Homework #6 - MAT 2250

Aden Letchworth

Due: December 1, 2022

Chapter 1.1

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]=C \tag{5}$$

This is an ordinary differential equation or ODE because there aren't partial derivatives with respect to more than one independent variable. The order is first-order because that is the highest-order derivative present. The equation is non-linear because of the quadratic present. The independent variables are {dx} and the dependent are {dy}.

Chapter 1.2

$$x = 2cos(t) - 3sin(t), \quad x'' + x = 0 \tag{4}$$

$$\begin{gathered} x' = -2sin(t) - 3cos(t) \\ x'' = -2cos(t) + 3sin(t) \\ (-2cos(t) + 3sin(t)) + (2cos(t) - 3sin(t)) = 0 \\ (2cos(t) - 2cos(t)) + (3sin(t) - 3sin(t)) = 0 \\ 0 + 0 = 0 \end{gathered}$$

So, x = 2cos(t)-3sin(t) is a solution to the initial value problem

$$y - ln(y) = x^2 - 1, \quad \frac{dy}{dx} = \frac{2xy}{y-1} \tag{10}$$

$$\begin{gathered} \frac{d}{dx}[y] - \frac{d}{dx}[ln(y)] = \frac{d}{dx}[x^2] - \frac{d}{dx}[-1] \\ \frac{dy}{dx} - \frac{1}{y} \times \frac{dy}{dx} = 2x \\ \frac{dy}{dx}(1 - \frac{1}{y}) = 2x \\ y(\frac{dy}{dx}(1 - \frac{1}{y})) = y(2x) \\ \frac{dy}{dx}(y-1) = 2xy \\ \frac{\frac{dy}{dx}(y-1)}{y-1} = \frac{2xy}{y-1} \\ \frac{dy}{dx} = \frac{2xy}{y-1} \end{gathered}$$

So, $y - ln(y) = x^2 - 1$ is a solution to our equation

$$\frac{dx}{dt} + cos(x) = sin(t), \qquad x(\pi) = 0 \tag{26}$$

$$\frac{dx}{dt} = sin(t) - cos(x)$$
$$f(t,x) = sin(t) - cos(x)$$
$$\frac{\partial f}{\partial x} = sin(x)$$

By the existence and uniqueness theorom we know f(t,x) is continuous given the initial values $x(\pi) = 0$

Chapter 2.2

$$\frac{dx}{dt} = 3xt^2 \tag{8}$$

$$dx[p(x)] = dt[g(t)]$$
$$\int \frac{1}{3x}dx = \int t^2 dt$$
$$\frac{1}{3}\ln(x) = \frac{t^3}{3} + C$$
$$\ln|x| = t^3 + 3C$$
$$\ln|x| = t^3 + K$$
$$x = e^{t^3+K}$$

$$x^2dx + 2ydy = 0, \qquad y(0) = 2 \tag{22}$$

$$x^2dx = -2ydy$$
$$\int x^2dx = \int = 2ydy$$
$$\frac{x^3}{3} = -y^2 + C$$
$$\frac{(0)^3}{3} = -(2)^2 + C$$
$$C = 4$$
$$\frac{x^3}{3} = -y^2 + 4$$
$$y = \sqrt{\frac{-x^3}{3} + 4}$$

(30)

As stated in this section, the separation of equation (2) on page 42 requires division by p(y), and this may disguise the fact that the roots of the equation p(y) = 0 are actually constant solutions to the differential equation.

$$\frac{dy}{dx} = (x-3)(y+1)^{\frac{2}{3}} \tag{a}$$

$$\begin{aligned}
dx[x-3] &= dy\left[\frac{1}{(y+1)^{\frac{2}{3}}}\right] \\
\int (x-3)dx &= \int \frac{1}{(y+1)^{\frac{2}{3}}}dy \\
\frac{x^2}{2} - 3x &= 3(y+1)^{\frac{1}{3}} \\
\frac{x^2}{6} - x + C &= (y+1)^{\frac{1}{3}} \\
(y+1) &= \left(\frac{x^2}{6} - x + C\right)^3 \\
y &= -1 + \left(\frac{x^2}{6} - x + C\right)^3
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= (x-3)(y+1)^{\frac{2}{3}} \\
0 &= (x-3)(-1+1) \\
0 &= 0
\end{aligned} \tag{b}$$

Chapter 2.3

$$\frac{dy}{dx} - \frac{y}{x} = xe^x; \qquad y(1) = e - 1 \tag{17}$$

$$\mu(x) = e^{\int -\frac{1}{x} dx}$$
$$\mu(x) = e^{-ln|x|}$$
$$\mu(x) = \frac{1}{x}$$
$$\mu(x)\frac{dy}{dx} - \mu(x)\frac{y}{x} = \mu(x)xe^x$$
$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\frac{y}{x} = \frac{1}{x}xe^x$$
$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$$
$$\int dx\left[\frac{1}{x}y\right] = \int e^x$$
$$\frac{y}{x} = e^x + C$$
$$y = e^x + Cx$$
$$(e-1) = e^{(1)} + C(1)$$
$$C = -1$$

$$y = xe^x - x$$

Chapter 3.2

(8)

A tank initially contains s_0 lb of salt dissolved in 200 gal of water, where s_0 is some positive number. Starting at time t = 0, water containing 0.5 lb of salt per gallon enters the tank at a rate of 4 gal/min, and the well stirred solution leaves the tank at the same rate. Letting c(t) be the concentration of salt in the tank at time t, show that the limiting concentration-that is, $\lim_{t\to\infty} c(t)$-is 0.5lb/gal

$$\left(4\frac{gal}{min}\right)\left(.5\frac{lb}{gal}\right) = 2\frac{lb}{min}$$
$$4\frac{gal}{min}\left[\frac{x(t)}{200}\frac{lb}{gal}\right] = \frac{x(t)}{50}\frac{lb}{min}$$

$$\frac{dx}{dt} = 2 - \frac{x}{50}$$
$$\frac{dx}{dt} = \frac{100-x}{50}$$
$$\int \frac{dx}{100-x} = \int \frac{dt}{50}$$
$$-ln|100-x| = \frac{1}{50}t + C$$
$$100 - x = e^{-\frac{1}{50}t-C}$$
$$x = -e^{-\frac{1}{50}t-C} + 1000 = -e^{-\frac{1}{50}0-C} + 100$$

$$C = -ln|100-x| - \frac{1}{50}t$$
$$C = -ln[100-(0)] - \frac{1}{50}(0)$$
$$C = -ln[100]$$

$$x = -e^{-\frac{1}{50}t-(-ln|100|)} + 100$$
$$x = -e^{-\frac{1}{50}t} * e^{ln|100|} + 100$$
$$x = -100e^{-\frac{1}{50}t} + 100$$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} -100e^{-\frac{1}{50}t} + 100$$
$$\lim_{t\to\infty} x(t) = 100$$
$$\frac{x(t)}{200} = \frac{100}{200}$$

This shows us that the limiting concentration of salt in the tank is 1/2 or .5 $\frac{lb}{gal}$.

Chapter 4.1

$$y'' + 2y' + 4y = 5sin3t, \qquad \Omega = 3 \tag{7}$$

$$y = Acos\Omega t + Bsin\Omega t$$
$$y = Acos3t + Bsin3t$$
$$y' = -3Asin3t + 3Bcos3t$$
$$y'' = -9Acos3t - 9Bsin3t$$

$$(-9Acos3t - 9Bsin3t) + 2(-3Asin3t + 3Bcos3t) + 4(Acos3t + Bsin3t) = 5sin3t$$
$$-9Acos3t - 9Bsin3t - 6Asin3t + 6Bcos3t + 4Acos3t + 4Bsin3t = 5sin3t$$
$$-5Acos3t - 5Bsin3t - 6Asin3t + 6Bcos3t = 5sin3t$$
$$-5A + 6B = 0$$
$$-5B + 6A = 5$$
$$A = -\frac{30}{61}, \qquad B = -\frac{25}{61}$$

$$y = \left(-\frac{30}{61}\right) cos3t + \left(-\frac{25}{61}\right) sin3t$$

Chapter 4.2

$$y'' - 4y' + 4y = 0; \qquad y(1) = 1,\ y'(1) = 1 \tag{20}$$

$$r^2 - 4r + 4 = 0$$
$$r = 2$$
$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$
$$y(t) = c_1 e^{2t} + c_2 t e^{2t}$$
$$y' = 2c_1 e^{2t} + 2c_2 t e^{2t} + c_2 e^{2t}$$
$$1 = c_1 e^{2(1)} + c_2(1) e^{2(1)}$$
$$1 = c_1 e^2 + c_2 e^2$$
$$1 = 2c_1 e^{2(1)} + 2c_2(1) e^{2(1)} + c_2 e^{2(1)}$$
$$1 = 2c_1 e^2 + 2c_2 e^2 + c_2 e^2$$
$$c_1 = \frac{2}{e^2}, \quad c_2 = -\frac{1}{e^2}$$

$$y(t) = 2e^{2t-2} - te^{2t-2}$$

$$y_1(t) = te^{2t}, \qquad y_2(t) = e^{2t} \tag{29}$$

$$y_1'(t) = 2te^{2t} + e^{2t}$$
$$y_2'(t) = 2e^{2t}$$

$$W(y_1, y_2,)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'(t) & y_2'(t) \end{vmatrix}$$

$$W(y_1, y_2)(t) = \begin{vmatrix} te^{2t} & e^{2t} \\ 2te^{2t} + e^{2t} & 2e^{2t} \end{vmatrix}$$

$$W(y_1, y_2)(t) = ((te^{2t})(2e^{2t})) - ((2te^{2t} + e^{2t})(e^{2t})$$
$$W(y_1, y_2)(t) = -e^{4t}$$
$$-e^{4t} \neq 0$$

The Wronskian of our functions y_1 and y_2 cannot equal zero therefore we can say they are linearly independent on the interval (0,1)

Chapter 4.3

$$u'' + 7u = 0 \tag{12}$$

$$r^2 + 7 = 0$$
$$r = i\sqrt{7},\ -i\sqrt{7}$$
$$\alpha = 0, \qquad \beta = \sqrt{7}$$
$$u(t) = c_1 e^{\alpha t} cos\beta t + c_2 e^{\alpha t} sin\beta t$$

$$u(t) = c_1 cos\sqrt{7}t + c_2 sin\sqrt{7}t$$

$$w'' - 4w' + 2w = 0; \qquad w(0) = 2,\ w'(0) = 1 \tag{23}$$

$$r^2 - 4r + 2 = 0$$
$$r = 2 + \sqrt{2},\ \ 2 - \sqrt{2}$$
$$w(t) = c_1 e^{rt} + c_2 e^{rt}$$
$$w(t) = c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2-\sqrt{2})t}$$
$$w'(t) = (2 + \sqrt{2})c_1 e^{(2+\sqrt{2})t} + (2 - \sqrt{2})c_2 e^{(2-\sqrt{2})t}$$
$$0 = c_1 e^{(2+\sqrt{2})(0)} + c_2 e^{(2-\sqrt{2})(0)}$$
$$c_1 + c_2 = 0$$
$$1 = (2 + \sqrt{2})c_1 e^{(2+\sqrt{2})0} + (2 - \sqrt{2})c_2 e^{(2-\sqrt{2})0}$$
$$1 = (2 + \sqrt{2})c_1 + (2 - \sqrt{2})c_2$$
$$c_1 = \frac{\sqrt{2}}{4},\ \ c_2 = -\frac{2}{4}$$

$$w(t) = \frac{\sqrt{2}}{4} e^{(2+\sqrt{2})t} - \frac{\sqrt{2}}{4} e^{(2-\sqrt{2})t}$$