Financial derivatives

Lecture 4: Option price bounds and put-call parity

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No-arbitrage bounds for option prices

Motivations

Options derive their values from their underlying assets. Before we **explicitly model** the price dynamics of underlying assets, we should first:

Model – free price boundaries.

- identify **conditions** that **option prices must satisfy regardless of the price evolution** of their **underlying assets**;
- use these **model-free** conditions as guidance for **no-arbitrage restrictions** on option prices.

Notations

We use the following notations for options:

- S_t: **market price of an underlying asset** at time t.
- K: **pre-specified strike (exercise) price** of an option.
- $T - t$: **time to maturity** of the option contract at time t.
- C_t: **market price of a call option** at time t. Option to buy
- P_t: **market price of a put option** at time t. option to sell.
- $C_{A,t}, C_{E,t}$: **market price of the American and European calls** at time t, respectively. (early exercise) only exercise at time T.
- $P_{A,t}, P_{E,t}$: **market price of the American and European puts** at time t, respectively. $P_A > P_E$, since more freedom
- $\mathrm{PV}(D)$: **present value** of **dividends receivable** over the life of an option.
- $\mathrm{PV}(K)$: **present value** of **receiving the strike price** at time T.

$$PV = \underbrace{\frac{K}{1+r}}_{\text{Discrete}} = \underbrace{Ke^{-r}}_{\text{cont}}$$

Option styles

American vs European options

The **buyer (holder)** of an option has the **right** to **buy (call)** or **sell (put)** the **underlying asset** at a **pre-specified price**.

- **American options**: (early exercise)
  The holder can **exercise** the option **at any time on or before the expiration date**.
- **European options**:
  The holder can **only exercise** the option **at expiration**.

American vs European options

$P_A > P_E$

Since American option holders have **more flexibility** than their European counterparts, we should expect:

$$C_{A,t} \geq C_{E,t} \text{ and } P_{A,t} \geq P_{E,t}$$

for contracts with **identical strike price** and **maturity**.

Maximum and minimum for call options

Upper bound for call options

Upper bound:

- A call option **must not be more expensive** than its underlying asset:

$$C_t \leq S_t$$

(UA)

No one will **pay more** than the price of the underlying asset to own an option to buy the underlying asset. → because then you can just buy UA directly.

Lower bounds for call options

Call payoff: max [$S_t - K$, 0]

Lower bounds:

1. The **minimum payoff** of a call option is **zero** so its price **must be positive**:

$$C_t > 0$$

(For both A, E)

early exercise.

2. If $S_t - K > C_{A,t}$, we can make an **arbitrage profit** by **buying the American call option** and **exercising it immediately**. So:

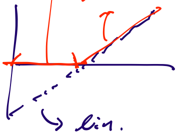
buy $C_{A,t}$, get $S_t - K$. (can't hold)

$$C_{A,t} \geq S_t - K$$

avoid loss.

3. The **non-linear payoffs** of a call option is **more valuable** than the **linear payoff** of buying the underlying asset and borrowing $\mathrm{PV}(K)$:

$$C_t \geq \underbrace{S_t - \mathrm{PV}(K) - \mathrm{PV}(D)}_{\text{Forward contract.}}$$



Lower bounds for call options

Table: Lower bound of an European call option

Investment	Cash flows at	(out) OTM.	(in) ITM,
	t	$T: S_T < K$	$T: S_T \geq K$
Call option	$-C_{E,t}$	0	$S_T - K$
Linear payoff	$\mathrm{PV}(K) - S_t$	$S_T - K < 0.$	$S_T - K$
	borrow PV of K. / buy stock.		repay K.

Since the European call option avoids the **negative payoff** when $S_T < K$, we must have:

(more expensive)

$-C_{E,t} < PV(K) - S_t$

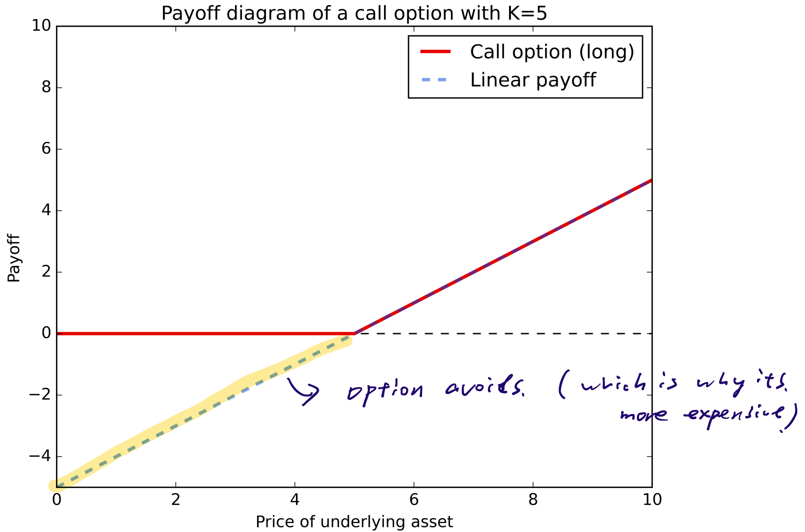
$\Rightarrow S_t - PV(K) < C_{E,t}.$

$$C_{E,t} \geq S_t - \mathrm{PV}(K)$$

Furthermore, we know that $C_{A,t} \geq C_{E,t}$ so:

$$C_t \geq S_t - \mathrm{PV}(K)$$

Lower bounds for call options



Lower bounds for call options

If the underlying asset pays dividend during $T-t$, we can borrow $\mathrm{PV}(D)$ now to offset D in the future:

Table: Lower bound of an European call option

Investment	Cash flows at		
	t	$T: S_T < K$	$T: S_T \geq K$
Call option	$-C_{E,t}$	0	$S_T - K$
Linear payoff	$\mathrm{PV}(K) + \mathrm{PV}(D) - S_t$	$(S_T + \cancel{D}) - K - \cancel{D}$	$S_T + D - K - D$

borrow

buy asset.

receive D and S_T at T.

Therefore, we have:

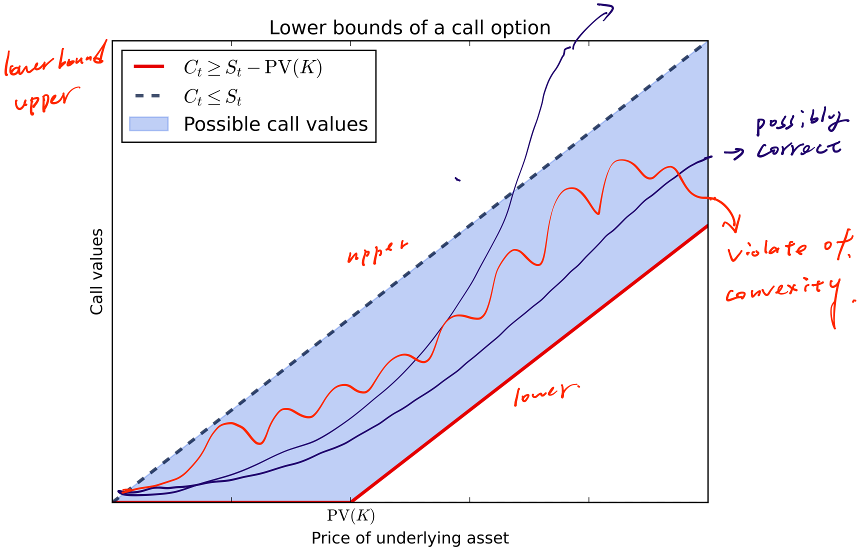
$$C_t \geq S_t - \mathrm{PV}(K) - \mathrm{PV}(D)$$

$-C_{E,t} < \mathrm{PV}(K) + \mathrm{PV}(D) - S_t$

$\Rightarrow S_t - \mathrm{PV}(K) - \mathrm{PV}(D) < C_{E,t}$

Lower bounds for call options

violate no arbitrage.



Summary for call option bounds

We have the following bounds on call option prices:

1. $C_{E,t} \leq S_t$ and $C_{A,t} \leq S_t$. (or we will buy S_t directly!)

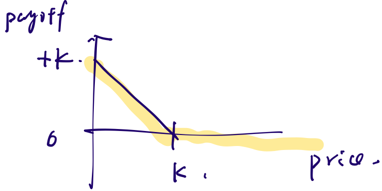
(E) 2. $C_{E,t} \geq 0$ and $C_{E,t} \geq S_t - \text{PV}(K) - \text{PV}(D)$
or equivalently $C_{E,t} \geq \max[0, S_t - \text{PV}(K) - \text{PV}(D)]$

(A) 3. $C_{A,t} \geq 0$, $C_{A,t} \geq S_t - K$ and $C_{A,t} \geq S_t - \text{PV}(K) - \text{PV}(D)$
or equivalently $C_{A,t} \geq \max[0, S_t - K, S_t - \text{PV}(K) - \text{PV}(D)]$

early exercise

Maximum and minimum for put options

Upper bound for put options



Upper bounds:

1. The **maximum payoff** of buying a **put option** is K when the price of the underlying asset drops to zero:

$$P_t \leq K$$ (early exercise!)

2. For an **European put option**, the maximum payoff can only occur at T. Therefore:

$$P_{E,t} \leq \mathrm{PV}(K)$$

Lower bounds for put options

Payoff: max. (K- St, 0)

Lower bounds:

1. The **minimum payoff** of a put option is **zero** so its price **must be positive**:

$$P_t \geq 0$$

early exercise

2. If $K - S_t > P_{A,t}$, we can make an **arbitrage profit** by buying the **American put option** and exercising it immediately. Hence:

put price.

$$P_{A,t} \geq K - S_t$$

3. The **non-linear payoffs** of a put option is **more valuable** than the **linear payoff** of shorting the underlying asset and lending $\mathrm{PV}(K)$:

$$P_t \geq \mathrm{PV}(K) + \mathrm{PV}(D) - S_t$$

Lower bounds for put options

Table: Lower bound of an European put option

Investment	Cash flows at		
	t	ITM. $T: S_T < K$	OTM. $T: S_T \geq K$
Put option	$-P_{E,t}$	$K - S_T > 0$	0 not exercise
Linear payoff	$S_t - \mathrm{PV}(K)$ short stocks (spot)	$K - S_T$ buy back stock.	$K - S_T < 0$

Since the European put option avoids the **negative payoff** when $S_T \geq K$, we must have:

invest PV(K).

$$P_{E,t} \geq \mathrm{PV}(K) - S_t \quad (\equiv -P_{E,t} < S_t - PV(K))$$

Moreover, we know that $P_{A,t} \geq P_{E,t}$ so:

$$P_t \geq \mathrm{PV}(K) - S_t$$

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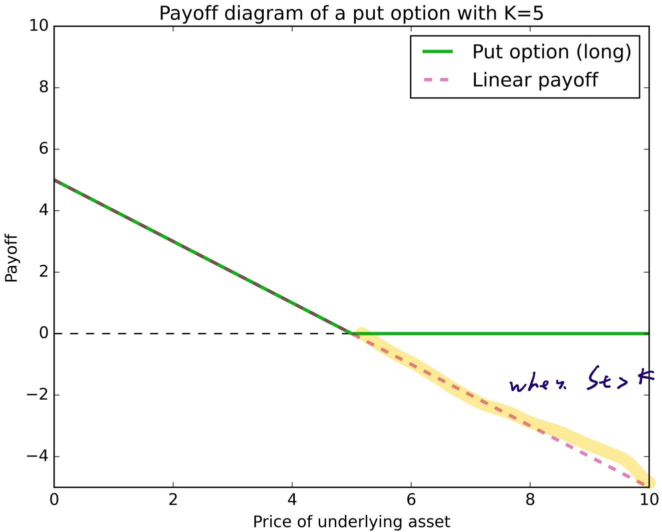
Lower bounds for put options



when $S_t > K$, put avoid loss

Lower bounds for put options

If the underlying asset pays dividend during $T-t$, we can lend $\mathrm{PV}(D)$ now to pay D back to lender in the future:

借.

Table: Lower bound of an European put option

Investment	Cash flows at		
	t	$T: S_T < K$	$T: S_T \geq K$
Put option	$-P_{E,t}$	$K - S_T$	0
Linear payoff	$S_t - \mathrm{PV}(K) - \mathrm{PV}(D)$	$(K - S_T - D) + D$	$K - S_T - D + D$

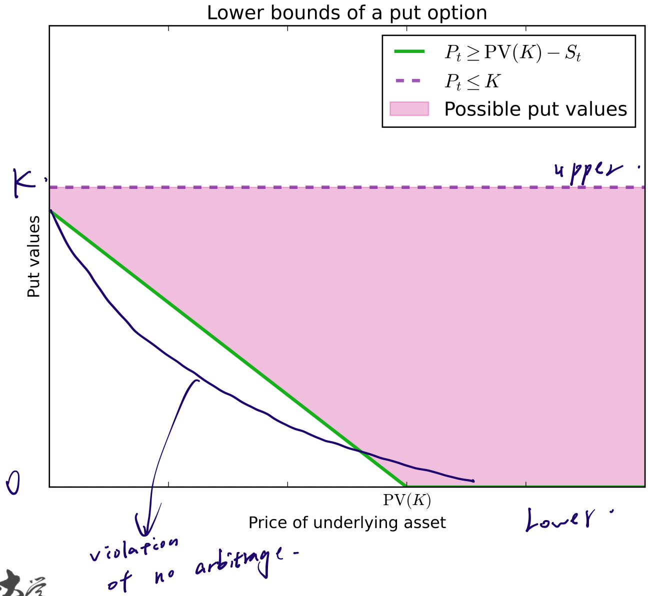
inverse D

Thus, we have:

$$P_t \geq \mathrm{PV}(K) + \mathrm{PV}(D) - S_t$$

* if we short stock, we have to repay stock tgt with D. (−D)

Lower bounds for put options



Summary for put option bounds

We have the following bounds on put option prices:

1. $P_{E,t} \leq \mathrm{PV}(K)$ and $P_{A,t} \leq K$.
2. $P_{E,t} \geq 0$ and $P_{E,t} \geq \mathrm{PV}(K) + \mathrm{PV}(D) - S_t$
   or equivalently $P_{E,t} \geq \max[0, \mathrm{PV}(K) + \mathrm{PV}(D) - S_t]$
3. $P_{A,t} \geq 0, P_{A,t} \geq K - S_t$ and $P_{A,t} \geq \mathrm{PV}(K) + \mathrm{PV}(D) - S_t$
   or equivalently $P_{A,t} \geq \max[0, K - S_t, \mathrm{PV}(K) + \mathrm{PV}(D) - S_t]$

Strike price and option value

Strike prices and call option values

max (St - K, 0)

If the **strike price increases**, the **call option** value must **fall**:

$$\text{If } K_1 < K_2, \text{ then } C_t(K_1) \geq C_t(K_2)$$

For **European options**, the **additional payoff** of having a K_1 call over a K_2 call is $K_2 - K_1$ but it can only be realized at T so:

suppose,

$$\left\{ \begin{array}{l} K_1 = 100 \\ K_2 = 105 \\ S_T = 120 \end{array} \right.$$

$$\text{If } \overset{100}{K_1} < \overset{105}{K_2}, \text{ then } \overset{20}{C_{E,t}(K_1)} - \overset{15}{C_{E,t}(K_2)} \leq \text{PV}(K_2 - K_1)$$

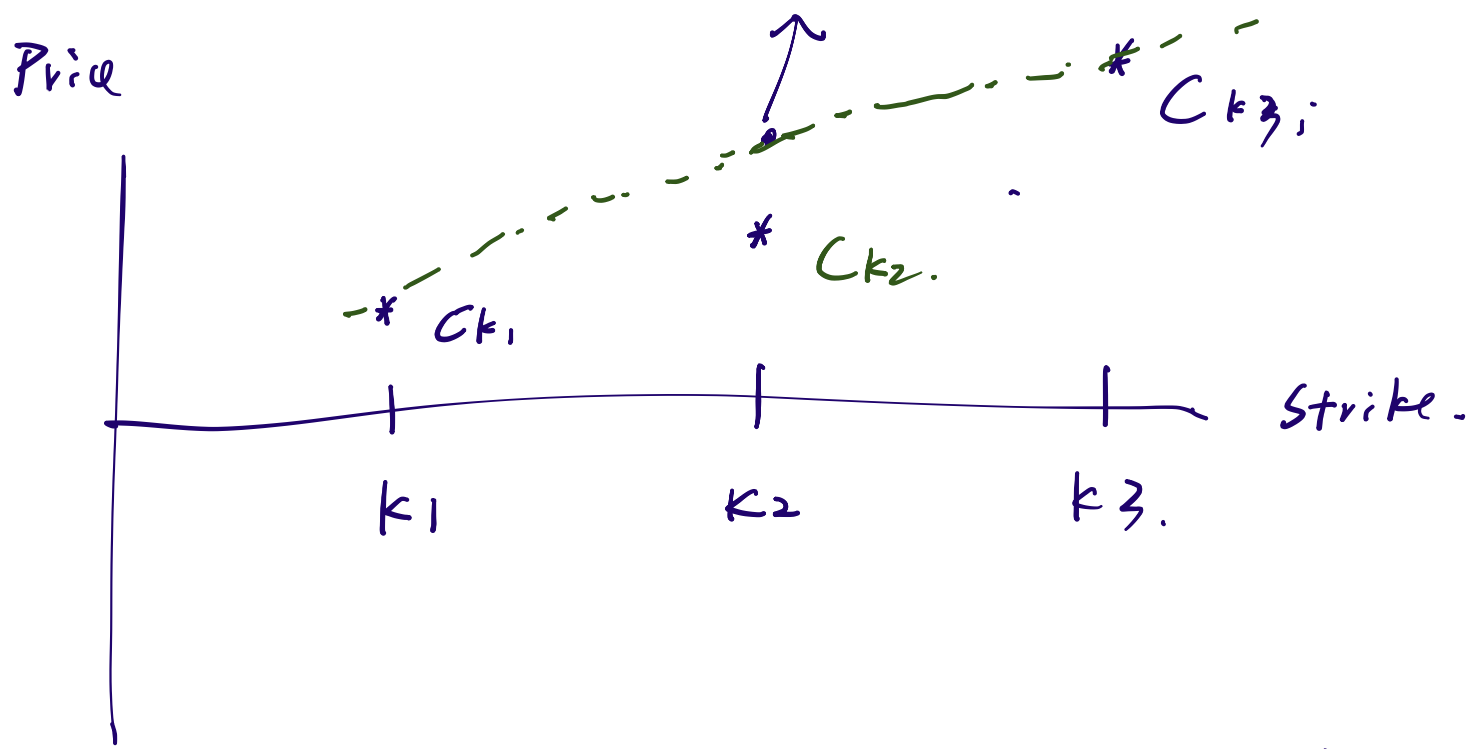
折回來. (we get $K_2 - K_1$ at time T).

Likewise, the **additional payoff** for **American options** of having a K_1 call over a K_2 call is $K_2 - K_1$ which may be realized before T:

$$\text{If } K_1 < K_2, \text{ then } C_{A,t}(K_1) - C_{A,t}(K_2) \leq K_2 - K_1$$

Linear:

$w C_{K1} + (1-w) C_{K3}$.



$* \; C_{K3} \geq C_{K2} \geq C_{K1}$.

Strike prices and call option values

Due to the **convexity** in option prices with respect to strike prices, we have:

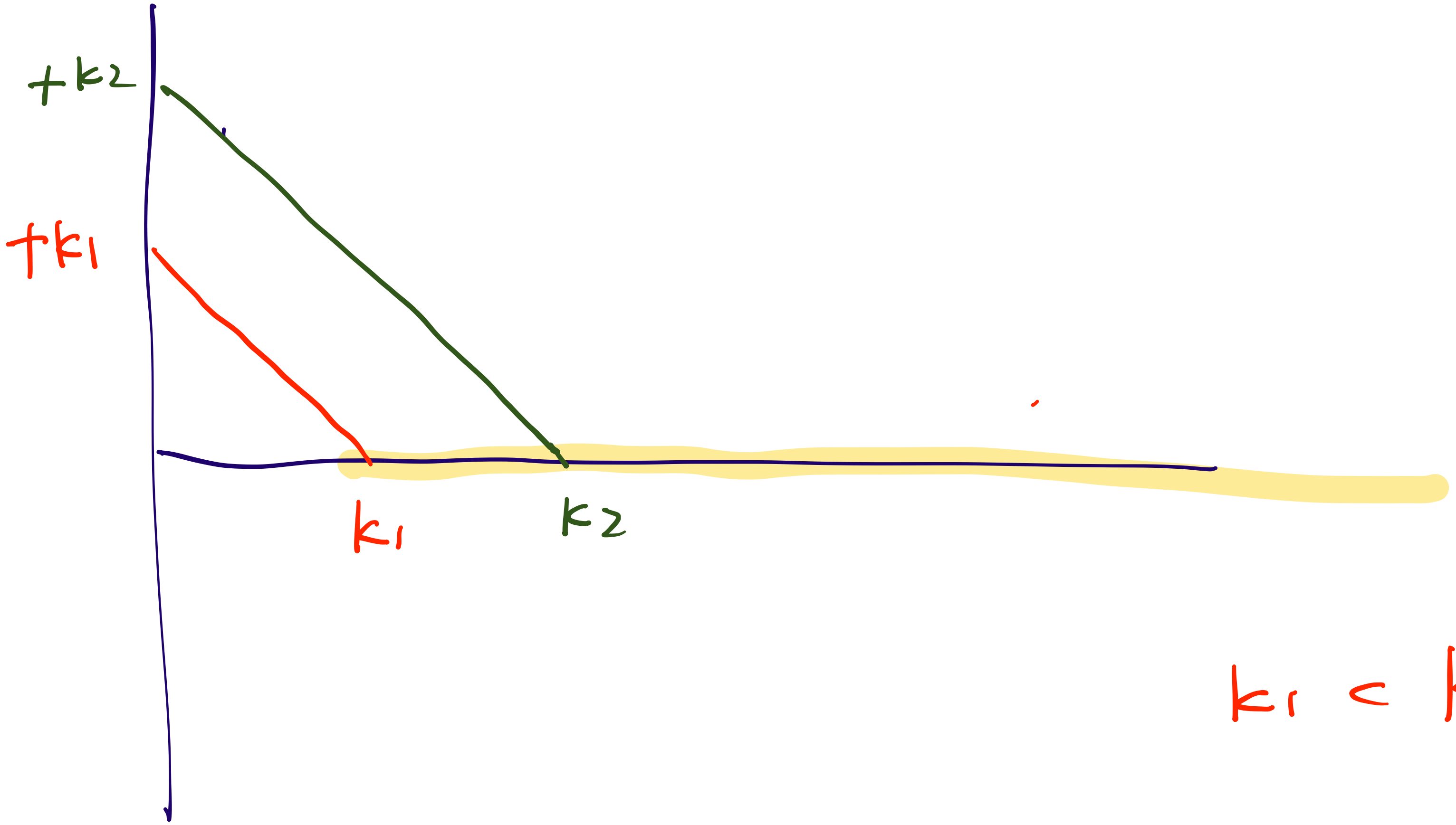
$$\text{If } K_1 < K_2 < K_3, \text{ then } wC_t(K_1) + (1-w)C_t(K_3) \geq C_t(K_2)$$

where:

$$w = \frac{K_3 - K_2}{K_3 - K_1}$$

$w \in [0, 1]$.

- In other words, the **linear combinations** of $C_t(K_1)$ and $C_t(K_3)$ is **more expensive** than $C_t(K_2)$.



$k_1 < k_2$.

Strike prices and put option values

If the **strike price increases**, the **put option** value must **increase**:

earn more.

$$\text{If } K_1 < K_2, \text{ then } P_t(K_1) < P_t(K_2)$$

For **European options**, the **additional payoff** of having a K_2 put over a K_1 put is $K_2 - K_1$ but it can only be realized at T so:

$$\text{If } K_1 < K_2, \text{ then } P_{E,t}(K_2) - P_{E,t}(K_1) \leq \mathrm{PV}(K_2 - K_1)$$

Likewise, the **additional payoff** for **American options** of having a K_2 put over a K_1 put is $K_2 - K_1$ which may be realized before T:

$$\text{If } K_1 < K_2, \text{ then } P_{A,t}(K_2) - P_{A,t}(K_1) \leq K_2 - K_1$$

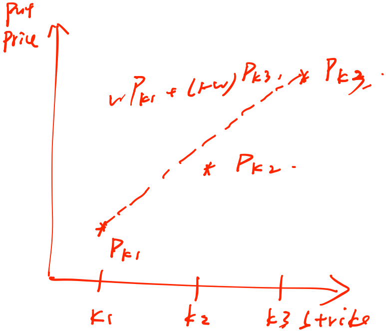
Strike prices and put option values

Similarly, the **convexity** in option prices with respect to strike prices applies to put options, we have:

$$\text{If } K_1 < K_2 < K_3, \text{ then } wP_t(K_1) + (1-w)P_t(K_3) \geq P_t(K_2)$$

where:

$$w = \frac{K_3 - K_2}{K_3 - K_1}$$



Time to maturity and option value

Time to maturity and call option value

If the **time to maturity increases**, the **call option** value must **increase**:

more time to wait until the call is ITM.

$$\text{If } T_1 < T_2, \text{ then } C_t(T_1) \leq C_t(T_2)$$

For **American options**, the call with a longer maturity can **always be exercised at the same time** as the one with a short maturity:

$$\text{If } T_1 < T_2, \text{ then } \overset{①}{C_{A,t}}(T_1) \leq \overset{②}{C_{A,t}}(T_2)$$

For **European options**, we know that:

$$\underbrace{C_{E,t}}_{\text{Call option}} \geq \underbrace{S_t - \mathrm{PV}(K)}_{\text{linear}} \quad \text{(from previous)}$$

When the call with maturity T_1 expires, its payoff is:

$$① \quad C_{E,T_1}(T_1) = \max(0, S_{T_1} - K) \qquad T_1 < T_2.$$

Which is smaller than the value of call with maturity T_2 at T_1:

$$② \quad C_{E,T_1}(T_2) = \max(0, S_{T_1} - \mathrm{PV}(K))$$

Time to maturity and put option value

For **American options**, the put with a longer maturity can **always be exercised at the same time** as the one with a short maturity:

$$\text{If } T_1 < T_2, \text{ then } P_{A,t}(T_1) \leq P_{A,t}(T_2)$$

→ ? ?

Nevertheless, **European put options** are bounded by:

$$P_{E,t} \leq \mathrm{PV}(K)$$

In the extreme case of having $T \to \infty$:

$$\lim_{T \to \infty} P_{E,t} = 0$$

$\lim_{T \to \infty} \frac{K}{(1+r)^T} = 0$

As a result, it is **possible** for **European put option** values to **drop** when **maturity increases.**

Decomposing option values

Decomposing call option prices

We know from previous slides that:

$$C_t \geq S_t - \mathrm{PV}(K) - \mathrm{PV}(D)$$

the cost to pay for avoiding losses.

Let's further define the **insurance value** as:

$$\mathrm{IV}(C_t) = \underbrace{C_t}_{\text{call}} - \underbrace{[S_t - \mathrm{PV}(K) - \mathrm{PV}(D)]}_{\text{linear payoff}} > 0. \quad \cdots ①$$

After rearranging, we have:

$$① \Rightarrow \quad C_t = S_t - \mathrm{PV}(K) - \mathrm{PV}(D) + \mathrm{IV}(C_t)$$

- By adding and subtracting K, we obtain:

$$C_t = \underset{(+,-)}{(S_t - K)} + \underset{(+)}{(K - \mathrm{PV}(K))} - \mathrm{PV}(D) + \underset{(+)}{\mathrm{IV}(C_t)}$$

Call value = Intrinsic value + Time value − Payout impacts + Insurance value (+)

Decomposing put option prices

We know from previous slides that:

$$P_t \geq \mathrm{PV}(K) + \mathrm{PV}(D) - S_t$$

Let's further define the **insurance value** as:

$$\mathrm{IV}(P_t) = \underbrace{P_t}_{\text{put}} - \underbrace{[\mathrm{PV}(K) + \mathrm{PV}(D) - S_t]}_{\text{linear.}}$$

After rearranging, we have:

$$P_t = \mathrm{PV}(K) + \mathrm{PV}(D) - S_t + \mathrm{IV}(P_t)$$

By adding and subtracting K, we obtain:

longer T
lower put

$$P_t = (K - S_t) - (K - \mathrm{PV}(K)) + \mathrm{PV}(D) + \mathrm{IV}(P_t)$$

Put value = Intrinsic value (+, −) [− Time value] + Payout impacts (+)

+ Insurance value (+)

Early exercise

Early exercise of American call options

When $S_t > K$, **American call option** holders can consider:

1. Exercise the call option and get $S_t - K$;
2. Sell the call at the prevailing market price of $C_{A,t}$;
3. Keep the call in the portfolio. (hold).

We know from previously that $C_{A,t} \geq S_t - K$. In the case of $C_{A,t} = S_t - K$, it is **optimal** to **early exercise** the call.

if $C_{A,t} = S_t - K$.

① sell

② hold.

American call options on non-dividend-paying assets

Conclusion:

- For non-div paying asset.
- American.
- One should never exercise it (Merton)

$PV(\overset{0}{\overset{\shortparallel}{D}}) = 0$

For non-dividend-paying assets, we have:

$$C_{A,t} = \underbrace{(S_t - K)}_{\text{exercise}} + \underbrace{(K - \mathrm{PV}(K))}_{\text{time}} + \underbrace{\mathrm{IV}(C_t)}_{\text{insurance}}$$

The **difference** between **selling the call** and **exercising it immediately** is: sell call receives.

$$C_{A,t} - \underbrace{(S_t - K)}_{\text{exercise}} = \overset{+}{(K - \mathrm{PV}(K))} + \overset{+}{\mathrm{IV}(C_t)} > 0$$

As the difference is **positive**, the call option holder is **better-off** selling the option rather than exercising it.

Assume no ~~trading~~ friction!

↓

i.e. Can sell call exact at $C_{A,t}$.

(liquidity issue in real case)

American call options on dividend-paying assets

For dividend-paying assets, we have:

$$C_{A,t} = (S_t - K) + (K - \mathrm{PV}(K)) + \mathrm{IV}(C_t) - \mathrm{PV}(D)$$

The **difference** between **selling the call** and **exercising it immediately** is:

$$\underbrace{C_{A,t}}_{\text{sell call}} - \underbrace{(S_t - K)}_{\text{early exercise}} = \overbrace{(K - \mathrm{PV}(K))}^{+} + \overbrace{\mathrm{IV}(C_t)}^{+} - \overbrace{\mathrm{PV}(D)}^{-}$$

If $\mathrm{PV}(D)$ is **large enough to offset** $(K - \mathrm{PV}(K)) + \mathrm{IV}(C_t)$, it is **optimal** to early exercise the call option from **receiving dividend payments**.

$C_{A,t} - (S_t - K) < 0. \quad (\mathrm{PV}(D) > K - \mathrm{PV}(K) + \mathrm{IV}(C_t))$

Early exercise of American put options

When $K > S_t$, **American put option** holders can consider:

1. Exercise the put option and get $K - S_t$;
2. Sell the put at the prevailing market price of $P_{A,t}$;
3. Keep the put in the portfolio. hold.

We know from previously that $P_{A,t} \geq K - S_t$. In the case of $P_{A,t} = K - S_t$, it is **optimal** to **early exercise** the put.

or it will have arbitrage opportunity

(or you'll face :

- time value ↓ (decrease over time
- fluctuations in underlying asset)

American put options on non-dividend-paying assets

For non-dividend-paying assets, we have:

$$P_t = (K - S_t) - (K - \mathrm{PV}(K)) + \mathrm{IV}(P_t)$$

The **difference** between **selling the put** and **exercising it immediately** is:

sell put   −   T↑, P↓   +

$$P_t - (K - S_t) = -(K - \mathrm{PV}(K)) + \mathrm{IV}(P_t)$$

exercise.   time.

The insurance value is positive while the $\mathrm{PV}(K) - K < 0$. Therefore, it can be **optimal** to early exercise the put. Early exercise is more likely when:

- **volatility is low** which reduces insurance value;   high vola. → high insurance value.
- **interest rate is high** which reduces the present value of K.

American put options on dividend-paying assets

For dividend-paying assets, we have:

$$P_t = (K - S_t) - (K - \mathrm{PV}(K)) + \mathrm{PV}(D) + \mathrm{IV}(P_t)$$

The **difference** between **selling the put** and **exercising it immediately** is:

$$\underbrace{P_t}_{\text{sell put}} - (\underbrace{K - S_t}_{\text{exercise.}}) = -(K - \mathrm{PV}(K)) + \overset{+}{\mathrm{PV}(D)} + \overset{+}{\mathrm{IV}(P_t)}$$

The insurance value and the present value of dividend payment are positive. When $\mathrm{PV}(K) - K$ is very low, it can be **optimal** to early exercise the put.

Early exercise or not?

Conclusion:
We early exercise American call option
in real life.

*"A classic result by Merton (1973) is that, **except just before expiration or dividend payments**, one should **never exercise a call option** and **never convert a convertible bond**. We show theoretically that this result is **overturned when investors face frictions**. Early option exercise can be optimal when it reduces short-sale costs, transaction costs, or funding costs. We provide consistent empirical evidence, **documenting billions of dollars of early exercise for options and convertible bonds using unique data on actual exercise decisions and frictions**. Our model can explain as much as 98% of early exercises by market makers and 67% by customers."*

- Mads Vestergaard Jensen and Lasse Heje Pedersen, Early option exercise: Never say never, Journal of Financial Economics, Volume 121, Issue 2, 2016, Pages 278–299.

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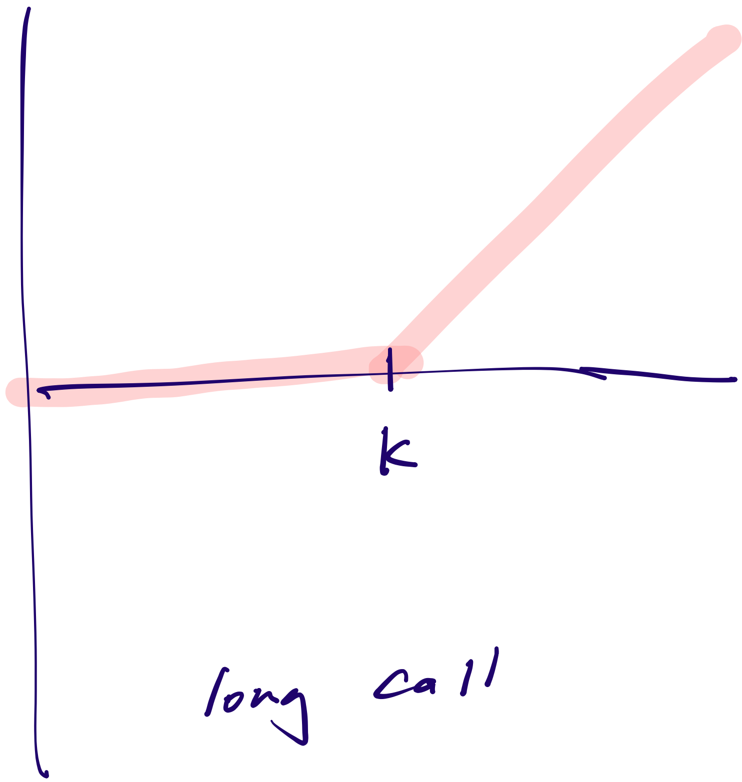
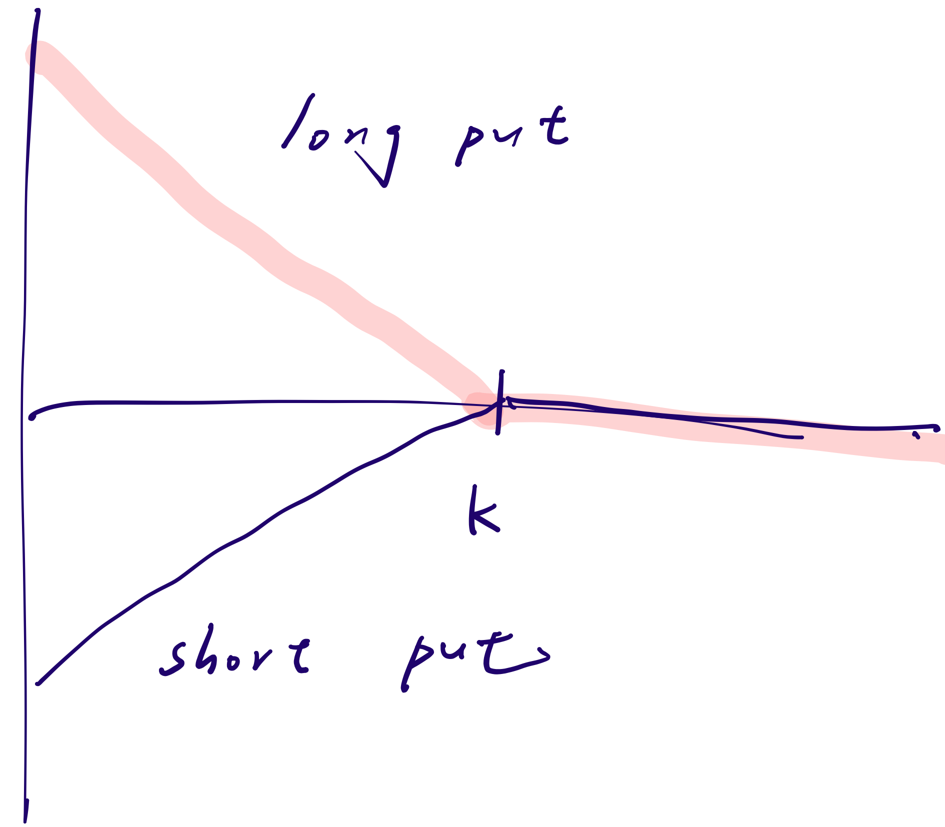
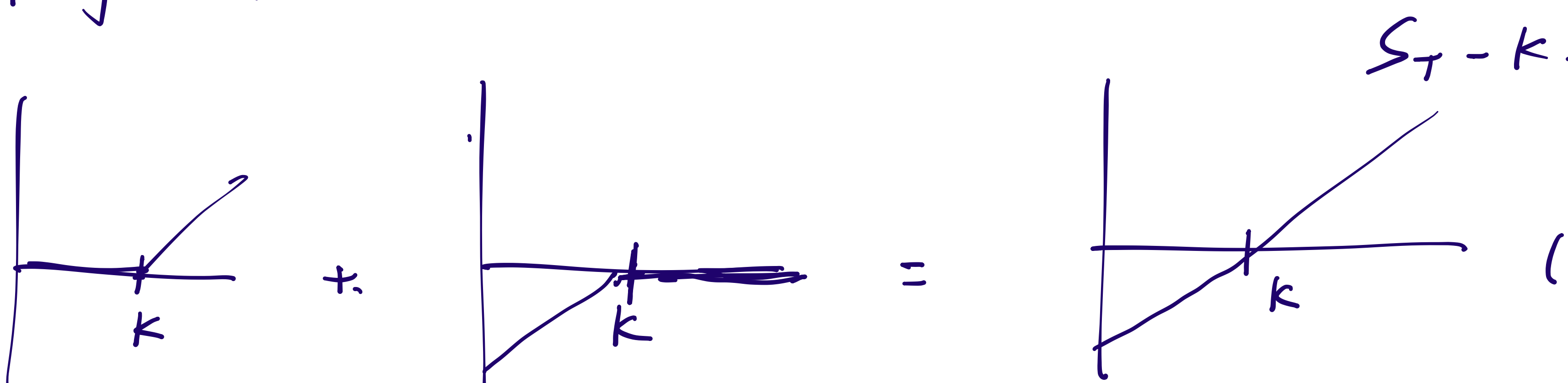
Put-call parity





* long call + short put (put call parity)



(Forward Contract)

$F_{0,T} = K$.

Buy call & sell put

$$= (+)\max[S_T - K, 0] \ (-) \max[K - S_T, 0]$$

$\downarrow$ buy $\qquad\qquad\qquad\qquad$ $\downarrow$ sell

$$\Rightarrow \quad S_T - K.$$

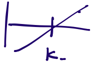
Cash flows of the put-call parity for European options

Table: Cash flows of the put-call parity

Investment	Cash flows at		
	t	$T: S_T < K$	$T: S_T \geq K$
Long call	$-C_{E,t}$ (buy)	0 OTM	$S_T - K$ ITM.
Invest $\mathrm{PV}(K)$	$-\mathrm{PV}(K)$ (buy)	K	K
Invest $\mathrm{PV}(D)$	$-\mathrm{PV}(D)$	D	D
Short put	$P_{E,t}$. (sell)	$S_T - K$. $-(K - S_T)$ ITM.	0 OTM
Short stock	S_t (sell)	$-S_T - D$	$-S_T - D$
	PV should be 0	sum = 0.	sum = 0.

FV = 0

Put-call parity for European options



Meaning:

$$\underbrace{C_{E,t} - P_{E,t}}_{\text{long call, short put}} = \underbrace{S_t - PV(K) - PV(D)}_{\text{synthetic forward}}$$

Since the net cash flows at T are always **zero**, by the **no-arbitrage condition**, we have:

$$S_t + P_{E,t} - C_{E,t} - \mathrm{PV}(K) - \mathrm{PV}(D) = 0 \quad \text{(column one)}$$

The **put-call parity** is then:

$$C_{E,t} + \mathrm{PV}(K) + \mathrm{PV}(D) = P_{E,t} + S_t$$

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Payoff diagram of the put-call parity