DYN L20

September 24, 2019 9:20 AM

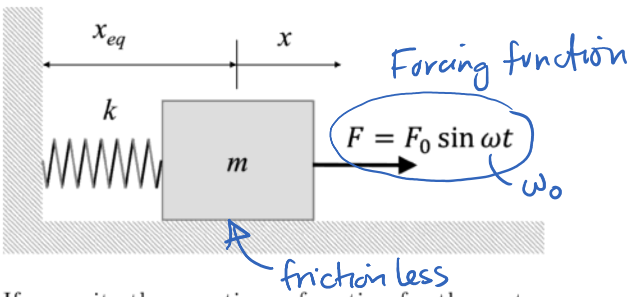
13 L20: Harmonic Force~~s~~d Undamped Vibration

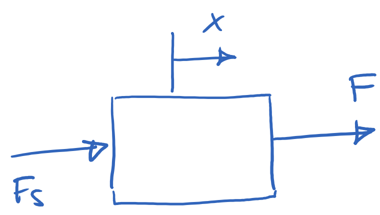
Readings

13.1 Objectives

- To model the behaviour of undamped systems under harmonic excitation.
- Identify the complementary, particular, and general solution of the equation of motion for the system.
- Observe how the amplitude of motion changes as a function of the relationship between the forced frequency and the system natural frequency.
- Observe the phenomena of beats, and resonance.

13.2 Mass Spring Model





$F_s = -kx$

If we write the equations of motion for the system:

$$\sum F_x = ma$$

$$\cancel{kxF_0 \sin \omega_0 t = m\ddot{x}} \qquad -kx + F_0 \sin \omega_0 t = m\ddot{x}$$

$$\Rightarrow \underbrace{m\ddot{x} + kx}_{x + \text{derivs}} = \underbrace{F_0 \sin \omega_0 t}_{\text{forcing func.}}$$

not a funct. of $x, \dot{x}, \ddot{x}$

The **general solution** of this equation is composed of the superposition of the **complementary solution** and the **particular solution**.

Std form

$$\ddot{x} + \underbrace{\frac{k}{m}}_{\omega_n^2} x = \frac{F_0 \sin \omega_0 t}{m}$$

For the complementary solution set $RHS = 0$.

free undamped vibration sol'n $\rightarrow$

$$x_c = C \sin(\omega_n t + \phi)$$

$$\text{where } \omega_n = \sqrt{\frac{k}{m}}$$

$F = F_0 \sin \omega_0 t$

For the particular solution, since the forcing function is periodic, assume a similar periodic solution:

$$\begin{aligned} x_p &= D \sin(\omega_0 t) \\ \ddot{x}_p &= -\omega_0^2 D \sin(\omega_0 t) \end{aligned}$$

$F = F_0 \cos(\omega_0 t + \psi)$
$x_p = D \cos(\omega_0 t + \psi)$

forcing frequency

ψ = psi
ϕ = phi

Substituting this back into our equation of motion:

$$-m\omega_0^2 D \sin\cancel{(\omega_0 t)} + kD \sin\cancel{(\omega_0 t)} = F_0 \sin\cancel{(\omega_0 t)}$$

$D(k - m\omega_0^2) = F_0$

We can then solve for D, as:

$$D = \frac{\frac{F_0}{k}}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2}$$

amplitude of forced response

Thus, the particular solution is:

$$x_p = \frac{\frac{F_0}{k}}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2} \sin(\omega_0 t)$$

And the general solution is:

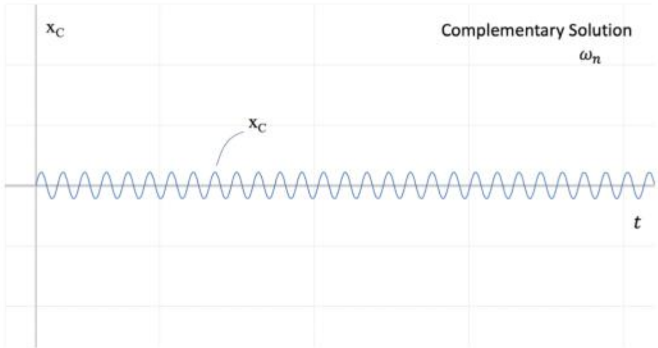
$$x_G(t) = \underbrace{\frac{\frac{F_0}{k}}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2}}_{\text{forced response}} \sin(\omega_0 t) + \underbrace{C \sin(\omega_n t + \phi)}_{\text{free response}}$$

Where C and ϕ are defined by the initial conditions.

OR $F = F_0 \cos(\omega_0 t + \psi)$
$x(t) = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \cos(\omega_0 t + \psi) + C \sin(\omega_n t + \phi)$

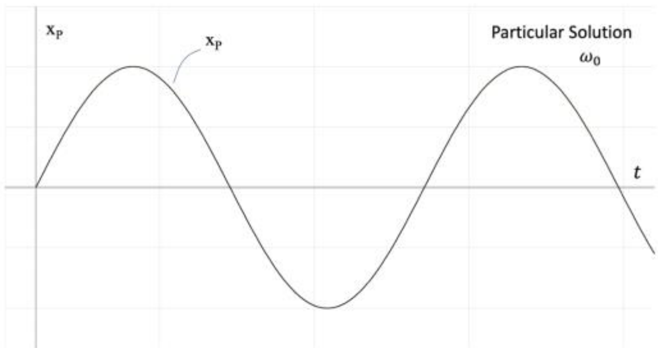
Free vibration - also called **transient vibrations** - will eventually die out due to friction/damping in the system.

Forced vibration - also called **steady-state vibrations** - remain as long as the forcing function continues.



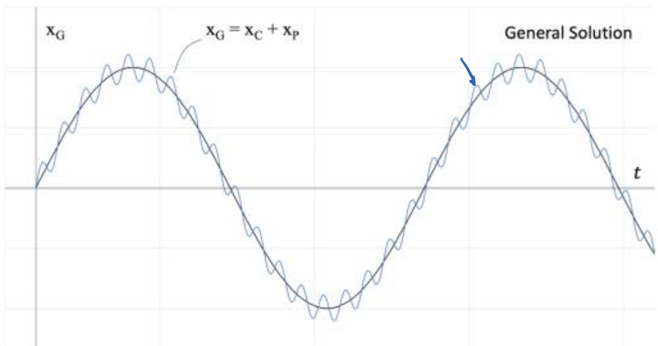
Free response

+



Forced response

=



Total response

13.3 Amplitude of Forced Vibration

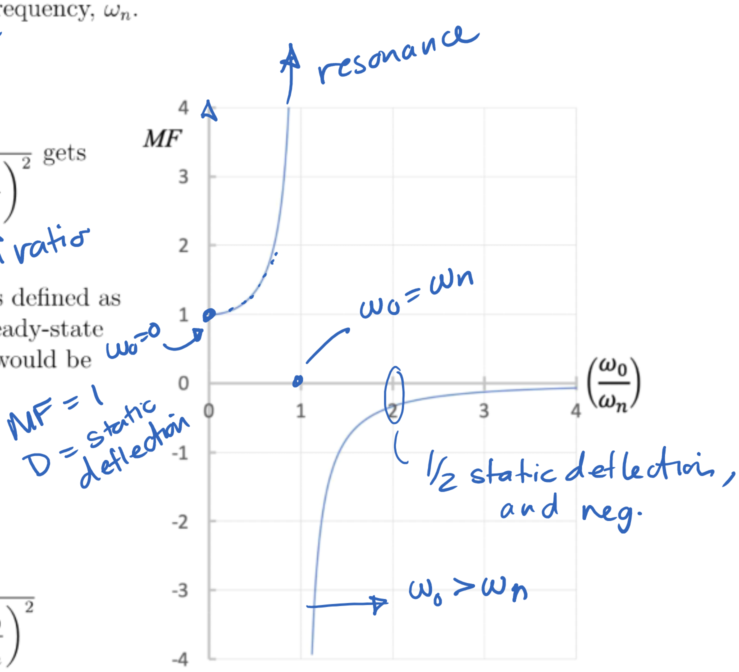
The amplitude of the steady-state (forced) vibration depends on the ratio of the frequency of the forced vibration, ω_0, to the natural frequency, ω_n.

amplitude of forced response

As ω_0 approaches ω_n, $D = \dfrac{\frac{F_0}{k}}{1-\left(\frac{\omega_0}{\omega_n}\right)^2}$ gets really BIG!

ratio

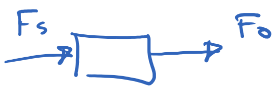
The **magnification factor**, MF, is defined as the ratio of the amplitude of the steady-state vibration to the displacement that would be achieved by static deflection:



$$MF = \frac{\dfrac{\frac{F_0}{k}}{1-\left(\frac{\omega_0}{\omega_n}\right)^2}}{\dfrac{F_0}{k}} = \frac{1}{1-\left(\frac{\omega_0}{\omega_n}\right)^2}$$

static deflection

Static deflection: if F_0 was applied without oscillation (i.e. no $\sin(\omega_0 t)$):



static position

static deflection

$\Sigma F_x: \; F_s + F_0 = 0$

$-kx + F_0 = 0 \quad \Rightarrow \quad x = \dfrac{F_0}{k}$

When $\omega_0 = \omega_n$, **resonance** occurs. The amplitude of vibration gets very large, causing stress and failure (not good!).

For $\omega_0 \approx 0$, $MF \approx 1$. The forcing function is changing quite slowly so it is not exciting a lot of natural vibration.

For $\omega_0 > \omega_n$, the magnification factor becomes negative. This means that the motion of the block is always in the opposite direction (out of phase) with the force. When the mass is displaced to the right, the force acts to the left and visa-versa.

At extremely high forcing frequencies, the force is changing direction too fast for the block's motion to respond.

13.4 Beats

For zero initial conditions, we can write the response of the system in for form:

forced free

$\omega_n \neq \omega_0$

$$x(t) = \frac{\frac{F_0}{k}}{1-\left(\frac{\omega_0}{\omega_n}\right)^2}\left(\cos(\omega_0 t) - \cos(\omega_n t)\right)$$

$x(0) = 0$

$\dot{x}(0) = 0$

same phase angle

initial conds.

This equation can be re-written using trig identities as:

$\omega^* =$ ave. freq.

$\omega_n - \omega_0 = \Delta\omega$

$$x(t) = \frac{\frac{F_0}{k}}{1-\left(\frac{\omega_0}{\omega_n}\right)^2}\sin\left(\frac{\omega_n - \omega_0}{2}t\right)\sin\left(\frac{\omega_n + \omega_0}{2}t\right)$$

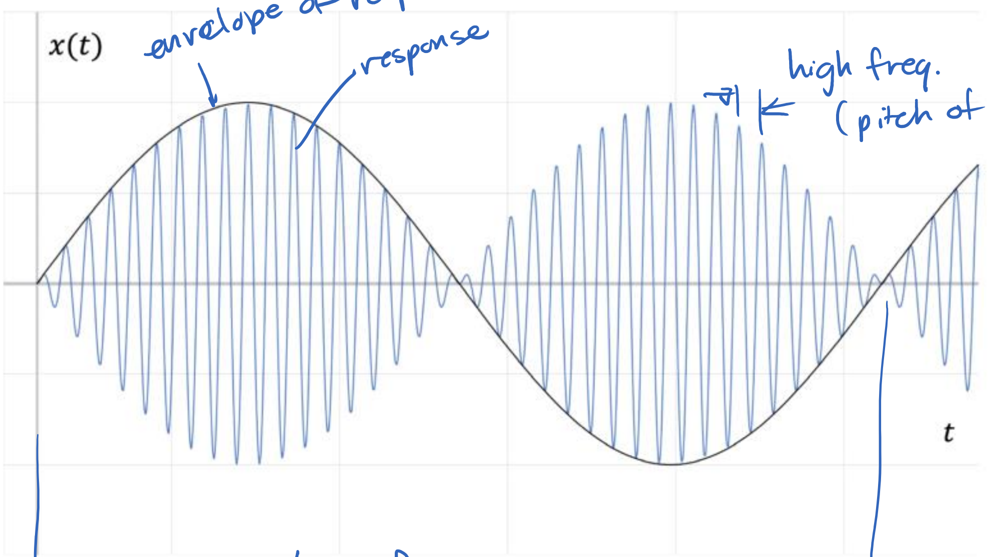
diff small

When ω_0 and ω_n are close to each other, the first sine term, $\sin\left(\frac{\omega_n - \omega_0}{2}t\right)$, represents a very low frequency oscillation, whereas, by comparison, $\sin\left(\frac{\omega_n + \omega_0}{2}t\right)$ is quite high frequency, near the frequency of both ω_0 and ω_n. The result is a low frequency wave enveloping a high frequency vibration. This is known as **beating**.

envelope of response

response

high freq. (pitch of note)

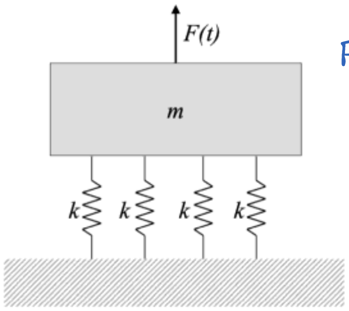


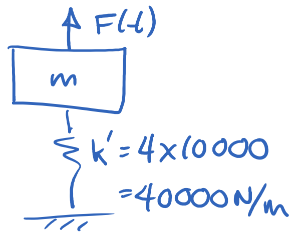
low freq. (volume swells)

13.4.1 Example 1

A ~~500~~ 50 kg block is supported on four springs (k = ~~5000~~ 10000 N/m). Find the magnification factor and maximum deflection for: (a) $F(t) = 750\cos(2t)$ (b) $F(t) = 750\cos(27t)$

F_o ω_o





$$\omega_n = \sqrt{\frac{k'}{m}} = \sqrt{\frac{40000\text{ N/m}}{50\text{ kg}}} = 28.3\text{ rad/s}$$

a) $$MF = \frac{1}{1-\left(\frac{\omega_o}{\omega_n}\right)^2} = \frac{1}{1-\left(\frac{2}{28.3}\right)^2} = 1.076$$

max deflect:

$$D = \frac{F_o}{k'} \cdot MF = \frac{750}{40000}(1.076) = 20.2\text{ mm}$$

b) $$MF = \frac{1}{1-\left(\frac{27}{28.3}\right)^2} = 11.14$$

max deflect:

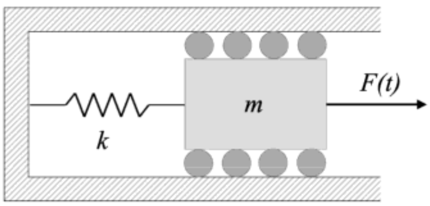
$$D = \frac{F_o}{k'} \cdot MF = \frac{750}{40000}(11.14) = 209\text{ mm}$$

much larger (close resonance)

13.4.2 Example 2

A mass-spring system ($m = 20$ kg, $k = 100$ N/m) is excited by a force, $F(t) = 8\cos 2t$, with t in seconds. Find the maximum speed of the block once friction causes the free vibrations to damp out. ← steady-state response (only $x_p(t)$)

F_o ω_o



$x(t) = D\cos\omega_o t$ ← same form as $F(t)$

$$D = \frac{\frac{F_o}{k}}{1-\left(\frac{\omega_o}{\omega_n}\right)^2}$$

$F_o = 8$ N

$\omega_n = \sqrt{\frac{k}{m}} = 2.24$ rad/s

$\omega_o = 2$ rad/s

Find D : $$D = \frac{8/100}{1-\left(\frac{2}{2.24}\right)^2} = 0.39\text{ m}$$

$\dot{x}(t) = -D\,\omega_o \sin(\omega_o t)$, max when $\sin(\omega_o t) = 1$

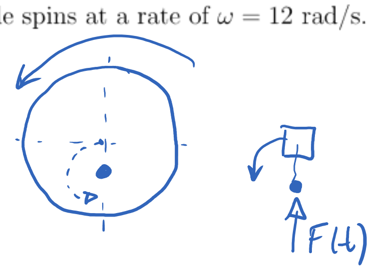
$$|V_{max}| = D\,\omega_o = 0.39\text{ m}(2\text{ rad/s}) = 0.79\text{ m/s}$$

www.menti.com 43 31 69

13.4.3 Example 3

A fan (mass 15 kg) is attached to the end of a horizontal beam (negligible mass). The fan blade was mounted eccentrically, and the eccentricity is equivalent to a mass of 3 kg located 100 mm from the axis of rotation. The static weight of the fan produces a deflection of of 25 mm. Find the steady-state amplitude of vibration when the fan blade spins at a rate of $\omega = 12$ rad/s.

$D = 11.6\text{ mm}$



static deflection:

equilibrium

$mg = 15 \cdot g$

25 mm

x mg F_s

static

$\Sigma F_x: \; F_s + mg = 0$

$F_s = -kx$

$\Rightarrow mg = kx$

$\Rightarrow k = \dfrac{mg}{x} = \dfrac{15(9.81)}{0.025\text{ m}}$

$= 5886\text{ N/m}$

$\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{5886\text{ N/m}}{15\text{ kg}}}$

$= 19.81\text{ rad/s}$

unbalanced rotation:

$\omega = 12\text{ rad/s}$ (const.)

3 kg

100 mm

F_o

F_o

$\Sigma F_y: \; F_o = m_{eq} a$

$= m_{eq} \omega^2 r$

$= (3\text{ kg})(12\text{ rad/s})(0.1\text{ m})$

$= 43.2\text{ N}$

$D = \dfrac{F_o/k}{1 - \left(\frac{\omega_o}{\omega_n}\right)^2} = \dfrac{43.2\text{N}/5886\text{ N/m}}{1 - \left(\frac{12\text{ rad/s}}{19.81\text{ rad/s}}\right)} = \boxed{11.6\text{ mm}}$