas even if stuck

5. Check units/reasonableness