DYN L1

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1 L1: Rigid Body Kinematics - Chasles' Theorem

Readings

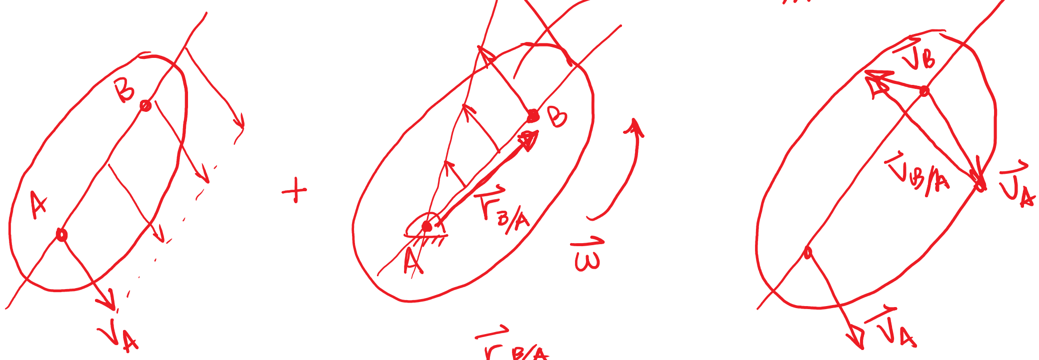
1.1 Objective

To describe the planar motion of a rigid body.

1.2 Introduction: Chasles' Theorem

The motion of rigid body can be described by the **translation** of ***one point*** plus the **rotation** about that same point.

$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$



v_A	+	$\omega \times \cancel{r_{A/B}}$ $\vec{r}_{B/A}$	=	v_B
Translational velocity at A	+	Velocity due to rotation of point B about A	=	Velocity at point B

Y + $\vec{\omega}$ x   right hand rule

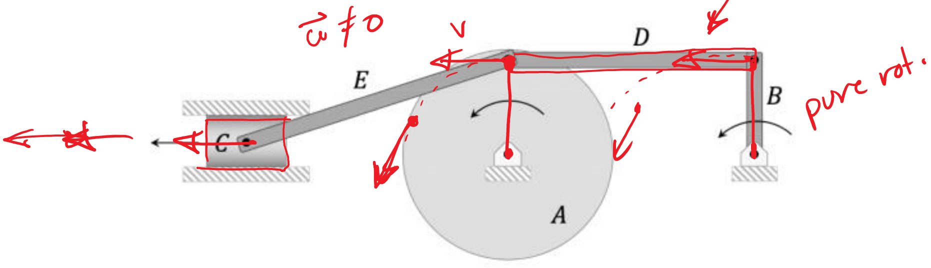
IMPORTANT NOTE: For a RIGID BODY, angular velocity is the SAME at all points on the body.

$\vec{V}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$ $|\vec{V}_{B/A}| = \omega r_{B/A}$

curvilinear trans

$\vec{V}_B = \vec{V}_A + \cancel{\vec{\omega}} \times \vec{r}_{B/A}$

1.3 Types of Rigid Body Motion



An example of bodies undergoing the three types of motion is shown in the above mechanism.

The wheel and crank (A and B) undergo **rotation about a fixed axis**. In this case, both axes of rotation are at the location of the pins and perpendicular to the plane of the figure.

The piston (C) undergoes **rectilinear translation** since it is constrained to slide in a straight line.

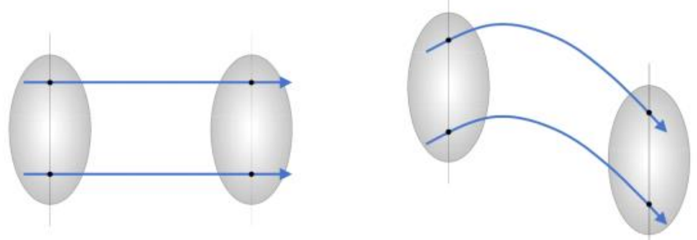
The connecting rod (D) undergoes **curvilinear translation**, since it will remain horizontal as it moves along a circular path.

The connecting rod (E) undergoes **general plane motion**

1.4 Pure Translation

If a body is translating only, the motion equations are the same as for particle motion – all points travel at the same velocity, acceleration, etc.

The angular velocity of the body, $\omega = 0$ angular acceleration $\vec{\alpha} = 0$

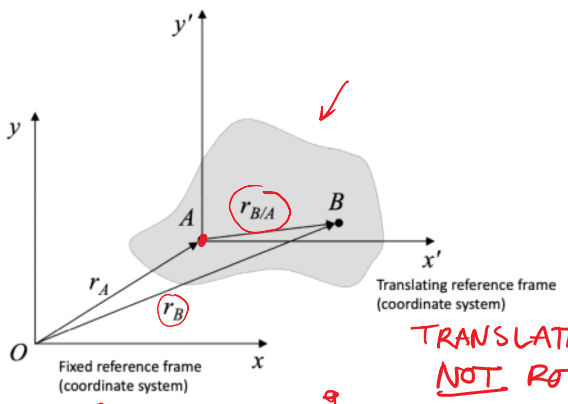


(a) Rectilinear translation (b) Curvilinear translation

Position: $\boldsymbol{r_B} = \boldsymbol{r_A} + \boldsymbol{r_{B/A}}$

Velocity: $\boldsymbol{v_B} = \boldsymbol{v_A}$

Acceleration: $\boldsymbol{a_B} = \boldsymbol{a_A}$

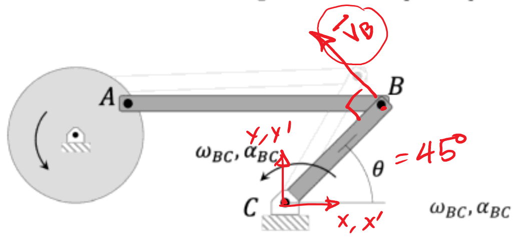


A is fixed $\vec{V}_B = \cancel{\vec{V}_A}^{0} + \vec{\omega} \times \vec{r}_{BA}$

1.5 Pure Rotation

When a rigid body is pinned it rotates about a fixed point, eg. P. Since $v_P = 0$, $a_P = 0$, we only need to consider the motion of the rest of the body with respect to point P.

Cranks, rotating shafts and bodies mounted to fixed hinges are all examples of pure rotation systems.



$\vec{\omega}, \vec{\alpha} \perp$ to plane of motion ⊙ ⊗

Consider the crank arm CB. It has angular velocity $\vec{\omega} = 1\, rad/s$ (ω_{BC}) and angular acceleration $\alpha = 1\, rad/s^2$ (α_{BC}). The length of CB is $1\, m$. Compute the velocity and acceleration of point B when the crank arm forms an angle $\theta = 45°$ with the horizontal plane, as shown.

By Chasles' theorem, we know:

$\vec{V}_B = \vec{V}_C + \vec{\omega} \times \vec{r}_{B/C}$ But $\vec{V}_C = 0$ (pinned)

$\therefore$ for pure rotation: $\vec{V}_B = \vec{\omega} \times \vec{r}_{B/C} = [\vec{\omega} \times \vec{r}_B]$

measured from pin

$$v_B = v_C + \omega \times r_{B/C}$$

ω_{BC} $\vec{\omega}_{BC} = \omega\hat{k}$

$$\vec{v}_B = \underbrace{1\ \text{rad/s}\ \hat{k}}_{\vec{\omega}} \times \underbrace{1\text{m}(\cos 45°\hat{i} + \cos 45°\hat{j})}_{\vec{r}_{B/C}}$$

$$= \frac{1}{\sqrt{2}}\hat{j} + \left(-\frac{1}{\sqrt{2}}\right)\hat{i}\ \text{m/s}$$

include units

$\cos 45 = \frac{1}{\sqrt{2}}$

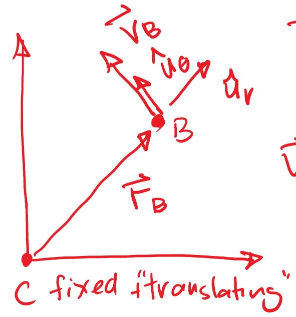
Compare with polar coordinate expression from particle kinematics:

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{u}}_{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{u}}_{\boldsymbol{\theta}}$$

$\dot{r} = 0$

$\dot{\theta} = \|\vec{\omega}\|$

$\frac{d}{dt}$



$\vec{r}_B = r\hat{u}_r$

$\vec{\omega} = \dot{\theta}\hat{k}$

$\vec{v}_B = \vec{\omega}\times\vec{r}_B$

$= r\dot{\theta}(\hat{k}\times\hat{u}_r)$

$= r\dot{\theta}\hat{u}_\theta$

For acceleration, recall:

rigid body rigid body

$$\boldsymbol{a} = (\cancel{\ddot{r}} - r\dot{\theta}^2)\hat{\boldsymbol{u}}_{\boldsymbol{r}} + (2\cancel{\dot{r}}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{u}}_{\boldsymbol{\theta}}$$

So, for pure rigid body rotation we are looking for an equation of the same form. Differentiate:

$\vec{v}_{B/C}$ pure rotation

$$\boxed{\boldsymbol{v}_{\boldsymbol{B}} = \boldsymbol{\omega}\times\boldsymbol{r}_{\boldsymbol{B/C}}}$$

$\vec{r}_B = \vec{r}_{B/C}$ for pure rot.

$\frac{d}{dt}$:

$$\vec{a}_B = \frac{d}{dt}(\vec{\omega})\times\vec{r}_{B/C} + \vec{\omega}\times\frac{d}{dt}\vec{r}_{B/C}$$

$$= \frac{d}{dt}(\omega)\hat{k}\times\vec{r}_B + \vec{\omega}\times\vec{v}_{B/C}$$

$\neq 0$ because direction changes

$$= \alpha\hat{k}\times\vec{r}_B + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{B/C})$$

$\vec{\omega} = \omega\hat{k}$

$\vec{\alpha} = \alpha\hat{k}$

PURE ROT: $$\boxed{\vec{a}_B = \vec{\alpha}\times\vec{r}_B + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{B/C})}$$

Components of acceleration for pure rotation:

rigid body particle

$\boldsymbol{\alpha}\times\boldsymbol{r}_{\boldsymbol{B/C}} \Longleftrightarrow r\alpha\boldsymbol{u}_{\boldsymbol{\theta}}$ - transverse component

$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{\boldsymbol{B/C}}) \Longleftrightarrow -\omega^2 r\boldsymbol{u}_{\boldsymbol{r}}$ - centripedal component

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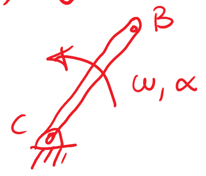
(Note: for $\boldsymbol{a} \perp \boldsymbol{b}$: $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b}) = -a^2\boldsymbol{b}$)

Thus:

pure rot., rigid body

$$\boxed{\boldsymbol{a_B} = \boldsymbol{\alpha} \times \boldsymbol{r_{B/C}} - \omega^2 \boldsymbol{r_{B/C}}}$$



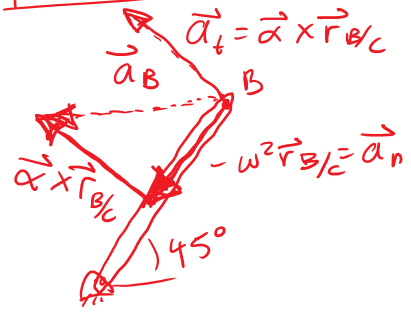
Solve for the acceleration of point B: $\vec{a}_B$?

$$\vec{r}_{B/C} = 1\,\text{m}\,(\cos 45\,\hat{\imath} + \cos 45\,\hat{\jmath})$$
$$= \left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right)\text{m}$$

$$\vec{\omega} = 1\,\text{rad/s}\,\hat{k}$$
$$\vec{\alpha} = 1\,\text{rad/s}^2\,\hat{k}$$

$$\vec{a}_B = (1\,\text{rad/s}^2\,\hat{k}) \times \left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right)\text{m}$$
$$- (1\,\text{rad/s})^2\left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right)\text{m}$$
$$= \cancel{\frac{1}{\sqrt{2}}\hat{\jmath}} - \frac{1}{\sqrt{2}}\hat{\imath} - \left(\frac{1}{\sqrt{2}}\hat{\imath} + \cancel{\frac{1}{\sqrt{2}}\hat{\jmath}}\right)\ \text{m/s}^2$$

$$\boxed{\vec{a}_B = -\sqrt{2}\,\hat{\imath}\ \text{m/s}^2}$$



1.6 Fixed Rotation: Angular Relations

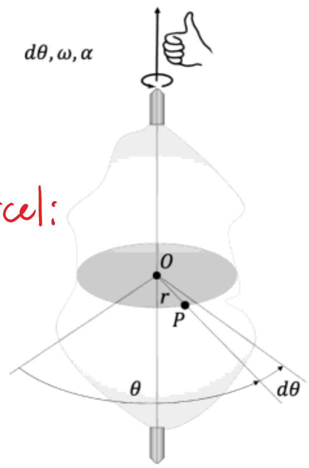
In this section, we will establish parallel relationships between distance/speed/acceleration and angle/angular speed/angular acceleration **in a single direction (scalar)**

always true

Angular:	**Rectilinear:**
$\omega = \frac{d\theta}{dt}$	$v = \frac{ds}{dt}$
$\alpha = \frac{d\omega}{dt}$	$a = \frac{dv}{dt}$

only true for constant accel:

For $\alpha = \alpha_c$ (constant):	For $a = a_c$ (constant):
$\omega = \omega_0 + \alpha_c t$	$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$	$v = v_0 + a_c t$
$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$	$v^2 = v_0^2 + 2a_c(s - s_0)$

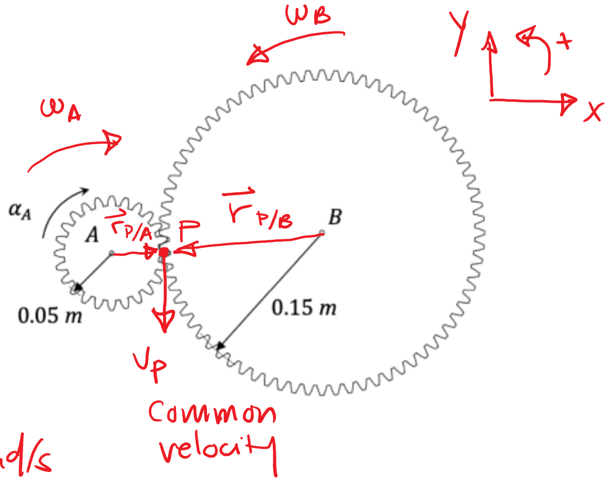


Important Note: Angular velocity and angular acceleration FOR A RIGID BODY is the SAME for EVERY POINT on the body

1.6.1 Example

Gear A and B mesh as shown. A starts from rest, with constant angular acceleration $\alpha_A = 2\,rad/s^2$. How long does it take B to reach $\omega_B = 50\,rad/s$?

velocities at $t=0$ $=0$



find ω_A when $\omega_B = 50$ rad/s

A: $\vec{v}_P = \vec{\omega}_A \times \vec{r}_{P/A} = -\omega_A \hat{k} \times 0.05\,m(\hat{i})$

$= -0.05\,\omega_A \hat{j}$ m/s

B: $\vec{v}_P = \vec{\omega}_B \times \vec{r}_{P/B} = \omega_B \hat{k} \times 0.15\,m(-\hat{i})$

$= -0.15\,\omega_B \hat{j}$ m/s

$\vec{v}_P = \vec{v}_P$: $-0.05\,\omega_A = -0.15\,\omega_B$ m/s

$$\boxed{\omega_A = 3\,\omega_B}$$

gear ratio: $\dfrac{|\omega_A|}{|\omega_B|} = \dfrac{r_B}{r_A}$

find t to get to ω_B: $\omega_B = 50$ rad/s $\quad \omega_A = 150$ rad/s

$\omega_A = \cancel{\omega_{A0}}^{\,0} + \alpha_{A\,const}\, t \quad$ 2 rad/s²

solve: $\boxed{t = 75\text{ sec}}$

