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6 L7: Absolute General Plane Motion

Reading

6.1 Objective

To relate rectilinear point motion with rotational motion on a rigid body.

6.2 Method

There are no "new equations". Use known geometric relationships. Try to relate angular rotation of a line fixed to the body, $\theta(t)$, $\dot{\theta}(t)$, $\ddot{\theta}(t)$, to the rectilinear (straight line) location of a point specified by a parameterized path, e.g.: $s = f(\theta)$.

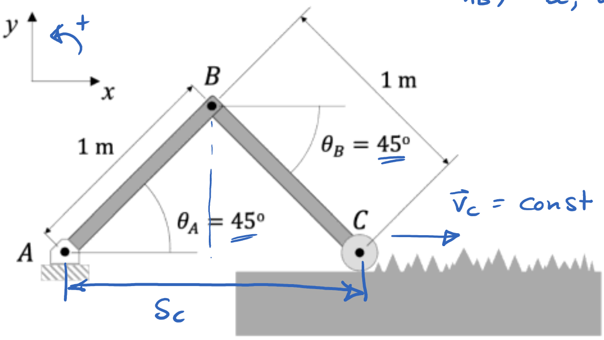
1. Draw it!
2. For a point P (translational point), define a position coordinate, s.
3. Draw a line through the body at reference angle θ.
4. Using geometry/trigonometry, find a parameterized relationship, $\boxed{s = f(\theta)}$.
5. Use time derivatives to relate:

$$v = f'(\theta)\dot{\theta} = f'(\theta)\omega$$
$$a = f''(\theta)\dot{\theta}^2 + f'(\theta)\alpha$$

6.2.1 Example

Consider the robotic grinder. Find the position, velocity and acceleration of the grinding wheel as a function of the angles θ_A and θ_B and their derivatives.

Find $\vec{\omega}_{AB}$, $\vec{\omega}_{BC}$, $\vec{\alpha}_{AB}$, $\vec{\alpha}_{BC}$



$S_C = 2\ (1\ (m))\cos\theta_A$

$v_C = \dot{S}_C = 2\ (-\sin\theta_A)\dot{\theta}_A$ ①

$a_C = \dot{v}_C = -2\cos\theta_A\,\dot{\theta}_A^2 - 2\sin\theta_A\,\ddot{\theta}_A$ ②

Isoceles triangle

$AB = BC$

$\theta_A = -\theta_B$

Set: $\theta_A = 45°$, $v_C = 1\ m/s$, $a_C = 0$
(same as before)

differentiate:

$\dot{\theta}_A = -\dot{\theta}_B$

$\ddot{\theta}_A = -\ddot{\theta}_B$

① $\dot{\theta}_A = \dfrac{v_C}{(-2)\sin\theta_A} = \dfrac{1\ m/s}{(-2)\frac{1}{\sqrt{2}}} = -\dfrac{1}{\sqrt{2}}\ rad/s$

as before

$$\boxed{\vec{\omega}_{AB} = -\frac{1}{\sqrt{2}}\ rad/s\ \hat{k}}$$

② $0 = (\cancel{-2})\cos\theta_A\dot{\theta}_A^2 + (\cancel{-2})\sin\theta_A\,\ddot{\theta}_A$

$= -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)^2 - \frac{1}{\sqrt{2}}\ddot{\theta}_A$

$\Rightarrow \ddot{\theta}_A = -\frac{1}{2}\ rad/s^2$ $\quad \boxed{\vec{\alpha}_{AB} = -\frac{1}{2}\ rad/s^2\ \hat{k}}$

$\dot{\theta}_A = -\dot{\theta}_B \Rightarrow \boxed{\vec{\omega}_{BC} = \frac{1}{\sqrt{2}}\ rad/s\ \hat{k}}$

$\ddot{\theta}_A = -\ddot{\theta}_B \Rightarrow \boxed{\vec{\alpha}_{BC} = \frac{1}{2}\ rad/s^2\ \hat{k}}$

6.3 Summary

Absolute general plane motion analysis can provide a quick, ad-hoc solution for 2-link problems with constrained motion. For more than 2 links, however, the math and geometry can get pretty ugly. Better to use relative frames.

Pros:

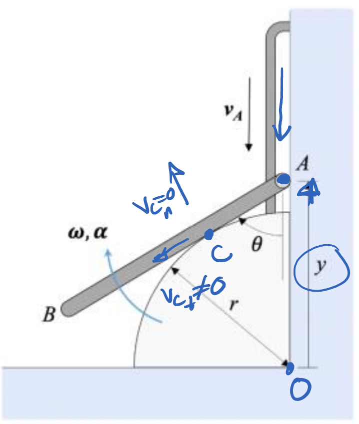
- Can yield a fast solution

Cons:

- Relationship may not be obvious
- If relationship is wrong, you've blown the whole problem (and it may be hard for the marker to give you part marks, unless they can understand what you have done)
- Method cannot easily be checked by vector analysis

6.3.1 Example

Point A at the end of a bar is moving downward in a slot with a constant velocity $\boldsymbol{v_A}$. Find the angular velocity and angular acceleration of the bar as a function of the vertical position of A, y.



$\vec{V}_A$ = constant Find ω & α as function of y

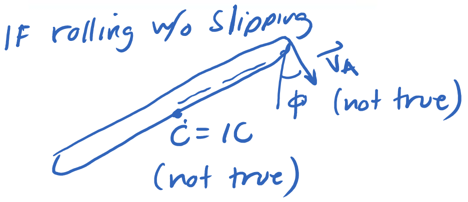
$\vec{a}_A = 0$

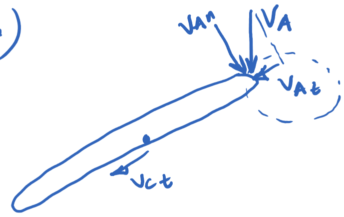
is C rolling w/o slipping? No

Has to slide because constrained by gravity and cam

V_c component along the bar (tangent to cam)

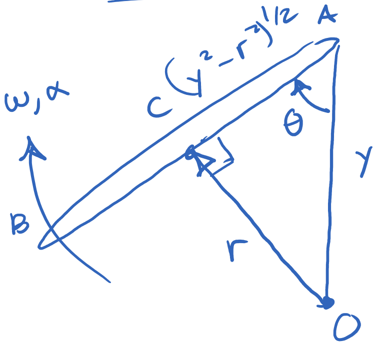
IF rolling w/o slipping





(Example continued)

ABSOLUTE MOTION



θ is in negative direction, but ω & α are also neg.

$r = y\sin\theta$ $\quad (r = y\sin(-\theta) = -y\sin\theta \rightarrow$ get same result$)$

$\frac{d}{dt}: \; 0 = \dot{y}\sin\theta + y\cos\theta\,\dot{\theta}$

$\Rightarrow y\dot{\theta}\cos\theta = -\dot{y}\sin\theta$ $\qquad \dot{y} = -v_A$

$\dot{\theta} = \dfrac{-\dot{y}\sin\theta}{y\cos\theta}$

$\dfrac{\sin\theta}{\cos\theta} = \tan\theta = \dfrac{r}{AC} = \dfrac{r}{(y^2-r^2)^{1/2}}$

$$\boxed{\dot{\theta} = \omega = \frac{v_A}{y}\cdot\frac{r}{(y^2-r^2)^{1/2}}}$$

$[\vec{\omega} = -\omega\hat{k}]$ neg. dir

$\frac{d}{dt}\left(0 = \dot{y}\sin\theta + y\cos\theta\,\dot{\theta}\right)$

v_A const

$\Rightarrow 0 = \cancel{\ddot{y}}^{\,0}\sin\theta + \underbrace{\dot{y}\cos\theta\,\dot{\theta} + \dot{y}\cos\theta\,\dot{\theta}} - y\sin\theta\,\dot{\theta}^2 + y\cos\theta\,\ddot{\theta}$

$0 = 2\dot{y}\dot{\theta}\cos\theta - y\dot{\theta}^2\sin\theta + y\ddot{\theta}\cos\theta$

$\cancel{y}\ddot{\theta}\cancel{\cos\theta} = \cancel{y}\dot{\theta}^2\dfrac{\sin\theta}{\cos\theta} - \dfrac{2\dot{y}}{y}\dot{\theta}\cancel{\cos\theta}$

recall: $\dot{y} = -v_A$, $\quad \dot{\theta} = \dfrac{v_A}{y}\dfrac{r}{(y^2-r^2)^{1/2}}$

$\ddot{\theta} = \dot{\theta}^2\tan\theta - \dfrac{2\dot{y}}{y}\dot{\theta}$

$= \left(\dfrac{v_A}{y}\dfrac{r}{(y^2-r^2)^{1/2}}\right)^2\left(\dfrac{r}{(y^2-r^2)^{1/2}}\right) - \dfrac{2(-v_A)}{y}\left(\dfrac{v_A}{y}\dfrac{r}{(y^2-r^2)^{1/2}}\right)$

$= \dfrac{v_A^2}{y^2}\left(\dfrac{r^3}{(y^2-r^2)^{3/2}} + \dfrac{2r}{(y^2-r^2)^{1/2}}\right) = \dfrac{v_A^2 r}{y^2}\left(\dfrac{r^2+2(y^2-r^2)}{(y^2-r^2)^{3/2}}\right)$

$$\boxed{\ddot{\theta} = \alpha = \frac{v_A^2 r}{y^2}\left(\frac{2y^2-r^2}{(y^2-r^2)^{3/2}}\right)}$$

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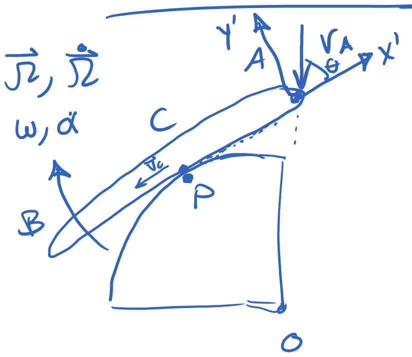
(Example continued)

ROTATING AXES REL. MOTION

C as a sticker on AB, same position as P (on cam)



$$\vec{v}_P = 0 = \vec{v}_A + \vec{\Omega} \times \vec{r}_{P/A} + (\vec{v}_{P/A})_{x'y'z'}$$

$$0 = -v_A \cos\theta \, \hat{i}' - v_A \sin\theta \, \hat{j}' + \left(-\Omega \hat{k}' \times \underbrace{(y^2-r^2)^{1/2}}_{AC}(-\hat{i}')\right) + (v_{P/A})_{x'y'z'} \, \hat{i}'$$

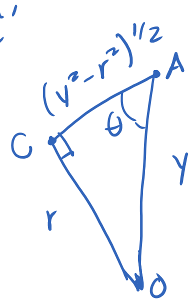
$$= -v_A \cos\theta \, \hat{i}' - v_A \sin\theta \, \hat{j}' + \Omega (y^2-r^2)^{1/2} \hat{j}' + (v_{P/A})_{x'y'z'} \hat{i}'$$

components:

$\hat{i}'$: $0 = -v_A \cos\theta + (v_{P/A})_{x'y'z'}$

$(v_{P/A})_{x'y'z'} = v_A \cos\theta$

$\hat{j}'$: $0 = -v_A \sin\theta + \Omega (y^2-r^2)^{1/2}$



$$\dot{\theta} = \Omega = \frac{v_A \sin\theta}{(y^2-r^2)^{1/2}} = \underline{\underline{\frac{v_A r}{y} \frac{1}{(y^2-r^2)^{1/2}}}} \qquad \sin\theta = \frac{r}{y}$$

same

$$\vec{a}_P = 0 = \cancel{\vec{a}_A}^{\,0} + \dot{\vec{\Omega}} \times \vec{r}_{P/A} - \Omega^2 \vec{r}_{P/A} + 2\vec{\Omega} \times (\vec{v}_{P/A})_{x'y'z'} + (\vec{a}_{P/A})_{x'y'z'}$$

$\dot{\vec{\Omega}} = -\dot{\Omega} \hat{k}'$, $\vec{\Omega} = -\Omega \hat{k}'$ $\qquad \vec{r}_{P/A} = -(y^2-r^2)^{1/2} \hat{i}'$

$(\vec{v}_{P/A})_{x'y'z'} = v_A \cos\theta \, \hat{i}'$

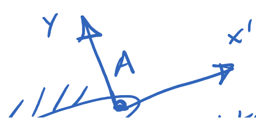
$(\vec{a}_{P/A})_{x'y'z'} = \underbrace{(a_{P/A})_t \, \hat{i}' - \Omega^2 r \, \hat{j}'}$ rolling up bar

- Cross products
- Components

$$\Rightarrow \quad \alpha = \dot{\Omega} = \frac{v_A^2 r}{y^2} \frac{(2y^2 - r^2)}{(y^2-r^2)^{3/2}}$$ same (see complete solution below)

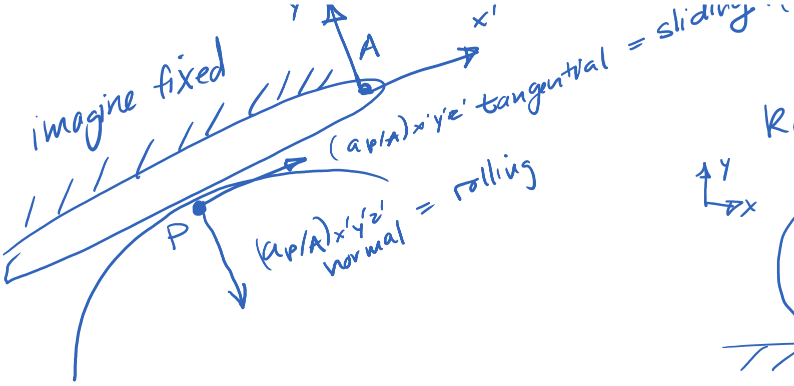
Details on how we get $(\vec{a}_{P/A})_{x'y'z'}$:

Observing from the rotating frame: it appears that we are stationary, and the cam is sliding and rolling toward us

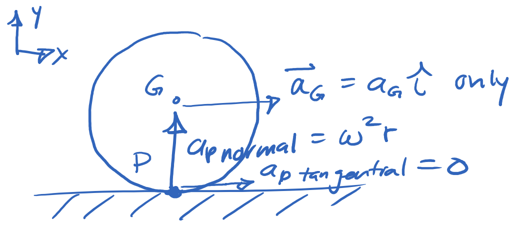


... fixed ... tangential = sliding. (recall that we have v_c along the bar) ... rolling w/o slipping

v_c along the bar)



Recall: rolling w/o slipping



$$\vec{a}_P = \vec{a}_G + \vec{\alpha} \times \vec{r}_{P/G} - \omega^2 \vec{r}_{P/G}$$

$$\vec{r}_{P/G} = -r\hat{j}$$

$$a_P \hat{j} = a_G \hat{i} + \alpha r \hat{i} + \omega^2 r \hat{j}$$

$$\underline{\underline{a_P = \omega^2 r}} \ (\uparrow)$$

Similarly, $|(a_{P/A})_{x'y'z'\,normal}| = \Omega^2 r$ and direction must be $-\hat{j}'$ since the cam cannot move through the bar and "rolling" will cause the cam point P to move away from the "fixed" bar.

Remainder of solution:

$$0 = -\dot{\Omega}\hat{k} \times \left(-(y^2-r^2)^{1/2}\right)\hat{i}' - \Omega^2\left(-(y^2-r^2)^{1/2}\right)\hat{i}' + 2\left(-\Omega\hat{k}' \times v_A\cos\theta\,\hat{i}'\right)$$
$$+ (a_{P/A})_{x'y'z'_t}\hat{i}' - \Omega^2 r\hat{j}'$$

$$= \dot{\Omega}(y^2-r^2)^{1/2}\hat{j}' + \Omega^2(y^2-r^2)^{1/2}\hat{i}' - 2\Omega v_A\cos\theta\,\hat{j} + (a_{P/A})_{x'y'z'_t}\hat{i}' - \Omega^2 r\hat{j}'$$

Components:

$$\hat{j}': \quad 0 = \dot{\Omega}(y^2-r^2)^{1/2} - 2\Omega v_A\cos\theta - \Omega^2 r$$

$$\dot{\Omega} = \frac{2\Omega v_A\cos\theta + \Omega^2 r}{(y^2-r^2)^{1/2}}$$

$$= \frac{2v_A\left(\frac{v_A r}{y^2}\right) + \frac{v_A^2 r^3}{y^2(y^2-r^2)}}{(y^2-r^2)^{1/2}}$$

$$= \frac{\frac{2v_A^2 r(y^2-r^2)}{y^2(y^2-r^2)} + \frac{v_A^2 r^3}{y^2(y^2-r^2)}}{(y^2-r^2)^{1/2}}$$

$$= \frac{v_A^2 r(2y^2-2r^2+r^2)}{y^2(y^2-r^2)^{3/2}}$$

recall: $\Omega = \dfrac{v_A r}{y}\dfrac{1}{(y^2-r^2)^{1/2}}$

and $\cos\theta = \dfrac{(y^2-r^2)^{1/2}}{y}$

$$\Rightarrow \Omega\cos\theta = \frac{v_A r}{y\,\cancel{(y^2-r^2)^{1/2}}} \cdot \frac{\cancel{(y^2-r^2)^{1/2}}}{y}$$

$$= \frac{v_A r}{y^2}$$

same as

$$\overline{\qquad y^2 \, (y^2 - r^2)^{3/2}}$$

$$\boxed{\dot{\Omega} = \alpha = \frac{v_A^2 r}{y^2} \frac{(2y^2 - r^2)}{(y^2 - r^2)^{3/2}}}$$

same as before