DYN L18

September 24, 2019 9:19 AM

11 L18: Free Undamped Vibrations

Readings

11.1 Objective

Introduce Single Degree of Freedom Vibration of a rigid body.

- Define terminology
- Establish model of single degree of freedom vibrations using Newton's Second Law
- Show examples in mechanical systems

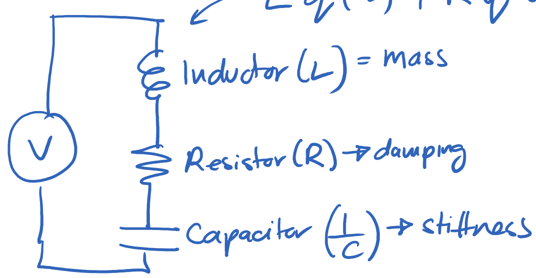
11.2 Free Vibration

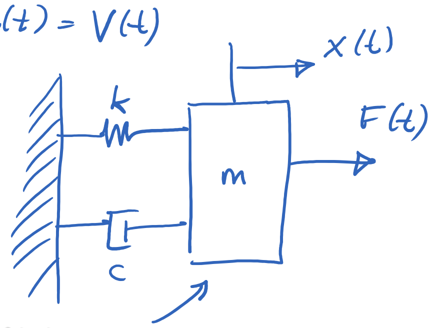
A vibrating mechanical system is a system that exchanges kinetic and potential energy. Therefore it must have:

1. Potential field (i.e., an element that can store and release potential energy)
2. Inertia (mass, or moment of inertia) – an element that can have kinetic energy

In the real world, mechanical systems also have energy losses – through friction and often (intentionally) through energy absorbing devices or materials called "dampers". To start we will ignore these losses and look at a simple vibrating mass-spring system, and some related models

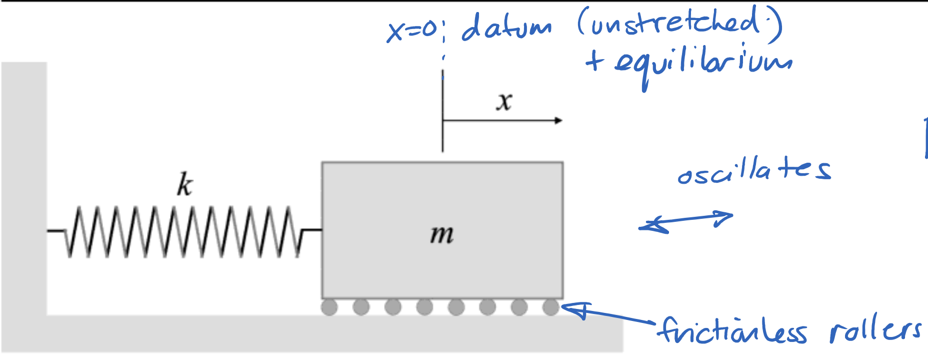
$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = V(t)$$

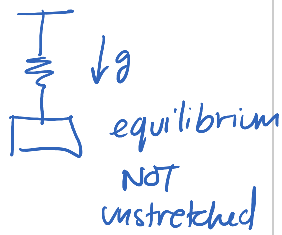




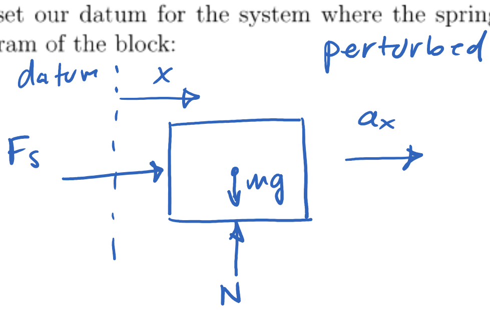
Consider the mass attached to a spring, moving along with no friction.

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

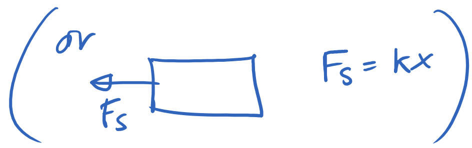




We set our datum for the system where the spring is unstretched. Then we draw the free body diagram of the block:



$F_s = -kx$



And write the equations of motion

ΣF_x: $F_s = m a_x$ ①

ΣF_y: $N - mg = 0$

We can express the spring force, using Hooke's Law:

$F_s = -kx$ force opposite displacement

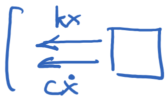
EQN OF MOTION

① $\Rightarrow$ $-kx = m a_x$ $\quad a_x = \ddot{x}$ ✓ all terms on LHS must have same sign

$-kx = m\ddot{x} \rightarrow m\ddot{x} + kx = 0$

So the equation of motion for the free motion – free vibration – of the system is a Homogeneous, second order, differential equation.

$$m\ddot{x}(t) + kx(t) = 0$$



$\Rightarrow m\ddot{x} + c\dot{x} + kx = 0$

FREE VIBRATION

The solution for this type of motion – simple harmonic motion – is of the form:

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$
$$\dot{x}(t) = A\omega\cos(\omega t) - B\omega\sin(\omega t)$$
$$\ddot{x}(t) = -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t) = -\omega^2 x(t)$$

$$= -\omega^2 \underbrace{(A\sin\omega t + B\cos\omega t)}_{x(t)}$$

Substituting back into the differential equation we get:

cancel out x(t)

$$m(\underbrace{-\omega^2 x(t)}_{\ddot{x}}) + kx(t) = 0$$

$-m\omega^2 + k = 0$

$\rightarrow \omega_n^2 = \frac{k}{m}$

ω that satisfies this equation is: $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$

ω_n is the **angular natural frequency** of the system, measured in *radians/second*. For example, $\omega_n = 2\pi$ implies that the system repeats itself every second.

$f = \frac{\omega_n}{2\pi}$ is the frequency of the system in $Hertz = cycles/second = s^{-1}$.

$T = \frac{1}{f} = \frac{2\pi}{\omega_n}$ is the time to complete one vibration cycle. = one wavelength

period

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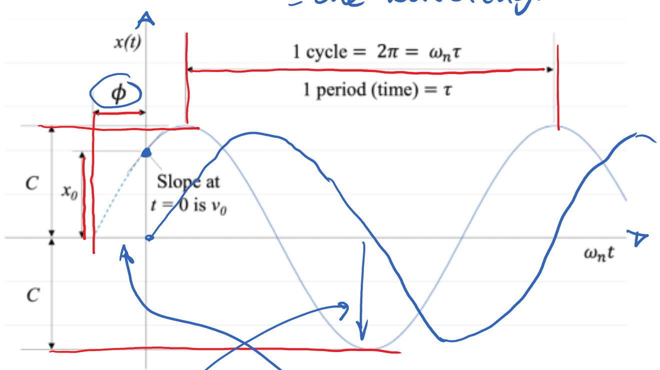


Another form of the solution: $x = C\sin(\omega_n t + \phi)$

Natural frequency of a system depends only on its physical properties – for spring type vibration it depends on the system inertia (mass) and stiffness.

$\omega_n = \sqrt{\frac{k}{m}}$	Increasing stiffness increases frequency of vibration Increasing mass decreases frequency of vibration

11.3 Initial Conditions

$$x(t) = A\sin(\omega_n t) + B\cos(\omega_n t)$$

Assume at $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = v_0$:

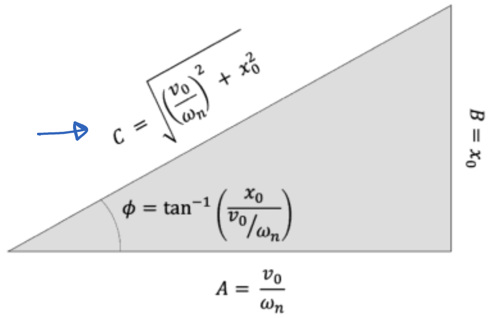
$$x(t): \quad x_0 = A\sin(0) + B\cos(0) \;\Rightarrow\; B = x_0$$

$$\dot{x}(t): \quad v_0 = \omega(A\cos(0) - B\sin(0)) \;\Rightarrow\; A = \frac{v_0}{\omega_n}$$

Thus:

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t)$$

To see the relationship between these constants we can construct a right triangle:

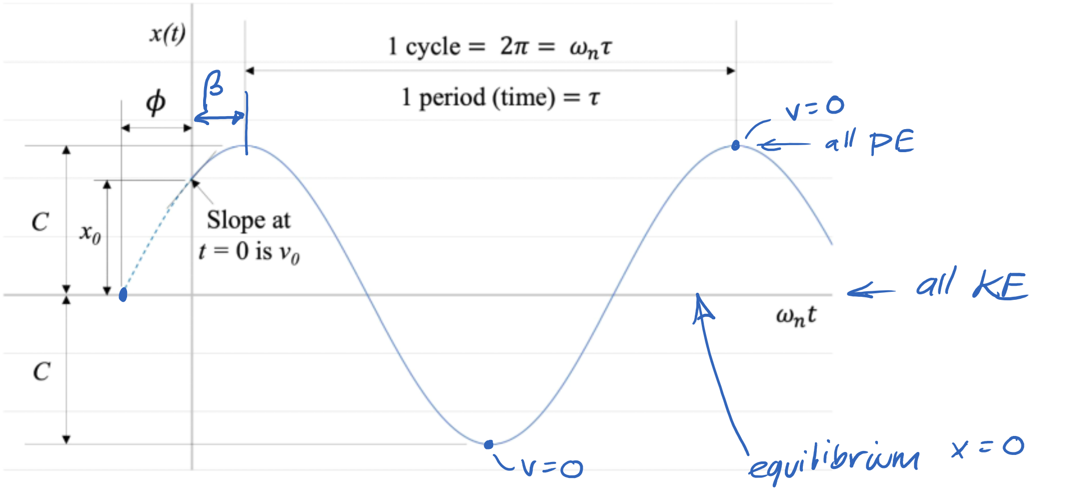


$C = \sqrt{A^2 + B^2}$ is the **amplitude** of the system. ϕ is the **phase angle**. These values are related to the initial conditions and also to the natural frequency (for non-zero v_0).

$$A = C\cos(\phi) \qquad\qquad x(t) = C\cos(\phi)\sin(\omega_n t) + C\sin(\phi)\cos(\omega_n t)$$

$$\Rightarrow$$

$$B = C\sin(\phi) \qquad\qquad x(t) = C\sin(\omega_n t + \phi)$$



"lead angle"

$$x(t) = C\sin(\omega_n t + \phi)$$
$$\dot{x}(t) = C\omega_n \cos(\omega_n t + \phi) = C\omega_n \sin\left(\omega_n t + \phi + \frac{\pi}{2}\right)$$
$$\ddot{x}(t) = -C\omega_n^2 \sin(\omega_n t + \phi) = C\omega_n^2 \sin(\omega_n t + \phi + \pi)$$

$x(t) = C\cos(\omega_n t - \beta)$

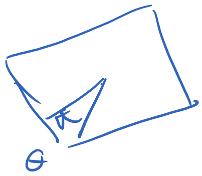
"lag angle"

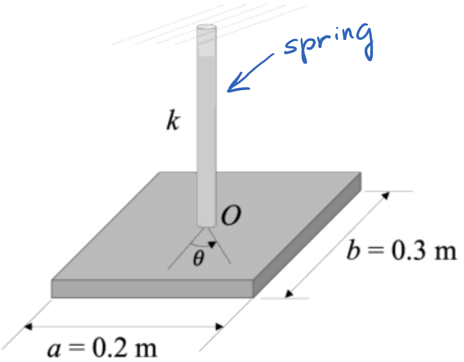
Process:

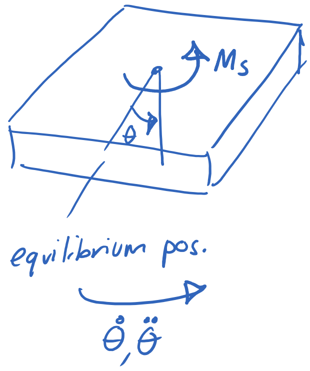
1. Draw our system at a small displacement from equilibrium (assume $x, \dot{x}, \ddot{x}$ or $\theta, \dot{\theta}, \ddot{\theta}$ are all in the same direction) +ve
2. Find a coordinate relating equilibrium position to displaced position (x or θ)
3. Equation(s) of motion (ΣF_x, ΣM)
4. Put in standard form ($\ddot{x} + \omega_n^2 x = 0$)

11.3.1 Example 1 - Torsional Shaft Vibration

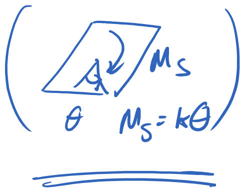
Find ω_n







$M_s = -k\theta$



$$\left(\sum M_G = I_G\alpha\right)$$

$$\sum M_G = \sum M_O : \; -k\theta = I_O\alpha = I_O\ddot{\theta} \qquad \circlearrowright +ve$$

$$\Rightarrow \ddot{\theta}(t) + \underbrace{\frac{k}{I_O}}_{\omega_n^2}\theta(t) = 0 \quad \text{(standard form)}$$

$$-I_O\omega_n^2\theta(t) + k\theta(t) = 0$$

$$\theta(t) = C\sin(\omega_n t + \phi)$$

$$\dot{\theta}$$

$$\ddot{\theta}$$

$$-I_O\omega_n^2 C\sin(\omega_n t + \phi)$$

char. eqn: $-I_O\omega_n^2 + k = 0$ (sin)

solve

$$\omega_n = \sqrt{\frac{k}{I_O}}$$

thin plate:

$$I_O = \frac{1}{12}m(a^2+b^2) \Rightarrow \omega_n = \sqrt{\frac{12k}{m(a^2+b^2)}}$$

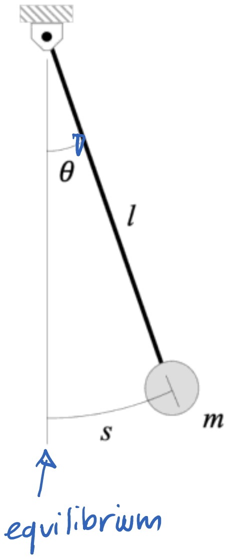
$\omega_n = 3.72$ rad/s

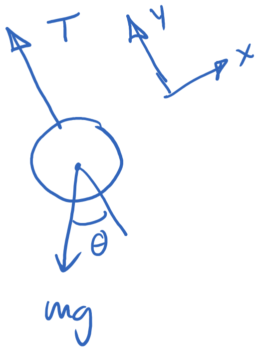
$(m = 10\text{ kg}, \; k = 1.5\text{ N}\cdot\text{m/rad})$

diff unit from linear spring

11.3.2 Example 2 - Swinging Pendulum (small θ)

Find ω_n





$\sum F_x: -mg\sin\theta = ma_x$ ①

$\sum F_y: T - mg\cos\theta = ma_y$

$a_y = -\omega^2 l \quad (= -\omega^2 r)$

$a_x = \alpha\, l \quad (= \alpha r) = \ddot{\theta} l$

① $\Rightarrow -mg\sin\theta = m\ddot{\theta} l$

for small angles (say $< 5°$, pretty good up to $\sim 30°$)

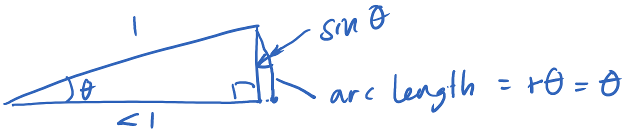
$\sin\theta \approx \theta$

$\cos\theta \approx 1$

$\Rightarrow -mg\,\theta = m\ddot{\theta} l$

$(\pm)\ \cancel{m} l\ddot{\theta} + \cancel{m} g\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \underbrace{\frac{g}{l}}_{\omega_n^2}\theta = 0$

$\omega_n = \sqrt{\frac{g}{l}}$



11.4 SHM Solution using Euler's Formula

Equation of motion:

$$m\ddot{x}(t) + kx(t) = 0 \tag{11.1}$$

Propose solution of the form: $x = ae^{rt}$

Substitute back into (11.1) and cancel out e^{rt} terms.

$$mr^2 + k = 0 \Rightarrow r = \pm\sqrt{-\frac{k}{m}} = \pm\omega_n i$$

where i is the imaginary number, $i = \sqrt{-1}$.

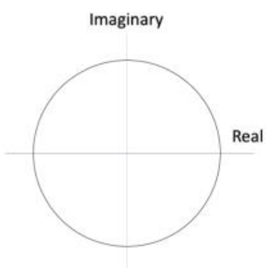
So our solution is of the form

$$x(t) = a_1 e^{i\omega_n t} + a_2 e^{-i\omega_n t}$$

another form

Euler's formulae:

$$e^{i\omega_n t} = \cos(\omega_n t) + i\sin(\omega_n t)$$
$$e^{-i\omega_n t} = \cos(\omega_n t) - i\sin(\omega_n t)$$



Since $x(t)$ is REAL, a_1 and a_2 must be complex conjugate numbers.

$$\begin{aligned} x(t) &= (a_1 + a_2)\cos(\omega_n t) + (a_1 - a_2)i\sin(\omega_n t) \\ &= B\cos(\omega_n t) + A\sin(\omega_n t) \\ &= C\sin(\omega_n t + \phi) \end{aligned}$$

Where:

$$a_1 = \frac{B - Ai}{2},\ a_2 = \frac{B + Ai}{2}$$

11.5 Summary

For undamped, single degree of freedom vibration, the form of the equation of motion is:

$$m\ddot{x}(t) + kx(t) = 0$$ ← check all +ve (or all −ve)

standard form $\ddot{x}(t) + \underbrace{\frac{k}{m}}_{\omega_n^2} x(t) = 0$

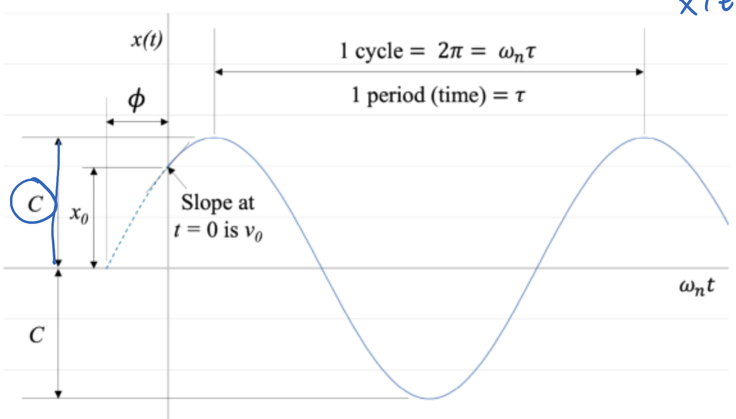
$\underbrace{\frac{c}{m}}_{2\omega_n \zeta}$

lead phase angle

$x(t) = C\sin(\omega_n t + \phi)$

$= A \sin \omega_n t + B \cos \omega_n t$

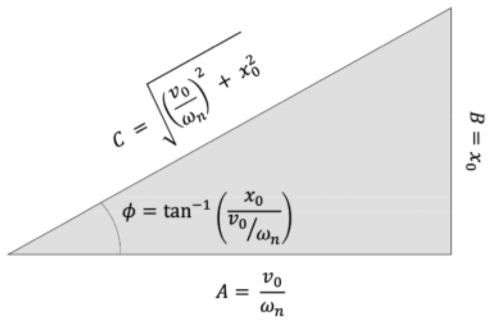
$\omega_n = \sqrt{\frac{k}{m}}$ is the (angular) natural frequency of the system. Natural frequency depends on physical system parameters and is independent of initial conditions.



Initial conditions specify the amplitude of vibration, C, and the phase angle, ϕ.

For $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$C = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + x_0^2}, \ \tan\phi = \frac{\omega_n x_0}{v_0}$$



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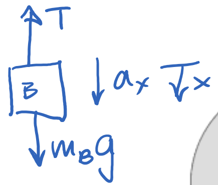
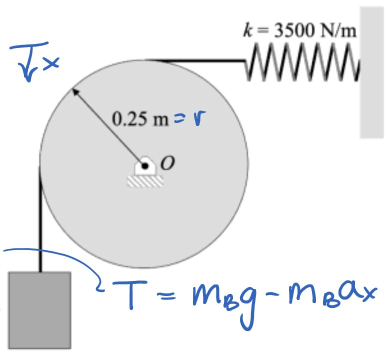
11.5.1 Example

A cord, running over a pulley (7.5 kg), connects a spring (k = 3500 N/m) with a hanging mass (5 kg). What is the natural period of vibration of the system?

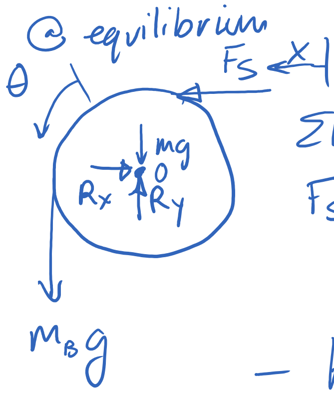


ΣF: $-T + m_B g = +m a_x$



$T = m_B g - m_B a_x$

@ equilibrium



$F_s = -k(r\theta_{eq})$, $r\theta_{eq} = x_{eq}$

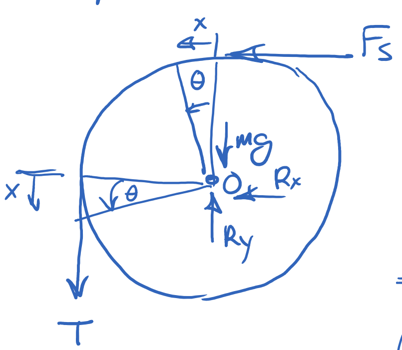
ΣM_O: $F_s \cancel{r} + m_B g \cancel{r} = 0$

$F_s = -k x_{eq}$

$-k x_{eq} + m_B g = 0$

$\Rightarrow k x_{eq} = m_B g$

@ perturbation



ΣM_O: $F_s r + T r = I\alpha$ $\qquad \Sigma M = I\alpha$

$F_s = -k(x + x_{eq})$

$-krx - \cancel{krx_{eq}} + \cancel{m_B g r} - m_B a_x r = I\alpha$

$\Rightarrow -kr^2\theta - m_B r^2 \ddot{\theta} = I\ddot{\theta}$ $\qquad I = \frac{1}{2} m_D r^2$

$\left(\frac{1}{2} m_D r^2 + m_B r^2\right)\ddot{\theta} + kr^2\theta = 0$

small angle:

$x \approx r\theta$

$a_x \approx r\alpha$

$(\alpha = \ddot{\theta})$

$$\ddot{\theta} + \underbrace{\frac{k \cancel{r^2}}{\left(\frac{1}{2} m_D \cancel{r^2} + m_B \cancel{r^2}\right)}}_{\omega_n^2}\theta = 0 \qquad \text{std form}$$

$$\omega_n = \sqrt{\frac{k}{\frac{m_D}{2} + m_B}} \qquad \tau = \frac{2\pi}{\omega_n} = 0.327\,s$$