Financial derivatives

Lecture 6: Binomial option pricing model

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Primer of option pricing

Option payoffs

↳ non-linear payoff

Similar to pricing of forwards, we have to **price options through payoffs replication**. However, there are a few salient differences:

Euro
- Some options may not be exercised before their **expiration**.

US-
- Some options may offer **flexibility** in the timing of exercise.
- **Payoffs** of some options can be path-dependent.

Therefore, we cannot replicate all the outcomes of an option without modeling the underlying price over the contract life.

Option payoffs

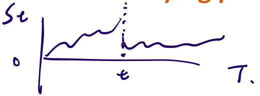
Non-path-dependent options:

European options: fluctuating is ignored, (S_T is enough)

- can only be **exercised at maturity**;
- suffice to **model terminal underlying prices**.

t: dividend payment.

Path-dependent options:



American options:

- can be **exercised anytime during their lifetimes**;
- need to know the **entire underlying price path**.

Asian options:

- payoffs depend on **lifetime averages of underlying prices**;
- need to know the **entire underlying price path**.

$$Avg = \frac{\sum_{i=1}^{T} S_i}{T}$$

Binomial model

$\boxed{\begin{array}{l} u: \text{upp} \\ d: \text{down} \end{array}}$

States

- $S_u = u S_0$, for $u > 1$.
- $S_d = d S_0$

No arbitrage condition: $u > (1 + \underbrace{r}_{\text{risk-free}}) > d$.
(only possible case)

How can we model underlying price movements?

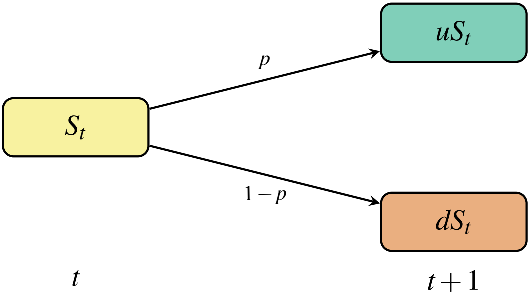
Using the **binomial model**, the underlying price follows:

$$S_{t+1} = \begin{cases} uS_t, & \text{with probability } p \\ dS_t, & \text{with probability } 1-p \end{cases}$$

In particular:

- p is the **physical probability** of being in the **up-state**;
- u is the multiplier of an **up move**;
- d is the multiplier of a **down move**.

From time t to $t+1$, the **underlying price** only goes up or down:



Restriction on u and d

The following **restriction must hold** for the binomial model:

$$u > 1 + r > d$$

If $1 + r > u$:

- the risk-free asset **dominates** the underlying asset;
- **arbitrage** by **buying the risk-free asset** and **short selling the underlying asset**.

If $d > 1 + r$:

- the underlying asset **dominates** the risk-free asset;
- **arbitrage** by **buying the underlying asset** and **borrowing at the risk-free rate**.

Pricing by replication

Underlying price movements (Stock Tree)

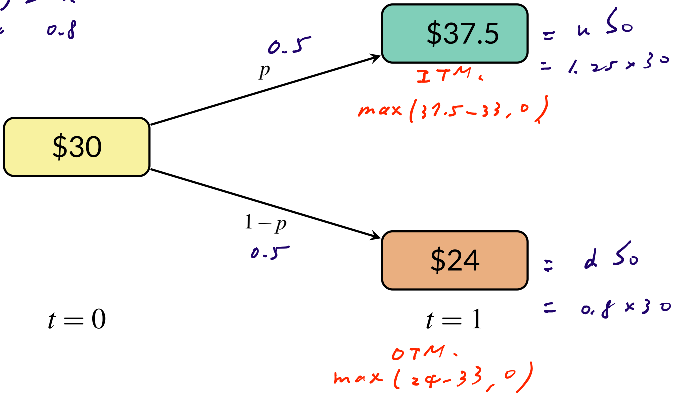
assumptions.

Suppose the price movements of a stock is given by a **one-period binomial model** with the following assumptions:

- $u = 1.25$; $d = 0.8$; (use $d = \frac{1}{u}$)
- $p = 0.5$; $r = 2.4\%$; and $S_0 = \$30$. → current market price.

$u > (1+r) > d.$

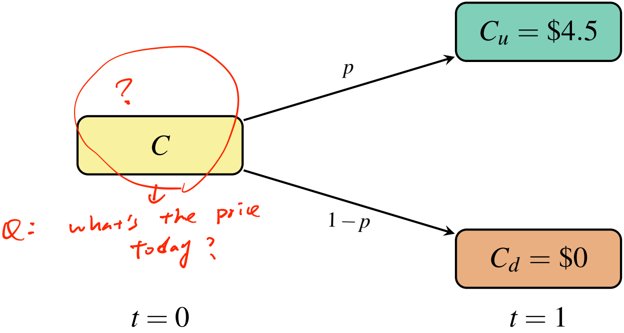
1.25 1.24 0.8



Replicating a call option

We want to **replicate a call option** with $K = \$33$. For instance:

- $C_u = \max[\$37.5 - \$33, 0] = \$\overset{4.5}{\cancel{5.35}}$;
- $C_d = \max[\$24 - \$33, 0] = \$0$.



Option Tree.

p : don't know

$E(r)$: estimation.

$$C_0 \begin{cases} u & 4.5 \\ d & 0 \end{cases}$$

$$C_0 = \frac{p(4.5) + (1-p)(0)}{1 + E(r)}$$

Let $C_0 = \Delta S_0 + B$. , let $r = 2.4\%$.

B ↓ dollar amount borrow or lend

$$\Rightarrow \Delta S_0 + B = C_0 \begin{cases} C_u = 4.5 = \Delta S_u + B(1+r) \\ C_d = 0 = \Delta S_d + B(1+r) \end{cases}$$

Solve:

$$\begin{aligned} & \Delta S_u + B(1+r) = 4.5 \\ -) \; & \Delta S_d + B(1+r) = 0 \quad \cdots \text{①} \\ \hline & \Delta (S_u - S_d) = 4.5 \end{aligned}$$

$S_u = 37.5$

$S_d = 24$

Hence $\Delta = \frac{1}{3}$

$\textcircled{D} \Rightarrow \left(\frac{1}{3}\right) S d + B(1+0.024) = 0$

$\Rightarrow B = \frac{-8}{1.024}$ #

replicating portfolio

求、 $C_0 = \Delta S_0 + B$, S_0 is given to be 30

$= \frac{1}{3} \times 30 - \frac{8}{1.024}$

$= 2.1875$ #

* Assume u & d.

prob p has no use at all

Replicating a call option

not "change"

To replicate the payoffs at $t = 1$, we form a portfolio consisting of:

- Δ units of the **underlying asset**;
- B dollars of the **risk-free asset** with return r.

In particular, we want the following **two equations** to hold:

$$\Delta \overset{S_u}{\underline{uS_0}} + (1+r)B = C_u$$

$$\Delta \underset{S_d}{\underline{dS_0}} + (1+r)B = C_d$$

After solving them:

$C_0 = \Delta S_0 + B.$

$$\Delta = \frac{C_u - C_d}{S_0(u-d)}$$

$$B = \frac{1}{(1+r)}\left[\frac{uC_d - dC_u}{u-d}\right]$$

Replicating a call option

for put :

$\Delta S_0 + B = P_0$

P_u : OTM,

P_d : ITM

Table: Replicating a call option

Investment	Cash flows at		
	$t=0$	$T=1: S_T<K$	$T=1: S_T \geq K$
Buy Δ underlying	$-\Delta S_t$	ΔS_T	ΔS_T
Invest B at r	$-B$	$B(1+r)$	$B(1+r)$
Net cash flows	$-(\Delta S_t+B)$	$\Delta dS_t+(1+r)B$	$\Delta uS_t+(1+r)B$

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Replicating a call option

Therefore, the **call option value** at $t = 0$ is:

$$C = \Delta S_0 + B$$
$$= \frac{C_u - C_d}{S_0(u-d)} S_0 + \frac{1}{(1+r)} \left[\frac{uC_d - dC_u}{u-d} \right]$$

By plugging in the numbers:

$$C = \frac{\$4.5}{\$30(1.25-0.8)} (\$30) + \frac{1}{(1.024)} \left[\frac{-0.8(\$4.5)}{1.25-0.8} \right]$$
$$= \frac{1}{3}(\$30) - \$7.8125$$
$$= \$2.1875$$ → the one we have done

In other words, we **replicate a call option** by:

- buying $\Delta = \frac{1}{3}$ units of the **underlying asset**; S.
- borrowing $B = -\$7.8125$ from the **risk-free rate.**

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Option delta Δ

The call option value is given by:

$$C = \Delta S + B$$

Call: $\Delta \geq 0 \rightarrow$ margin buy

$P = \Delta S + B$

Put: $\Delta \leq 0$

- **Delta Δ** is the number of shares of an **underlying asset** required to **replicate the option**.
- √ One share of option **embeds** Δ shares of the underlying asset.
- Δ also measures the **sensitivity** of the **option price** C with respect to a **change in the underlying asset price** S.

If we **differentiate** the option value against S:

· Option $= \Delta S + B$

$\Delta \downarrow$ const, $S \downarrow$ r.v., $B \downarrow$ const

$$\frac{\partial C}{\partial S} = \Delta$$

· To get hedge position:

Option $- \Delta S = B$ (const)

Arbitrage from an under/overvalued call option

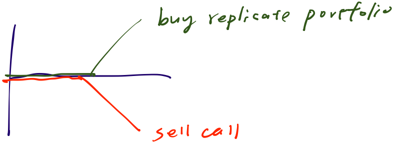
Fair value : 2.1875

Suppose the same call option is trading at \$2.25 (**overvalued**), we earn an **arbitrage profit** by:

- **selling the call option** at \$2.25; (earn)
- **buying the replicating portfolio** at \$2.1875. (cost) → 百己組合

If its price is \$2 (**undervalued**), we make an **arbitrage profit** by:

- **buying the call option** at \$2;
- **selling the replicating portfolio** at \$2.1875.



spread : 2.25 − 2.1875
(per transaction)

Risk-neutral probability

$C_0 = \Delta S_0 + B$

knowing $\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{C_u - C_d}{S_0(u-d)}$

$C_u = \Delta S_u + (1+r) B$

$\Rightarrow C_u = \frac{C_u - C_d}{S_0(u-d)} S_u + (1+r) B.$

$\Rightarrow C_u - \frac{C_u - C_d}{S_0(u-d)} S_u = (1+r) B.$

$S_u.$

$\Rightarrow B = \frac{1}{1+r}\left[\frac{S_0 C_u(u-d) - C_u S_u + C_d S_u}{S_0(u-d)}\right].$

$= \frac{1}{1+r}\left[\frac{C_u(\cancel{S_u} - S_d) - \cancel{C_u S_u} + C_d S_u}{S_0(u-d)}\right]$

$\Rightarrow \frac{1}{1+r}\left[\frac{\cancel{S_0}(C_u \times -d) + \cancel{S_0}(C_d \times u)}{\cancel{S_0}(u-d)}\right]$

$\Rightarrow \frac{1}{1+r}\left[\frac{u C_d - d C_u}{(u-d)}\right]$ #

Result:

$$C_0 = \Delta S_0 + B$$

$$= \frac{C_u - C_d}{\cancel{S_0}(u-d)} \cancel{S_0} + \frac{1}{1+r}\left[\frac{u C_d - d C_u}{(u-d)}\right]$$

$$\Rightarrow \frac{(1+r)}{(1+r)} \frac{C_u - C_d}{(u-d)} + \frac{1}{1+r}\left[\frac{u C_d - d C_u}{(u-d)}\right]$$

$$\Rightarrow \frac{C_u(1+r-d) + C_d(u-1-r)}{(1+r)(u-d)}$$

$$\Rightarrow \frac{1}{1+r}\left[\frac{(1+r-d)}{u-d} C_u + \frac{(u-1-r)}{u-d} C_d\right]$$

Define:

$$q = \frac{1+r-d}{u-d},$$

$$\Rightarrow 1-q = 1 - \frac{(1+r-d)}{u-d} = \frac{u-1-r}{u-d}. \quad \#$$

Simplify: $C_0 = \frac{1}{1+r}\left[q C_u + (1-q) C_d\right]$

$\downarrow$

target

$\Rightarrow \quad C_0 = \frac{1}{1+r} \; E^{Q}(C_1)$ ... structure.

$\frac{1}{1+r}$ ↓ discounting factor

$E^{Q}(C_1)$ ↓ expected cash flow

However, C_0 still $= \Delta S_0 + B$.

Hence, it's not actually a prob func. (no uncertainty)

And we call q : risk-neutral prob.

$$q = \frac{1+r-d}{u-d}$$

, Look back to prob laws:

① $\begin{cases} q \geq 0 \\ 1-q \geq 0 \end{cases}$

② $\sum_{i=1}^{n} q_i = 1.$

(pf.)

$u > (1+r) > d$

$\frac{1+r-d}{u-d} > 0, \quad \frac{u-(1+r)}{u-d} > 0$

$q + (1-q) = 1$

Risk-neutral probability

So far, the general formula for pricing a call option is:

$$
\begin{aligned}
C &= \Delta S + B \\
&= \frac{C_u - C_d}{(u-d)} + \frac{1}{(1+r)}\left[\frac{uC_d - dC_u}{u-d}\right] \\
&= \frac{(1+r)C_u - dC_u}{(1+r)(u-d)} + \frac{uC_d - (1+r)C_d}{(1+r)(u-d)} \\
&= \frac{1}{1+r}\left[\underbrace{\frac{(1+r)-d}{u-d}}_{(q)} C_u + \underbrace{\frac{u-(1+r)}{u-d}}_{(1-q)} C_d\right]
\end{aligned}
$$

By defining the **risk-neutral probability** as:

$$q = \frac{(1+r)-d}{u-d} \quad \text{and} \quad 1-q = \frac{u-(1+r)}{u-d}$$

When $u > 1+r > d$, q behaves just like probability as:

$$1 > q > 0 \quad \text{and} \quad q + (1-q) = 1$$

Change of measure :

Replace P (physical prob) with

Measure Q (martingale measure)

$\downarrow$

risk neutral.

Risk-neutral probability

discounting factor; originally : $\frac{1}{1+E(r)} = ?$

As we **replicate the option future payoffs perfectly state by state**:

- we can use the **risk-free rate** to discount the **risk-neutral expectation** of C_u and C_d;
- we can replace the physical probability measure $\mathbb{P}$ with the equivalent martingale measure $\mathbb{Q}$ in pricing derivatives.

Therefore, we have:

$$S_t = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_T] = \frac{1}{1+r+\lambda}\mathbb{E}^{\mathbb{P}}[S_T]$$

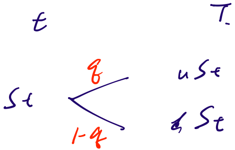
observed price

Risk premium. (eg. CAPM)

In a risk-neutral world:

- **Risk premium** λ is **always zero**.
- **Expected returns** of all assets are **always the risk-free rate**. r_f.

Risk-neutral probability



Under the **risk-neutral measure** $\mathbb{Q}$, we can show that the **price** of the **underlying asset** is expected to grow at the **risk-free rate**:

S_t

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[S_T] &= quS_t + (1-q)dS_t \\
&= \frac{(1+r)-d}{u-d}uS_t + \frac{u-(1+r)}{u-d}dS_t \\
&= \frac{(1+r)(u-d)S_t}{u-d} \\
&= (1+r)S_t
\end{aligned}
$$

$\Rightarrow \quad E^{Q}\left(\frac{S_T}{S_t}\right) = (1+r)$

$\downarrow$

growing rate

Risk-neutral option pricing

Risk-neutral option pricing

Hence, we have the **risk-neutral pricing formula** for call options:

$$C = \frac{1}{1+r}\left[qC_u + (1-q)C_d\right]$$

If we generalize the model further by defining:

- T: **time to maturity** of an option;
- $h = T/n$: **time step** used in the binomial model; n: steps.
- r: **continuously compounded risk-free rate**;
- δ: **continuously dividend rate**;
- S: **current price** of the underlying asset.

The formula becomes:

$$C = e^{-rh}\left[qC_u + (1-q)C_d\right]$$

where:

$$q = \frac{e^{(r-\delta)h} - d}{u-d}.$$

Risk-neutral option pricing

Table: Replicating a call option

Investment	Cash flows at		
	t	$T : S_T < K$	$T : S_T \geq K$
Buy Δ underlying	$-\Delta S_t$	$\Delta d S_t e^{\delta h}$	$\Delta u S_t e^{\delta h}$
Invest B at r	$-B$	Be^{rh}	Be^{rh}
Net cash flows	$-(\Delta S_t + B)$	$\Delta d S_t e^{\delta h} + Be^{rh}$	$\Delta u S_t e^{\delta h} + Be^{rh}$

We need to solve the following **two equations**:

$$\Delta u S_t e^{\delta h} + Be^{rh} = C_u$$

$$\Delta d S_t e^{\delta h} + Be^{rh} = C_d$$

Risk-neutral option pricing

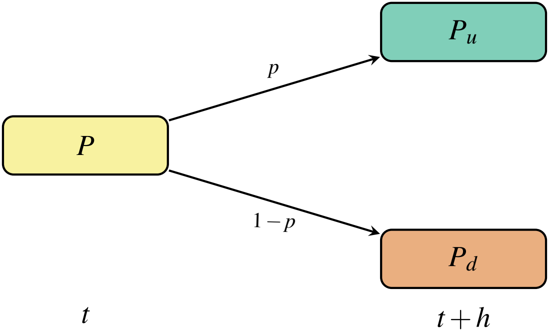
Similarly, the **risk-neutral pricing formula** for put options is:

$$P = \frac{1}{1+r}\left[qP_u + (1-q)P_d\right]$$

or:

$$P = e^{-rh}\left[qP_u + (1-q)P_d\right]$$

The binomial tree is given by:



Binomial tree

Cox-Ross-Rubinstein (CRR) tree

Following the **CRR** approach, we can form a n-period binomial tree with maturity of T by:

$h = \frac{T}{n}.$

$$u = e^{\sigma\sqrt{T/n}} = e^{\sigma\sqrt{h}}$$

σ: only source of risk

define: $d = \frac{1}{u} = e^{-\sigma\sqrt{h}}$

$$q = \frac{e^{rh} - d}{u - d} \approx \frac{(1+r) - d}{u - d}$$

(we have derived!)

The ratio of u/d reflects the **volatility** of the underlying asset:

$$\sigma = \left[\frac{1}{2\sqrt{h}}\right] \ln\left[\frac{u}{d}\right]$$

$\frac{u}{d} = e^{2\sigma\sqrt{h}} \Rightarrow \ln\left(\frac{u}{d}\right) = 2\sigma\sqrt{h}.$

$\Rightarrow \ln\left(\frac{u}{d}\right)\frac{1}{2\sqrt{h}} = \sigma$ #

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One-period binomial tree

half year contract

Suppose $S = \$10$, $T = 0.5$, $n = 1$, and $\sigma = 0.2$:

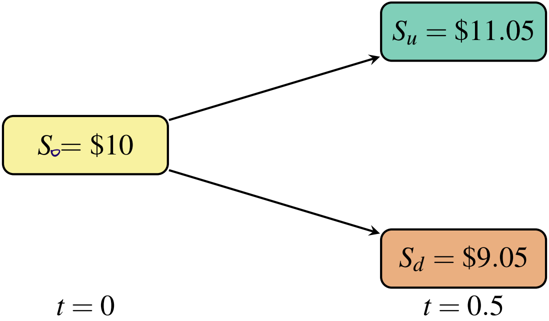
$h = \frac{0.5}{1} = 0.5$

$$S_u = \$10e^{0.2\sqrt{0.5}} = \$11.05$$

$$S_d = \$10e^{-0.2\sqrt{0.5}} = \$9.05$$

$S_u = S_0 e^{\sigma\sqrt{h}}$

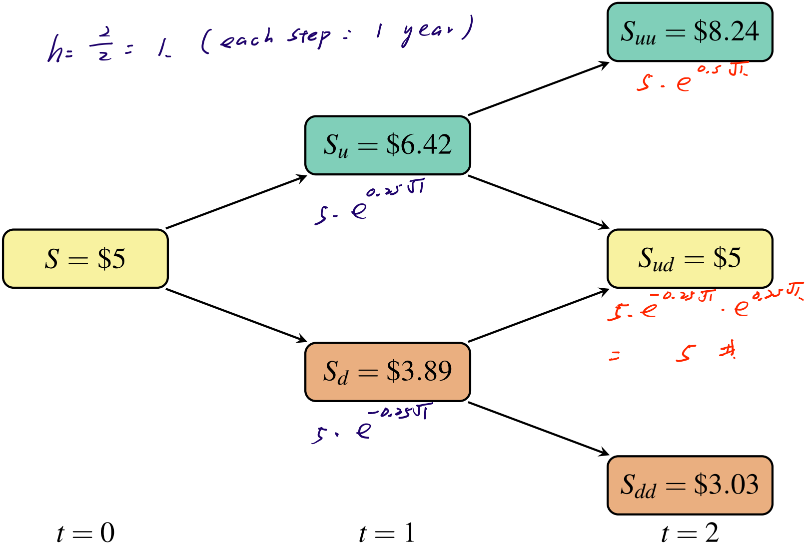
$S_d = S_0 e^{-\sigma\sqrt{h}}$



Two-period binomial tree

Suppose $S = \$5$, $T = 2$, $n = 2$, and $\sigma = 0.25$:

$h = \frac{2}{2} = 1$ (each step : 1 year)

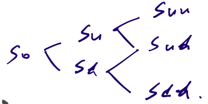


Binomial tree and option pricing

Using the values on the n-period binomial tree, we can use the **risk-neutral pricing formula** to price options through **backward induction:** (start at T, work back to $t=0$)

1. Determine the $n+1$ **terminal payoffs** of an option at T.
2. Calculate the **risk-neutral probability** q.
3. Compute the $T-h$ **period option value** from the payoffs.
4. Check for **early exercise** in **American options** by comparing the **exercise value** with the **European option value**.
5. **Repeat** the same process until $t=0$.

1: $n=2$, 3 terminal payoffs.

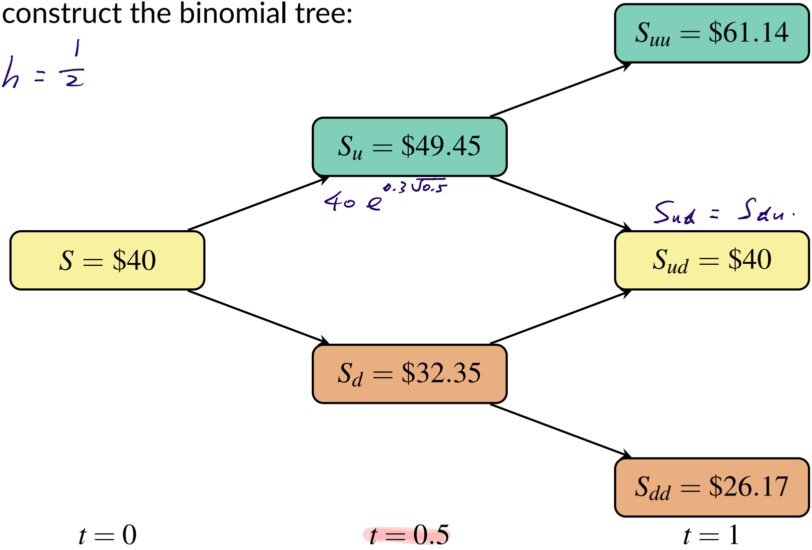


European options

European call options

Given that $S = \$40$, $r = 8\%$, $\sigma = 30\%$, $T = 1$, and $n = 2$, we first construct the binomial tree:

$h = \frac{1}{2}$



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European call options

To price a European call option with $K = \$40$, we compute the **risk-neutral probability** as:

$$u = e^{(30\%)\sqrt{0.5}} = 1.236$$

$$d = e^{-(30\%)\sqrt{0.5}} = 0.809$$

$$q = \frac{\overbrace{e^{0.5(8\%)}}^{e^{rh}} - 0.809}{1.236 - 0.809} = 0.543$$

The **terminal payoffs** are:

$S_{uu} - K.$

ITM $\quad C_{uu} = \$61.14 - \$40 = \$21.139$ $\quad C_{uu} = \max(S_{uu} - K, 0)$

ATM $\quad C_{ud} = C_{du} = \$0$ $\quad C_{ud} = \max(40 - 40, 0) = 0$

OTM, $\quad C_{dd} = \$0$

$$C_u \quad \begin{cases} q: & C_{uu} = 21.14 \\ 1-q: & C_{ud} = 0 \end{cases}$$

$$C_d \quad \begin{cases} q: & C_{ud} = 0 \\ 1-q: & C_{dd} = 0 \end{cases}$$

$$C_u = e^{-rh}\left[q\, C_{uu} + (1-q)\, C_{ud} \right].$$

$r = 0.08$

$h = 0.5$

$$= e^{-\frac{0.08}{2}}\left[0.543 \times 21.139 + 0 \right].$$

$C_d = 0$ since C_{ud} & C_{dd} both $= 0$

Hence,

$$C = e^{-\frac{0.08}{2}}\left[0.543 \times 11.021 + 0 \right] = 5.746$$

European call options

The **intermediate option prices** are:

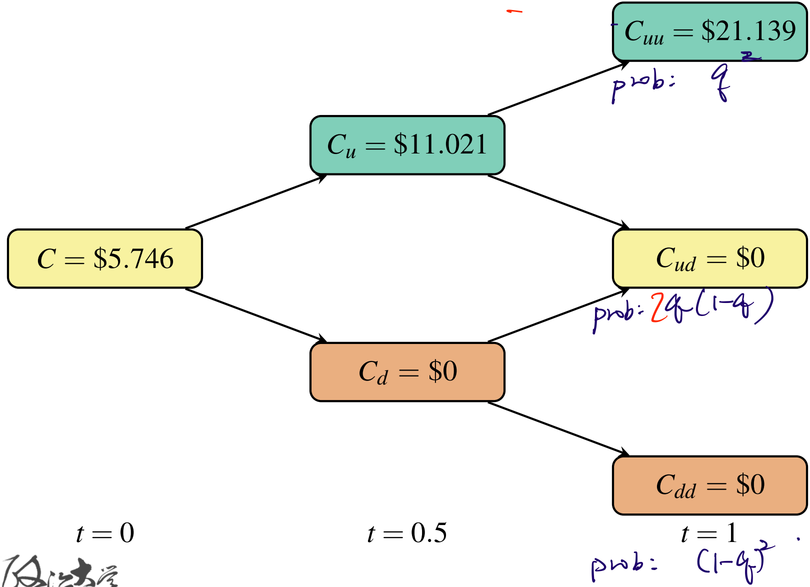
$$C_u = e^{-0.5(8\%)}[0.543(\$21.139)] = \$11.021$$
$$C_d = \$0$$

The **current option price** is:

$$C = e^{-0.5(8\%)}[0.543(\$11.021)] = \$5.746$$

European call options

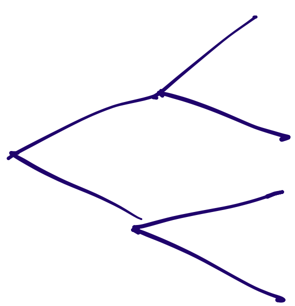
Lastly, the pricing tree of the European call option is:



$\sum$ Terminal State prob

$$= q^2 + 2q(1-q) + (1-q)^2$$

$$= [q + (1-q)]^2 = 1.$$



$\to c_2 = 1$

$\to c_1 = 2$

$\to c_0 = 1.$

c_n : n 个 up step.

$$C_0 = e^{-rT} [$$

European put options

To price a European put option with $K = \$41$, we determine its **terminal payoffs** to be:

$$P_{uu} = \$0$$
$$P_{ud} = P_{du} = \$1$$
$$P_{dd} = \$14.830$$

61.14
Puu = (41 − 61.14, 0)
= 0

40
Pud = (41 − 40, 0) = 1

26.17
Pdd = (41 − 26.17, 0)
= 14.83

The **intermediate option prices** are:

h r (1 − q) · Pud

$$P_u = e^{-0.5(8\%)}[0.457(\$1)] = \$0.439$$

$$P_d = e^{-0.5(8\%)}[0.543(\$1) + 0.457(\$14.830)] = \$7.038$$

q Pud 1 − q Pdd

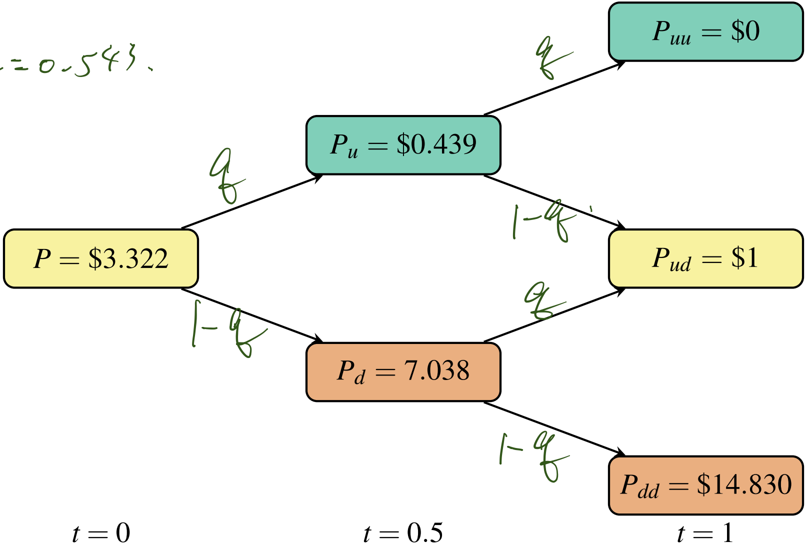
The **current option price** is:

$$P = e^{-0.5(8\%)}[0.543(\$0.439) + 0.457(\$7.038)] = \$3.322$$

European put options

Finally, the pricing tree of the European put option is:

$q = 0.543.$



American options

American options

For **American options**, we need to **check for early exercise** at each node. Specifically, we compare the **immediate exercise value** with the associated **European option value**:

- For call options:

$\tau < T.$

$$C_{A,\tau} = \max[S_\tau - K, C_{E,\tau}]$$

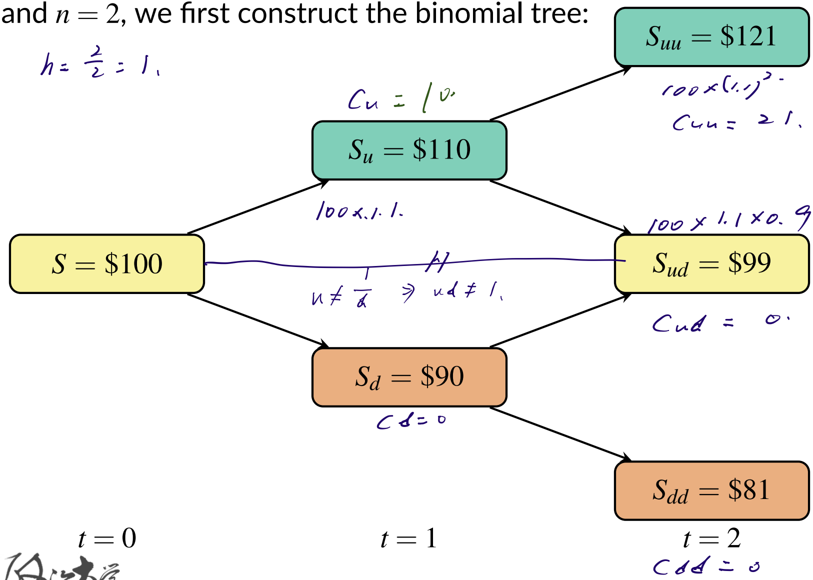
- For put options:

$$P_{A,\tau} = \max[K - S_\tau, P_{E,\tau}]$$

American call options

discrete

Given that $S = \$100$, $1 + r = 1.02$, $u = 1.1$, $d = 0.9$, $\delta = 0.05$, $T = 2$, and $n = 2$, we first construct the binomial tree:



American call options

continuous : $\dfrac{e^{(r-\delta)h} - d}{u - d}$ (for exam)

To price an American call option with $K = 100$, we compute the **risk-neutral probability** as:

$$\text{(discrete)} \quad q = \frac{1.02(1-0.05)-0.9}{1.1-0.9} = 0.345 \quad \frac{(1+r)(1-\delta)-d}{u-d}$$

The terminal payoffs are:

$$C_{A,uu} = \$121 - \$100 = \$21$$
$$C_{A,ud} = C_{A,du} = \$0$$
$$C_{A,dd} = \$0$$

American call options

The **intermediate European call prices** are:

$$C_{E,u} = \frac{1}{1.02}[0.345(\$21)] = \$7.103 \quad = \frac{1}{1+r}[q\, C_{uu} + (1-q)\, C_{ud}]$$

$$C_{E,d} = \$0$$

So it is **optimal to early exercise the American call** in the up-state:

$$C_{A,u} = \max[\underbrace{\overbrace{\$110 - \$100}^{C_{A,u}}}_{\text{early exercise}}, \underbrace{\overbrace{\$7.103}^{C_{E,u}}}_{\text{waiting value}}] = \$10$$

$$C_{A,d} = \$0$$

The **current American call price** is:

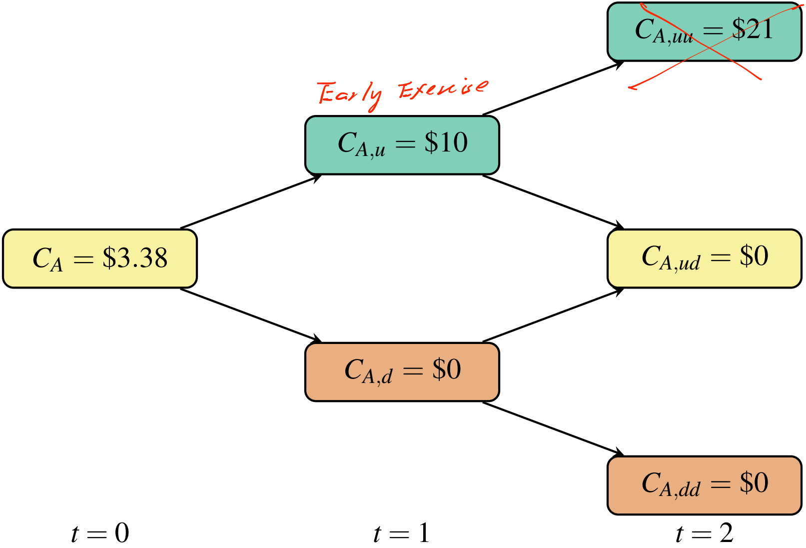
back to $t=0$

$$C_A = \frac{1}{1.02}[0.345(\$10)] = \$3.38 > C_E = \$2.403$$

$$\frac{1}{1+r}[q \cdot C_{A,u}]. \qquad\qquad \frac{1}{1+r}[q \cdot 7.103].$$

American call options

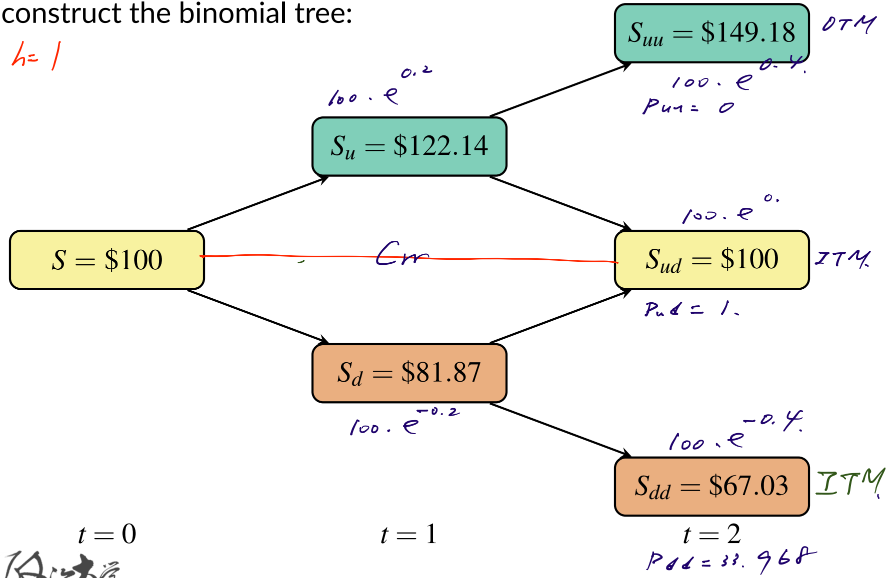
As a result, the pricing tree of the American call option is:



American put options

Given that $S = \$100$, $r = 0.05$, $\sigma = 0.2$, $T = 2$, and $n = 2$, we first construct the binomial tree:

$h = 1$



American put options

To price an **American put option** with $K = \$101$, we compute the **risk-neutral probability** as:

$$q = \frac{e^{0.05} - 0.8187}{1.2214 - 0.8187} = 0.5775 \qquad q = \frac{e^{rh} - d}{u - d}$$

Its **terminal payoffs** are:

$$P_{A,uu} = \$0$$
$$P_{A,ud} = P_{A,du} = \$1$$
$$P_{A,dd} = \$33.968$$

The **intermediate European put prices** are:

$$P_{E,u} = e^{-0.05}[0.4225(\$1)] = \$0.4019$$
$$P_{E,d} = e^{-0.05}[0.5775(\$1) + 0.4225(\$33.968)] = \$14.2009$$

American put options

$p = 0.4225$ chance to get 0.4225

Again, it is **optimal to early exercise the American put** in the down-state:

$$P_{A,u} = \$0.4019$$

$P_{E,d}$.

$$P_{A,d} = \max[\$101 - \$81.8731, \$14.2009] = \$19.1269$$

The **current American put price** is:

$$P_A = e^{-0.05}[0.5775(\$0.4019) + 0.4225(\$19.1269)]$$
$$= \$7.9079 > P_E = \$5.9282$$

American put options

Ultimately, the pricing tree of the American put option is: