Financial derivatives

Lecture 7: Applications of binomial option pricing

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A generalized binomial model for European options

n-period binomial tree (CRR)

One way of constructing a recombinant binomial tree is by defining the **up factor** u and the **down factor** d as:

$$u = \frac{1}{d}$$

Assume $u, d \in$ const :
hence $ud = du$.

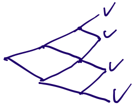
- the recombinant binomial tree has $n+1$ terminal nodes.

In contrast, if u and d change over time,

- the non-recombinant binomial tree has 2^n terminal nodes that is **computationally challenging** as n becomes larger.

To simplify the exposition, we will focus on the **binomial option pricing model** under the **recombinant tree**.

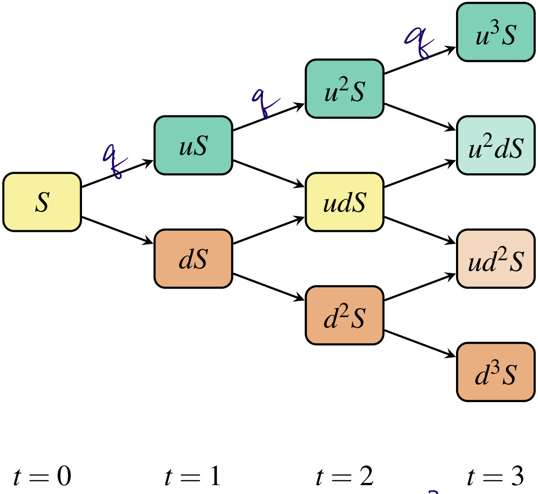
$n=3$ terminal: 4.



n-period binomial tree

C^n_r

$\begin{cases} n: \text{Total.} \\ r: \text{upstep.} \end{cases}$



Prob

$C^3_3 \cdot q^3.$

$C^3_2 \; q^2(1-q).$

$C^3_2 \; q(1-q)^2.$

$C^3_1 \; (1-q)^3.$

$\Sigma \text{ Prob} = 1 = [q+(1-q)]^3 = 1.$

n-period binomial model

u: ups. d: down.

After n periods, there will be $n+1$ **terminal nodes**:

$S_{T,m} = u^m d^{(n-m)}$

function of m

$$S_T = \begin{cases} u^n S & n=m. \\ u^{n-1} dS & \\ \vdots & \\ d^n S & m=0. \end{cases}$$

based

Let $C_T(m)$ be the **terminal payoff** of an **European call option** given m ups and $n-m$ downs:

$$C_T(m) = \max[\underbrace{u^m d^{n-m}}_{S_{T,m}} S - K, 0]$$

Similarly, let $P_T(m)$ be the **terminal payoff** of an **European put option** given m ups and $n-m$ downs:

$$P_T(m) = \max[K - \underbrace{u^m d^{n-m}}_{S_{T,m}.} S, 0]$$

Pascal's triangle

To determine the **risk-neutral probability** of each node, we need the **number of different combinations** of m ups and $n-m$ downs.

$n=0$						1					
$n=1$					1		1				
$n=2$				1		2		1			
$n=3$			1		3		3		1		
$n=4$		1		4		6		4		1	
$n=5$	1		5		10		10		5		1

Mathematically, we can calculate it by:

$$n\mathbf{C}m = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

where:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Risk-neutral probabilities for n-period binomial model

Define: (simplify)

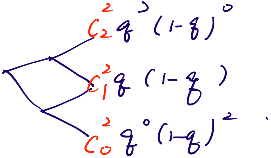
$1+r = R.$

The **risk-neutral probability** of an up:

$$q = \frac{(1+r)-d}{u-d} = \frac{R-d}{u-d}$$

The **risk-neutral probability** of m ups and $n-m$ downs is:

$$q^m(1-q)^{n-m}$$



while the **total risk-neutral likelihood** is:

$$Q(m) = \binom{n}{m} q^m (1-q)^{n-m}$$

Terminal payoffs for n-period binomial model

Thus, the **expected payoff** of a call at maturity is:

worst case   best case,

$$\mathbb{E}^{\mathbb{Q}}[C_T] = \underbrace{Q(0)}_{\text{prob}}\underbrace{C_T(0)}_{\text{exp(payoff)}} + \cdots + Q(n-1)C_T(n-1) + Q(n)C_T(n)$$

Since the call is **in-the-money** only when $S_T \geq K$, we can further define its **terminal payoffs** as:

$$C_T(m) = \begin{cases} \overset{\text{(OTM)}}{0}, & \text{if } m < m^* \\ \underbrace{u^m d^{n-m} S}_{S_T} - K, & \text{if } m \geq m^* \end{cases}$$

less than m^*-ups.

As a result, the **expected payoff** of a call becomes:

$$\mathbb{E}^{\mathbb{Q}}[C_T] = Q(m^*)C_T(m^*) + Q(m^*+1)C_T(m^*+1) + \cdots + Q(n)C_T(n)$$

$$= \sum_{m=m^*}^{n} Q(m)[\underbrace{u^m d^{n-m} S}_{S_{T,m}} - K]$$

Option pricing for n-period binomial model

We can find the **call price** by **discounting** the above payoff:

$$C_0 = \frac{1}{R^n}\mathbb{E}^{\mathbb{Q}}[C_T] = \frac{1}{(1+r)^n} E^{Q}[C_T]$$

since we let $R = 1+r$.

$$= \frac{1}{R^n}\sum_{m=m^*}^{n} Q(m)[u^m d^{n-m} S - K]$$

const

$$= S\sum_{m=m^*}^{n} Q(m)[\frac{u^m d^{n-m}}{R^n}] - \underbrace{\frac{K}{R^n}}_{PV(K)} \underbrace{\sum_{m=m^*}^{n} Q(m)}_{\Sigma \text{ all prob when (ITM)}}$$

Recall the **risk-neutral likelihood** following the **binomial distribution** with probability q:

$$\sum_{m=0}^{n} Q(m) = \sum_{m=0}^{n}\binom{n}{m} q^m (1-q)^{n-m} = 1$$

We can also express it as:

$$\sum_{m=m^*}^{n} Q(m) = \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{m\geq m^*}] = \text{Prob}^{Q}[S_T > K]$$

ITM.

$$uq + d(1-q)$$

$$= d + q(u-d), \quad \text{since } q = \frac{R-d}{u-d}$$

$$= d + R - d = R$$

Recall R is defined as: $uq + d(1-q) = R$.

Option pricing for n-period binomial model

In particular, note that:

q

$$uq + d(1-q) = d + (u-d)\boxed{\frac{R-d}{u-d}} = R$$

We can write **risk-neutral likelihood** for the **first term** following the **binomial distribution** with probability $\tilde{q}$ as:

$$\sum_{m=0}^{n} Q(m)[\frac{u^m d^{n-m}}{R^n}] = \sum_{m=0}^{n} \binom{n}{m} q^m (1-q)^{n-m} [\frac{u^m d^{n-m}}{R^n}] \quad = R^m \cdot R^{n-m},$$

prob

$$= \sum_{m=0}^{n} \binom{n}{m} \left[\frac{uq}{R}\right]^m \left[\frac{d(1-q)}{R}\right]^{n-m} \qquad d(1-q) = R-uq$$

$Q(m) = C^n_m q^m (1-q)^{n-m}$

$$= \sum_{m=0}^{n} \binom{n}{m} \left[\frac{uq}{R}\right]^m \left[\frac{R-uq}{R}\right]^{n-m}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \left[\frac{uq}{R}\right]^m \left[1-\frac{uq}{R}\right]^{n-m} \qquad \tilde{q} = \frac{uq}{R}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \tilde{q}^m (1-\tilde{q})^{n-m} = 1$$

(Adjusted Risk neutral)

pf: $\sum C_m^n \tilde{q}^m \cdot (1-\tilde{q})^{n-m} = 1$,

First show $\tilde{q} = \frac{uq}{R} > 0$

Since $u > 0$, $q > 0$, $R > 0$,

Hence $\tilde{q} > 0$ #

Option pricing for n-period binomial model

Similarly, we can write:

$$\sum_{m=m^*}^{n} \binom{n}{m} \tilde{q}^m (1-\tilde{q})^{n-m} = \sum_{m=m^*}^{n} \tilde{Q}(m) = \mathbb{E}^{\tilde{\mathbb{Q}}}[\mathbb{1}_{m \geq m^*}]$$

$= \text{Prob}^{\tilde{Q}}(S_T > K)$

The **call price** under the **generalized binomial model** is:

$$\begin{aligned} C_t &= S_t \sum_{m=m^*}^{n} Q(m) \left[\frac{u^m d^{n-m}}{R^n} \right] - \frac{K}{R^n} \sum_{m=m^*}^{n} Q(m) \\ &= S_t \sum_{m=m^*}^{n} \binom{n}{m} \tilde{q}^m (1-\tilde{q})^{n-m} - \frac{K}{R^n} \sum_{m=m^*}^{n} \binom{n}{m} q^m (1-q)^{n-m} \\ &= S_t \mathbb{E}^{\tilde{\mathbb{Q}}}[\mathbb{1}_{m \geq m^*}] - \frac{K}{R^n} \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{m \geq m^*}] \end{aligned}$$

$= S_t \,\text{Prob}^{\tilde{Q}}[S_T > K] - \frac{K}{R^n} \text{Prob}^{Q}[S_T > K]$

$PV(K)$

Which looks **remarkably similar** to the **Black-Scholes formula**:

$$C_t = S_t \mathcal{N}(d_1) - Ke^{-r(T-t)} \mathcal{N}(d_2)$$

$PV(K)$

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Option pricing for n-period binomial model

For a put option, the **expected payoff** at maturity is:

$$\mathbb{E}^{\mathbb{Q}}[P_T] = Q(0)P_T(0) + \cdots + Q(n-1)P_T(n-1) + Q(n)P_T(n)$$

Since the put is **in-the-money** only when $S_T \leq K$, we can further define its **terminal payoffs** as:

$K - S_T$

$$P_T(m) = \begin{cases} K - u^m d^{n-m} S, & \text{if } m < m^* \quad \text{(ITM)} \\ 0, & \text{if } m \geq m^* \quad \text{(OTM)} \end{cases}$$

m: # of up-steps

Option pricing for n-period binomial model

Similarly, the **put price** is:

$$P_t = \frac{1}{R^n}\sum_{m=0}^{m^*-1} Q(m)[K - u^m d^{n-m} S_t]$$

$$= \frac{K}{R^n}\sum_{m=0}^{m^*-1} Q(m) - S_t \frac{1}{R^n}\sum_{m=0}^{m^*-1} Q(m)[u^m d^{n-m}]$$

$$= \frac{K}{R^n}\underbrace{\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{m<m^*}]}_{\text{Prob}^{Q}[K > S_T]} - S_t \underbrace{\mathbb{E}^{\tilde{\mathbb{Q}}}[\mathbb{1}_{m<m^*}]}_{\text{Prob}^{\tilde{Q}}[K > S_T]}$$

Again, it is **similar** to the **Black-Scholes formula**:

$$P_t = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S_t\mathcal{N}(-d_1)$$

Radon-Nikodym derivative and change of measure

(RND)

So far, we have $\mathbb{E}^{\tilde{\mathbb{Q}}}[\mathbb{1}_{m \geq m^*}]$ under the $\tilde{\mathbb{Q}}$ measure:

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[\mathbb{1}_{m \geq m^*}] = \sum_{m=m^*}^{n} \tilde{Q}(m) = \sum_{m=m^*}^{n} Q(m) \boxed{\frac{u^m d^{n-m}}{R^n}}$$

RND

and $\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{m \geq m^*}]$ under the $\mathbb{Q}$ measure:

$$\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{m \geq m^*}] = \sum_{m=m^*}^{n} Q(m)$$

By defining the **Radon-Nikodym derivative** as:

$$\mathbb{Z}(m) = \frac{u^m d^{n-m}}{R^n}$$

we can **change** from the $\mathbb{Q}$ measure to the $\tilde{\mathbb{Q}}$ measure:

$$\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{m \geq m^*} \mathbb{Z}(m)] = \mathbb{E}^{\tilde{\mathbb{Q}}}[\mathbb{1}_{m \geq m^*}]$$

Radon-Nikodym derivative and change of measure

For a **single discrete event** m^*, we can write:

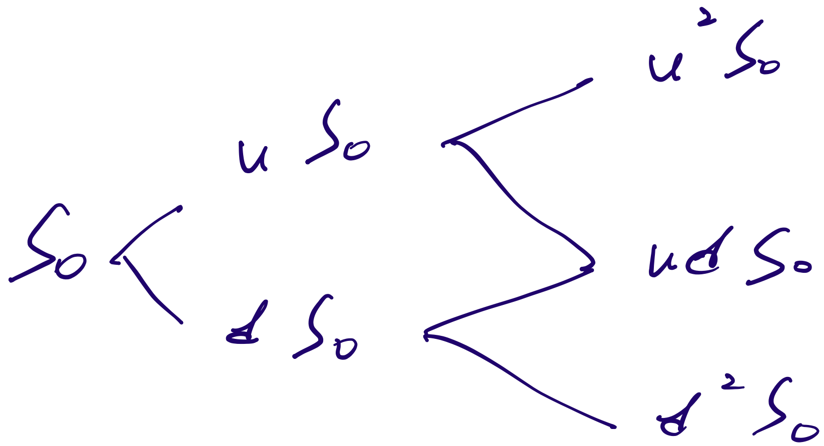
$\text{Prob}^{\tilde{Q}}[S_T = Su^{m^*} \cdot d^{n-m^*}]$.

$$\mathbb{Z}(m^*) = \frac{\mathbb{E}^{\tilde{\mathbb{Q}}}[\mathbb{1}_{m=m^*}]}{\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{m=m^*}]} = \frac{\tilde{Q}(m^*)}{Q(m^*)} = \frac{\cancel{Q(m^*)}\frac{u^{m^*}d^{n-m^*}}{R^n}}{\cancel{Q(m^*)}}$$

The **Radon-Nikodym derivative** for **continuous distribution** is:

$$\mathbb{Z}(\omega) = \frac{d\mathbb{Q}(\omega)}{d\mathbb{P}(\omega)} = \frac{\text{PDF under } Q.}{\text{PDF under } P}$$

$$Z(m) = \frac{u^m d^{n-m}}{R^n}$$



$Q(m)$	$Z(m)$	$\tilde{Q}(m)$
q^2	$\frac{u^2}{R^2}$	$\left(\frac{uq}{R}\right)^2 = \tilde{q}^2$
$q(1-q)$	$\frac{ud}{R^2}$	$2\tilde{q}(1-\tilde{q})$
$(1-q)^2$	$\frac{d^2}{R^2}$	$(1-\tilde{q})^2$

Log-normal distribution

$X \sim N(0, 1)$

$Y = e^{X}$.

$\Rightarrow \ln Y = X \sim N(0, 1)$

Why should we use log-normal distribution?

To price option contracts, we need to model the **price dynamics** of the underlying asset. Specifically,

- S_t **cannot be negative** for financial assets;
- If $\ln S_t$ is **normally distributed**, S_t only takes **positive values**.

Therefore, we can express **log-returns** as :

net return,

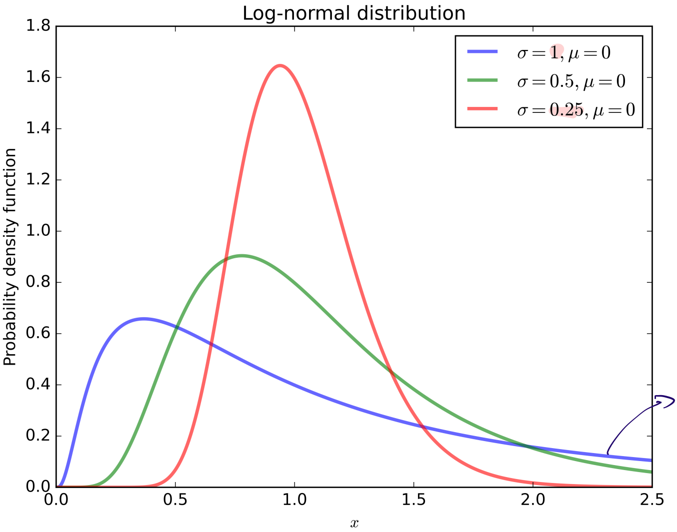
$$\ln\left(\frac{S_T}{\cancel{S_{T-1}}} \times \cancel{\frac{S_{T-1}}{S_{T-2}}} \times \cdots \times \cancel{\frac{S_1}{S_0}}\right) = \ln\left(\frac{S_T}{S_0}\right) = \ln(\underline{R_{0,T}}) = \ln(\underline{1 + r_{0,T}})$$

gross return,

Equivalently, this implies the **continuously compounded returns**:

$$e^{\ln}\frac{S_T}{S_0} = e^{\ln(R_{0,T})} = e^{\ln(1+r_{0,T})}$$

Log-normal distribution



high vol

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Log-normal distribution

The **log-normal distribution** assumes that the **natural log** of these returns follows the **normal distribution**.

- Let S_T be the underlying price in T years and S_0 be the current underlying price. The **log-return** follows:

$$\ln\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}(\mu T, \sigma^2 T)$$

$\mathcal{N}(\mu, var)$

- The **expected log-return** is:

$$\mathbb{E}\left[\ln\left(\frac{S_T}{S_0}\right)\right] = \mu T$$

- The **variance of log-return** is:

$$\mathrm{Var}\left[\ln\left(\frac{S_T}{S_0}\right)\right] = \sigma^2 T$$

Log-returns and gross returns

Recall the **probability density function** of a **normal distribution**:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

discrete

$E(x)$

$= \Sigma\, x_i f(x_i)$

Let $X \sim \mathcal{N}(0,1)$ and $a \in \mathbb{R}$, then we have:

$$\begin{aligned}
\mathbb{E}[e^{aX}] &= \int_{-\infty}^{\infty} e^{ax} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{ax} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-a)^2 + \frac{1}{2}a^2} dx \\
&= e^{\frac{1}{2}a^2} \underline{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-a)^2} dx} \\
&= e^{\frac{1}{2}a^2}
\end{aligned}$$

$\leftarrow$ const

CDF of $N(a, 1)$

$= 1.$

Log-returns and gross returns

$\mathbb{E}[e^{ax}] = e^{\frac{1}{2}a^2}$.

Since the **log-return** is **normally distributed**:

$$\ln\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}(\mu T, \sigma^2 T)$$

scaling factor

Var = $\sigma^2 T$

std = $\sigma\sqrt{T}$.

with ε denoting a **standard normal noise**, we have:

Data generating function (DGF)

(known) (random).

$$\ln\left(\frac{S_T}{S_0}\right) = \underbrace{\mu T}_{\text{avg}} + \underbrace{\sigma\sqrt{T}\varepsilon}_{\text{diffusion (stochastic)}}$$

The **expected gross return** under log-normal distribution is:

$$\mathbb{E}\left[\frac{S_T}{S_0}\right] = \mathbb{E}\left[e^{\ln\left(\frac{S_T}{S_0}\right)}\right] = \mathbb{E}\left[e^{\overbrace{\mu T}^{\text{const}}+\sigma\sqrt{T}\varepsilon}\right]$$

$$= e^{\mu T}\mathbb{E}\left[e^{\underbrace{\sigma\sqrt{T}}_{a}\underbrace{\varepsilon}_{x}}\right] = e^{\mu T} e^{\frac{1}{2}(\sigma\sqrt{T})^2}$$

$$= e^{\mu T + \frac{1}{2}\sigma^2 T}$$

$\mathbb{E}[e^{ax}] = e^{\frac{1}{2}a^2}$.

Log-returns and gross returns

Under the **risk-neutral** distribution, the **expected gross return** should equal the **risk-free rate**:

$$e^{\mu T + \frac{1}{2}\sigma^2 T} = e^{rT}$$

Thus, we have:

$\mu T + \frac{1}{2}\sigma^2 T = rT. \Rightarrow \mu + \frac{1}{2}\sigma^2 = r.$

$$\Rightarrow \quad \mu = r - \frac{1}{2}\sigma^2$$

If there is no volatility ($\sigma = 0$), the **log-return** (gross) becomes:

$$\ln\left(\frac{S_T}{S_0}\right) = \mu T$$

or equivalently:

$$S_T = S_0 e^{\mu T}$$

Since $\mathrm{Var}[X] = \mathbb{E}[X^2] - [\mathbb{E}[X]]^2$, the **variance of the gross return** is:

$$\mathrm{Var}\left[\frac{S_T}{S_0}\right] = e^{2\mu T + 2\sigma^2 T} - e^{2\mu T + \sigma^2 T}$$

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$$u = \left(r - \frac{1}{2}\sigma^2\right)$$

$(\mathbb{P} \Rightarrow \mathbb{Q})$

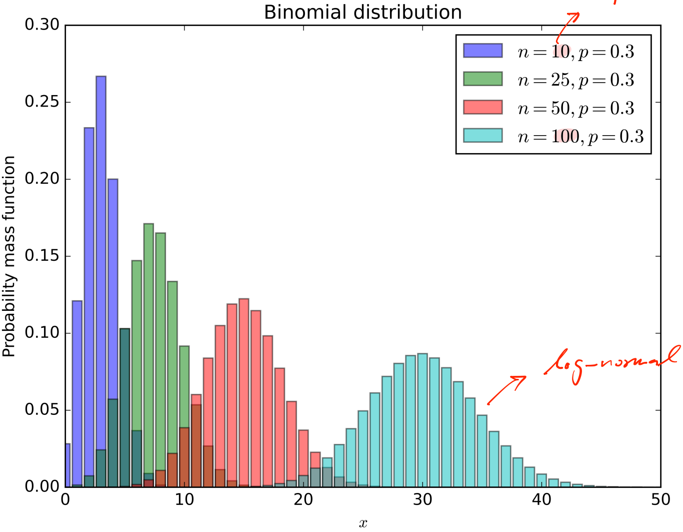
physical prob

risk-neutral

Binomial approximations of the normal distribution

Binomial approximations of the normal distribution

steps in Tree.



log-normal dist.

Binomial approximations of the normal distribution

In a n-period binomial tree, each step length is:

$$h = T/n$$

$R = \frac{S_u}{S_0} = \frac{uS_0}{S_0} = u$

For each step, the gross return of u occurs with a probability of p while the gross return of d occurs with a probability of $1-p$:

$R = \frac{S_d}{S_0} = \frac{dS_0}{S_0} = d.$

$$R_{t,t+h} = \begin{cases} \ln u, & \text{with probability } p \\ \ln d, & \text{with probability } 1-p \end{cases}$$

The **expected log-return** is:

$$E(x) = \quad p\ln u + (1-p)\ln d$$

with **variance** of:

$$Var(x) = p(1-p)[\ln u - \ln d]^2$$

$$\mathrm{Var}(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = (\ln u)^2 \cdot p + (\ln d)^2 (1-p),$$

$$(E(x))^2 = p^2 (\ln u)^2 + (1-p)^2 (\ln d)^2$$

$$+ 2p(1-p)(\ln u)(\ln d)$$

Binomial approximations of the normal distribution

i.i.d.

Since each step is **independent** to each other, we can aggregate them over n steps. To match the **mean** and **variance** of the log-normal distribution, the following two equalities should hold:

$$n[p\ln u + (1-p)\ln d] = \mu T$$

$\ln x \sim N(\mu T, \sigma^2 T)$

$$np(1-p)[\ln u - \ln d]^2 = \sigma^2 T$$

After dividing both sides by n, we have:

$$\Rightarrow \quad p\ln u + (1-p)\ln d = \mu h$$

$$\Rightarrow \quad p(1-p)[\ln u - \ln d]^2 = \sigma^2 h$$

As a result, we have to choose u, d, and p that fulfill the above two equations.

The Cox-Ross-Rubinstein approach

Cox-Ross-Rubinstein (CRR) calibrate the binomial model using:

Use This !!!

$$u = e^{\sigma\sqrt{h}}$$

$$d = \frac{1}{u} = e^{-\sigma\sqrt{h}} \qquad \frac{T}{n\uparrow} = h \downarrow$$

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{h}$$

(subjective physical prob)

The **expected value** and the **variance** are:

sub u, d, p ⇒

$$p \ln u + (1-p)\ln d = \mu h \qquad \text{(match)}$$

$$p(1-p)[\ln u - \ln d]^2 = \sigma^2 h - \mu^2 h^2 \qquad \text{(not-match)}$$

Given a large n, h^2 becomes **very small** so the variance will **converge towards** the variance of a log-normal distribution of $\sigma^2 h$.

The Jarrow-Rudd approach

Jarrow-Rudd (JR) calibrate the binomial model using:

$$u = e^{\mu h + \sigma\sqrt{h}}$$

$$d = e^{\mu h - \sigma\sqrt{h}}$$

$$p = \frac{1}{2}$$

(physical prob)

The **expected value** and the **variance** are:

sub $\Rightarrow$

$$p \ln u + (1-p) \ln d = \mu h$$

(match)

$$p(1-p)[\ln u - \ln d]^2 = \sigma^2 h$$

(match)

As $n \to \infty$, this binomial approximation will **converge** to a log-normal distribution with mean μT and variance $\sigma^2 T$.

$$\frac{1}{2}\left(uh+\cancel{\sigma\sqrt{h}}\right)+\frac{1}{2}\left(uh-\cancel{\sigma\sqrt{h}}\right)=uh$$

(p) $\qquad\qquad\qquad\qquad$ $(1-p)$