Financial derivatives

Lecture 6: Binomial option pricing model

Douglas Chung

Primer of option pricing

Option payoffs

Similar to pricing of forwards, we have to **price options through payoffs replication**. However, there are a few salient differences:

- Some options may not be exercised before their **expiration**.
- Some options may offer **flexibility** in the timing of exercise.
- **Payoffs** of some options can be path-dependent.

Therefore, we cannot replicate all the outcomes of an option without modeling the underlying price over the contract life.

Option payoffs

Non-path-dependent options:

European options:

- can only be **exercised at maturity**;
- suffice to **model terminal underlying prices**.

Path-dependent options:

American options:

- can be **exercised anytime during their lifetimes**;
- need to know the **entire underlying price path**.

Asian options:

- payoffs depend on **lifetime averages of underlying prices**;
- need to know the **entire underlying price path**.

Binomial model

How can we model underlying price movements?

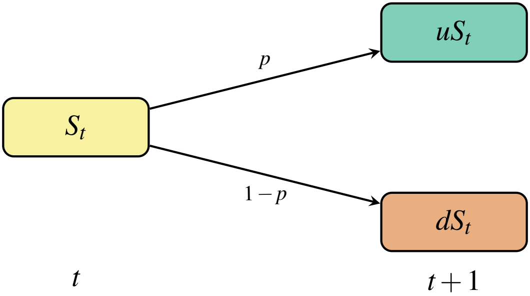
Using the **binomial model**, the underlying price follows:

$$S_{t+1} = \begin{cases} uS_t, & \text{with probability } p \\ dS_t, & \text{with probability } 1-p \end{cases}$$

In particular:

- p is the **physical probability** of being in the **up-state**;
- u is the multiplier of an **up move**;
- d is the multiplier of a **down move**.

From time t to $t+1$, the **underlying price** only goes up or down:



Restriction on u and d

The following **restriction must hold** for the binomial model:

$$u > 1 + r > d$$

If $1 + r > u$:

- the risk-free asset **dominates** the underlying asset;
- **arbitrage** by **buying the risk-free asset** and **short selling the underlying asset**.

If $d > 1 + r$:

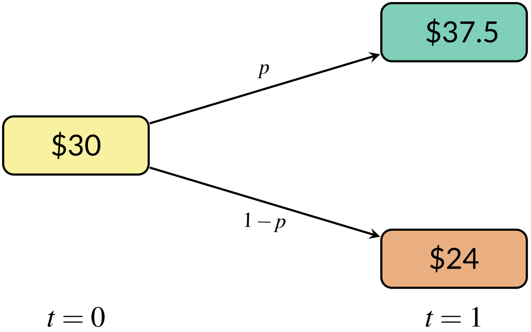
- the underlying asset **dominates** the risk-free asset;
- **arbitrage** by **buying the underlying asset** and **borrowing at the risk-free rate**.

Pricing by replication

Underlying price movements

Suppose the price movements of a stock is given by a **one-period binomial model** with the following assumptions:

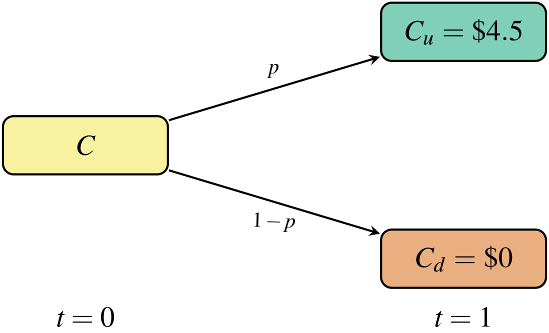
- $u = 1.25$; $d = 0.8$;
- $p = 0.5$; $r = 2.4\%$; and $S_0 = \$30$.



Replicating a call option

We want to **replicate a call option** with $K = \$33$. For instance:

- $C_u = \max[\$37.5 - \$33, 0] = \$5.35$;
- $C_d = \max[\$24 - \$33, 0] = \$0$.



Replicating a call option

To replicate the payoffs at $t = 1$, we form a portfolio consisting of:

- Δ units of the **underlying asset**;
- B dollars of the **risk-free asset** with return r.

In particular, we want the following **two equations** to hold:

$$\Delta u S_0 + (1+r)B = C_u$$
$$\Delta d S_0 + (1+r)B = C_d$$

After solving them:

$$\Delta = \frac{C_u - C_d}{S_0(u-d)}$$
$$B = \frac{1}{(1+r)}\left[\frac{uC_d - dC_u}{u-d}\right]$$

Replicating a call option

Table: Replicating a call option

Investment	**Cash flows at**		
	$t=0$	$T=1: S_T < K$	$T=1: S_T \geq K$
Buy Δ underlying	$-\Delta S_t$	ΔS_T	ΔS_T
Invest B at r	$-B$	$B(1+r)$	$B(1+r)$
Net cash flows	$-(\Delta S_t + B)$	$\Delta d S_t + (1+r)B$	$\Delta u S_t + (1+r)B$

Replicating a call option

Therefore, the **call option value** at $t = 0$ is:

$$
\begin{aligned}
C &= \Delta S_0 + B \\
&= \frac{C_u - C_d}{S_0(u-d)} S_0 + \frac{1}{(1+r)} \left[\frac{uC_d - dC_u}{u-d} \right]
\end{aligned}
$$

By plugging in the numbers:

$$
\begin{aligned}
C &= \frac{\$4.5}{\$30(1.25-0.8)} (\$30) + \frac{1}{(1.024)} \left[\frac{-0.8(\$4.5)}{1.25-0.8} \right] \\
&= \frac{1}{3} (\$30) - \$7.8125 \\
&= \$2.1875
\end{aligned}
$$

In other words, we **replicate a call option** by:

- buying $\Delta = \frac{1}{3}$ units of the **underlying asset**;
- borrowing $B = -\$7.8125$ from the **risk-free rate**.

Option delta Δ

The call option value is given by:

$$C = \Delta S + B$$

- **Delta** Δ is the number of shares of an **underlying asset** required to **replicate the option**.
- One share of option **embeds** Δ shares of the underlying asset.
- Δ also measures the **sensitivity** of the **option price** C with respect to a **change in the underlying asset price** S.

If we **differentiate** the option value against S:

$$\frac{\partial C}{\partial S} = \Delta$$

Arbitrage from an under/overvalued call option

Suppose the same call option is trading at \$2.25 (overvalued), we earn an **arbitrage profit** by:

- selling the call option at \$2.25;
- buying the replicating portfolio at \$2.1875.

If its price is \$2 (undervalued), we make an **arbitrage profit** by:

- buying the call option at \$2;
- selling the replicating portfolio at \$2.1875.

Risk-neutral probability

Risk-neutral probability

So far, the general formula for pricing a call option is:

$$
\begin{aligned}
C &= \Delta S + B \\
&= \frac{C_u - C_d}{(u-d)} + \frac{1}{(1+r)}\left[\frac{uC_d - dC_u}{u-d}\right] \\
&= \frac{(1+r)C_u - dC_u}{(1+r)(u-d)} + \frac{uC_d - (1+r)C_d}{(1+r)(u-d)} \\
&= \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}C_u + \frac{u-(1+r)}{u-d}C_d\right]
\end{aligned}
$$

By defining the **risk-neutral probability** as:

$$q = \frac{(1+r)-d}{u-d} \quad \text{and} \quad 1-q = \frac{u-(1+r)}{u-d}$$

When $u > 1+r > d$, q behaves just like probability as:

$$1 > q > 0 \quad \text{and} \quad q + (1-q) = 1$$

Risk-neutral probability

As we **replicate the option future payoffs perfectly state by state**:

- we can use the **risk-free rate** to discount the **risk-neutral expectation** of C_u and C_d;
- we can replace the physical probability measure $\mathbb{P}$ with the equivalent martingale measure $\mathbb{Q}$ in pricing derivatives.

Therefore, we have:

$$S_t = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_T] = \frac{1}{1+r+\lambda}\mathbb{E}^{\mathbb{P}}[S_T]$$

In a risk-neutral world:

- **Risk premium** λ is **always zero**.
- **Expected returns** of all assets are **always the risk-free rate**.

Risk-neutral probability

Under the risk-neutral measure $\mathbb{Q}$, we can show that the **price** of the **underlying asset** is expected to grow at the **risk-free rate**:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[S_T] &= quS_t + (1-q)dS_t \\
&= \frac{(1+r)-d}{u-d}uS_t + \frac{u-(1+r)}{u-d}dS_t \\
&= \frac{(1+r)(u-d)S_t}{u-d} \\
&= (1+r)S_t
\end{aligned}
$$

Risk-neutral option pricing

Risk-neutral option pricing

Hence, we have the **risk-neutral pricing formula** for call options:

$$C = \frac{1}{1+r}\left[qC_u + (1-q)C_d\right]$$

If we generalize the model further by defining:

- T: **time to maturity** of an option;
- $h = T/n$: **time step** used in the binomial model;
- r: **continuously compounded risk-free rate**;
- δ: **continuously dividend rate**;
- S: **current price** of the underlying asset.

The formula becomes:

$$C = e^{-rh}\left[qC_u + (1-q)C_d\right]$$

where:

$$q = \frac{e^{(r-\delta)h} - d}{u-d}.$$

Risk-neutral option pricing

Table: Replicating a call option

Investment	**Cash flows at**		
	t	$T: S_T < K$	$T: S_T \geq K$
Buy Δ underlying	$-\Delta S_t$	$\Delta d S_t e^{\delta h}$	$\Delta u S_t e^{\delta h}$
Invest B at r	$-B$	Be^{rh}	Be^{rh}
Net cash flows	$-(\Delta S_t + B)$	$\Delta d S_t e^{\delta h} + Be^{rh}$	$\Delta u S_t e^{\delta h} + Be^{rh}$

We need to solve the following **two equations**:

$$\Delta u S_t e^{\delta h} + Be^{rh} = C_u$$

$$\Delta d S_t e^{\delta h} + Be^{rh} = C_d$$

Risk-neutral option pricing

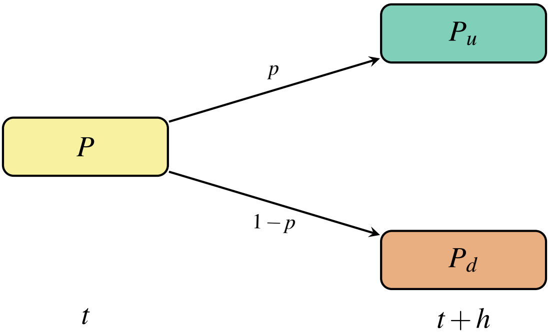
Similarly, the **risk-neutral pricing formula** for put options is:

$$P = \frac{1}{1+r}\left[qP_u + (1-q)P_d\right]$$

or:

$$P = e^{-rh}\left[qP_u + (1-q)P_d\right]$$

The binomial tree is given by:



Binomial tree

Cox-Ross-Rubinstein (CRR) tree

Following the **CRR** approach, we can form a n-period binomial tree with maturity of T by:

$$
\begin{aligned}
u &= e^{\sigma\sqrt{T/n}} = e^{\sigma\sqrt{h}} \\
d &= \frac{1}{u} = e^{-\sigma\sqrt{h}} \\
q &= \frac{e^{rh} - d}{u - d}
\end{aligned}
$$

The ratio of u/d reflects the **volatility** of the underlying asset:

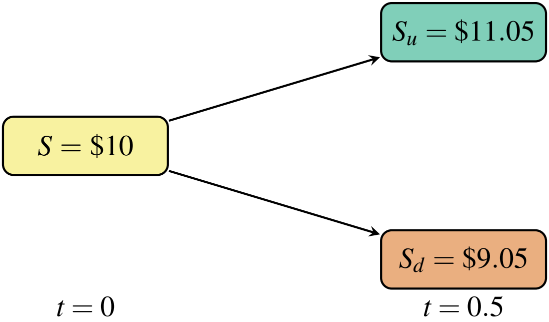
$$\sigma = \left[\frac{1}{2\sqrt{h}}\right] \ln\left[\frac{u}{d}\right]$$

One-period binomial tree

Suppose $S = \$10$, $T = 0.5$, $n = 1$, and $\sigma = 0.2$:

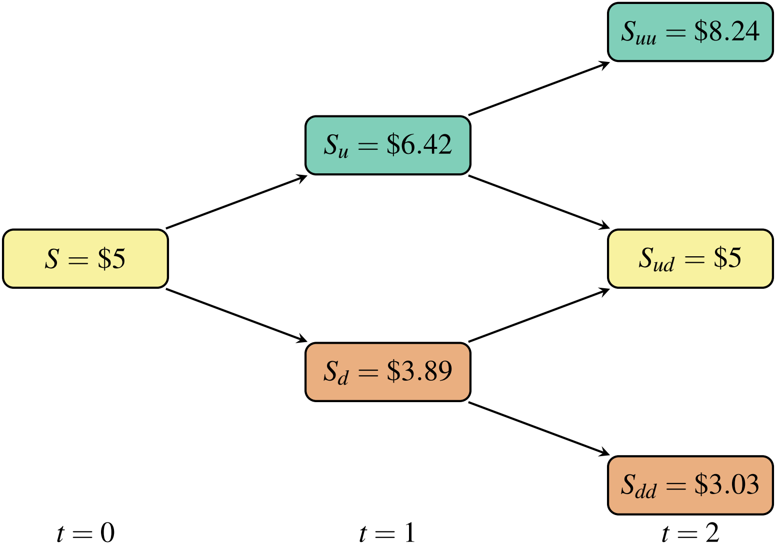
$$S_u = \$10e^{0.2\sqrt{0.5}} = \$11.05$$

$$S_d = \$10e^{-0.2\sqrt{0.5}} = \$9.05$$



Two-period binomial tree

Suppose $S = \$5$, $T = 2$, $n = 2$, and $\sigma = 0.25$:



Binomial tree and option pricing

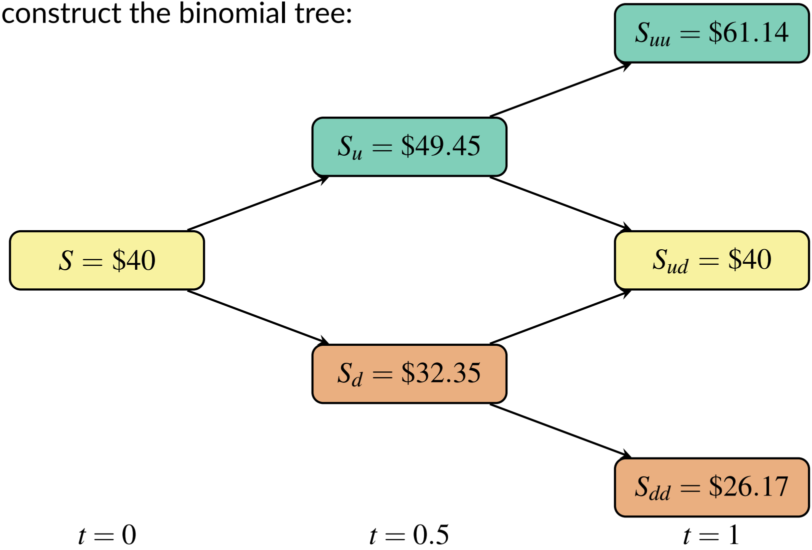
Using the values on the n-period binomial tree, we can use the **risk-neutral pricing formula** to price options through **backward induction**:

1. Determine the $n+1$ **terminal payoffs** of an option at T.
2. Calculate the **risk-neutral probability** q.
3. Compute the $T-h$ **period option value** from the payoffs.
4. Check for **early exercise** in **American options** by comparing the **exercise value** with the **European option value**.
5. **Repeat** the same process until $t=0$.

European options

European call options

Given that $S = \$40$, $r = 8\%$, $\sigma = 30\%$, $T = 1$, and $n = 2$, we first construct the binomial tree:



European call options

To price a European call option with $K = \$40$, we compute the **risk-neutral probability** as:

$$u = e^{(30\%)\sqrt{0.5}} = 1.236$$
$$d = e^{-(30\%)\sqrt{0.5}} = 0.809$$
$$q = \frac{e^{0.5(8\%)} - 0.809}{1.236 - 0.809} = 0.543$$

The **terminal payoffs** are:

$$C_{uu} = \$61.14 - \$40 = \$21.139$$
$$C_{ud} = C_{du} = \$0$$
$$C_{dd} = \$0$$

European call options

The **intermediate option prices** are:

$$C_u = e^{-0.5(8\%)}[0.543(\$21.139)] = \$11.021$$
$$C_d = \$0$$

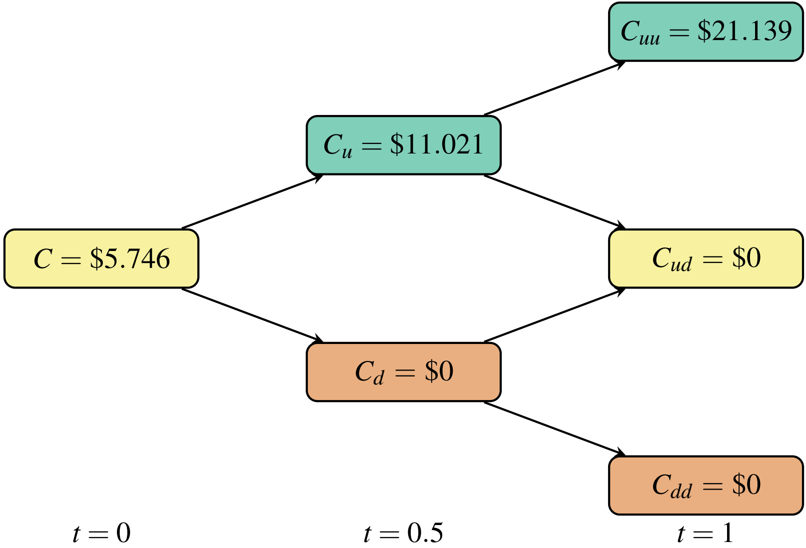
The **current option price** is:

$$C = e^{-0.5(8\%)}[0.543(\$11.021)] = \$5.746$$

NATIONAL CHENGCHI UNIVERSITY

European call options

Lastly, the pricing tree of the European call option is:



European put options

To price a European put option with $K = \$41$, we determine its **terminal payoffs** to be:

$$P_{uu} = \$0$$
$$P_{ud} = P_{du} = \$1$$
$$P_{dd} = \$14.830$$

The **intermediate option prices** are:

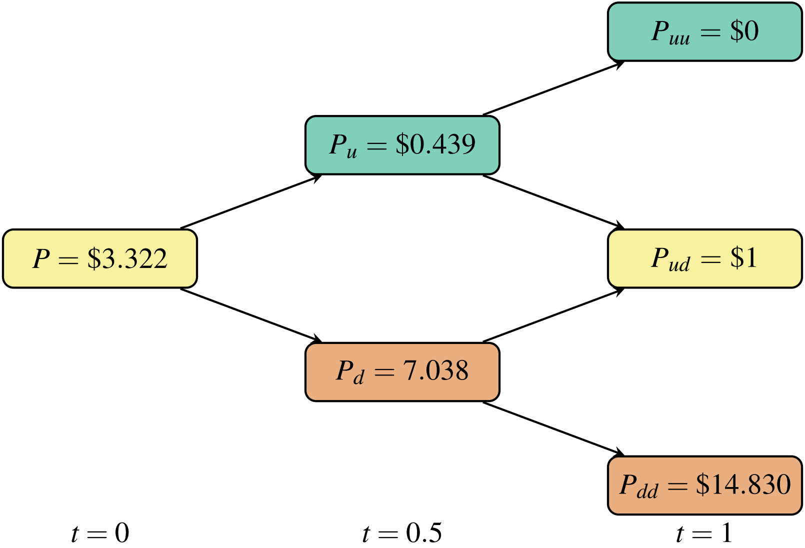
$$P_u = e^{-0.5(8\%)}[0.457(\$1)] = \$0.439$$
$$P_d = e^{-0.5(8\%)}[0.543(\$1) + 0.457(\$14.830)] = \$7.038$$

The **current option price** is:

$$P = e^{-0.5(8\%)}[0.543(\$0.439) + 0.457(\$7.038)] = \$3.322$$

European put options

Finally, the pricing tree of the European put option is:



American options

American options

For **American options**, we need to **check for early exercise** at each node. Specifically, we compare the **immediate exercise value** with the associated **European option value**:

- For call options:

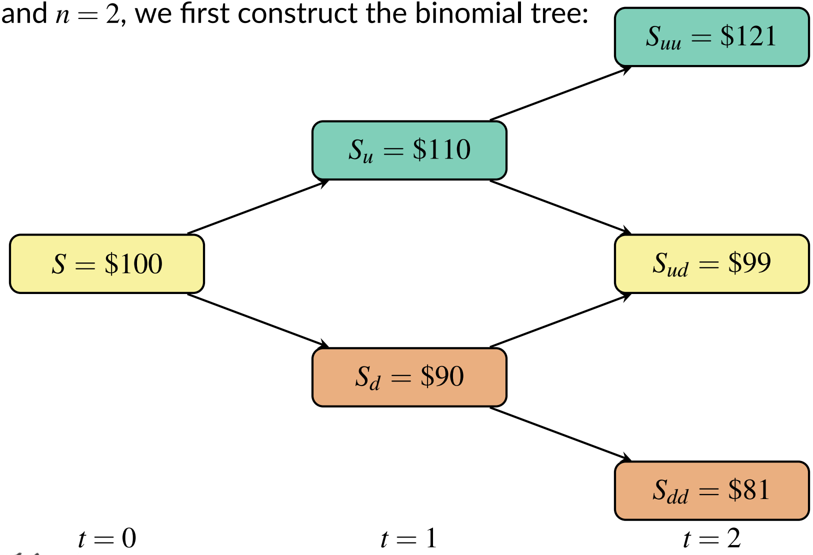
$$C_{A,\tau} = \max[S_\tau - K, C_{E,\tau}]$$

- For put options:

$$P_{A,\tau} = \max[K - S_\tau, P_{E,\tau}]$$

American call options

Given that $S = \$100$, $1 + r = 1.02$, $u = 1.1$, $d = 0.9$, $\delta = 0.05$, $T = 2$, and $n = 2$, we first construct the binomial tree:



American call options

To price an American call option with $K = 100$, we compute the **risk-neutral probability** as:

$$q = \frac{1.02(1-0.05)-0.9}{1.1-0.9} = 0.345$$

The terminal payoffs are:

$$C_{A,uu} = \$121 - \$100 = \$21$$
$$C_{A,ud} = C_{A,du} = \$0$$
$$C_{A,dd} = \$0$$

American call options

The **intermediate European call prices** are:

$$C_{E,u} = \frac{1}{1.02}[0.345(\$21)] = \$7.103$$
$$C_{E,d} = \$0$$

So it is **optimal to early exercise the American call** in the up-state:

$$C_{A,u} = \max[\$110 - \$100, \$7.103] = \$10$$
$$C_{A,d} = \$0$$

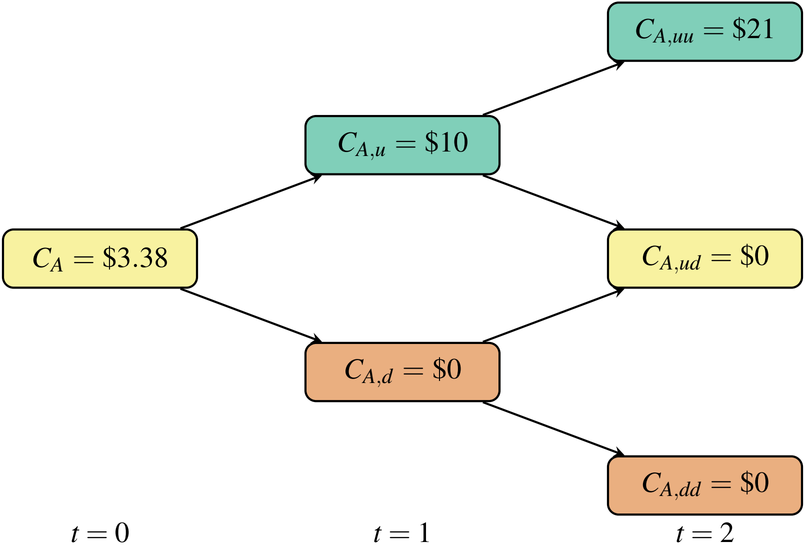
The **current American call price** is:

$$C_A = \frac{1}{1.02}[0.345(\$10)] = \$3.38 > C_E = \$2.403$$

NATIONAL CHENGCHI UNIVERSITY

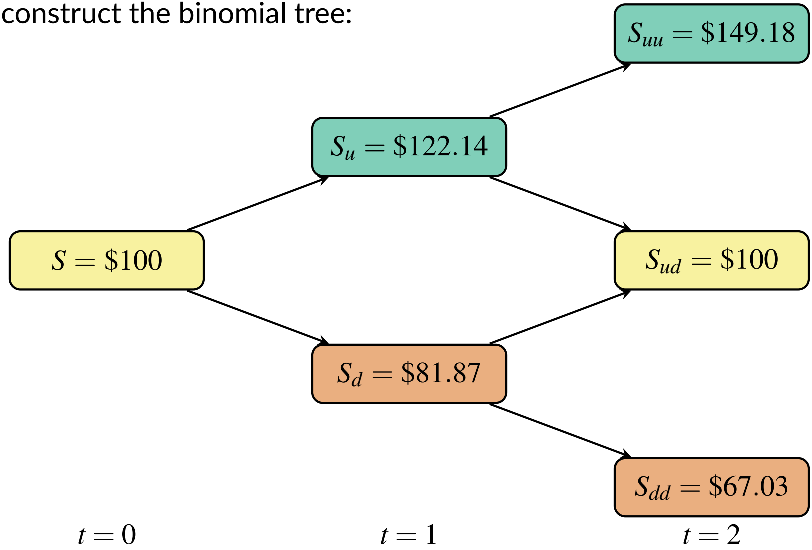
American call options

As a result, the pricing tree of the American call option is:



American put options

Given that $S = \$100$, $r = 0.05$, $\sigma = 0.2$, $T = 2$, and $n = 2$, we first construct the binomial tree:



American put options

To price an **American put option** with $K = \$101$, we compute the **risk-neutral probability** as:

$$q = \frac{e^{0.05} - 0.8187}{1.2214 - 0.8187} = 0.5775$$

Its **terminal payoffs** are:

$$P_{A,uu} = \$0$$
$$P_{A,ud} = P_{A,du} = \$1$$
$$P_{A,dd} = \$33.968$$

The **intermediate European put prices** are:

$$P_{E,u} = e^{-0.05}[0.4225(\$1)] = \$0.4019$$
$$P_{E,d} = e^{-0.05}[0.5775(\$1) + 0.4225(\$33.968)] = \$14.2009$$

NATIONAL CHENGCHI UNIVERSITY

American put options

Again, it is **optimal to early exercise the American put** in the down-state:

$$P_{A,u} = \$0.4019$$
$$P_{A,d} = \max[\$101 - \$81.8731, \$14.2009] = \$19.1269$$

The **current American put price** is:

$$\begin{aligned} P_A &= e^{-0.05}[0.5775(\$0.4019) + 0.4225(\$19.1269)] \\ &= \$7.9079 > P_E = \$5.9282 \end{aligned}$$

American put options

Ultimately, the pricing tree of the American put option is: