DYN L21 22

September 24, 2019 9:20 AM

14 L21/L22: Damped Free Vibrations

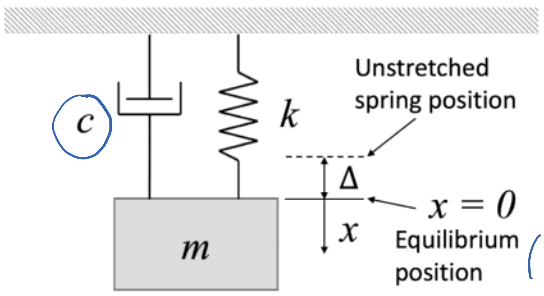
Readings

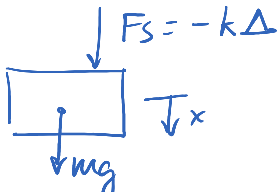
14.1 Objectives

- Introduce **damped** single degree of freedom vibration of a rigid body
- Discuss linear and non-linear damping
- ~~Consider damping effects from a work and energy perspective~~ [Not in this year's course]
- Show examples in mechanical systems

14.2 Damped Free Vibration

In the real world, mechanical systems also have energy losses – through friction and often (intentionally) through energy absorbing devices or materials called dampers. For example, a piston moving through a fluid filled chamber will absorb energy and "damp out" the system. The figure below shows a schematic mass-spring-damper system, where the damper is modeled as a piston moving through a fluid.





$\Sigma F_x : -k\Delta + mg = 0$ (no motion unless perturbed)

$k\Delta = mg$

There are various models for damping. Viscous damping is damping proportional to speed: $\dot{x}$

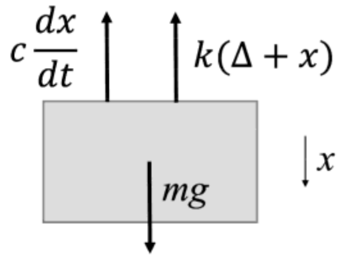
$$F_v = c\dot{x}$$

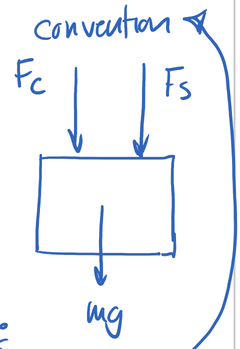
linear

This is the model for the viscous force on a body (such as simply a loose fitting piston in an oil-filled cylinder) moving **slowly** through a liquid. The constant c depends on the viscosity of the fluid and the shape of the body.

When we construct the free body diagram for the system, for free vibration (no excitation forces) we see, similar to free vibration:

$$\sum F_x = -k(\cancel{\Delta} + x) - c\dot{x} + \cancel{mg} = m\ddot{x}$$





$F_c = -c\dot{x}$
$F_s = -k(x+\Delta)$

Where, at equilibrium,

$$-k\Delta + mg = 0$$

Thus, the equation of motion for viscous-damped free vibration of the form:

$$m\ddot{x} + c\dot{x} + kx = 0$$

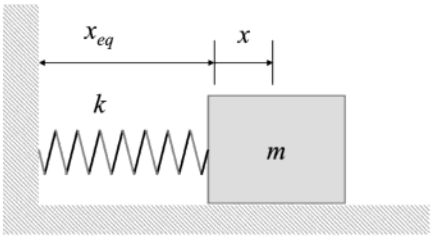
same sign

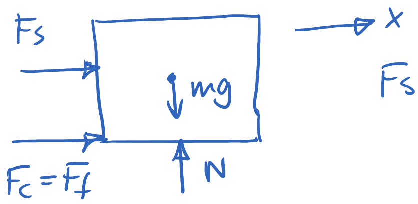
14.2.1 Example

not covered this year

Find the equation of motion for a spring mass system with frictional damping.

will not cover this year, sol'n included for interest





$F_s = -kx$

ΣF_x: $F_s + F_f = ma_x$

$-kx + F_f = m\ddot{x}$ ①

ΣF_y: $-mg + N = 0 \Rightarrow N = mg$

$F_f = -\mu N \operatorname{sgn}(\dot{x})$ ← "sign" function

$$\operatorname{sgn}(\dot{x}) = \begin{cases} -1 & \text{if } \dot{x} > 0 \\ 1 & \text{if } \dot{x} < 0 \end{cases}$$

dir. of $\dot{x}$: → ←

F_f NOT proportional to speed, merely takes DIRECTION of motion into account

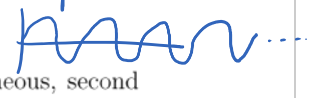
① ⇒ $m\ddot{x} - F_f + kx = 0$

$m\ddot{x} + \mu N \operatorname{sgn}(\dot{x}) + kx = 0$ ← non-linear differential eq'n → solve numerically

(check - all positive)

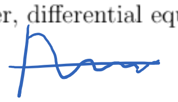
14.3 Solution of the Linear (Viscous) Damping Equation

undamped



The equation of motion for the viscously damped free vibration is a linear homogeneous, second order, differential equation.

damped



$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (14.1)$$

Find $x(t)$

Because of the damping, we know that the solution will not simply be sinusoidal. Energy is being taken out of the system so the amplitude of the vibration must decrease. We propose a solution of the form:

$$x(t) = ae^{rt}$$

$\dot{x}(t) = are^{rt}$

$\ddot{x}(t) = ar^2e^{rt}$

Substituting this solution back into (14.1) and cancel out e^{rt} terms yields:

into EOM: $m(\cancel{a}r^2\cancel{e^{rt}}) + c(\cancel{a}r\cancel{e^{rt}}) + k(\cancel{a}\cancel{e^{rt}}) = 0$

$$mr^2 + cr + k = 0$$

This is the **characteristic equation** for the differential equation (DE). The roots of this equation are:

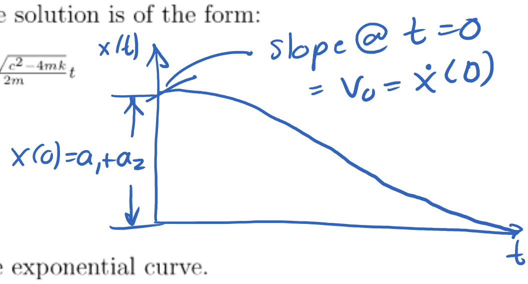
$$\Rightarrow r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

The term under the root sign determines the type of solution, and the behaviour of the system:

FIRST CASE

If $c^2 - 4mk > 0$, the roots are both real, and negative. The solution is of the form:

$$x(t) = a_1 e^{\frac{-c+\sqrt{c^2-4mk}}{2m}t} + a_2 e^{\frac{-c-\sqrt{c^2-4mk}}{2m}t}$$



- The system is overdamped.
- There is no vibration.
- The system moves back to equilibrium along a negative exponential curve.
- The larger the damping, the slower the motion back to equilibrium as the damping force quickly absorbs the initial system energy (when the system velocity is high), and then becomes smaller as the system velocity drops.
- a_1 and a_2 are REAL numbers determined by the system initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$.

$x(0) = x_0 = a_1 \cancel{e^{0}}^{1} + a_2 \cancel{e^{0}}^{1} \Rightarrow a_1 = x_0 - a_2$

$\dot{x}(0) = V_0 = a_1 r_1 \cancel{e^{0}}^{1} + a_2 r_2 \cancel{e^{0}}^{1}$

$V_0 = a_1 r_1 + a_2 r_2$

$(x_0 - a_2) r_1 + a_2 r_2 = V_0$

r_1 & r_2 only depend on c, k, m

$$\boxed{a_2 = \frac{v_0 - r_1 x_0}{r_2 - r_1}}$$

$$a_1 = x_0 - a_2$$
$$= x_0 - \frac{v_0 - r_1 x_0}{r_2 - r_1} = \frac{x_0(r_2 - r_1) - v_0 + r_1 x_0}{(r_2 - r_1)}$$

$$\boxed{a_1 = \frac{-v_0 + r_2 x_0}{r_2 - r_1}}$$

overdamped system

SECOND CASE

If $c_2 - 4mk = 0$, we have repeated roots:

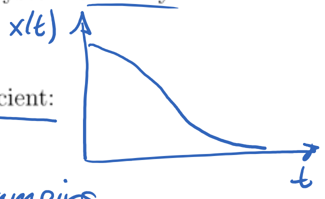
$$\Rightarrow r_{1,2} = \frac{-c}{2m} = -\frac{\sqrt{4mk}}{2m} = -\sqrt{\frac{k}{m}} = -\omega_n$$

form of solution e^{rt}

The solution is then:

$$x(t) = (A + Bt)e^{-\omega_n t}$$

- The system does not vibrate, but moves back to equilibrium as fast as possible without vibration.
- When selecting a damper it is often desirable to select ~~on~~ one that makes the system critically damped.



We denote this situation as critical damping and define the critical damping coefficient:

$$c_c = 2\sqrt{mk} = 2m\omega_n$$

damping coeff. to obtain critical damping

Once again, A and B can be solved for using the initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$.

$$x(0) = x_0 = (A + B(0))e^{0} \qquad \boxed{A = x_0}$$

$$\dot{x}(t) = Be^{-\omega_n t} - (A+Bt)\,\omega_n e^{-\omega_n t}$$

$$\dot{x}(0) = v_0 = B\cancel{e^{0}}^{\,1} - (A + \cancel{B(0)}^{\,0})\,\omega_n \cancel{e^{0}}^{\,1}$$

$$\boxed{B = v_0 + x_0\omega_n}$$

$$x(t) = (x_0 + (v_0 + x_0\omega_n)t)\,e^{-\omega_n t}$$

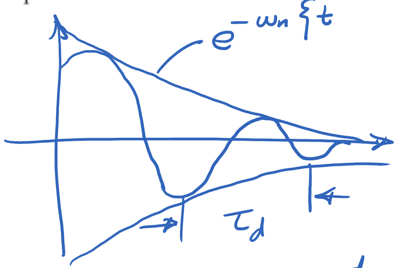
critically damped.

THIRD CASE

If $\mathbf{c_2}$–$\mathbf{4mk < 0}$, The roots of the characteristic equation are complex numbers:

$$\Rightarrow r_{1,2} = \frac{-c}{2m} \pm \frac{i\sqrt{4mk - c^2}}{2m}$$

The system is underdamped. The system vibrates with a decaying amplitude. We introduce a non-dimensional number called the **damping ratio**:



$\zeta = \xi =$ zeta

what you've got

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} = \frac{c}{2m\omega_n}$$

c for critically damped

We use this term to simplify the expression for the roots:

$$\frac{c}{2m} = \omega_n\zeta$$

$$\frac{\sqrt{4mk - c^2}}{2m} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_n^2 - \frac{4m^2\omega_n^2\zeta^2}{4m^2}} = \omega_n\sqrt{1-\zeta^2}$$

ω_d = damped natural freq.

Thus, the roots can be rewritten in terms of the natural frequency and the damping ratio:

$$r_{1,2} = -\omega_n\zeta \pm \underbrace{i\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

The solution then takes the form:

e^{rt}

$$\begin{aligned}
x(t) &= a_1 e^{(-\omega_n\zeta + i\omega_n sqrt1-\zeta^2)t} + a_2 e^{(-\omega_n\zeta - i\omega_n sqrt1-\zeta^2)t} \\
&= e^{-\omega_n\zeta t}(a_1 e^{i\omega_n\sqrt{1-\zeta^2}t} + a_2 e^{-i\omega_n\sqrt{1-\zeta^2}t}) \\
&= Ae^{-\omega_n\zeta t}\sin(\omega_n\sqrt{1-\zeta^2}t + \phi)
\end{aligned}$$

EOM: $m\ddot{x} + c\dot{x} + kx = 0$

std form: $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \Leftrightarrow \ddot{x} + \overbrace{2\zeta\omega_n}^{\frac{c}{m}}\dot{x} + \overbrace{\omega_n^2}^{\frac{k}{m}}x = 0$

Forced eqn:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \Rightarrow \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$

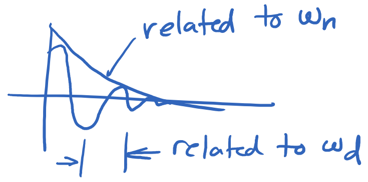
We can make this solution a bit nicer looking by introducing the **damped natural frequency**, ω_d.

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

function of c, k, m

This is the frequency of the vibration of the system which has been shifted from the natural frequency due to the damping (more damping, bigger change in frequency).

Then the solution for the damped free vibration has the form:



$$x(t) = Ae^{-\omega_n\zeta t}\sin(\omega_d t + \phi)$$

n d

Where A and ϕ are determined from the system initial conditions, $x(0) = x_0$, $\dot{x}(0) = v_0$. If there was no damping ($\zeta = 0$), then our solution collapses back to the undamped free vibration case.

14.3.1 Exercise

Show that for:

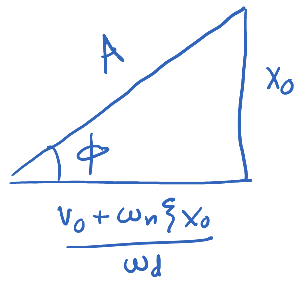
$$x(t) = Ae^{-\omega_n\zeta t}\sin(\omega_d t + \phi)$$

$$x(0) = x_0,\ \dot{x}(0) = v_0$$

The amplitude and phase are described by:

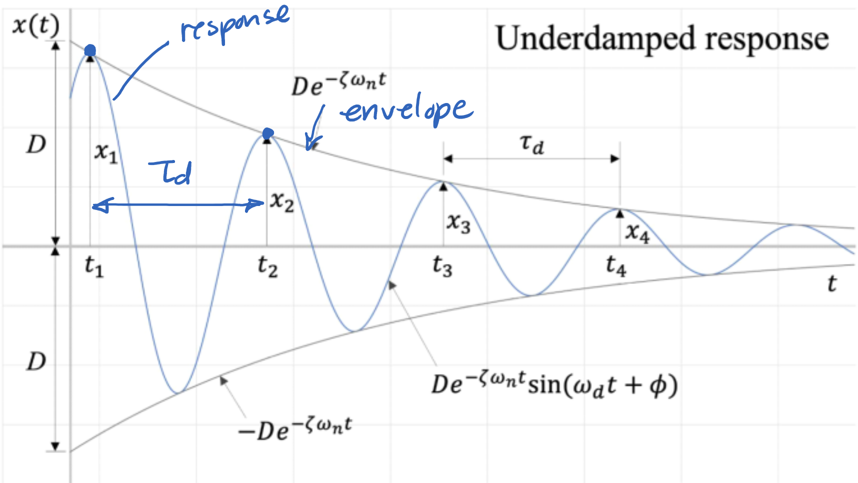
$$A = \sqrt{\frac{(v_0 + \omega_n\zeta x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$

$$\phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \omega_n\zeta x_0}\right]$$



14.4 Measuring Damping

One can estimate the damping of the system by observing the decaying amplitude of vibration.



$\xi = \frac{c}{c_c} = \frac{c}{\sqrt{4km}}$

$t = t_i$ $\qquad x_i = Ae^{-\omega_n \zeta t_i} \sin(\omega_d t_i + \phi)$ damped period

$t = t_{i+1}$ $\qquad$
$$\begin{aligned} x_{i+1} &= Ae^{-\omega_n \zeta (t_i + \tau_d)} \sin(\omega_d (t_i + \tau_d) + \phi) \\ &= Ae^{-\omega_n \zeta (t_i + \tau_d)} \sin(\omega_d t_i + 2\pi + \phi) \end{aligned}$$

Where τ_d is the period for the damped vibration, $\tau_d = \frac{2\pi}{\omega_d}$.

$\tau_d = \frac{2\pi}{\omega_d}$

The ratio of these two amplitudes is:

$$\frac{x_i}{x_{i+1}} = e^{-\zeta\omega_n t - (-\zeta\omega_n(t+\tau_d))} = e^{\zeta\omega_n\tau_d}$$

$e^{\xi \omega_n \tau_d}$

Taking natural logs on both sides we see:

ξ is a function of c

$$\delta = \ln\left(\frac{x_i}{x_{i+1}}\right) = \zeta\omega_n\tau_d = \zeta\omega_n\left(\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

damping you've got

δ is called the **logarithmic decrement**. For **lightly damped systems**, $\zeta = \frac{c}{c_c} < 0.2$, ω_d ω_n, and solving for ζ:

damping needed for critically-damped sys.

$$\zeta \approx \frac{\delta}{2\pi}, \text{ thus: } \quad c = c_c\zeta \approx 2\sqrt{km}\frac{\delta}{2\pi} = \frac{\delta\sqrt{km}}{\pi} = \frac{\delta k}{\pi\omega_n}$$

For all ζ, one can show that:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

relationship to logrithmic decrement

underdamped systems

14.5 Frictional Damping (not covering)

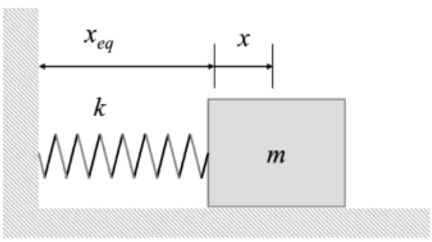
not included

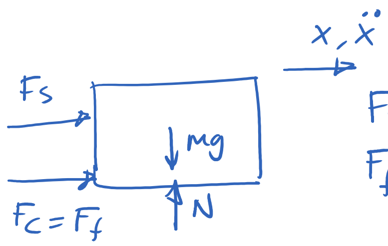
As discussed in Section 2, not all damping is linear. As in the example, dry sliding friction, (coloumb friction) is constant in absolute value, but direction dependant:

$$F_d = F_d(\dot{x}) = \left\{ \begin{array}{ll} -\mu N, & \dot{x} > 0 \\ 0, & \dot{x} = 0 \\ \mu N, & \dot{x} < 0 \end{array} \right\} = -\mu N \text{sgn}(\dot{x})$$

And the equation of motion is:

$$m\ddot{x} + \mu mg\text{sgn}(\dot{x}) + kx = 0$$



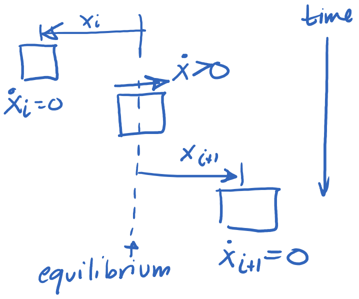


$F_s = -kx$

$F_f = -\mu mg \text{ sgn}(\dot{x})$

We can get some insight into the solution to this equation if we consider work and energy principles. If we look at the absolute values of the displacements for the motion, we see that for two peaks, the change in potential energy is equal to the friction force multiplied by the displacement (work taken out of system):

V = potential energy
U = work



$$V_i + U_{i\to i+1} = V_{i+1}$$
$$V_i - V_{i+1} = -U_{i\to i+1}$$
$$\frac{1}{2}kx_i^2 - \frac{1}{2}kx_{i+1}^2 = \underbrace{\mu mg\left(|x_i| + |x_{i+1}|\right)}_{F\cdot d}$$
$$\frac{1}{2}k\cancel{\left(|x_i| + |x_{i+1}|\right)}\left(|x_i| - |x_{i+1}|\right) = \mu mg\cancel{\left(|x_i| + |x_{i+1}|\right)}$$
$$\Rightarrow \frac{1}{2}k\left(|x_i| - |x_{i+1}|\right) = \mu mg$$
$$\Rightarrow |x_i| - |x_{i+1}| = -\frac{2\mu mg}{k}$$

} Δx = diff. in amplitude between peaks (pos + neg)

$x_{i+1} < x_i$ (energy removed, moves less distance)

Divide by Δt:

slope of envelope →

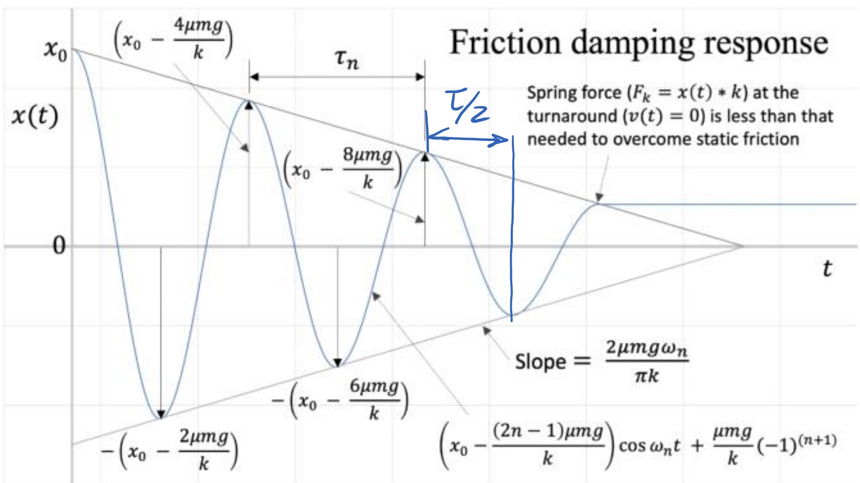
$$\frac{\Delta x}{\Delta t} = -\frac{2\mu mg}{k\left(\frac{T}{2}\right)} = -\frac{2\mu mg\omega_n}{k\pi}$$

T/2 ← time between x_i and x_{i+1} = half period

T/2

Where $\frac{T}{2} = \frac{\pi}{\omega_n}$ is half the period (time between two absolute peaks). This implies that the slope of the envelope surrounding the motion is a negative, constant value.

Note $\tau \neq \tau_d$



For coloumb damping, the coefficients of the oscillating motion change for each direction change – each $\frac{1}{2}$ period of the system. However, the frequency of oscillation is the same as the natural frequency of the system:

(unlike with viscous damping)

system response

$$x(t) = \left(x_0 - \frac{\overbrace{(2n-1)}^{=1,3,5\ldots}\mu mg}{k}\right)\cos\omega_n t + \frac{\mu mg}{k}(-1)^{\overbrace{n+1}^{=2,3,4,5\ldots}}$$

$$n = \begin{cases} 1, & \omega_n t < \pi \\ 2, & \pi \le \omega_n t < 2\pi \\ \vdots & \vdots \end{cases}$$

half cycles

The motion eventually stops ... where?

eqn of envelope (upper):

$$x_{env}(t) = x_0 - \frac{2\mu mg\,\omega_n}{\pi k}t$$

Stops when the remaining stored spring energy is not enough to overcome static friction at the turn-around point ($\mu_s > \mu_k$, typically).

stopping condition: $k\,x(t) < \mu mg$ AND $\dot{x}(t) = 0$

max spring force

no KE, only PE (stopped, ∴ needs to overcome μ_s to start)

n = number of direction changes

stop when: $$\left|x_{env}\Big(\underbrace{\frac{n\pi}{\omega_n}}_{\frac{n\tau}{2}}\Big)\right| = x_0 - \left(\frac{2\mu mg\,\cancel{\omega_n}}{\cancel{\pi}k}\right)\Big(\underbrace{\frac{n\cancel{\pi}}{\cancel{\omega_n}}}_{\frac{n\tau}{2}}\Big) \le \frac{\mu mg}{k}$$

$$x_0 - \left(\frac{2\mu mg}{k}\right)n \le \frac{\mu mg}{k}$$

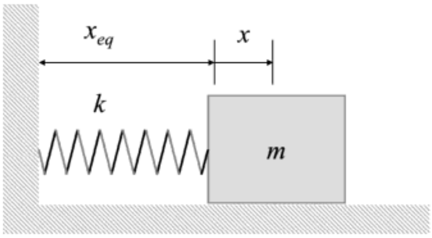
$$\frac{x_0 k}{2\mu mg} - \frac{1}{2} \le n$$

number of half cycles before stopping

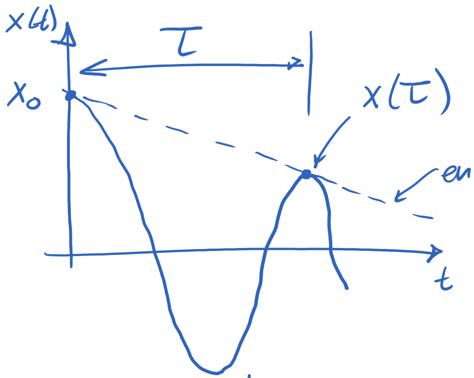
frictional damping

14.5.1 Example 1

The 5 kg block experiences frictional damping where the force of friction is 6 N. The spring constant is $k = 9 \times 10^3$ N/m, and the initial displacement is $x_0 = 4$ cm. Find the displacement, x, one cycle later.



$$\omega_d = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9\times10^3\,N/m}{5\,kg}} = 42.4\ rad/s$$
(frictional)



$n = 2$

recall:
linear damping:
$m\ddot{x} + c\dot{x} + kx = 0$
frictional damping:
$m\ddot{x} + c\,\mathrm{sgn}(\dot{x}) + kx = 0$
non-linear

Slope of envelope:

$$\frac{\Delta x}{\Delta t} = \frac{-2\mu m g}{k\frac{\tau}{2}} \qquad \Delta t = \tau$$

$$\Rightarrow \frac{\Delta x}{\cancel{\tau}} = \frac{-2\mu m g}{k\frac{\cancel{\tau}}{2}} = \frac{-4\overbrace{\mu m g}^{F_f}}{k} = \frac{-4F_f}{k} \quad \text{for one cycle}$$

$$X(t=\tau) = X_0 - \frac{4F_f}{k} = 0.04\,m - 4\left(\frac{6\,N}{9\times10^3\,N/m}\right)$$

$$\boxed{X = 0.037\,m = 3.7\,cm}$$

not included this year

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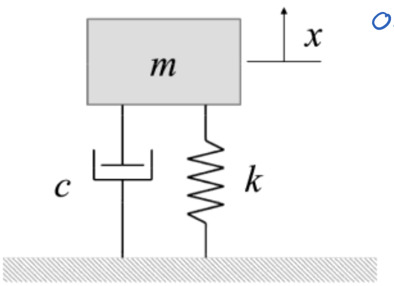
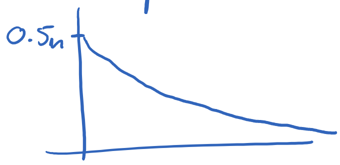
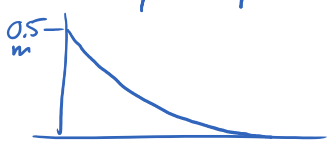
14.5.2 Example 2

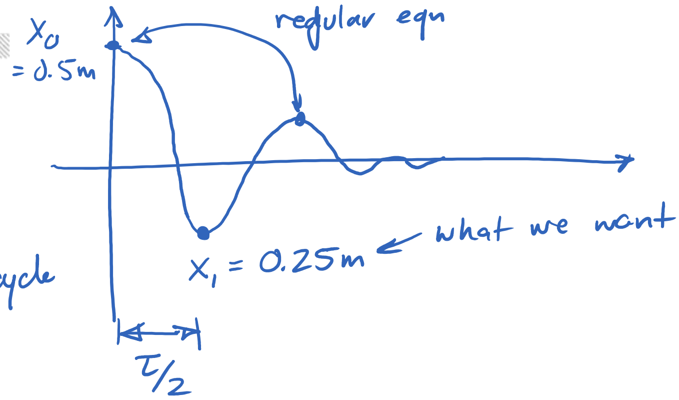
The 500 kg mass is a supported on a spring (k = 12500 N/m) and damper. Find the damping constant, c, such that if the mass is displaced upward from equilibrium by 0.5 m, it will drop exactly 0.25 m below equilibrium.







system must be underdamped



$$\frac{\delta}{2} = \ln\left(\frac{|x_0|}{|x_1|}\right)$$

$\frac{1}{2}\tau$

$$= \zeta\,\omega_n\left(\frac{\tau}{2}\right) \leftarrow \tfrac{1}{2}\text{ cycle}$$

$$= \frac{\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\zeta = f(k, c, m) \qquad \zeta = \frac{c}{2m\omega_n}$$

ω_n ← $f(k,m)$

$$\ln\left(\frac{0.5}{0.25}\right) = \frac{\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\ln(2) = \frac{c\pi}{\sqrt{4m^2\omega_n^2 - c^2}}$$

$$(\ln(2))^2 = \frac{c^2\pi^2}{4m^2\omega_n^2 - c^2}$$

$$4m^2\omega_n^2(\ln 2)^2 - c^2(\ln 2)^2 = c^2\pi^2$$

solve for c

$$c = \sqrt{\frac{4m^2\omega_n^2(\ln 2)^2}{\pi^2 + (\ln 2)^2}}$$

$$\boxed{c = 1077.6\ \frac{\text{N}\cdot\text{s}}{\text{m}}}$$

14.6 Vibrations Summary

What you should be able to do:

1. Explain/identify the necessary physical conditions for a system to vibrate ~~vibration~~ in terms ~~of work and energy principles~~.
2. Generate equation of motion (differential equation) using:
   (a) Newton's Second Law (undamped and damped equations)
   (b) ~~Work and Energy principles (undamped)~~ [Not in this year's course]
3. Classify vibrating systems as overdamped, critically damped or underdamped
4. From the equation of motion, find:
   (a) System natural frequency and period
   (b) Damping ratio
   (c) Damped frequency and period
5. Solve for the coefficients of the solution for the equation of motion (amplitude and phase angle) from initial (or other) conditions.
6. Label all parts of a vibrating motion trace and compute damping from the decaying peaks. log. decrement
7. ~~Explain how friction damping behaves using work and energy principles.~~
8. Compute magnification factor and displacement for forced, undamped, vibrations.