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10 L15/L16/L17: Work and Energy

Readings

10.1 Objective

To use the concepts of Work and Energy in an Equilibrium Equation to solve MOTION (dynamics) problems.

10.2 Key Concepts

STATE EQUATIONS Newton's Second Law equations are INSTANTANEOUS equations "At this instant" find α, a - Must INTEGRATE to find position, velocity at a LATER time (state).

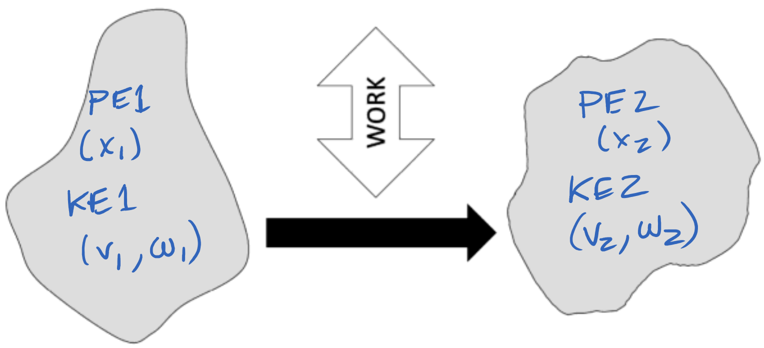
$$\int F\,dx = \int ma\,dx \Rightarrow F(x_2 - x_1) = \int_{x_1}^{x_2} m \underbrace{\frac{dv}{dt}}_{a} dx = \int_{x_1}^{x_2} m\,dv \cdot \underbrace{\frac{dx}{dt}}_{v}$$

$\int (F=ma)dx$

Work $= \vec{F} \cdot \vec{s}$ $+ \Delta PE$

$$= \int_{x_1}^{x_2} m v\,dv = \underbrace{\frac{1}{2} m (v_2^2 - v_1^2)}_{\Delta KE}$$

Work and Energy equations are STATE EQUATIONS



STATE 1 at t = t_1 STATE 2 at t = t_2

Start with ENERGY of system at STATE 1 (time t_1). Add or subtract WORK. Yields ENERGY of system at STATE 2 (time t_2).

$$KE \rightarrow \underbrace{T_1 + \overset{PE}{V_1}}_{\text{State 1}} + \overset{\text{work}}{\sum U_{1\rightarrow 2}} = \underbrace{T_2 + V_2}_{\text{State 2}}$$

10.3 Work

Calorie

Energy is a measure of the **stored ability to do work**. Work and Energy are both measured in the same units – the SI units for work and energy are Joules. 1 *Joule* = 1 *N-m*. Other units include *ft-lb*, *kcal*, *BTU*, *KWh*, and *ergs*.

as opposed to lb-ft (moment)

Right away we see that work and energy relate to force and distance. And in fact, work done, U, by a force, F, can be measured as force applied over distance, i.e.:

scalar

vectors (dot product)

$$dU = \boldsymbol{F} \cdot \boldsymbol{dr}$$

$$U = \int_S F \cos\theta ds$$

Here, force and distance are both vector quantities and θ is the angle between these vectors. For a constant force applied at a constant angle along a distance, $s_2 - s_1$, we have.

$$U = F\cos\theta(s_2 - s_1)$$

missing line

Work done by a torque: $dU = \boldsymbol{M} \cdot \boldsymbol{d\theta}$

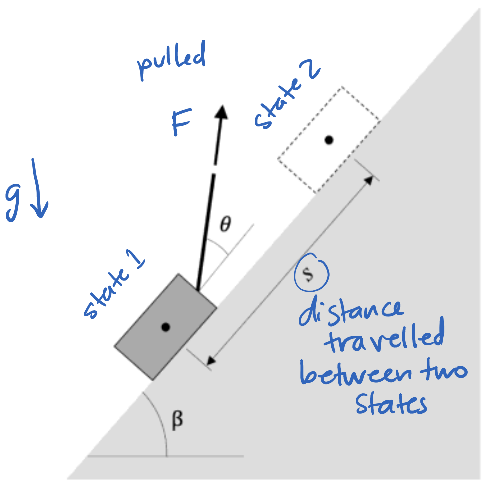
For planar motion, since angle and torque are both about the same axis (the axis perpendicular to the plane), for constant torque we can write

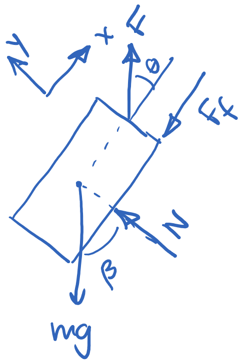
$$U = M(\theta_2 - \theta_1)$$

(NOTE: θ is in radians!)

Work is a scalar value. If it is positive, this means that work is being done by the force being considered. If it is negative then work is being done against the force being considered. Friction, for example, never does positive work since it always opposes motion (does negative work). Thus, friction is always associated with a loss of energy (loss of the stored ability to do work).

10.3.1 Examples - Work Done by Various Forces





ΣF_x: $F\cos\theta - F_f - mg\sin\beta = ma_{Gx}$ ← motion along x

ΣF_y: $-mg\cos\beta + F\sin\theta + N = 0$ $\quad a_{Gy} = 0$

no motion along y

NO WORK DONE ALONG y

$(U = \vec{F}\cdot\vec{s})$

$s_y = 0$

gravity: $U_G = -mg\sin\beta \cdot s$ $\quad (-mgh)$

tension/pull force : $U_F = F\cos\theta \cdot s$ ← positive

friction : $U_f = -f\cdot s$ alway negative.

normal : $U_N = 0$ (no motion in dir. of normal force)

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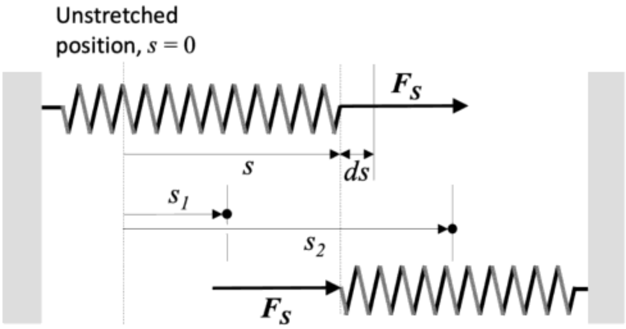
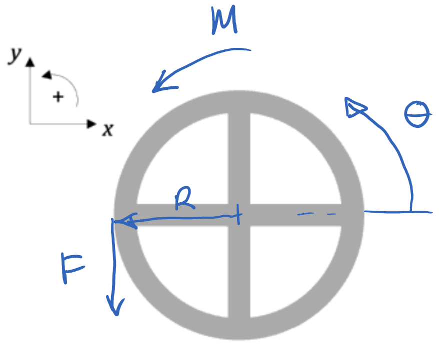


$F_s = ks$ $\left(U = \int \vec{F} \cdot d\vec{s}\right)$

U on spring $= \int_{s_1}^{s_2} \underbrace{ks}_{F}\, ds = \frac{1}{2}k\left(s_2^2 - s_1^2\right)$

$=$ energy stored by the spring

Recall, (Hooke's Law) k is the spring constant (in SI units k is in N/m).



$U = M\theta$

if $M = FR$

$U = (FR)\theta$

$= F(R\theta)$

$= Fs$

10.4 Power

Power is the rate at which work is done. The higher the power the faster work is done.

$$P = \frac{dU}{dt}$$

Since:

$$dU = \boldsymbol{F} \cdot \boldsymbol{ds}$$

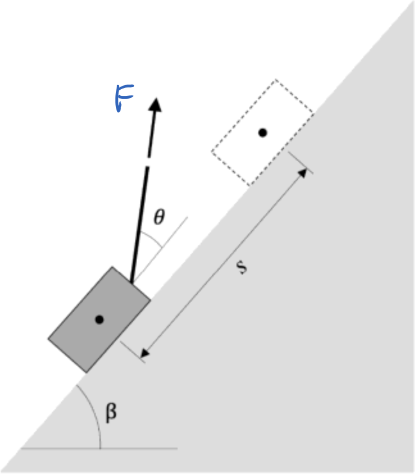
$$P = \frac{dU}{dt} = \boldsymbol{F} \cdot \frac{\boldsymbol{ds}}{dt} = \boldsymbol{F} \cdot \underline{\underline{\boldsymbol{v}}}$$

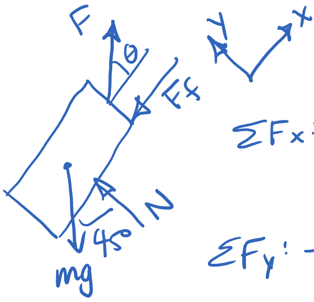
Power is also a SCALAR Value. SI units for power are Watts. $1\,W = 1\,J/s = 1\,N\text{-}m/s$.

10.4.1 Example - Power

Find the power required to move the block of mass $2\,kg$ up the 45° ramp with a constant speed $2\,m/s$, assuming the coefficient of friction between the block and the ramp is $\mu_k = 0.3$, and $\theta = 15°$.

rate at which it is moving





① $\Sigma F_x: F\cos\theta - F_f - mg\sin\beta = 0$ constant

② $\Sigma F_y: -mg\cos\beta + N + F\sin\theta = 0$ no motion in y

$F_f = \mu_k N$ ③

$Power = \vec{F}\cdot\vec{v} = Fv\cos\theta$

①+③: $F\cos\theta = \mu N + mg\frac{1}{\sqrt{2}}$

②: $\mu\left(F\sin\theta = -N + mg\frac{1}{\sqrt{2}}\right)$

+

$F(\cos\theta + \mu\sin\theta) = \frac{mg}{\sqrt{2}}(1+\mu)$

$\Rightarrow F = \frac{mg}{\sqrt{2}}\frac{(1+\mu)}{(\cos\theta + \mu\sin\theta)}$

$$\boxed{P = \frac{mg}{\sqrt{2}}\frac{(1+\mu)\cos\theta}{(\cos\theta+\mu\sin\theta)}\cdot(2\,m/s)}$$

Where did power go? $P_G + P_f = -P_F$

Gravity: $P_G = \frac{1}{\sqrt{2}}(-mg)(2\,m/s)$ $(P = \vec{F}\cdot\vec{v})$

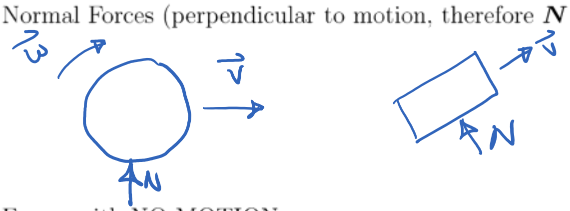
Friction: need to find F_f, ∴ we need to find N

$$N = \frac{mg}{\sqrt{2}}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \mu\sin\theta}\right) \quad F_f = \mu N$$

$$P_f = -\mu N(2\,m/s) = -\frac{\mu mg}{\sqrt{2}}\left(\frac{\cos\theta - \sin\theta}{\cos\theta+\mu\sin\theta}\right)\cdot 2\,m/s$$

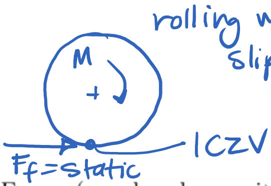
10.5 Forces That Don't Do Work

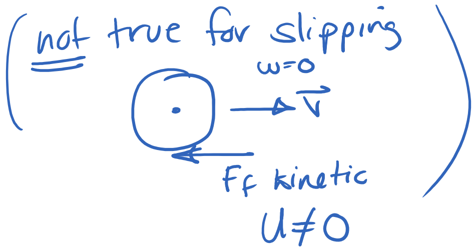
Normal Forces (perpendicular to motion, therefore $\boldsymbol{N} \cdot d\boldsymbol{s} = 0$)



Forces with NO MOTION

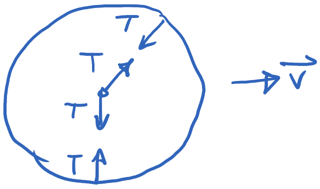
rolling w/o slipping





Internal Forces (equal and opposite)

Bicycle Spokes



10.6 Mechanical Energy

x, θ, h

Energy is the stored ability to do work. Mechanical Energy refers to energy stored by a mechanism.

Mechanical Energy can be stored as Potential Energy (P.E.), denoted with the symbol, V, or Kinetic Energy (K.E.), denoted with the symbol T.

v, ω

10.7 Kinetic Energy

Kinetic Energy is energy stored in motion. Any body that is translating and or rotating has kinetic energy. In the absence of energy input to the system, this energy can be:

- Maintained by the body maintaining it's motion (if there are no losses of energy due to friction)
- Dissipated – due to work done against friction
- Used to do (useful) work
- Transferred and stored as potential energy (e.g. stored in a mechanical or electrical storage component – spring storage, gravitational storage, battery storage)

For a body that is only translating, Kinetic Energy is computed as:

$$T = \frac{1}{2} m v_G^2$$

where v_G is the velocity of the centre of mass of the body. For a body that is only rotating, Kinetic Energy is computed as:

$$T = \frac{1}{2} I_G \omega^2$$

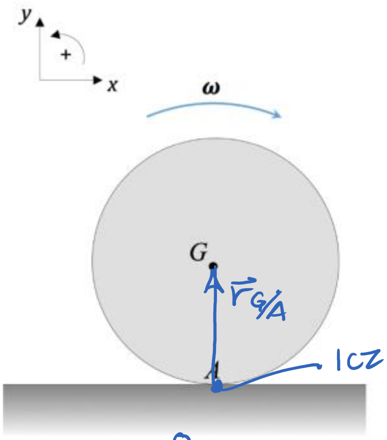
where I_G is the mass moment of inertia of the body around its **centre of mass**. THE REFERENCE TO CENTRE OF MASS IS IMPORTANT (inertia is different around different axes!).

For a body that is both translating and rotating, the Kinetic Energy is the sum of both rotational and translational energies:

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

10.7.1 Example 1 - Kinetic Energy

Find the kinetic energy of a wheel rolling without slipping:



$\vec{r}_{G/A}$

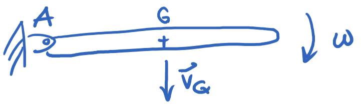
ICZV $\vec{v}_A = 0$

$$\vec{v}_G = \cancel{\vec{v}_A}^{\,0} + \vec{\omega} \times \vec{r}_{G/A}$$
$$= -\omega \hat{k} \times r \hat{j}$$
$$= \omega r \hat{i}$$

$$I_G = \frac{1}{2} m r^2$$

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$
$$= \frac{1}{2} m (\omega r)^2 + \frac{1}{2}\left(\frac{1}{2} m r^2\right)\omega^2$$
$$= \frac{3}{4} m r^2 \omega^2$$
$$= \frac{1}{2}\left(\frac{3}{2} m r^2\right)\omega^2$$
$$= \frac{1}{2}\left(\frac{1}{2} m r^2 + m r^2\right)\omega^2$$
$$= \frac{1}{2}\underbrace{\left(I_G + m r^2\right)}_{I_A}\omega^2$$

special case:
rotating about pin
or ICZV



$T = \frac{1}{2} I_A \omega^2$

$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

Note: for a body rotating around a point that is not moving (around a point P – a PIN or the ICZV):

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 = \frac{1}{2}(m(\omega r_{G/P})^2 + I_G \omega^2) = \frac{1}{2}(m r_{G/P}^2 + I_G)\omega^2$$

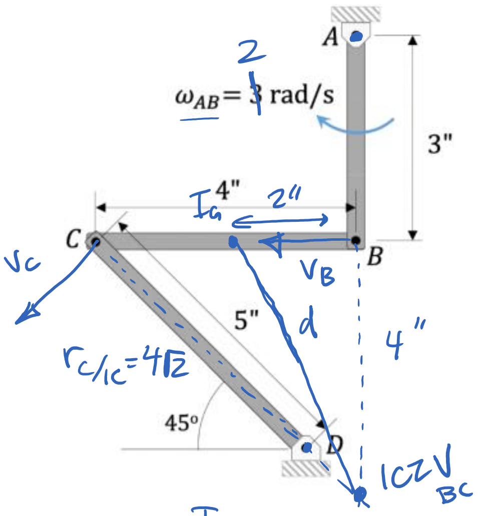
$$T = \frac{1}{2} I_P \omega^2$$

if pinned or ICZV ONLY   otherwise I_G

USE WITH CAUTION – ONLY VALID AROUND A PIN OR ICZV!

10.7.2 Example 2 - Kinetic Energy

Find the kinetic energy of the mechanism shown, use $\rho_{bar} = 0.5\, lb/in$.



kinematics

$\omega_{AB} = 2\, rad/s$

$v_B = 2\, rad/s\,(3\, in) = 6\, in/s$

$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{6\, in/s}{4''} = 1.5\, rad/s$

$v_C = \omega_{BC} \cdot r_{C/IC} = 1.5\, rad/s \cdot 4\sqrt{2}\, in = 6\sqrt{2}\, in/s$

$\omega_{CD} = \frac{v_C}{r_{C/D}} = \frac{6\sqrt{2}\, in/s}{5''} = \frac{6\sqrt{2}}{5}\, rad/s$

kinetic energy

$T_{TOT} = T_{AB} + T_{BC} + T_{CD}$

$T_{AB} = \frac{1}{2}\big(\overbrace{\tfrac{1}{3} m_{AB} l_{AB}^2}^{I_A}\big)\omega_{AB}^2 = \frac{1}{6}\rho\, l_{AB}^3\, \omega_{AB}^2$ $(m_{AB} = \rho\, l_{AB})$

$T_{BC} = \frac{1}{2}\big[\underbrace{\tfrac{1}{12} m_{BC} l_{BC}^2 + m_{BC}((2in)^2 + (4in)^2)}_{I_{IC}}\big]\omega_{BC}^2$

pound-force

$\frac{lb}{g} \Rightarrow slug$

$T_{CD} = \frac{1}{2}\big(\underbrace{\tfrac{1}{3} m_{CD} l_{CD}^2}_{I_D}\big)\omega_{CD}^2$

$T_{TOT} = 0.045\ ft \cdot lb$

10.8 Potential Energy

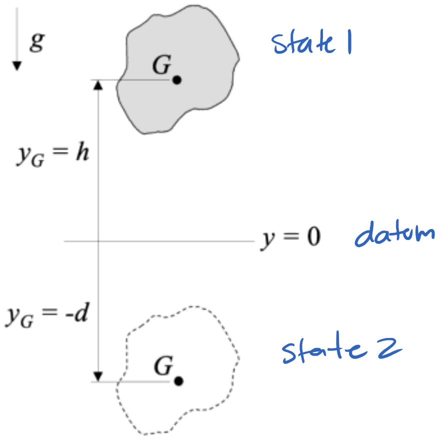
The amount of potential energy stored is dependant on location **with respect to a datum**. Change in potential energy depends only on the change of position, and is **independent of path**.

Potential energy can be stored as **gravitational potential energy**:

$$V_g = mgy_G$$

measured against a datum

where m is the mass of the body, g is the acceleration due to gravity and y_G is the distance up measured from a datum to the **centre of mass** of the body.



In the figure, the body of mass, m, has potential energy:

$$V_{g_1} = mgh$$

positive

at the top point, and

$$V_{g_2} = -mgd$$

neg.

at the bottom point (i.e. it has negative potential energy with respect to some arbitrary datum. The change in potential energy as the body falls between top and bottom is:

$$\underset{\text{now}}{V_{g_2}} - \underset{\text{then}}{V_{g_1}} = -mgd - mgh = -mg(d+h)$$

lost potential energy

That is the body has lost potential energy. If the body is moved up from the bottom to the top, they it is said to have gained potential energy:

2 → 1

$$V_{g_1} - V_{g_2} = mgh - (-mgd) = mg(d+h)$$

☺ gained potential energy

Note that, an increase in stored gravitational potential energy results from work being done against the force of gravity (gravity did negative work – i.e. something else did positive work to lift the

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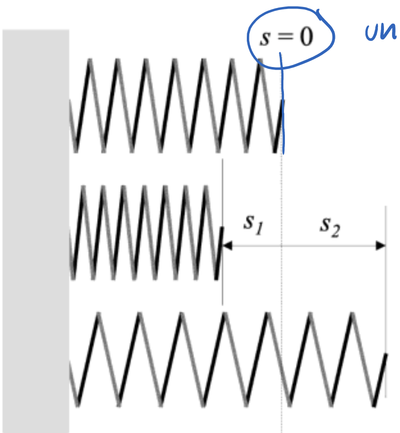
mass up).

weight diff. in position

$$\begin{aligned} U_{g(1\to2)} &= \boldsymbol{W} \cdot \Delta \boldsymbol{r_G} \\ &= -(mg)\hat{\boldsymbol{j}} \cdot (h - (-d))\hat{\boldsymbol{j}} \\ &= -mg(h+d) \end{aligned}$$

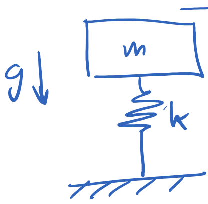
work done by gravity is neg.

Potential Energy can also be stored in a spring. Again we set a datum, this time, the unstretched length of the spring, $s = 0$.



unstretched length MUST be datum (not arbitrary)

datum = unstretched length
no movement



already have energy in the system

ΣF_y: $mg = kx_{eq}$

Can't cancel out mg w. eq. stretch

As the spring is stretched or compressed, work is done against the spring, and energy is stored in the spring.

The stored potential is:

distance from unstretched

$$V_s = \frac{1}{2}ks^2$$

units of k are $\frac{N}{m}$ linear spring

(recall k is the spring constant and s is measured from unstretched length.) Note that, the potential energy in a spring is always greater than zero. However, a spring can gain or lose potential energy.

positive

When a spring does work, it loses potential:

positive work, F_s is in pos. direction of motion

$$U_{1\to2} = \int \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{s_1}^{s_2} -kx\,dx = -\frac{1}{2}k(s_1^2 - s_2^2)$$

A torsional spring works along the same principle except that instead of measuring linear displacement, s, from the unstretched length, we measure angular displacement from the untwisted state. The displacement is measured in radians, and the spring constant is measured in units of force x distance (SI units N-m).

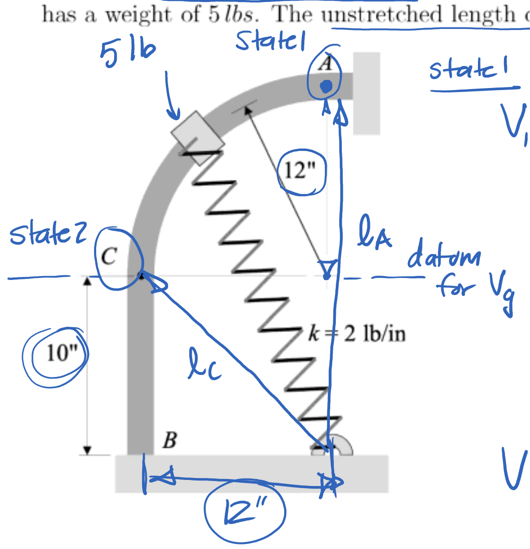
radians units of k_θ are N-m/rad

$$V_s = \frac{1}{2}k_\theta\theta^2$$

10.8.1 Example - Potential Energy

Find the change in potential energy of the mechanical system from point A to point C. The collar has a weight of $5\,lbs$. The unstretched length of the spring is $12\,inches$.

$l_0 = 12''$



State 1

$$V_1 = V_{As} + V_{Ag}$$

$$V_{Ag} = mg\,y_G = 5\,lb \cdot 1\,ft = 5\,ft\cdot lb$$

$l_A = 12'' + 10'' = 22''$

$$V_{As} = \frac{1}{2}k\,(l_A - l_0)^2$$

$$= \frac{1}{2}(2\,lb/in) \times \frac{12\,in}{ft}\left(\frac{22-12\,in}{12\,in/ft}\right)^2$$

$$= \frac{100}{12}\,ft\cdot lb$$

$$V_1 = 5\,ft\cdot lb + \frac{100}{12}\,ft\cdot lb = 13\tfrac{1}{3}\,ft\cdot lb$$

State 2

$$V_2 = V_{Cs} + \cancel{V_{Cg}} \quad 0 \text{ because datum @C } y_{GC} = 0$$

$$V_{Cs} = \frac{1}{2}k\,(l_C - l_0)^2 \qquad l_C = \sqrt{10''^2 + 12''^2}$$

$$= \frac{1}{2}\left(2\,\frac{lb}{in}\cdot\frac{12\,in}{ft}\right)\left(\frac{\sqrt{10^2+12^2} - 12\,in}{12\,in/ft}\right)^2 = 1.1\,ft\cdot lb$$

$$\Delta V = \underset{\text{now}}{V_C} - \underset{\text{then}}{V_A} = 1.1 - 13\tfrac{1}{3}\,ft\cdot lb = \underline{\underline{-12.2\,ft\cdot lb}}$$

N.B Cannot say $\Delta V_s = \frac{1}{2}k\,(l_C^2 - l_A^2)$ X

"Nota Bene"

10.9 Conservative Forces

Potential Energy is related to Conservative Forces. These are forces for which the work done is **independent of path** – depends only on the starting and ending points.

Compare with friction for which the (negative) work done is **path dependant**.

Potential Energy and Conservative Forces are related in the through the gradient operator, "del":

"del"

you will see next term

$$\boldsymbol{F} = -\nabla V = -\left(\frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) V$$

takes 1st derivative wrt each coordinate

Thus, for gravitational potential (assuming y is "up") we see

only y component

$$\begin{aligned}\boldsymbol{F}_g &= -\nabla V_g = -\left(\frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) mgy_G \\ &= \frac{\partial}{\partial y}(mgy_G)\hat{\boldsymbol{j}} \\ &= -mg\hat{\boldsymbol{j}}\end{aligned}$$

For spring potential assuming the spring stretches along x, where $x = 0$ at the unstretched length:

$$\begin{aligned}\boldsymbol{F} &= -\nabla V_s = -\left(\frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) \frac{1}{2}kx^2 \\ &= -\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right)\hat{\boldsymbol{i}} \\ &= -kx\hat{\boldsymbol{i}}\end{aligned}$$

10.10 Principle of Work and Energy

We can bring all of these energy terms together into the following relationship for a system:

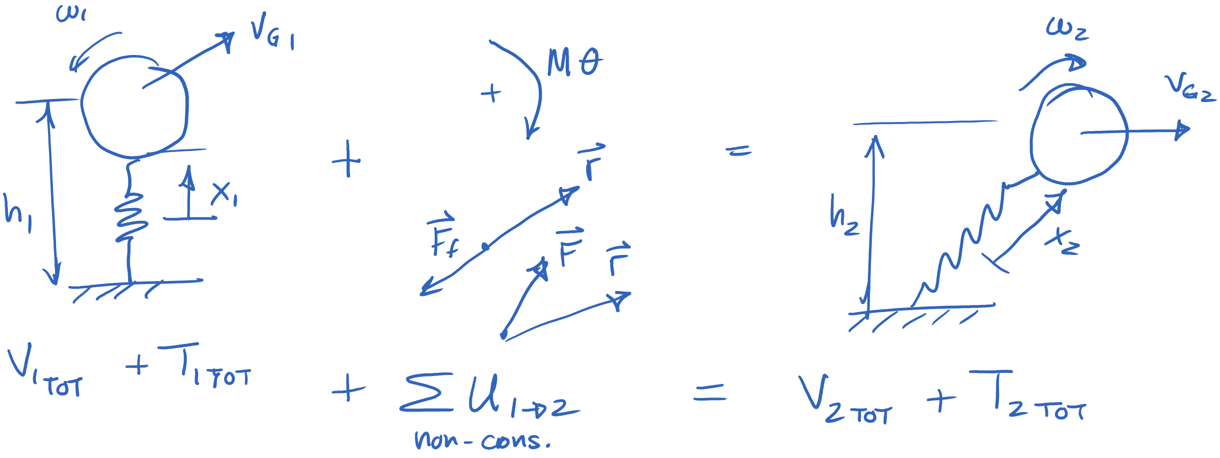
$$T_1 + V_1 + \sum_{non-conservative\,work} U_{1\to 2} = T_2 + V_2$$

if only have conservative forces: $T_1 + V_1 = T_2 + V_2$

N.B.: $\sum U_{1\to 2} = \sum_{non-cons.} U_{1\to 2} - \Delta PE$

In other words, in state 1, the system starts out with a quantity of potential and kinetic energy (that can be added together). As the system moves from state 1 to state 2, if no work is added or subtracted from the system (input from a motor, or losses from friction) then the total quantity of potential and kinetic energy at state 2 will be the same (although their relative amounts may change).

However, in most mechanical systems, there will be "non-conservative" losses due to friction. Thus, the total amount of kinetic and potential energy at state 2 will be reduced.



10.11 Method

1. Identify all bodies in the system
2. **For each body**,
   - Draw the free body diagram to identify:
     - Conservative forces (related to changes in **potential energy – path independent changes in energy**)
     - Non-conservative forces (related to the addition or subtraction of energy in the form of **work**)
   - Consider Potential Energy

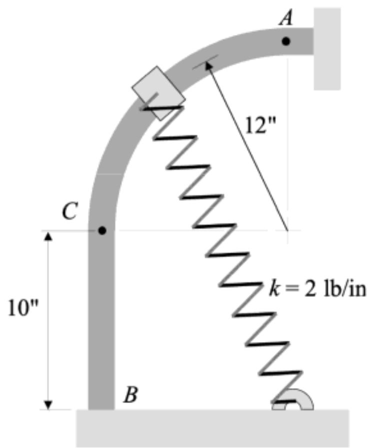
     state1 state 2
     - Draw two diagrams showing the body located at its initial and final state.
     - Identify a datum and find the potential energy at state 1 and state 2 for each conservative force. Gravitational potential energy ($V_G = mgh$) can be positive or negative. Spring potential energy ($V_s = \frac{1}{2}ks^2$) is always positive.
     - Potential Energy at each state is the SUM of the potential energy related to all conservative forces in the system.
   - Consider Work by non-conservative forces over the path. Between state 1 and state 2, integrate all sources of work related to non-conservative forces:
     - $dU = \boldsymbol{F} \cdot d\boldsymbol{r}$
     - $dU = \boldsymbol{M} \cdot d\boldsymbol{\theta}$
     - Work done from state 1 to state 2 can be positive or negative. Friction always does negative work.
   - Consider Kinetic Energy at each state: $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$
   - Principle of Work and Energy: $\boxed{T_1 + V_1 + \sum_{non-conservative\ work} U_{1\to2} = T_2 + V_2}$

10.11.1 Example 1 – Principle of Work and Energy

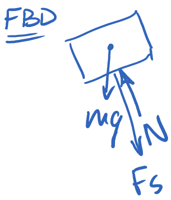
state 1 info — non. cons. force/work — no friction

The 5 *lb* collar is $\boxed{\text{released from rest at } A}$ and travels along a smooth guide. Find its speed when it reaches $\boxed{\text{point } C}$. The unstretched length of the spring is 12 *inches*. How would the solution change if there was friction between the collar and the guide?

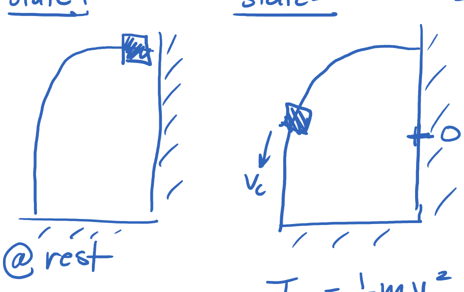
State 2



FBD



state 1 — state 2



$T = \frac{1}{2} I_G \omega^2 = 0$ pt. mass

@ rest

$T_1 = 0$ $\quad$ $T_2 = \frac{1}{2} m v_c^2$

$V_1 = V_A$ $\quad$ $V_2 = V_c$

from previous: $V_1 = V_A = 13\frac{1}{3}$ ft · lb

$V_2 = V_c = 1.1$ ft · lb

$$V_1 + \cancel{T_1}^{\,0} + \underbrace{\cancel{\Sigma U_{1\to 2}}^{\,0}}_{\text{non-cons.}} = V_2 + T_2$$

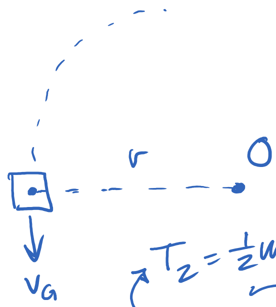
$$T_2 = V_1 - V_2 = \frac{1}{2} m v_G^2$$

speed $v_c^2 = \dfrac{(12.2) \text{ ft} \cdot \text{lb}}{\frac{1}{2}\Big[\underbrace{5\text{lb}/32.2\text{ft}/\text{s}^2}_{m}\Big]} \Rightarrow \underline{\underline{v_c = 12.5 \text{ ft}/\text{s}}}$

What happens if friction is added?

$T_2 = V_1 - V_2 + U_f$ ← always neg

$v_c \downarrow$ with friction



$T_2 = \frac{1}{2} m v_G^2 + \cancel{\frac{1}{2} I_G \omega_2^2}^{\,0}$

$T_2 = \frac{1}{2} I_O \omega_2^2$

$I_G \hat{=} 0$ because $r \simeq 0$

$I_O = \cancel{I_G}^{\,0} + mr^2$

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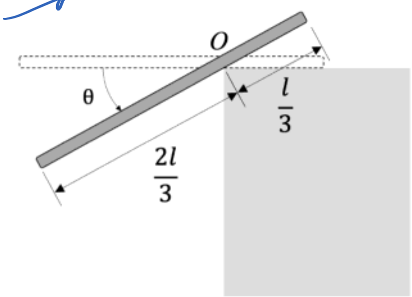
10.11.2 Example 2 – Principle of Work and Energy

position

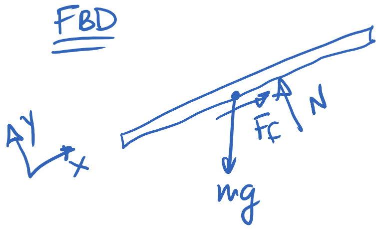
state 1

The uniform bar has mass m and length l. If it is $\boxed{\text{released from rest when } \theta = 0^\circ}$, determine the $\boxed{\text{angle } \theta \text{ at which it first begins to slip.}}$ The coefficient of static friction is $\mu_s = 0.3$.

state 2



FBD

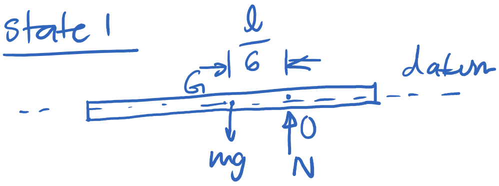


$$\sum U_{1\to 2}^{\text{non cons}} = 0$$

$$\vec{F}_f \cdot \vec{r} = 0$$

no motion

state 1



energy

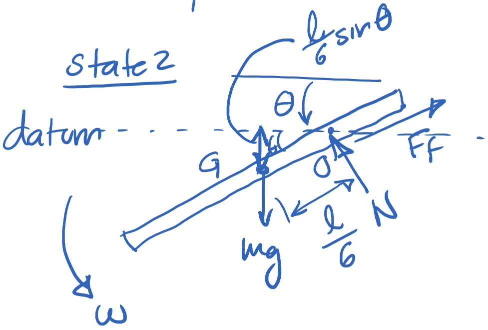
$T_1 = 0$ from rest

$V_1 = 0$ datum through G

$v_{G_1} = 0$

$\omega_1 = 0$

state 2



just before slipping
O acts like a pin (ICZV)

O is ICZV

energy

$$T_2 = \frac{1}{2} I_O \omega^2$$

$$\left(= \frac{1}{2} I_G \omega^2 + \frac{1}{2} m v_G^2\right)$$

$$V_2 = -mg\frac{l}{6}\sin\theta$$

$$I_O = \frac{1}{12} m l^2 + m\left(\frac{l}{6}\right)^2 = \frac{1}{9} m l^2$$

cons. of energy: $T_1 + V_1 = T_2 + V_2$ (recall $\sum U_{1\to 2}^{\text{non cons}} = 0$)

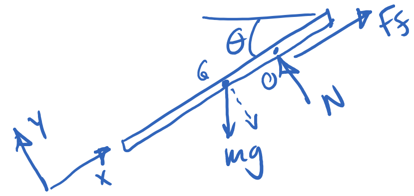
$$0 = \frac{1}{2}\left(\frac{1}{9} m l^2\right)\omega^2 - mg\frac{l}{6}\sin\theta$$

$$\Rightarrow \omega^2 = \frac{3g}{l}\sin\theta \quad (4)$$

2 unknowns, 1 eqn

(Example continued)

NSL @ θ_{slip}



ΣF_x: $F_f - mg\sin\theta = m a_{Gx}$ ①

ΣF_y: $N - mg\cos\theta = m a_{Gy}$ ②

ΣM_0: $mg\cos\theta \frac{l}{6} = I_0 \alpha$ ③ (acts like a pin)

unknowns: $F_f, N, a_{Gx}, a_{Gy}, \alpha, \theta, \omega$ 7 unknowns, 4 eqn.

$F_f = \mu_s N$ ⑤ (5 eqn)

kinematic constraints

O acts like a pin, $\vec{a}_O = 0$

$$\vec{a}_G = \cancel{\vec{a}_O}^{\,0} + \vec{\alpha} \times \vec{r}_{G/O} - \omega^2 \vec{r}_{G/O}$$

$$\vec{r}_{G/O} = \frac{l}{6}(-\hat{\imath})$$

$$\vec{a}_G = \alpha\hat{k} \times \frac{l}{6}(-\hat{\imath}) - \omega^2\left(\frac{l}{6}(-\hat{\imath})\right)$$

$$= \underbrace{-\frac{\alpha l}{6}}_{a_{GY}}\hat{\jmath} + \underbrace{\frac{\omega^2 l}{6}}_{a_{GX}}\hat{\imath}$$

SOLVE

($\Rightarrow \theta = 8.53°$)

③ $\Rightarrow \alpha = \dfrac{\cancel{m}g\cos\theta \frac{\cancel{l}}{6}}{\frac{1}{9}\cancel{m}l^{\cancel{2}}} \Rightarrow \alpha = \dfrac{3}{2}\dfrac{g}{l}\cos\theta$

$\Rightarrow a_{Gy} = -\dfrac{\alpha l}{6} = -\dfrac{\cancel{l}}{6}\left(\dfrac{3}{2}\dfrac{g}{\cancel{l}}\cos\theta\right) = \dfrac{-1}{4}g\cos\theta \overset{②}{\Rightarrow} N = \dfrac{3}{4}mg\cos\theta$

⑤ $\Rightarrow F_f = \mu_s \frac{3}{4}mg\cos\theta$ (just before slipping ∴ no inequality) set equal

① $\Rightarrow F_f = m\left(g\sin\theta + \underbrace{\dfrac{3g}{\cancel{l}}\sin\theta}_{\omega^2} \cdot \dfrac{\cancel{l}}{\cancel{6}\,2}\right) = \dfrac{3}{2}mg\sin\theta$

$F_f = F_f \Rightarrow \dfrac{\cancel{3}}{2}\cancel{mg}\sin\theta = \mu \dfrac{\cancel{3}}{\cancel{4}\,2}\cancel{mg}\cos\theta \Rightarrow \tan\theta = \dfrac{\mu}{2} = \dfrac{0.3}{2} \Rightarrow \boxed{\theta = 8.53°}$

10.12 Summary

Work SCALARS	$dU = \boldsymbol{F} \cdot d\boldsymbol{r}$ $dU = \boldsymbol{M} \cdot d\boldsymbol{\theta}$	Force along a path, Moment through an angle
Kinetic Energy	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$ $T = \frac{1}{2} I_O \omega^2$ if pinned @ O	Energy stored as motion
Potential Energy		Path independent change of the stored ability for a conservative force to do work
	$V_G = mgh$	Gravitational potential energy can be positive or negative
	$V_s = \frac{1}{2}ks^2$	Spring potential energy is always positive
Principle of Work and Energy	$T_1 + V_1 + \sum_{non-conservative\,work} U_{1\to 2} = T_2 + V_2$	

$$T_1 + V_1 + \sum_{non.cons} U_{1\to 2} = T_2 + V_2$$

If we compare this to Newton's Second Law Equation (for a point mass), $\boldsymbol{F} = m\boldsymbol{a}$, and we consider F to be a force which we integrate between two states, we see:

$$\sum_{all\,work} U_{1\to 2} = \int \boxed{\boldsymbol{F}} \cdot d\boldsymbol{s} = \int \boxed{m\boldsymbol{a}} \cdot d\boldsymbol{s} = \int m\frac{d\boldsymbol{v}}{dt} \cdot d\boldsymbol{s} = \int m\boldsymbol{v} \cdot d\boldsymbol{v} = \frac{1}{2}m(v_2^2 - v_1^2) = T_2 - T_1$$

distance θ, x, y, etc.

In other words, the Principle of Work and Energy is the same as integrating Newton's Second Law over the motion from state 1 to state 2.