DYN L8 9

September 24, 2019 9:17 AM

7 L8/L9: Centroids, Centres of Mass, Moments of Inertia, and Mass Moments of Inertia

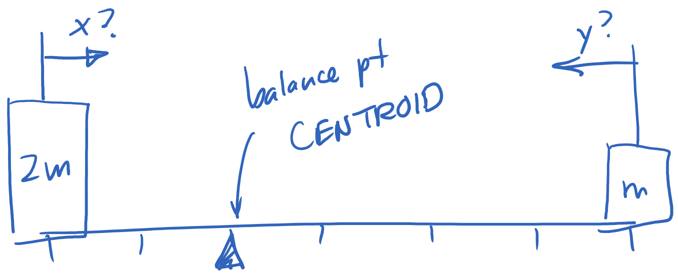
Readings

7.1 Objective

To describe what the Centroids, Centres of Mass, Moments of Inertia and Mass Moments of Inertia are, to see examples where these quantities are used, and to compute them.

7.2 Centroids

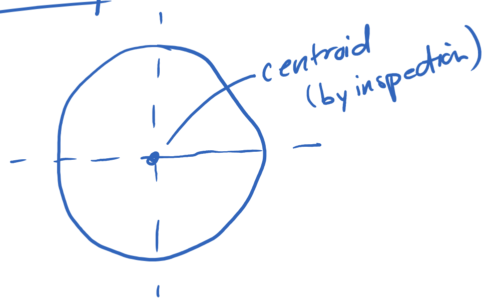
The centroid of a region can be thought of as the region's **balance point** or **point of symmetry** of the object. The centroid is located at the intersection of the **axes of symmetry** of the body.

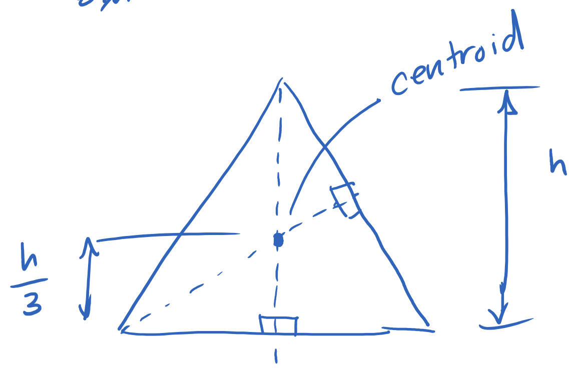


$$\left.\begin{aligned} 2m\,x) &= m(y) \\ x + y &= 6 \end{aligned}\right\} \Rightarrow \quad 2m(6-y) = my$$

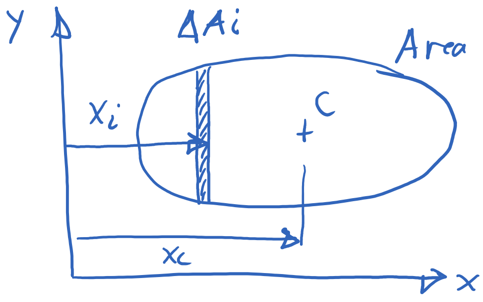
$$y = \frac{2\cancel{m}(6)}{3\cancel{m}} = 4$$

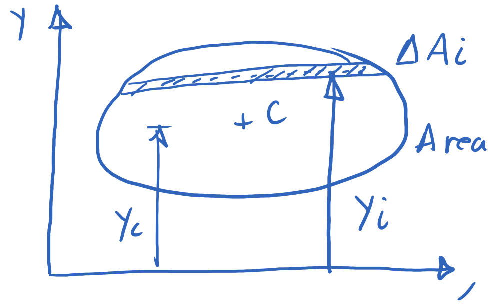
symmetry:





Consider a uniform mass distributed over a planar surface: if placed on a pivot at the centroid, the surface would be balanced on the pivot. (Centroid located at (x_C, y_C)).





If we identify the centroid by the letter C, and we consider each portion, ΔA_i, of the area (where $\sum_i \Delta A_i = A$, we can find the x, y location of the centroid as follows:

$$x_C = \frac{\sum_i x_i \Delta A_i}{\sum_i \Delta A_i} = \frac{\sum_i x_i \Delta A_i}{A} \qquad y_C = \frac{\sum_i y_i \Delta A_i}{\sum_i \Delta A_i} = \frac{\sum_i y_i \Delta A_i}{A}$$

total area

In integral form this is written as:

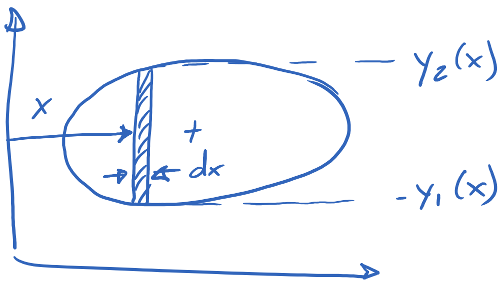
don't worry

$$x_C = \frac{\iint_A x\,dA}{\iint_A dA} = \frac{\int_{x_1}^{x_2}\int_{y_1(t)}^{y_2(t)} x\,dy\,dx}{\int_{x_1}^{x_2}\int_{y_1(t)}^{y_2(t)} dy\,dx} = \frac{\int_{x_1}^{x_2} x\,dx \int_{y_1(t)}^{y_2(t)} dy}{\int_{x_1}^{x_2} dx \int_{y_1(t)}^{y_2(t)} dy} = \frac{\int_{x_1}^{x_2} x\,dx[y_2(x) - y_1(x)]}{\int_{x_1}^{x_2} dx[y_2(x) - y_1(x)]}$$

total area

$$x_C = \frac{\int_{x_1}^{x_2} x[y_2(x) - y_1(x)]dx}{A}$$

single integral form for centroid

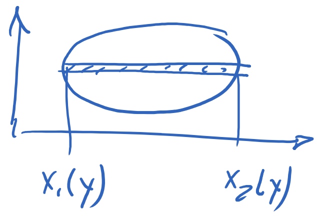


x^1

Note: the term $\iint x dA$ is called the **first moment of area** about the x-axis (i.e. x^1).

Similarly,

$$\underset{=}{y_C} = \frac{\iint_A y\,dA}{\iint_A dA} = \frac{\int_{y_1}^{y_2} y[x_2(\overset{y}{\cancel{x}}) - x_1(\overset{y}{\cancel{x}})]dy}{A}$$



If we think of $\boldsymbol{r}$ as a vector, $\boldsymbol{r} = (x, y)$, we can write the location of the centroid as:

$$\boldsymbol{r_C} = \frac{\sum_i \boldsymbol{r_i}\Delta A_i}{\sum_i \Delta A_i} = \frac{\sum_i \boldsymbol{r_i}\Delta A_i}{A}$$

Or, in integral form:

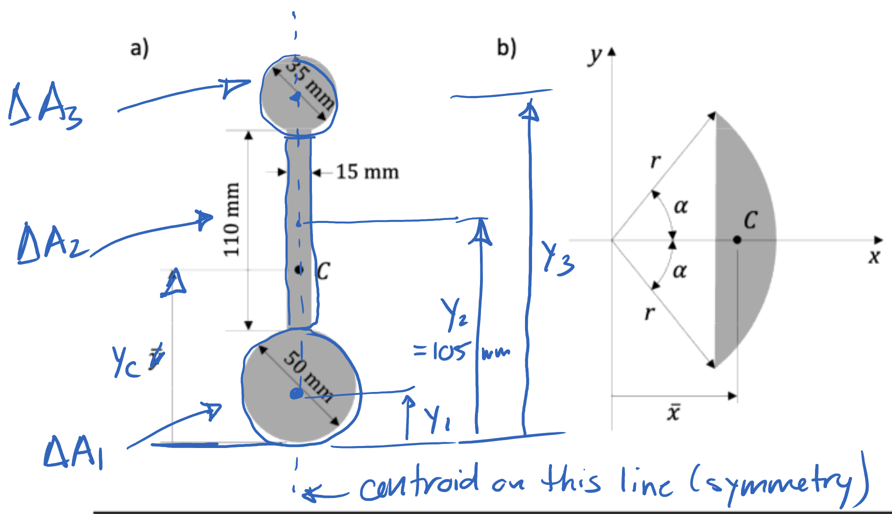
$$\boldsymbol{r_C} = \frac{\iint_A \boldsymbol{r_i}dA}{\iint_A dA} = \frac{\iint_A \boldsymbol{r_i}dA}{A}$$

where the the location of $\boldsymbol{r_C} = (x_C, y_C)$ can be computed using the equations for x_C and y_C above.

NOTE: for centroids of 3D objects, we compute $\boldsymbol{r_C} = (x_C, y_C, z_C)$ in a similar manner, except instead of considering area, we consider volume, $dV = dxdydz$.

7.2.1 Example

Compute the location of the centroid, $\boldsymbol{r_C}$, for the shapes shown.



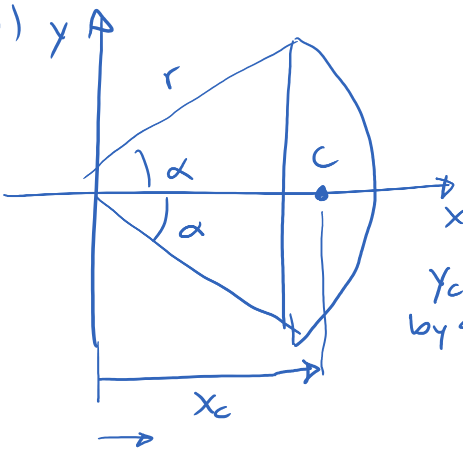
1 $A = \Delta A_1 + \Delta A_2 + \Delta A_3 = \pi(25\,\text{mm})^2 + (110)(15) + \pi(17.5\,\text{mm})^2$

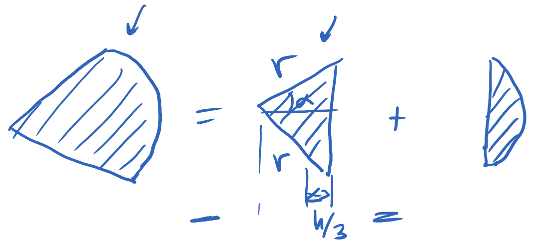
$= 4575\,\text{mm}^2$

$$y_c = \frac{\Sigma y_i \Delta A_i}{A} = \frac{\overset{y_1}{(25\,\text{mm})}\,\pi(25\,\text{mm})^2 + \overset{y_2}{(50+55)}(110)(15) + \overset{y_3}{(110+50+17.5)}\,\pi(17.5)^2}{A}$$

$y_c = 86\,\text{mm}$

(b)





$y_c = 0$ by symmetry

$$A_P = A_T + A_S$$

$$\alpha r^2 = \frac{1}{2}(\underbrace{r\cos\alpha}_{h})(\underbrace{2r\sin\alpha}_{b}) + A_S$$

$$A_S = \alpha r^2 - r^2\cos\alpha\sin\alpha$$

$$A_P\bar{x}_P = A_T\bar{x}_T + A_S\bar{x}_S \leftarrow \text{want to find}$$

from formula sheet

$$\left\{\begin{aligned} \bar{x}_P &= \frac{2}{3}r\frac{\sin\alpha}{\alpha} \\ \bar{x}_T &= \frac{2}{3}h = \frac{2}{3}r\cos\alpha \end{aligned}\right.$$

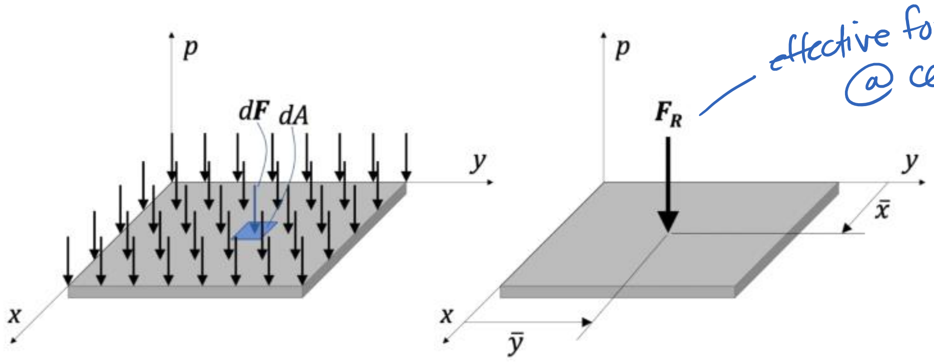
$$\bar{x}_S = \frac{A_P\bar{x}_P - A_T\bar{x}_T}{A_S} = \frac{\alpha r^2\left(\frac{2}{3}r\frac{\sin\alpha}{\alpha}\right) - r^2\cos\alpha\sin\alpha\left(\frac{2}{3}r\cos\alpha\right)}{\alpha r^2 - r^2\cos\alpha\sin\alpha}$$

$$\Rightarrow \bar{x}_S = \frac{2}{3}\frac{r\sin^3\alpha}{(\alpha - \cos\alpha\sin\alpha)}$$

(Example continued)

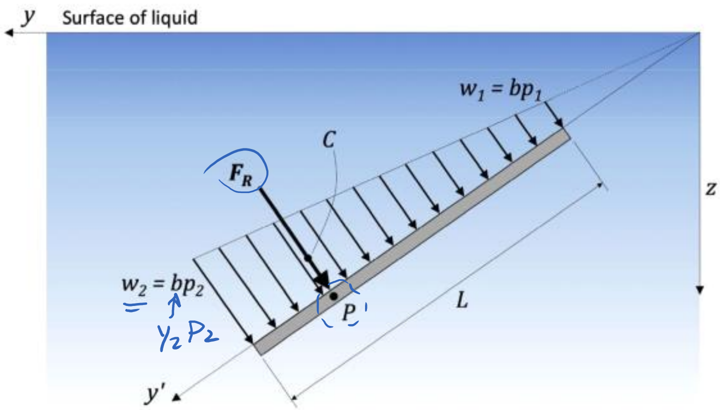
7.3 Applications of Centroids

For a uniform distributed load over a surface: e.g. a uniform pressure distribution, or a uniformly distributed normal load, the effective force acts through the centroid of the surface.



effective force @ centroid

For a non-uniform pressure or stress distribution, the resultant force acts through the centroid of the distribution:



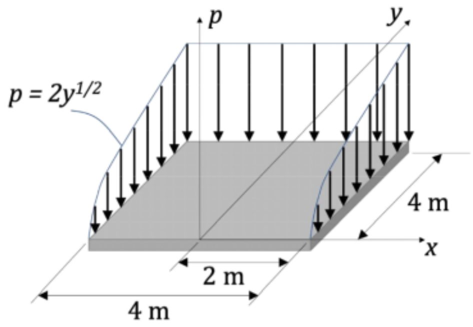
$$P = \frac{\int y\,dF}{F} = \frac{\int_{y_1}^{y_2} y \overbrace{p}^{w}\,dy}{\int_{y_1}^{y_2} p\,dy}$$

p = expression for pressure times width (into page)

In other words, the resultant force acts through the balance point at which there is an equal moment on each side of the point.

7.3.1 Example

Find the location of the resultant force for the pressure distribution applied to the flat plate.



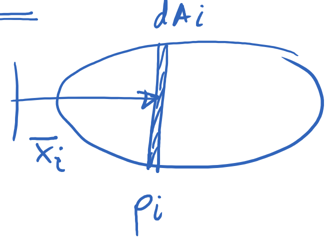
7.4 Centre of Mass

The centre of mass located at $\boldsymbol{r_G} = (x_G, y_G, z_G)$ is computed in a very similar manner to the centroid, where we replace ΔA or ΔV (dA or dV) with Δm (dm) as the variable of summation (integration).

$$x_G = \frac{\sum_i x_i \Delta m_i}{\sum_i \Delta m_i} = \frac{\sum_i x_i \Delta m_i}{m} \qquad y_G = \frac{\sum_i y_i \Delta m_i}{\sum_i \Delta m_i} = \frac{\sum_i y_i \Delta m_i}{m}$$

However, when computing the centre of mass, we consider that the body has a density $\rho(x, y, z)$ that could vary over the body. For planar bodies, we will just consider $\rho(x, y)$.

Thus we have: $dm = \rho(x, y)dA = \rho(x, y)dxdy$



$$x_G = \frac{\iint_m x dm}{\iint_m dm} = \frac{\int_{x_1}^{x_2} \int_{y_1(t)}^{y_2(t)} x\rho(x)dxdy}{\int_{x_1}^{x_2} \int_{y_1(t)}^{y_2(t)} \rho(x)dxdy}$$

If $\rho = constant$, then it can be cancelled out from the top and bottom of the above equation and centre of mass is the same as the centroid, $\boxed{x_G = x_C}$. ρ constant

Similarly $y_G = \dfrac{\iint_m y dm}{m}$ and, for 3D bodies, $z_G = \dfrac{\iiint_m z dm}{m}$. In general, $\boldsymbol{r_G} = \dfrac{\iiint_m \boldsymbol{r} dm}{m}$.

In statics and dynamics, we resolve forces and moments applied to a body down to a **resultant force acting at the center of mass, and a resultant moment**.

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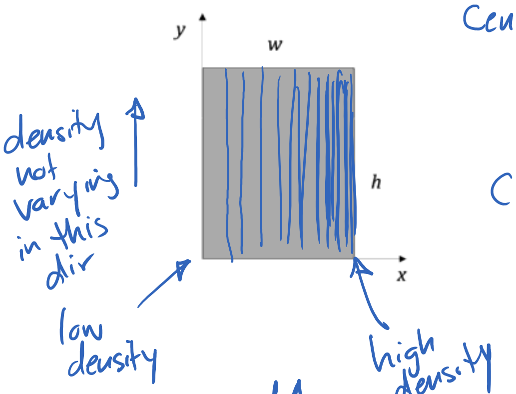
7.4.1 Example

Find the centre of mass of a plate with density distribution $\rho = \nu x$.

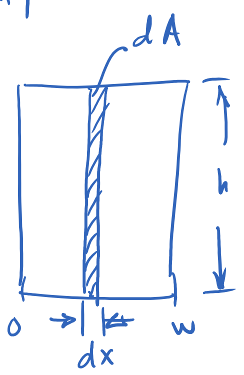


Centroid (x_c, y_c)

$y_c = \frac{h}{2}$, $x_c = \frac{w}{2}$ by symmetry

Centre of mass $(\bar{x}, \bar{y})$

$\bar{y} = \frac{h}{2}$ by symmetry

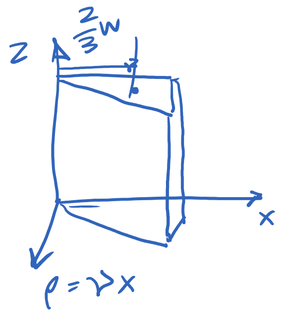


$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^w x(\overbrace{\cancel{\nu} x \cancel{h}\,dx}^{dm})}{\int_0^w \cancel{\nu} x \cancel{h}\,dx} = \frac{\int_0^w x^2\,dx}{\int_0^w x\,dx} = \frac{\frac{x^3}{3}\Big|_0^w}{\frac{x^2}{2}\Big|_0^w}$$

$dm = \rho\,dA$
$= \rho\,h\,dx$
$= (\nu x)\,h\,dx$

ν = "nu"

$$\bar{x} = \frac{2}{3} w$$



Centre of mass $\left(\frac{2}{3}w, \frac{h}{2}\right)$

centroid $\left(\frac{w}{2}, \frac{h}{2}\right)$

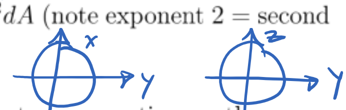
not the same when density varies

7.5 Moments of Inertia – Second Moment of Area

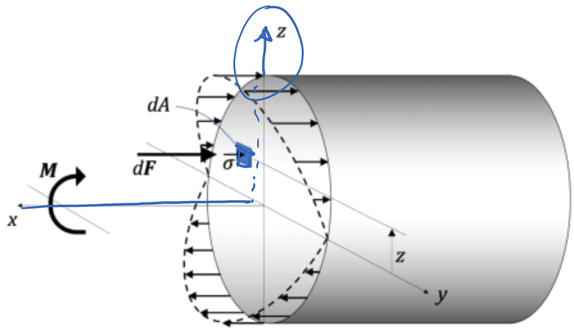
The term "moment of inertia" for an area is commonly used by engineers, but actually incorrectly applied. For an area, the correct term is "second moment of area". However, due to their similarity with integrals of **mass moments of inertia**, the term sticks.

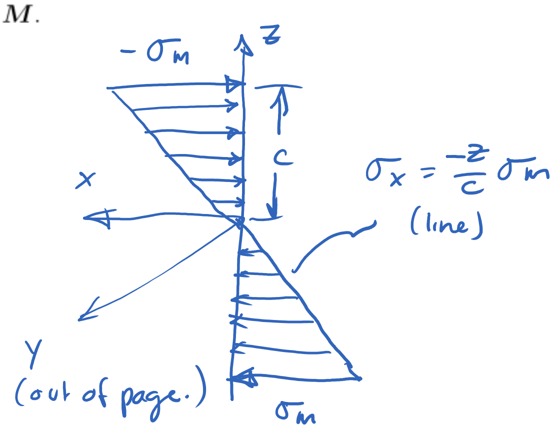
x^1

The second moment of area is **computed about an axis** as $\int\int_A x^2 dA$ (note exponent 2 = second moment), where x is measured perpendicular to the axis.



One considers the second moment of area when relating the normal stress, σ, acting on the cross-section of a beam related to an applied external moment $\boldsymbol{M}$.





$\sigma_x = -\frac{z}{c}\sigma_m$ (line)

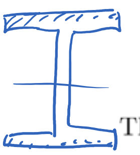
$dF = \sigma_x dA = -\frac{z}{c}\sigma_m dA$

Under bending, the stress, σ, varies linearly with distance from the centroidal axis, y, as $\sigma_x = -\frac{z}{c}\sigma_m$.

The moment (torque), M, about the y-axis is

F·d dF

$$M = \int -zdF = \int -z(-\frac{z}{c}\sigma_m)dA$$



Thus:

$$M = \frac{\sigma_m}{c}\underbrace{\int z^2 dA}_{I_y} = \frac{\sigma_m}{c}I_y$$

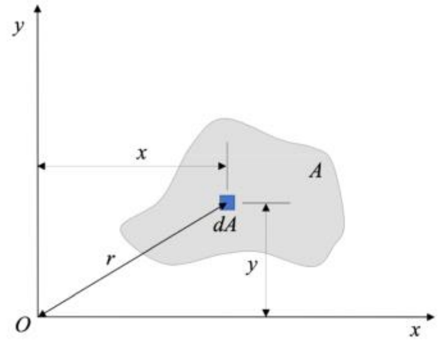
I_y = 2nd moment of Area about y-axis

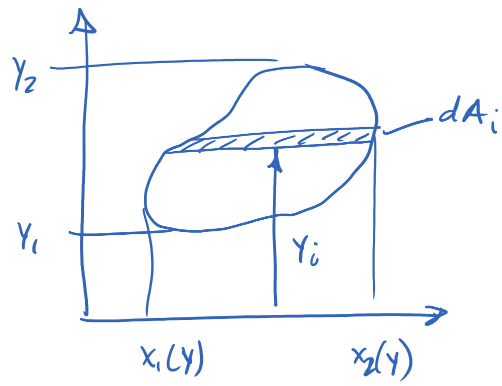
Rearranging, and using our first equation we get: $\sigma_m = -\frac{Mz}{I_y}$

= resistance to bending

The second moment of area shows up in mechanics of materials, fluid mechanics, machine design, etc.

How do we find this quantity? For an area, A, we can compute the second moment of area around the x- and y- axes as:





diff fr centroid

$$I_x = \iint_A y^2 dA = \int_{y_1}^{y_2}\int_{x_1}^{x_2} y^2 dx dy = \int_{y_1}^{y_2} y^2[x_2(y) - x_1(y)]dy$$

$$I_y = \iint_A x^2 dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} x^2 dy dx = \int_{x_1}^{x_2} x^2[y_2(x) - y_1(x)]dx$$

Around the z axis, we compute the second moment of area about the "pole" coming out of point O, and this is referred to as the **polar moment of inertia**.

$$J_O = \iint_A \boldsymbol{r}^2 dA = \iint_A (x^2 + y^2)dA = \underbrace{I_x + I_y}$$

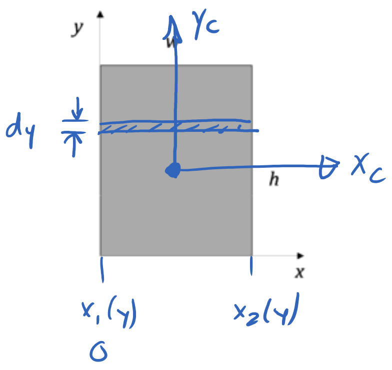
resistance to torsion

Notes:

- The value of the moment of area depends on the axis about which it is computed! Often, these values are compute about centroidal axes (axes passing through the centroid).
- Moments of areas about an axis can be added and subtracted to compute the moments of complex shapes

7.5.1 Example

Find the second moment of area (moment of inertia) and polar moment of inertia of a rectangular plate about the centroidal axes parallel to the x, y, and z axes shown.



Find I_{x_c}, I_{y_c}

$$I_{x_c} = \iint_A y^2\,dA \qquad dA = \underbrace{(w - 0)}_{(x_2(y) - x_1(y))}\,dy$$

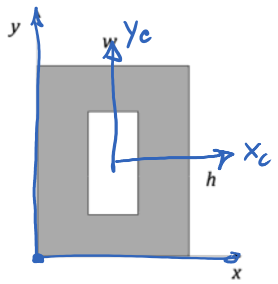
$$= \int_{-\frac{h}{2}}^{h/2} y^2 \underbrace{w\,dy}_{dA} = \frac{wy^3}{3}\Bigg|_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{wh^3}{12}$$

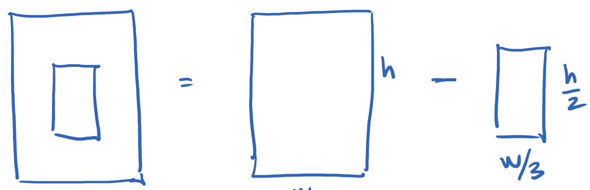
Standard formula in formula sheet.

similarly: $I_{y_c} = \dfrac{hw^3}{12}$

$$J_o = I_{x_c} + I_{y_c} = \frac{hw}{12}(w^2 + h^2)$$

A rectangle of height $\frac{h}{2}$ and width $\frac{w}{3}$ is cut out of the plate. Repeat the computation above.





$$I_{x_c} = \frac{wh^3}{12} - \frac{\left(\frac{w}{3}\right)\left(\frac{h}{2}\right)^3}{12}$$

$$= wh^3\left(\frac{1}{12} - \frac{1}{12}\cdot\frac{1}{24}\right) = \boxed{\frac{23}{(24)(12)}\,wh^3}$$

similarly:

$$I_{y_c} = \frac{hw^3}{12} - \frac{\left(\frac{h}{2}\right)\left(\frac{w}{3}\right)^3}{12}$$

$$\boxed{I_{y_c} = hw^3\,\frac{53}{(54)(12)}}$$

$$J_o = I_{x_c} + I_{y_c}$$

7.6 Mass Moment of Inertia

The mass moment of inertia is a measure of resistance of a body to angular acceleration, the same way that mass is a measure of a body's resistance to acceleration:

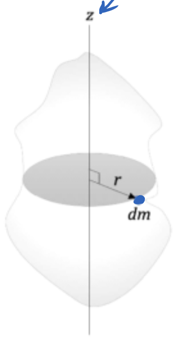
analogous to

$$\boldsymbol{M} = I_G \boldsymbol{\alpha} \qquad \boldsymbol{F} = m\boldsymbol{a}$$

The mass moment of inertia is the integral of the second moment about an axis of all the elements of mass, dm, which compose the body, $I = \iint_m \boldsymbol{r}^2 dm$.

mass MOI about axis

recall 2nd mom. of area
$I_y = \iint_A x^2 dA$



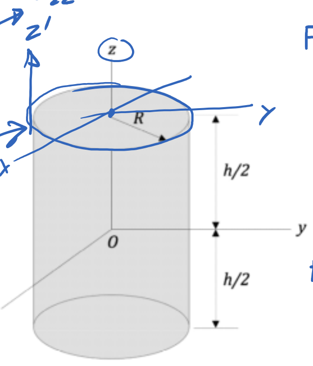
Again, we can compare this to the second moment of area if we consider a body with constant depth, h, and constant density, ρ:

$$I_z z = \iint_m \boldsymbol{r}^2 dm = \iint_m \boldsymbol{r}^2 \rho h dx dy = \rho h \iint_A (x^2 + y^2) dx dy = \rho h (I_x + I_y) = \rho h J_O$$

I_{zz}

double subscript

2nd mom. of area considers crossect.



From table:
$I_{zz} = \frac{1}{2} m r^2 = \rho h J_o$

$m = \rho \underbrace{h \pi r^2}_{\text{Volume}}$

sub in, solve for J_o

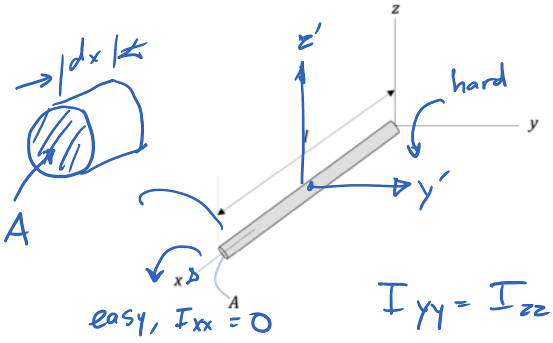
$J_o = \frac{1}{2}(\pi r^4)$ also in table

From table (circle):
$I_x = \frac{1}{4}\pi r^4$
$I_y = \frac{1}{4}\pi r^4$
$J_o = I_x + I_y = \frac{1}{2}\pi r^4$

The mass moment of inertia has units of $mass \times distance^2$, such as $kg\text{-}m^2$ or $slug\text{-}ft^2$. It is always positive.

7.6.1 Example

Find the mass moments of inertia about the end, and the center of mass, of a uniform thin bar.



$I_{xx} = 0$ (thin)

symmetry about x-axis

$$I_{yy} = I_{zz} = \iint_m x^2 \, dm$$

$$dm = \rho \underbrace{A \, dx}_{dV}$$

$$m = \rho A l$$

$$I_{yy} = I_{zz} = \rho A \int_0^l x^2 \, dx = \frac{l^3}{3} \rho A = \underbrace{(\rho A l)}_{m} \frac{l^2}{3}$$

$$I_{yy} = I_{zz} = \frac{ml^2}{3}$$

$$I_{y'y'} = I_{z'z'} = \rho A \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \, dx = \rho A \frac{x^3}{3} \Big|_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{1}{12} \rho A l^3$$

$$= \frac{1}{12} m l^2$$

change limits of integration

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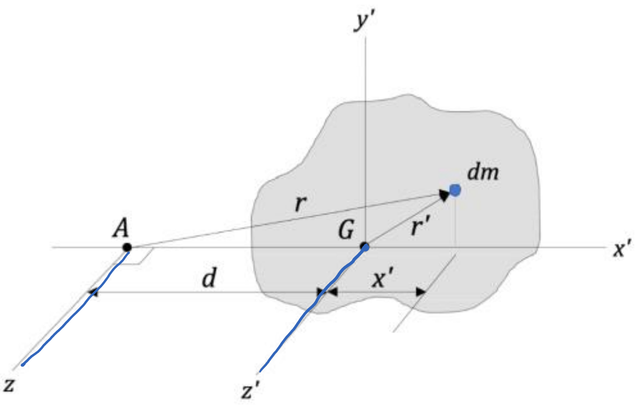
7.7 Parallel Axis Theorem

If the moment of inertia, I_G, of a body of mass, m, about an axis passing through the body's centre of mass, G, is known, then the moment about any other parallel axis, distance, d, from G can be found as:

centre of mass ALWAYS

$$I_{parallelaxis} = I_G + md^2$$

I_{zz} $I_{z'z'}$ distance btwn // axes



I_G always minimum MoI

This theorem can also be applied to second moments of area, i.e.:

about centroid ALWAYS

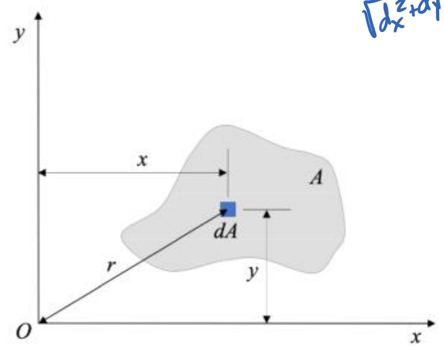
$$I_x = I_{Cx'} + Ad_y^2$$

instead of mass

$$I_y = I_{Cy'} + Ad_x^2$$

$$J_O = J_C + Ad^2$$

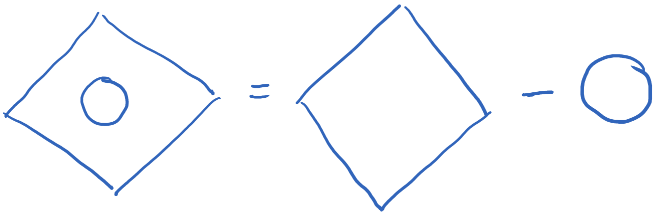
$\sqrt{d_x^2+d_y^2}$

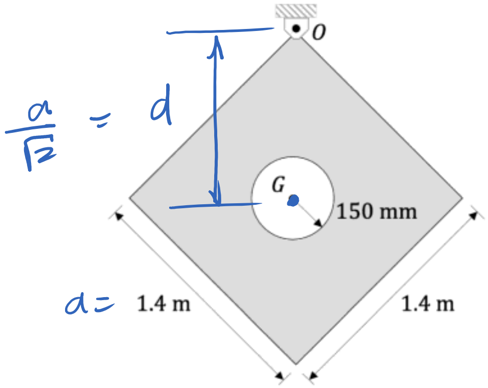


7.7.1 Example

Find the mass moment of inertia of the plate about point O. The plate has thickness $50\,mm$ and density $50\,kg/m^3$.

I_G:





$$I_{G\diamond} = \frac{1}{12} m_\diamond (a^2+b^2) = \frac{1}{12}(\rho A_\diamond t)(1.4^2+1.4^2) = 0.64\rho t$$

$$- I_{G\circ} = \frac{1}{2} m_\circ r^2 = \frac{1}{2}(\rho \pi r^2 t) r^2 = 0.000795\rho t$$

$$I_G \approx 0.64\rho t$$

want I_O

$$I_O = I_G + md^2 = I_G + \underbrace{\rho t(\overbrace{a^2-\pi r^2}^{A_\diamond})}_{m}\overbrace{\left(\frac{a}{\sqrt{2}}\right)^2}^{d^2}$$

$$= 2.5\rho t = 6.27\ \text{kg-m}^2$$

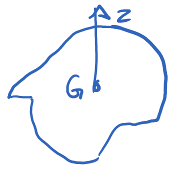
much bigger than I_G

7.8 Radius of Gyration

@ centre of gravity

The radius of gyration, k_G, is a method of specifying the inertia of a body without including the mass. It is specified in units of length. Using this measure, the mass moment of inertia of a body is:

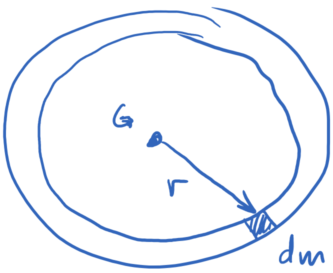
$$I = mk_G^2 \qquad or \qquad k_G = \sqrt{\frac{I}{m}}$$



I_{zz} head reported as k_G

One can specify the radius of gyration for the second moment of area by substituting area, A, for mass m.

For a circular ring the radius of gyration about the centroidal axis normal to the ring is the same as the radius of the ring.

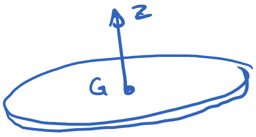


$$I_G = \iint_m r^2\, dm$$

$$= r^2 \underbrace{\iint_m dm}_{m}$$

$$= mr^2$$

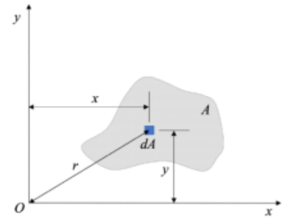
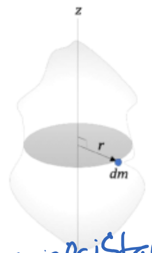
$$k_G = r$$



$$I_{zz_G} = \frac{1}{2}mr^2 = mk_G^2$$

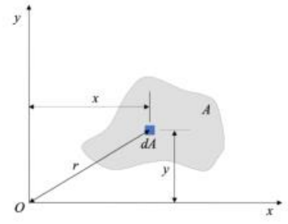
$$\Rightarrow k_G = \frac{r}{\sqrt{2}}$$

7.9 Summary

Centroid Centre of Mass	$r_C = \dfrac{\iint_A r\,dA}{A}$ $r_G = \dfrac{\iint_m r\,dm}{\cancel{A}\ m}$	Same for uniform density material
resistance to bending SOLID MECH Second Moment of Area (moment of inertia) about an axis (area in the x,y plane)	$I_x = I_{Cx'} + Ad_y^2$ $I_y = I_{Cy'} + Ad_x^2$ $J_O = J_C + Ad^2$	
resistance to torsion DYN Mass Moment of Inertia about an axis	resistance to Δ angular acceleration $I = \iint_m r^2 dm$	 (mass resistance to Δ linear acceleration)
For a planar body of uniform thickness and density	$I_z = \rho h(I_x + I_y) = \rho h J_O$	

MMOI 2nd MOA PMOA

DYN SM

Parallel Axis Theorem	2nd MOA $I_x = I_{Cx'} + Ad_y^2$ $I_y = I_{Cy'} + Ad_x^2$ $J_O = J_C + Ad^2$ MMOI $I_{parallelaxis} = I_G + md^2$	
Radius of Gyration MMOI	$I = mk_G^2$ or $k_G = \sqrt{\frac{I}{m}}$	For second moment of area, replace m with A