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5 L5/L6: Relative Plane Motion - Rotating Frame

Readings

5.1 Objective

To describe the planar velocity and acceleration of one rigid body which is moving with general plane motion with respect to another moving body. For example, the body is:

- Sliding with respect to another moving body.
- Moving with general plane motion with respect to another moving body.

5.2 Reference Frames

Up until now, we have dealt with two frames: fixed reference frame O_{xyz} and translating reference frame $A_{x'y'z'}$.

Position:

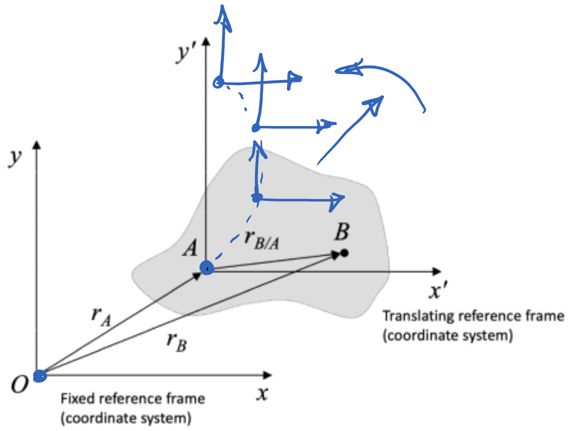
$$\boldsymbol{r_B} = \boldsymbol{r_A} + \boldsymbol{r_{B/A}}$$

Velocity:

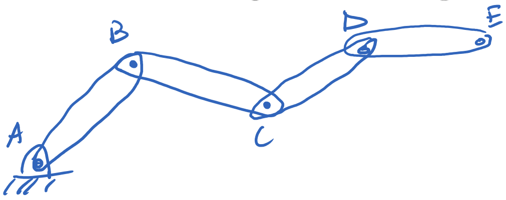
$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{B/A}}$$

Acceleration:

$$\boldsymbol{a_B} = \boldsymbol{a_A} + \boldsymbol{\alpha} \times \boldsymbol{r_{B/A}} - \omega^2 \boldsymbol{r_{B/A}}$$



This is useful for the problems where rigid bodies are pinned to each other.



$$\vec{V}_B = \vec{\omega}_{AB} \times \vec{r}_{B/A}$$
$$\vec{V}_C = \vec{V}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$$
$$\vec{V}_D = \vec{V}_C + \vec{\omega}_{CD} \times \vec{r}_{D/C}$$
$$\ldots$$

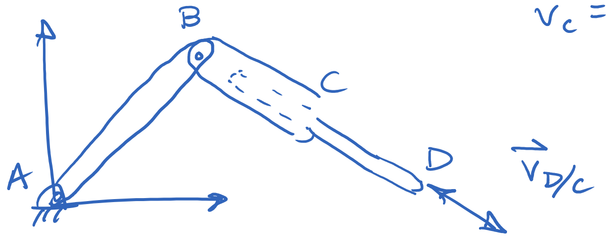
$$\vec{V}_E = \vec{\omega}_{AB} \times \vec{r}_{B/A} + \vec{\omega}_{BC} \times \vec{r}_{C/B} + \vec{\omega}_{CD} \times \vec{r}_{D/C} + \vec{\omega}_{DE} \times \vec{r}_{E/D}$$

However, when we wish to analyze problems where rigid bodies are connected with sliding joints, or analyze two related but unconnected bodies, it is convenient to allow the moving frame to rotated as well as translate.

In the case of a sliding joint on a rotating body, this allows the sliding motion to be described along a single coordinate direction.

$$\vec{V}_C = \vec{\omega}_{AB} \times \vec{r}_{B/A} + \vec{\omega}_{BC} \times \vec{r}_{C/B}$$

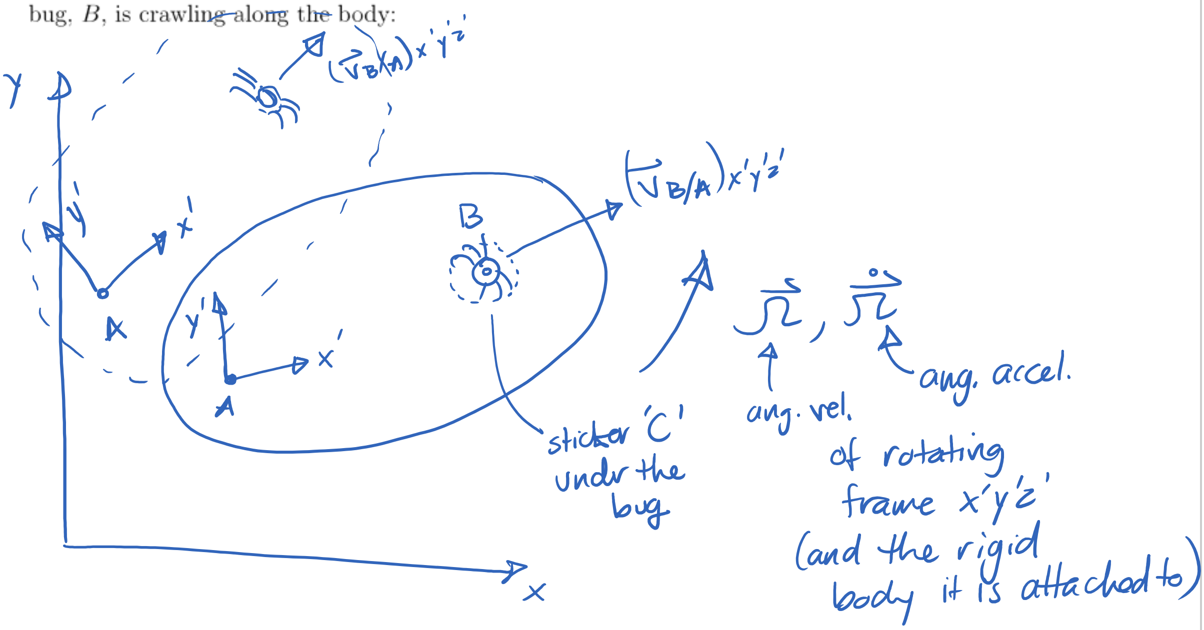
$\vec{V}_D$?



In the case of general motion, this allows the motion of one object to be described with respect to an observer (e.g. a camera) fixed to another moving object that is both translating and rotating.

In such cases, we consider the frame on the moving body to be **translating and rotating** with the body. This will introduce extra terms in our formulations, in order to correctly account for the rotating frame.

Consider a rigid body with frame $x'y'z'$ at point A. $x'y'z'$ translates and rotates with the body. A bug, B, is crawling along the body:

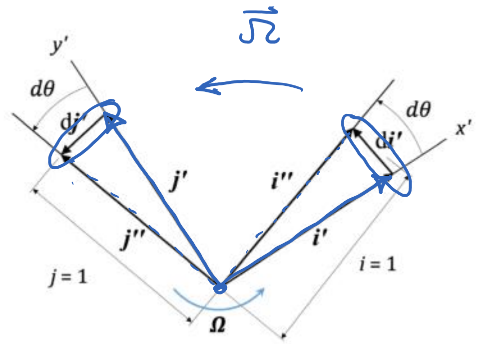


- The body and the frame $x'y'z'$ rotate with angular velocity $\boldsymbol{\Omega}$ and angular acceleration $\dot{\boldsymbol{\Omega}}$ in the $\hat{\boldsymbol{k}}$ direction.
- Viewed from the $x'y'z'$ frame, the bug, B, moves with velocity $(\boldsymbol{v_{B/A}})_{\boldsymbol{x'y'z'}}$ (velocity of B with respect to A, as viewed from the $x'y'z'$ frame).
- Point A on the body and the frame $x'y'z'$ are translating with velocity $\boldsymbol{v_A}$ and acceleration $\boldsymbol{a_A}$.

Describe the motion of point B:

Position (same as usual): $\boldsymbol{r_B} = \boldsymbol{r_A} + \boldsymbol{r_{B/A}}$

Velocity: since $\boldsymbol{r_{B/A}}$ is described in a reference frame that is rotating, $\boldsymbol{r_{B/A}} = \boldsymbol{r_x}\hat{\boldsymbol{i}}' + \boldsymbol{r_y}\hat{\boldsymbol{j}}'$ we need to differentiate both the magnitude and the direction of this vector.



As before, when we see coordinate frames change direction due to rotation, we find that differentiating the direction components is effectively a cross product:

$$\frac{d\hat{\imath}'}{dt} = \Omega \hat{k} \times \hat{\imath}' = \Omega \hat{\jmath}'$$

$$\frac{d\hat{\jmath}'}{dt} = \Omega \hat{k} \times \hat{\jmath}' = -\Omega \hat{\imath}'$$

(recall: trans. frame $\frac{dr_{B/A}}{dt} = v_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$)

Thus we have:

$$\frac{d\boldsymbol{r}_{B/A}}{dt} = \underbrace{(\dot{r}_x \hat{\imath}' + \dot{r}_y \hat{\jmath}')}_{\text{new term}} + \underbrace{\boldsymbol{\Omega} \times (r_x \hat{\imath}' + r_y \hat{\jmath}')}_{\text{as before}}$$

Ω = ang. vel. of rot. frame

$$\frac{d\boldsymbol{r}_{B/A}}{dt} = \underbrace{(\boldsymbol{v}_{B/A})_{x'y'z'}}_{\text{motion of B as seen by an observer fixed at } x'y'z'} + \underbrace{\boldsymbol{\Omega} \times \boldsymbol{r}_{B/A}}_{\text{motion due to the rotation of } x'y'z'}$$

Thus:

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\Omega} \times \boldsymbol{r}_{B/A} + \underbrace{(\boldsymbol{v}_{B/A})_{\cancel{x'y'z'}\,rot}}_{\text{new}}$$

most general relative velocity equation

vel. of B as seen from fixed frame

wrt fixed frame

$\hat{\imath}, \hat{\jmath} \Leftrightarrow \hat{\imath}', \hat{\jmath}'$

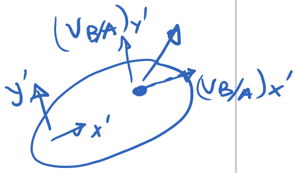
Acceleration: Now, let's differentiate again:

$$\boldsymbol{a_B} = \frac{d\boldsymbol{v_A}}{dt} + \frac{d\boldsymbol{\Omega}}{dt}\times \boldsymbol{r_{B/A}} + \boldsymbol{\Omega}\times\frac{d\boldsymbol{r_{B/A}}}{dt} + \frac{d(\boldsymbol{v_{B/A}})_{x'y'z'}}{dt}$$

$$\vec a_B = \vec a_A + \dot{\vec\Omega}\times\vec r_{B/A} + \vec\Omega\times\underbrace{\left((\vec v_{B/A})_{x'y'z'} + \vec\Omega\times\vec r_{B/A}\right)}_{\text{from previous}} + \frac{d}{dt}(\vec v_{B/A})_{x'y'z'}$$

(analogous to $\vec\alpha\times\vec r_{B/A}$)

last term: $(\vec v_{B/A})_{x'y'z'} = (v_{B/A})_{x'}\hat i' + (v_{B/A})_{y'}\hat j'$



$$\frac{d(\vec v_{B/A})_{x'y'z'}}{dt} = (a_{B/A})_{x'}\hat i' + (a_{B/A})_{y'}\hat j' + \underbrace{\vec\Omega\times(\vec v_{B/A})_{x'y'z'}}_{(v_{B/A})_{x'}\left(\frac{d\hat i'}{dt}\right) + (v_{B/A})_{y'}\left(\underbrace{\frac{d\hat j'}{dt}}_{\text{rotating by } \vec\Omega}\right)}$$

$$= (\vec a_{B/A})_{x'y'z'} + \vec\Omega\times(\vec v_{B/A})_{x'y'z'}$$

final:

$$\vec a_B = \vec a_A + \dot{\vec\Omega}\times\vec r_{B/A} + 2\vec\Omega\times(\vec v_{B/A})_{x'y'z'} + \vec\Omega\times(\vec\Omega\times\vec r_{B/A}) + (\vec a_{B/A})_{x'y'z'}$$

Coriolis accel. describe the interaction of mtn of object + mtn of rot. frame

describes mtn of object wrt. rotating frame

describes motion of axes (rotating frame)

For planar motion: $\vec\Omega \perp \vec r_{B/A}$

$$\vec\Omega\times(\vec\Omega\times\vec r_{B/A}) = -\Omega^2\vec r_{B/A}$$

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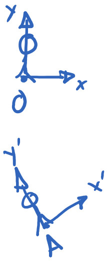
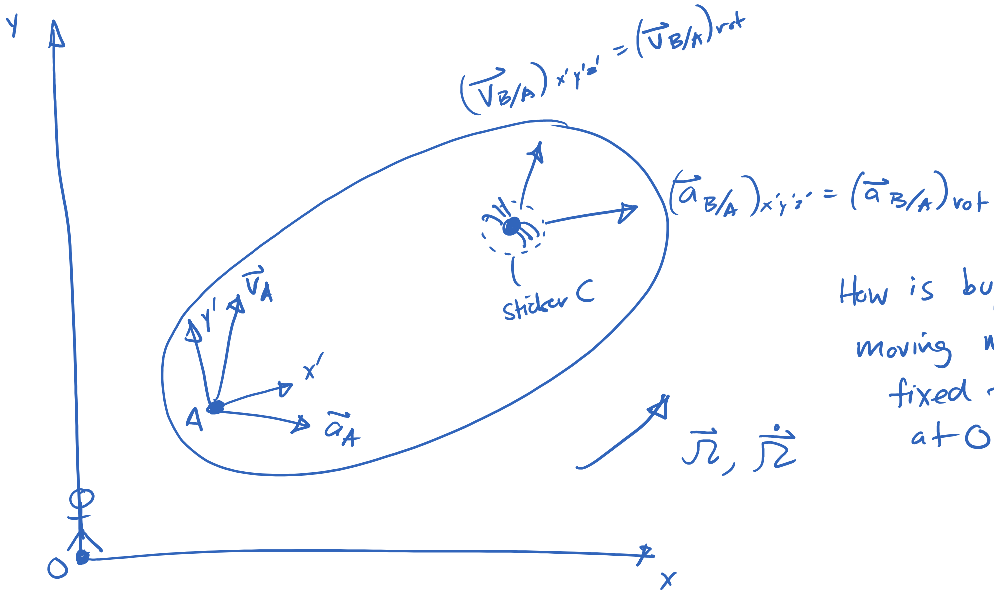
5.3 Summary



$\boldsymbol{a_A}$	Absolute acceleration of point A (referenced to a fixed non-moving frame)
$\boldsymbol{a_B}$	Absolute acceleration of point B (referenced to a non-moving frame)
$\boldsymbol{r_{B/A}}$	Relative location of point B w.r.t. A
$\boldsymbol{\Omega}$	Angular velocity of rotating $x'y'z'$ frame w.r.t. fixed XYZ frame.
$\dot{\boldsymbol{\Omega}}$	Angular acceleration of rotating $x'y'z'$ frame w.r.t. fixed XYZ frame.
$(\boldsymbol{v_{B/A}})_{x'y'z'}$	Velocity of point B w.r.t. A, as seen by an observer attached to the translating and rotating $x'y'z'$ frame.
$(\boldsymbol{a_{B/A}})_{x'y'z'}$	~~Velocity~~ acceleration of point B w.r.t. A, as seen by an observer attached to the translating and rotating $x'y'z'$ frame.



How is bug B moving wrt. fixed frame at O?

VERY IMPORTANT NOTE: The components of all terms must be computed along the same $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ directions. If the XYZ and $x'y'z'$ frames are not aligned, you must pick ONE FRAME and express all terms along the **directions** of that ONE FRAME.

Since the $x'y'z'$ frame rotates w.r.t. the XYZ frame, it is likely that these two frames will not be aligned.

$\boldsymbol{a}_B$ Fixed frame

$=$

$$\vec{a}_c \left\{ \begin{array}{l} \boldsymbol{a}_A \\ + \\ \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{B/A} \\ + \\ \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{B/A}) \end{array} \right.$$

acceleration of point on rigid body (same point as B at this instant)

$+$

$$\left. \begin{array}{l} 2\boldsymbol{\Omega} \times (\boldsymbol{v}_{B/A})_{x'y'z'} \\ + \\ (\boldsymbol{a}_{B/A})_{x'y'z'} \end{array} \right\}$$

relative acceleration of moving object B w.r.t rigid body

5.3.1 Example

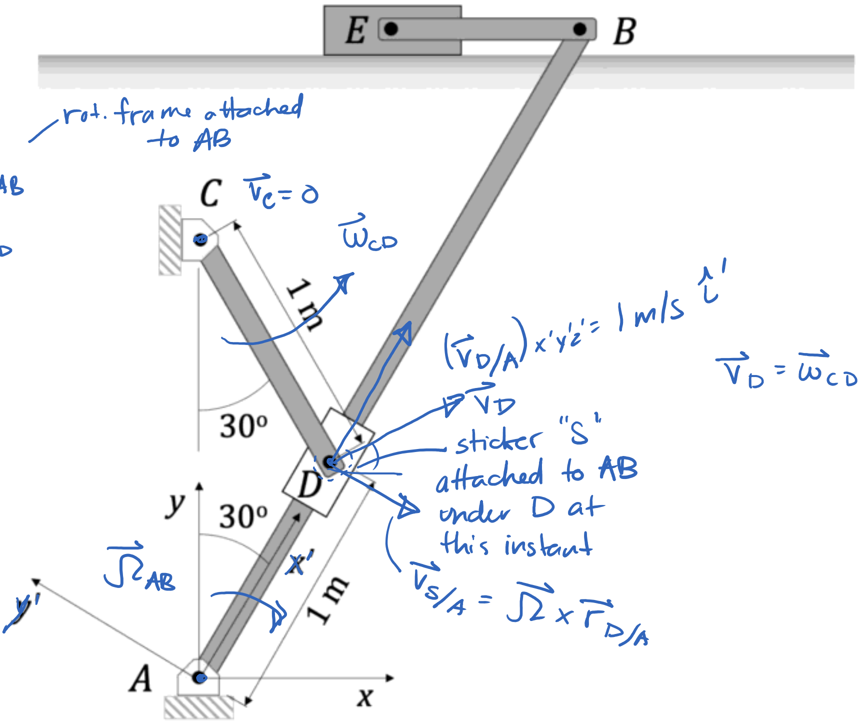
Consider the quick return mechanism shown in the following figure. There are pin joints at A, B, C, D, and E. Block D slides along bar AB. Block E is constrained to slide horizontally. The rotation of the crank CD is the input motion and the output is the sliding motion of E.

If the relative velocity of block D with respect to the rotating bar AB, $(\boldsymbol{v}_{D/A})_{x'y'z'}$ is a constant $1\,m/s$, find the angular velocities and angular accelerations of bars AB and CD.

Find $\vec{\Omega}_{AB}, \dot{\vec{\Omega}}_{AB}$

$\vec{\omega}_{CD}, \vec{\alpha}_{CD}$

rot. frame attached to AB



$\vec{v}_D = \vec{\omega}_{CD} \times \vec{r}_{D/C}$

$\vec{v}_A = 0$

$$\underbrace{\underbrace{\vec{v}_D = \cancel{\vec{v}_A}^{\,0} + \vec{\Omega}_{AB}}_{} \times \vec{r}_{D/A}}_{\vec{v}_S} + (\vec{v}_{D/A})_{x'y'z'}$$

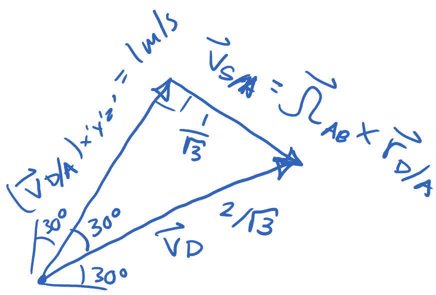
$\vec{\Omega}_{AB} = \vec{\omega}_{AB}$, $\vec{r}_{D/A} = \vec{r}_{S/A}$

Chasles' for S

$(\vec{v}_D = \vec{v}_D)$

$$\vec{\omega}_{CD} \times \vec{r}_{D/C} = \vec{\Omega}_{AB} \times \vec{r}_{D/A} + (\vec{v}_{D/A})_{x'y'z'}$$

$\vec{\omega}_{CD}$	$\vec{r}_{D/C}$	$\vec{\Omega}_{AB}$	$\vec{r}_{D/A}$	$(\vec{v}_{D/A})_{x'y'z'}$
m ✗	m ✓	m ✗	m ✓	m ✓
d ✓	d ✓	d ✓	d ✓	d ✓



$$|\vec{v}_D| = \frac{1 \text{ m/s}}{\cos 30} = \frac{2}{\sqrt{3}} \Rightarrow \vec{\omega}_{CD} = \frac{2}{\sqrt{3}}\hat{k} \text{ rad/s}$$

$$|\vec{\Omega}_{AB} \times \vec{r}_{D/A}| = \frac{1}{\sqrt{3}} \Rightarrow \vec{\Omega}_{AB} = -\frac{1}{\sqrt{3}}\hat{k} \text{ rad/s}$$

Find $\vec{\alpha}_{CD}$, $\dot{\vec{\Omega}}_{AB}$

(Example continued)

all in x'-y' terms

ACCEL: $\vec{a}_D = \vec{a}_A + \dot{\vec{\Omega}}_{AB} \times \vec{r}_{D/A} + \overbrace{\vec{\Omega}_{AB} \times (\vec{\Omega}_{AB} \times \vec{r}_{D/A})}^{= -\Omega_{AB}^2 \vec{r}_{D/A}} + 2\vec{\Omega}_{AB} \times (\vec{v}_{D/A})_{x'y'z'} + (\vec{a}_{D/A})_{x'y'z'}$

$\vec{r}_{D/A} = 1\,\hat{i}'\ \text{m}$

$\vec{r}_{D/C} = (-\sin 30\,\hat{i}' - \cos 30\,\hat{j}')\,\text{m} = \left(-\frac{1}{2}\hat{i}' - \frac{\sqrt{3}}{2}\hat{j}'\right)\text{m}$

$\vec{\omega}_{CD} = \frac{2}{\sqrt{3}}\hat{k}'\ \text{rad/s}, \quad \omega_{CD}^2 = \frac{4}{3}\ \text{rad}^2/\text{s}^2$

$\vec{\Omega}_{AB} = -\frac{1}{\sqrt{3}}\hat{k}'\ \text{rad/s}, \quad \Omega_{AB}^2 = \frac{1}{3}\ \text{rad}^2/\text{s}^2$

$(\vec{v}_{D/A})_{x'y'z'} = 1\ \text{m/s}\ \hat{i}'$ constant $\Rightarrow (\vec{a}_{D/A})_{x'y'z'} = 0$

$\vec{a}_A = 0, \quad \vec{a}_C = 0$

LHS: $\vec{a}_D = \cancel{\vec{a}_C}^{\,0} + \vec{\alpha}_{CD} \times \vec{r}_{D/C} - \omega_{CD}^2 \vec{r}_{D/C}$

$= \alpha_{CD}\hat{k}' \times \left(-\frac{1}{2}\hat{i}' - \frac{\sqrt{3}}{2}\hat{j}'\right) - \frac{4}{3}\left(-\frac{1}{2}\hat{i}' - \frac{\sqrt{3}}{2}\hat{j}'\right)$

$= -\frac{\alpha_{CD}}{2}\hat{j}' + \frac{\sqrt{3}\,\alpha_{CD}}{2}\hat{i}' + \frac{2}{3}\hat{i}' + \frac{2\sqrt{3}}{3}\hat{j}'$

RHS: $\vec{a}_D = \cancel{\vec{a}_A}^{\,0} + \dot{\vec{\Omega}}_{AB} \times \vec{r}_{D/A} - \Omega_{AB}^2 \vec{r}_{D/A} + 2\vec{\Omega}_{AB} \times (\vec{v}_{D/A})_{x'y'z'} + \cancel{(\vec{a}_{D/A})_{x'y'z'}}^{\,0}$

$= \dot{\Omega}_{AB}\hat{k}' \times 1\,\hat{i}' - \frac{1}{3}(1\,\hat{i}') + 2\left(-\frac{1}{\sqrt{3}}\hat{k}'\right) \times 1\ \text{m/s}\ \hat{i}'$

$= \dot{\Omega}_{AB}\hat{j}' - \frac{1}{3}\hat{i}' - \frac{2}{\sqrt{3}}\hat{j}'$

equate: $\hat{i}'$: $\underbrace{\frac{2}{3} + \frac{\sqrt{3}}{2}\alpha_{CD}}_{\text{LHS}} = \underbrace{-\frac{1}{3}}_{\text{RHS}} \Rightarrow \vec{\alpha}_{CD} = -\frac{2}{\sqrt{3}}\ \text{rad/s}^2\ \hat{k}'$

$\hat{j}'$: $\underbrace{-\frac{\alpha_{CD}}{2} + \frac{2\sqrt{3}}{3}}_{\text{LHS}} = \dot{\Omega}_{AB} - \frac{2}{\sqrt{3}} \Rightarrow \dot{\vec{\Omega}}_{AB} = \frac{5}{\sqrt{3}}\hat{k}'$