DYN L24 25 26

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16 L24/L25/L26: Momentum and Impact

Readings

16.1 Objectives

In this section, we will

- Summarize impulse and momentum for a rigid body for both linear and angular motion.
- Review conservation of linear and angular momentum and solve rigid body motion problems using this equation.
- Apply the principles of impulse and momentum to impacts.

16.2 Summary for Impulse and Momentum

toolbox

Linear Momentum:

$$\boldsymbol{L} = m\boldsymbol{v_G}$$

Angular Momentum:

$$\begin{aligned}
\boldsymbol{H_G} &= I_G\boldsymbol{\omega} \text{ (about centre of mass)} \\
\boldsymbol{H_O} &= I_G\boldsymbol{\omega} + \boldsymbol{r_G} \times m\boldsymbol{v_G} \\
&= \boldsymbol{H_G} + \boldsymbol{r_G} \times \boldsymbol{L} \text{ (for other points)} \\
\boldsymbol{H_O} &= I_G\boldsymbol{\omega} \text{ (for pinned points)}
\end{aligned}$$

Principle of Momentum and Impulse

$$\left(\sum \boldsymbol{L_{sys}}\right)_{t1} + \left(\sum \int_{t1}^{t2} \boldsymbol{F_{ext,sys}} dt\right) = \left(\sum \boldsymbol{L_{sys}}\right)_{t2}$$

$$\left(\sum \boldsymbol{H_{O,sys}}\right)_{t1} + \left(\sum \int_{t1}^{t2} \boldsymbol{M_{O,ext,sys}} dt\right) = \left(\sum \boldsymbol{H_{O,sys}}\right)_{t2}$$

apply to sys. of bodies

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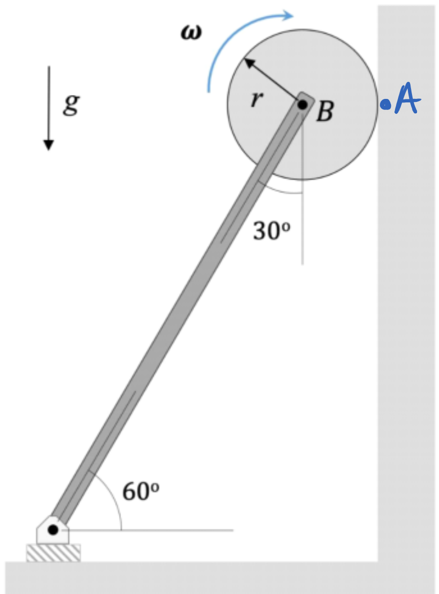
16.2.1 Example

state1

The disk has a mass of 20 kg and is originally spinning at the end of the strut, with an angular velocity of $\omega = 60$ rad/s. If it is then placed against the wall, for which $\mu = 0.3$, determine the time required for the motion to stop. Disk radius is 150 mm. Use $g = 10$ m/s^2.

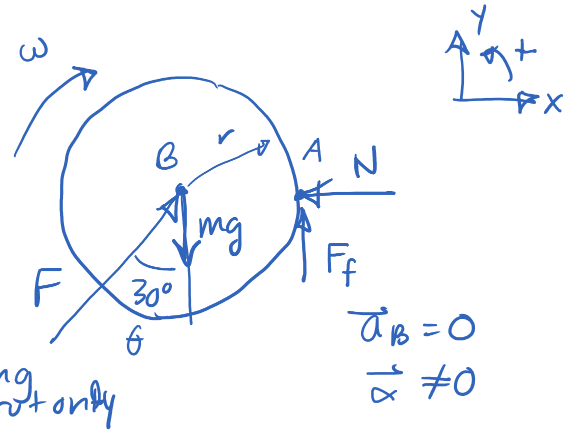
state2

Solve this problem using (a) Momentum and Impulse, (b) Work and Energy, (c) Newton's second law.



- $\vec{a}_B = 0$ pt. B is not moving
  $\vec{v}_B = 0$
- Disc is spinning (slipping on the wall @ A
- Disc is slowing down due to friction with the wall

FBD



$\vec{a}_B = 0$
$\vec{\alpha} \neq 0$

Momentum solution

$\vec{\alpha} \neq 0 \Rightarrow \Sigma M_B \neq 0$
$\Rightarrow$ angular momentum is NOT conserved ($\vec{H}_B$)
(is an angular impulse)

$\vec{a}_B = 0 \Rightarrow \Sigma \vec{F} = 0 \Rightarrow$ linear momentum is conserved ($\vec{L}$)
(no linear impulse)

Find impulses

$\Sigma M_B = F_f r$ $\quad F_f = \mu_k N$

$\Sigma F_x: -N + F \sin\theta = 0 \Rightarrow N = F \sin 30$

$\Sigma F_y: F\cos 30 - mg + \underbrace{\mu N}_{F_f} \Rightarrow mg = F\cos 30 + \mu N$

solve for F

(Momentum solution continued)

$$mg = \frac{F\sqrt{3}}{2} + \mu\frac{F}{2} \Rightarrow F = \frac{2mg}{\sqrt{3}+\underbrace{0.3}_{\mu}} = \frac{2(20kg)(10m/s^2)}{\sqrt{3}+0.3}$$

$$= 197N$$

$$N = \frac{F}{2} = 98N$$

$$F_f = \mu_k N = 30N \Rightarrow \underline{\Sigma M_B = F_f r = 4.5N\cdot m}$$

constant (not a function of time)

MOMENTA

state 1 just before disc touches the wall, spinning @ ω_1

$$H_{B_1} = -I_G\omega_1 = -\frac{1}{2}mr^2\omega_1$$

$$(\curvearrowright \omega_1 \text{ neg})$$

state2 just after it has stopped

$$H_{B2} = 0 \quad (\omega_2 = 0)$$

PRINCIPLE OF IMPULSE-MOMENTUM

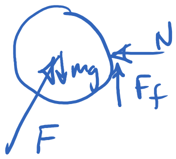
$$H_{B_1} + \int_{t_1}^{t_2} \Sigma M_B dt = H_{B2}$$

$$-\frac{1}{2}mr^2\omega_1 + \int_0^t 4.5Nm\,dt = 0$$

$$4.5(t-0) = \frac{1}{2}mr^2\omega_1 = \frac{1}{2}(20kg(0.15m)^2(60rad/s))$$

$$\boxed{t = 3s}$$

Work-energy solution



① FBD – done!

② Forces that do work

F_f only

$U_{1\to2} = -F_f \cdot s$ $s = r\,\Delta\theta$ $\Delta\theta = \theta_2 - \theta_1$

$= -30\,N \cdot r \cdot \Delta\theta$

③ energy

ΔV ? No.

K.E. $T_1 = \frac{1}{2} I_B \omega_1^2$ $I_B = \frac{1}{2} m r^2$

$= \frac{1}{4} m r^2 \omega_1^2$

$T_2 = 0 \quad (\omega_2 = 0)$

④ work-energy

$$T_1 + \cancel{V_1} + \sum_{\text{non-cons.}} U_{1\to2} = T_2 + \cancel{V_2}$$

$$\frac{1}{4} m r^2 \omega_1^2 - 30\, r\, \Delta\theta = 0$$

$$\Delta\theta = \frac{\frac{1}{4}(20\,kg)(0.15\,m)(60\,rad/s)^2}{30\,N} = 90 \text{ radians}$$

⑤ kinematics

Since $\Sigma M_B = \text{constant}$, α is constant (not a funct. of time)

only true for const α

$\Rightarrow \omega_f = \omega_0 + \alpha t \Rightarrow \alpha = \frac{\omega_f - \omega_0}{(t_f - t_0)}$ $t_0 = 0$

$\omega_f^2 = \omega_0^2 + 2\alpha\Delta\theta$

factored

90 rad

$$(\omega_f^2 - \omega_0^2) = \frac{2(\omega_f - \omega_0)}{\Delta t}(\Delta\theta) \Rightarrow \frac{\Delta\theta}{\Delta t} = \frac{(\omega_f + \omega_0)\cancel{(\omega_f - \omega_0)}}{2\cancel{(\omega_f - \omega_0)}} \Rightarrow \Delta t = \frac{2\Delta\theta}{\underbrace{\omega_f + \omega_0}_{0+60}} = \boxed{3s}$$

Newton's second law

① FBD – done!

② EOM – done!

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_B = I_G \alpha \qquad I_G = \frac{1}{2} m r^2$$

found earlier: $\curvearrowleft$ $4.5\ N \cdot m = \frac{1}{2} m r^2 \alpha$

solve for α: $\alpha = \dfrac{2(4.5\,N\cdot m)}{(20kg)(0.15m)^2} = 20\ rad/s^2$

③ kinematics

α const

$\therefore\ \omega_f = \omega_0 + \alpha t$

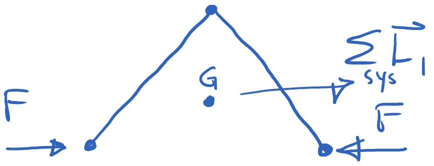
$0 = -60\ rad/s + 20\ rad/s^2 \cdot t \Rightarrow \boxed{t = 3\ s}$

16.3 Conservation of Momentum

When all external linear impulses, $\sum \int_{t1}^{t2} \boldsymbol{F} dt$ sum to zero (internal forces always cancel out), then linear momentum is conserved.

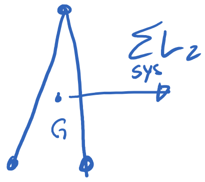
$$\sum_{sys} \boldsymbol{L_1} = \sum_{sys} \boldsymbol{L_2}$$

state 1



$$\sum_{sys} \vec{L}_1 = \sum_{sys} \vec{L}_2$$

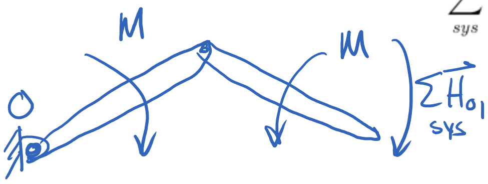
state 2



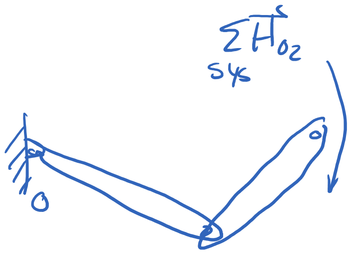
$\sum_{sys} F_{ext} = 0 \quad \therefore$ no linear impulse

When all external angular impulses, $\sum \int_{t1}^{t2} \underline{\boldsymbol{M}_{ext}} dt$ sum to zero (internal forces always cancel out), then angular momentum is conserved.

$$\sum_{sys} \boldsymbol{H_{O1}} = \sum_{sys} \boldsymbol{H_{O2}}$$



$$\sum_{sys} \vec{H}_{O1} = \sum_{sys} \vec{H}_{O2}$$



$\sum_{sys} M_{O ext} = 0 \quad \therefore$ no angular impulse

NOTE: Both linear moment and impulse and angular moment and impulse laws must be obeyed!

recall: $\vec{H}_G = I_G \vec{\omega}$

OR $\vec{H}_P = \underbrace{I_G \vec{\omega}}_{\vec{H}_G} + \vec{r}_{G/P} \times \overbrace{\vec{L}}^{m\vec{v}_G}$

OR $= I_P \vec{\omega} + \vec{r}_{G/P} \times m\vec{v}_P$

OR pinned $\vec{H}_O = I_O \vec{\omega}$

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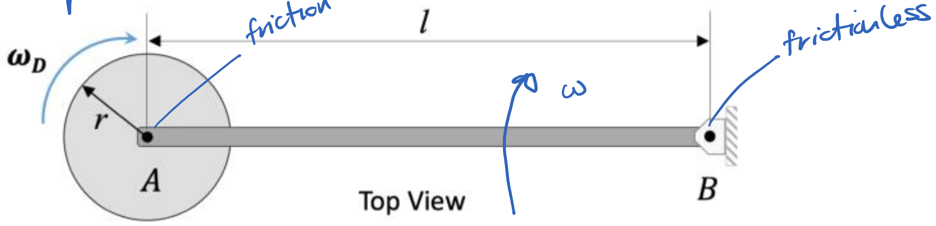
16.3.1 Example

state 1

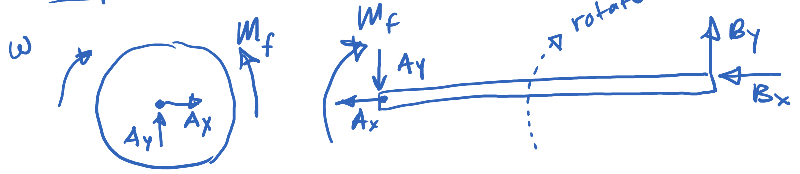
The 5 lb ($l = 3$ ft) rod AB supports the 3 lb disk ($r = 0.5$ ft) at its end. B is a frictionless pin joint. If the disk is given an angular velocity $\omega_D = 8$ rad/s, while the rod is held stationary and then released, determine the angular velocity of the rod after the disk has stopped spinning RELATIVE to the rod, due to friction at bearing A. Motion is in the **horizontal plane**.

state 2

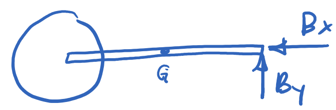
horizontal plane



why will the rod rotate?



system $\quad \Sigma M_P \neq 0$ for any pt except B



B is the only point with external forces
∴ only point about which $\vec{H}$ is conserved

Find ω_{AB} when $\omega_D = 0$

Cons. of angular momentum about B for sys:

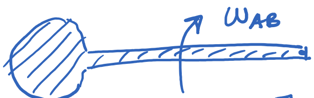
$$\vec{H}_{B1\,sys} = \vec{H}_{B2\,sys}$$

$$\vec{H}_{B1\,sys} = \vec{H}_{B1\,disc} + \cancel{\vec{H}_{B1\,rod}}^{\,0} \quad \text{not moving } \omega_{AB} = 0$$

$$\vec{H}_{B1\,disc} = \underbrace{\vec{H}_{A\,disc}}_{I_A\,\omega_D} + \cancel{\vec{r}_{A/B} \times m\vec{v}_{A1}}^{\,0} \quad \text{A is CoG of disc}$$

$I_A = I_D$ $\rightarrow I_A\,\omega_D$ $\qquad \vec{v}_{A1} = 0$ (starts w. rod @ rest)

$\vec{H}_{B2\,sys}$



acts like one solid body (no relative motion btwn parts)

$$= \left[\underbrace{(I_D + m_D l^2)}_{\substack{\text{parallel axes} \\ I_{B\,disc}}} + I_{rod\,B}\right]\omega_{AB}$$

$I_D = \frac{1}{2} m_D r_D^2$

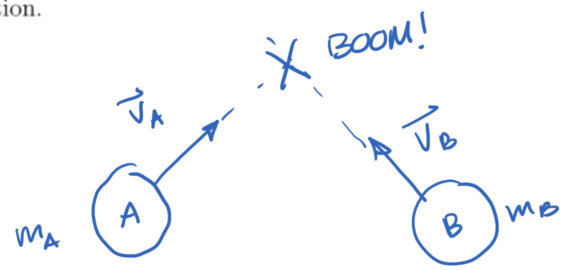
$I_{rod\,B} = \frac{1}{3} m_R l^2$

$$\Rightarrow \underbrace{I_D\,\omega_D}_{\text{state 1}} = (I_D + m_D l^2 + I_{B\,rod})\,\overset{\text{state}}{\omega_{AB}}$$

solve for $\omega_{AB} = 0.071$ rad/s

16.4 Impact

For particles, impact problems involve conservation of linear momentum equations and energy information.



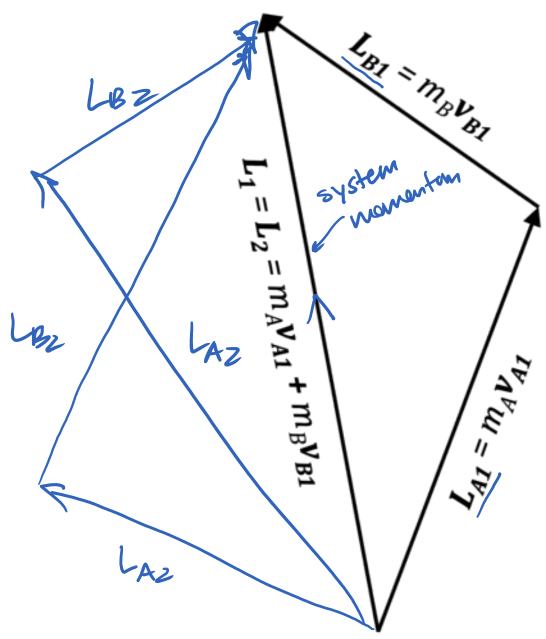
Conservation of linear momentum says:
system

$$\sum \boldsymbol{L}_1 = \sum \boldsymbol{L}_2$$

$$\underbrace{m_A \boldsymbol{v}_{\boldsymbol{A},\mathbf{1}} + m_B \boldsymbol{v}_{\boldsymbol{B},\mathbf{1}}}_{\text{before impact}} = \underbrace{m_A \boldsymbol{v}_{\boldsymbol{A},\mathbf{2}} + m_B \boldsymbol{v}_{\boldsymbol{B},\mathbf{2}}}_{\text{after impact}}$$

only true for system where impulsive force @ impact is internal

We draw a vector diagram:



many solutions to individual particle momentum

To get the post-impact solution we need either:

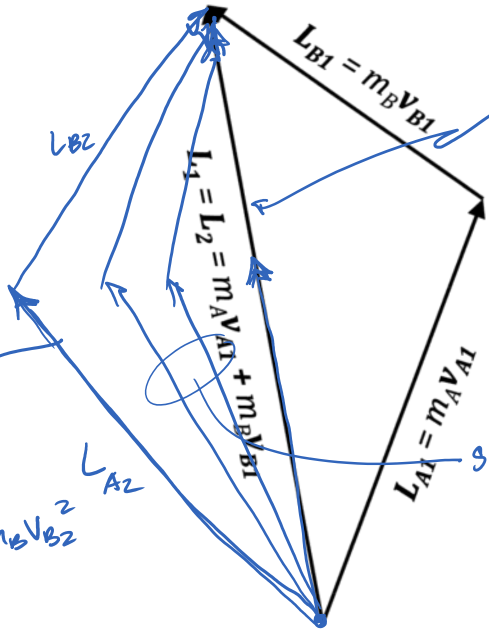
a) Post impact directions of A and B (solve for magnitudes of velocities)

b) Post impact speeds (magnitude of velocities) of A and B (solve for directions)

c) Post impact speed and velocity of one of A OR B

max energy loss (this path

plastic $e = 0$



elastic $e = 1$

no energy loss

$\frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2 = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2$

some energy loss

OR

d) Energy information: **Coefficient of Restitution**: e

how much energy is lost in an impact

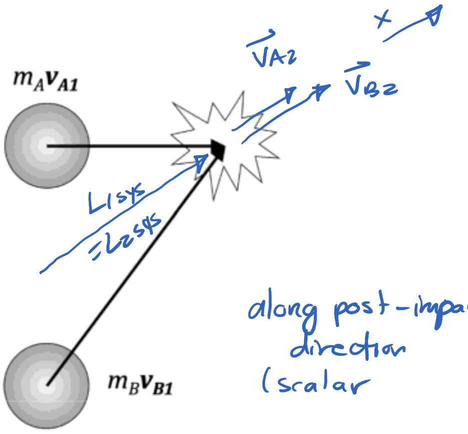
$e = 1$ **Completely elastic collision** – no energy loss: can use conservation of energy to solve the problem (but not necessarily fun or easy for oblique collisions without some further guidance…):

$$\frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 = \frac{1}{2} m_A v_{A,2}^2 + \frac{1}{2} m_B v_{B,2}^2$$

$e = 0$ **Completely plastic collision** – maximum energy loss – particles stick together immediately after collision. Therefore, outgoing motion must be along system momentum direction (minimum kinetic energy).

16.4.1 Example - Perfectly Plastic Collision

We can show that the two particles have the same post-impact velocity if $e = 0$.



cons. of momentum for system:

$$\vec{L}_{1_{sys}} = m_A \vec{v}_{A_1} + m_B \vec{v}_{B_1} = \vec{L}_{2_{sys}}$$

for min energy, dir $\vec{v}_{A2}$ = dir $\vec{v}_{B2}$
(from momentum vector diagram)

along post-impact direction (scalar: $v_{A2} = \dfrac{L_{1sys} - m_B v_{B2}}{m_A}$

$$T_{2_{sys}} = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2 = \frac{1}{2} m_A \left[\frac{L_{1sys} - m_B v_{B2}}{m_A}\right]^2 + \frac{1}{2} m_B v_{B2}^2$$

minimize energy wrt v_{B2}:

$$\frac{dT_2}{dv_{B2}} = 0 \Rightarrow \cancel{\tfrac{1}{2}}(\cancel{2})\cancel{m_A}\left[\frac{L_A - m_B v_{B2}}{m_A}\right]\left(\frac{-m_B}{\cancel{m_A}}\right) + \cancel{\tfrac{1}{2}}(\cancel{2}) m_B v_{B2} = 0$$

$$\Rightarrow \left[-\frac{L_1}{m_A} + \frac{m_B}{m_A} v_{B2} + v_{B2} = 0\right] \times m_A \rightarrow -L_1 + m_B v_{B2} + m_A v_{B2} = 0$$

min energy: $m_A v_{B2} + m_B v_{B2} = L_1$

RECALL:
Cons. mom: $m_A v_{A2} + m_B v_{B2} = L_1$

$\therefore v_{B2} = v_{A2}$

particles are stuck together, same vel. (dir + mag)

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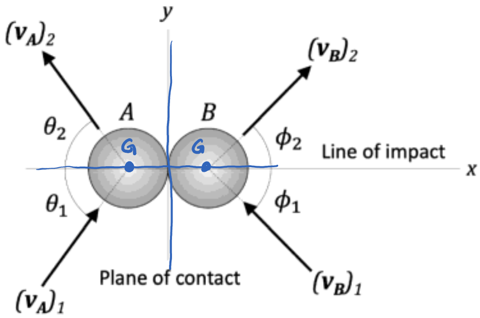
16.5 General Solution for Oblique Collisions for Particles

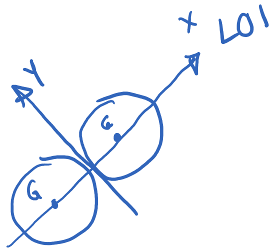
To solve oblique collisions in general, with e ranging from 0 to 1 we use a construction called the **line of impact** and make the following assumptions:

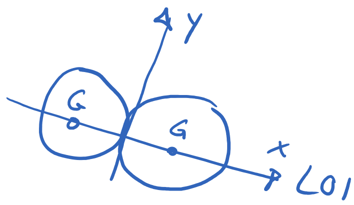
- The colliding bodies are effectively smooth.
- The impulses due to any frictional contact are negligible.

The line of impact is the line perpendicular to the contacting surfaces.

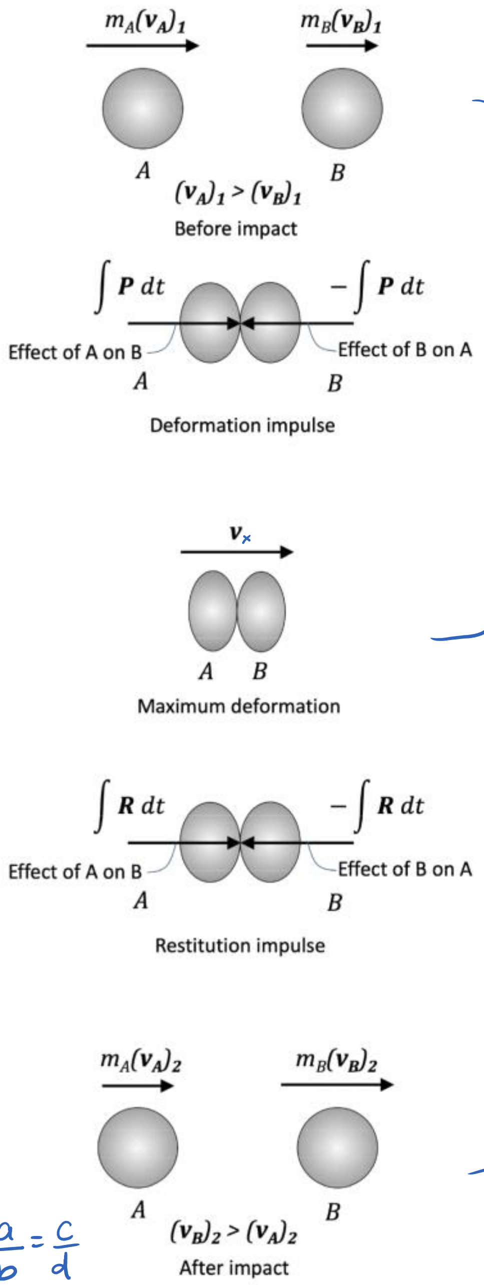


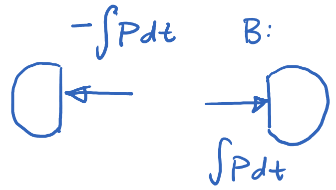
For particles – since they are assumed to be perfect spheres, it follows that the line of impact passes through the mass centres. (This is not the general case for rigid bodies, as we shall see later.)





Looking at the motion **along the line of impact** we construct the following model for how the collision occurs:



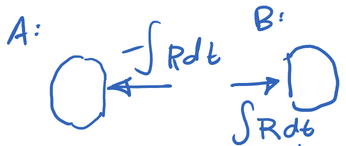


equal + opposite impulsive forces

impulse-momentum for each particle

$$m_A (v_{Ax})_1 - \int P\,dt = m_A v_x$$
$$m_B (v_{Bx})_1 + \int P\,dt = m_B v_x$$

same velocity at max deform.



impulse-momentum for each particle:

$$m_A v_x - \int R\,dt = m_A (v_{Ax})_2$$
$$m_B v_x + \int R\,dt = m_B (v_{Bx})_2$$

$$e = \frac{\overbrace{\int R\,dt}^{\text{restoration}}}{\underbrace{\int P\,dt}_{\text{deformation}}} = \frac{\cancel{m_A} v_x - \cancel{m_A} (v_{Ax})_2}{\cancel{m_A} (v_{Ax})_1 - \cancel{m_A} v_x} = \frac{\cancel{m_B} (v_{Bx})_2 - \cancel{m_B} v_x}{\cancel{m_B} v_x - \cancel{m_B} (v_{Bx})_1}$$

$$e = \frac{v_x - (v_{Ax})_2}{(v_{Ax})_1 - v_x} = \frac{(v_{Bx})_2 - v_x}{v_x - (v_{Bx})_1}$$

$$= \frac{\cancel{v_x} - (v_{Ax})_2 + (v_{Bx})_2 - \cancel{v_x}}{(v_{Ax})_1 - \cancel{v_x} + \cancel{v_x} - (v_{Bx})_1}$$

if $\frac{a}{b} = \frac{c}{d}$

then $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

$$e = \frac{\int R\,dt}{\int P\,dt} = \frac{(v_{Bx})_2 - (v_{Ax})_2}{(v_{Ax})_1 - (v_{Bx})_1}$$

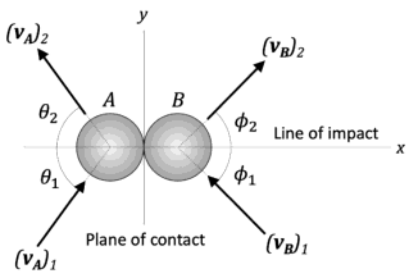
separation velocity

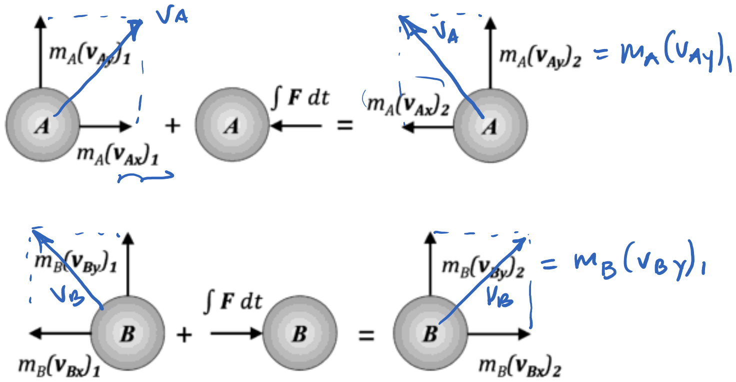
closing velocity

Thus along the line of impact - *defined as our x-axis* - we have:

$$e = \frac{(v_{B,x})_2 - (v_{A,x})_2}{(v_{A,x})_1 - (v_{B,x})_1} = \frac{\int R dt}{\int P dt} = \frac{\text{restoration impulse}}{\text{deformation impulse}}$$

Along the y-axis (defined normal to the line of impact, but still in the plane of motion), there is assumed to be NO IMPULSE, so the pre-and post-impact momentum of *each particle* does not change *in the y direction.* (cons. of momentum along y-dir





Collisions:

So, for impact we have established the following equations:

Conservation of Momentum applies to collisions: ↙ system

$$m_A(\boldsymbol{v_A})_1 + m_B(\boldsymbol{v_B})_1 = m_A(\boldsymbol{v_A})_2 + m_B(\boldsymbol{v_B})_2$$
$$\Rightarrow m_A(v_{A,x})_1 + m_B(v_{B,x})_1 = m_A(v_{A,x})_2 + m_B(v_{B,x})_2$$

← along LOI impulse is internal to system

Because there is no impulse normal to the line of impact (y-direction):

cons. of moment in y-only for each particle

$$m_A(v_{A,y})_1 = m_A(v_{A,y})_2 \qquad m_B(v_{B,y})_1 = m_B(v_{B,y})_2$$

The ratio of separation velocity over approach velocity along the line of impact (x-direction):
(closing)

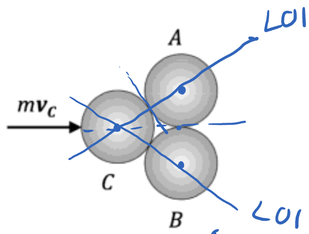
coefficient of restitution

$$e = \frac{(v_{B,x})_2 - (v_{A,x})_2}{(v_{A,x})_1 - (v_{B,x})_1} = \frac{\int R dt}{\int P dt} = \frac{\text{restoration impulse}}{\text{deformation impulse}}$$

only along LOI

16.5.1 Example

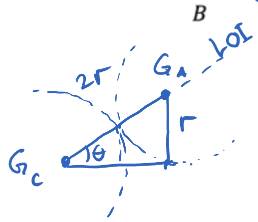
Consider a symmetrical collision between 3 billiard balls. All balls are the same mass m, A and B (state 1) are at rest, and C is incoming as shown. If the incoming ball remains at rest post impact (state 2), find the coefficient of restitution of the collision.



$e = \dfrac{\Delta v_{sep.}}{\Delta v_{closing}}$ along LOI

state 1 (pre-impact)

$V_{A_1} = 0, \quad V_{B_1} = 0$



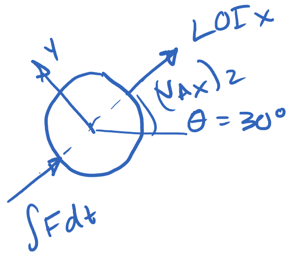
$\sin\theta = \dfrac{r}{2r} = \dfrac{1}{2}$

$\theta = 30°$

state 2 (post-impact) A

momentum conserved in y-dir

$m_A (V_{Ay})_1 = 0 = m_A (V_{Ay})_2$ no y-dir vel.



$(V_{Ax})_2 = \vec{V}_{A2}$

linear impulse-mom along LOI (x):

$0 + \int F\,dt = m\vec{V}_{A2}$

translate y, x: $\vec{V}_{A2} = V_{A2}(\cos 30\,\hat{i} + \sin 30\,\hat{j})$

similarly: $\vec{V}_{B2} = V_{B2}(\cos 30\,\hat{i} - \sin 30\,\hat{j})$ move downward

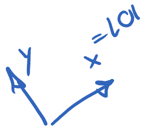
Cons. of mom. of system $(m_c = m_A = m_B = m)$

y, x

$m V_c \hat{i} = m V_{A2}\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) + m V_{B2}\left(\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + m \cancel{V_{c2}}^{\,0}$

$\hat{j}: \quad V_{A2} = V_{B2}$

$\hat{i}: \quad V_{c1} = \frac{\sqrt{3}}{2}V_{A2} + \frac{\sqrt{3}}{2}V_{A2} = \sqrt{3}\,V_{A2} \qquad V_{A2} = \dfrac{V_{c1}}{\sqrt{3}}$



$$e = \frac{\Delta v_{sep}}{\Delta v_{clos.}} \quad \text{along LOI}$$

$$\Delta v_{sep} = v_{A2} - \cancel{v_{C2x}}^{\,0} = v_{A2} = \frac{v_{C1}}{\sqrt{3}}$$

$$\Delta v_{clos} = v_{C1}\cos 30 - \cancel{v_{A1}}^{\,0} = v_{C1}\cos 30 = v_{C1}\frac{\sqrt{3}}{2}$$

$$e = \frac{\Delta v_{sep}}{\Delta v_{clos}} = \frac{\frac{\cancel{v_{C1}}}{\sqrt{3}}}{\frac{\cancel{v_{C1}}\sqrt{3}}{2}} = \frac{2}{3}$$

16.6 Eccentric Impact - Rigid Bodies

For collisions with rigid bodies, more often than not, the centres of mass do not lie on the line of impact (not all rigid bodies are circles or spheres).

If they do happen to line up nicely, we can apply the conservation of linear momentum equations just developed.

If they don't, then we have to consider the effects on **angular momentum** related to the collision.

Again, we make our assumptions that the bodies have **smooth surfaces** and thus all the **collision impulses are normal to the contacting surfaces along the line of impact**. Thus, we neglect any impulses due to frictional contact.

We will show that, in general, along the line of impact at the **point of impact, *P***:

Δv_{sep} @ P

$$e = \frac{(v_{P_{B,x}})_2 - (v_{P_{A,x}})_2}{(v_{P_{A,x}})_1 - (v_{P_{B,x}})_1} = \frac{\int R dt}{\int P dt} = \frac{\text{restoration impulse}}{\text{deformation impulse}}$$

Δv_{clos} @ P along LOI

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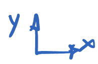
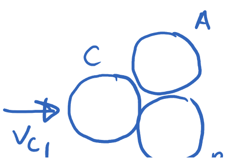
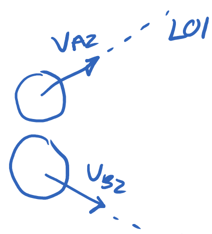
PART 2 $e = 1$, find v_{C2}

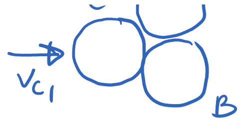
pre-impact : $\vec{v}_{A1} = \vec{v}_{B1} = 0$

$\vec{v}_{C1} \neq 0$

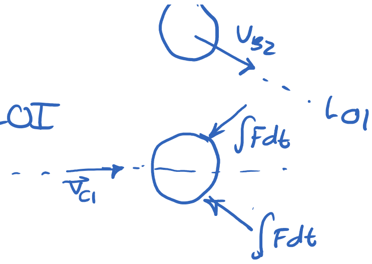








$\vec{V}_{C_1} \neq 0$



post impact: $\vec{V}_{A2}, \vec{V}_{B2}$ along LOI

As before:

$$\vec{V}_{A2} = V_{A2}(\cos 30\,\hat{i} + \sin 30\,\hat{j})$$
$$\vec{V}_{B2} = V_{B2}(\cos 30\,\hat{i} - \sin 30\,\hat{j})$$

NEW $\vec{V}_{C2} = V_{C2}(-\hat{i})$ (external impulse on C cancel in y-dir) (as sys)

Cons. of momentum of system: $\vec{L}_{1sys} = \vec{L}_{2sys}$

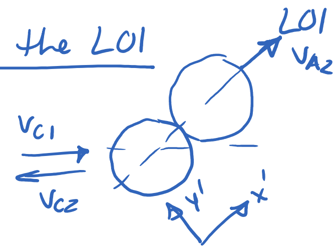
$$\cancel{m} V_{C_1}\hat{i} = \cancel{m} V_{A2}(\cos 30\,\hat{i} + \sin 30\,\hat{j}) + \cancel{m} V_{B2}(\cos 30\,\hat{i} - \sin 30\,\hat{j}) + \cancel{m}(-V_{C2}\hat{i})$$

$\hat{j}$: $V_{A2} = V_{B2}$

$\hat{i}$: $V_{C_1} = V_{A2}\frac{\sqrt{3}}{2} + V_{B2}\frac{\sqrt{3}}{2} - V_{C2} = \cancel{2} V_{A2}\frac{\sqrt{3}}{\cancel{2}} - V_{C2} = \sqrt{3}V_{A2} - V_{C2}$ (1)

know $e = 1$

need closing + separation velocities along the LOI



$$V_{closing} = V_{C_1}\cos 30 - \cancel{V_{A1}}^{\,0} = \frac{\sqrt{3}}{2}V_{C_1}$$

$$V_{separation} = V_{A2} - \underbrace{(-V_{C2}\cos 30)}_{-\hat{i}'}$$

$$= V_{A2} + \frac{\sqrt{3}}{2}V_{C2}$$

$$e = \frac{V_{sep}}{V_{clos.}} = \frac{V_{A2} + \frac{\sqrt{3}}{2}V_{C2}}{\frac{\sqrt{3}}{2}V_{C1}} = 1 \Rightarrow V_{C_1} = \frac{2}{\sqrt{3}}V_{A2} + V_{C2} \quad (2)$$

(1) = (2) : $V_{C_1} = \sqrt{3}\,V_{A2} - V_{C2} = \frac{2}{\sqrt{3}}V_{A2} + V_{C2} \Rightarrow V_{A2} = 2\sqrt{3}\,V_{C2}$

sub into V_{C_1} eq.

$$\Rightarrow \boxed{\begin{aligned} V_{C2} &= \frac{1}{5}V_{C1} \leftarrow \\ V_{A2} &= \frac{2\sqrt{3}}{5}V_{C1} \nearrow \\ &= V_{B2} \searrow \end{aligned}}$$

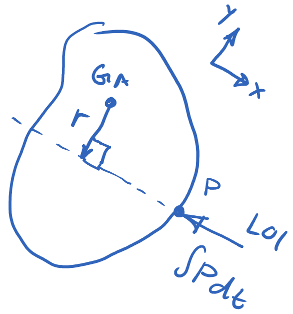
OR if $e = 1$, cons. of energy

$$T_1 = \frac{1}{2}mV_{C1}^2$$
$$T_2 = 2\left(\frac{1}{2}mV_{A2}^2\right) + \frac{1}{2}mV_{C2}^2$$

$T_1 = T_2 \rightarrow$ same ans.

Proof:

Consider Body A -Applying linear impulse and momentum along the line of impact:



linear mom.

@ CoM (CoG) always.

incoming, deformation, collision

$$m_A(v_{G_{A,x}})_1 - \int Pdt = m v_{G_{A,x}}$$

collision, restoration, outgoing

$$m v_{G_{A,x}} - \int Rdt = m_A(v_{G_{A,x}})_2$$

can write

$$e = \frac{\int Rdt}{\int Pdt}$$

for both equation pairs

Considering angular momentum:

incoming, deformation, collision

$$I_{G_A}\omega_{A_1} - r\int Pdt = I_{G_A}\omega_A$$

collision, restoration, outgoing

$$I_{G_A}\omega_A - r\int Rdt = I_{G_A}\omega_{A_2}$$

From both of these pairs of equations we can compute the coefficient of restitution:

$$e = \underbrace{\frac{v_{G_{A,x}} - (v_{G_{A,x}})_2}{(v_{G_{A,x}})_1 - v_{G_{Ax}}}}_{\text{from linear}} = \underbrace{\frac{r\omega_A - r\omega_{A_2}}{r\omega_{A_1} - r\omega_A}}_{\text{angular}}$$

Note: if $\frac{a}{b} = \frac{c}{d} \Rightarrow a = \frac{bc}{d} \Rightarrow \frac{a+c}{b+d} = \frac{\frac{bc}{d} + \frac{dc}{d}}{b+d} = \frac{\frac{c}{d}(b+d)}{b+d} = \frac{c}{d} = \frac{a}{b}$

So, we can write:

$$e = \frac{\overbrace{[v_{G_{A,x}} + r\omega_A]}^{\text{Chasles'}} - [(v_{G_{A,x}})_2 + r\omega_{A_2}]}{[(v_{G_{A,x}})_1 + r\omega_{A_1}] - [v_{G_{A,x}} + r\omega_A]}$$

But since $v_{P_x} = v_{G_{A,x}} + r\omega_A$ (velocity of common point), velocities @ P

$$e = \frac{v_{P_x} - (v_{P_{A,x}})_2}{(v_{P_{A,x}})_1 - v_{P_x}}$$

Using the same steps we can come up with the corresponding expression for Body B:

$$e = \frac{v_{P_x} - (v_{P_{B,x}})_2}{(v_{P_{B,x}})_1 - v_{P_x}}$$

Putting these two last expressions together we get, as required:

Coefficient of restitution @ P

$$\boxed{e = \frac{(v_{P_{B,x}})_2 - (v_{P_{A,x}})_2}{(v_{P_{A,x}})_1 - (v_{P_{B,x}})_1} = \frac{\int Rdt}{\int Pdt} = \frac{\text{restoration impulse}}{\text{deformation impulse}}}$$

16.6.1 Example 1

impulse-mom. state 1

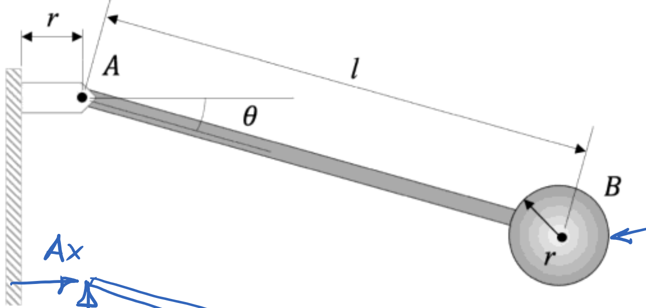
A pendulum, consisting of a 10 kg ball ($r = 0.3$ m) and a 4 kg rod ($l = 2$ m) is released from rest at $\theta_1 = 0°$. Find the maximum angle, θ_2 the ball rebounds to after wall impact. $e = 0.6$.

work-energy

state 4

b4 + after impact state 2 + 3

loss of energy

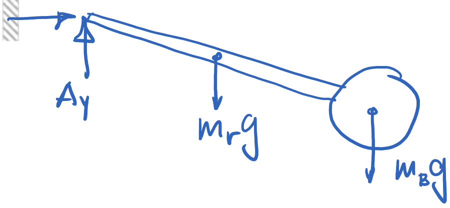


using work-energy AND impulse-momentum

point mass (same ans. as rigid body)

$I_B \sim r^2 = 0.09$

$I_{AB} \sim l^2 = 4$



- no non-cons. work
- yes, impulsive force – mg (neg. compared to impact impulse)

State 1

$T_1 = 0$ released from rest $(v_1 = 0, \omega_1 = 0)$

$V_1 = 0$

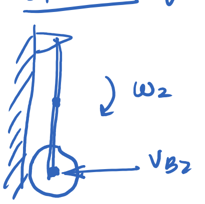
energy is conserved $1 \to 2$

state 2 = just before impact

$$\begin{cases} T_{2R} = \frac{1}{2} I_A \omega_2^2 \\ T_{2B} = \frac{1}{2} m_B V_{B_2}^2 \end{cases}$$

$I_A = \frac{1}{3} m_R l^2$

$\omega_2 = V_{B2}/l$



$T_2 = T_{2R} + T_{2B}$

$T_2 = \left(\frac{1}{2} m_B + \frac{1}{6} m_R\right) V_{B2}^2$

$V_2 = -m_B g l - m_R g \frac{l}{2} = -\left(m_B + \frac{m_R}{2}\right) g l$

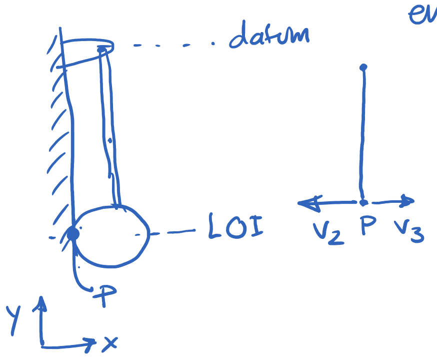
(Example continued)

cons. energy (1 → 2) $\Rightarrow \cancel{T_1} + \cancel{V_1}^{0} = T_2 + V_2$

$\Rightarrow T_2 = -V_2$

state 3 just after impact

energy NOT cons. 2 → 3



$$e \underset{@P}{=} \frac{V_{sep}}{V_{clos.}} = \frac{V_3 - 0}{0 - (-V_2)} = \frac{V_3}{V_2} = 0.6$$

state of energy

$$V_3 = V_2$$

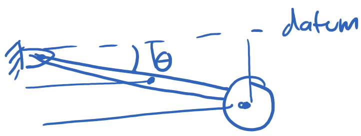
$$T_3 = \left(\tfrac{1}{2}m_B + \tfrac{1}{6}m_R\right)(\underbrace{0.6V_2}_{V_3})^2$$

$$= 0.6^2\, T_2$$

$V_2 + T_2 \neq V_3 + T_3$

energy lost to plastic deformation

state 4



$T_4 = 0$

$V_4 = -\left(m_B + \frac{m_R}{2}\right) g\, l \sin\theta$

energy is conserved 3 → 4

$$T_3 + V_3 = \cancel{T_4}^{0} + V_4$$

$$0.6^2(-V_2) + V_2 = V_4$$

$$(1 - 0.6^2)\, V_2 = V_4$$

$$(1 - 0.6^2)\left(-\cancel{\left(m_B + \frac{m_R}{2}\right) g\, l}\right) = -\cancel{\left(m_B + \frac{m_R}{2}\right) g\, l} \sin\theta$$

$$1 - 0.6^2 = \sin\theta$$

$$\Rightarrow \boxed{\theta = 39.8^\circ}$$

1 → 2: $T_2 = -V_2$

2 → 3: $T_3 = 0.6^2\, T_2$

$V_2 = V_3$

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16.6.2 Example 2

State 1

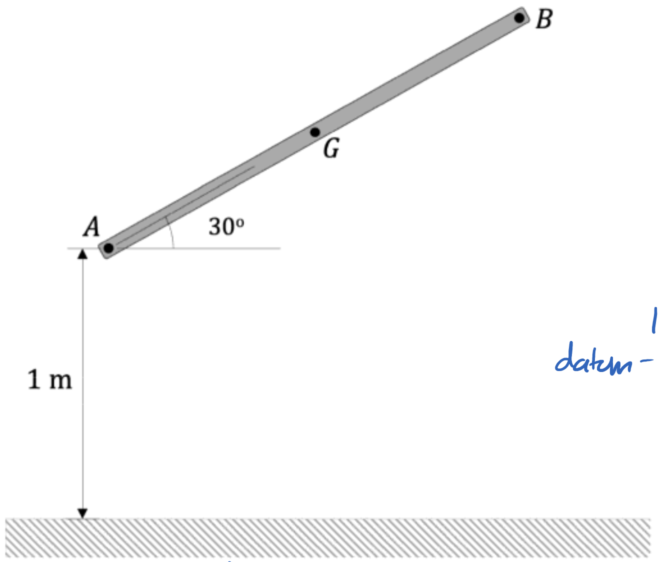
A 10 kg rod, length 1 metre, is dropped at an angle of 30° and travels 1 m before hitting the floor. Find the post impact angular velocity for:

State 2

state 3

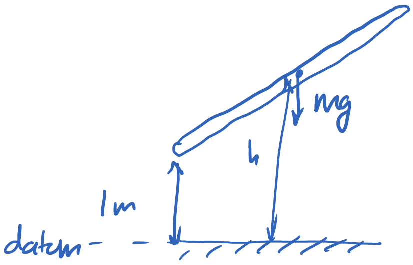
a) A completely plastic impact on a rough floor (assume no slipping) @A

b) A completely elastic impact on a smooth floor (slipping @A)



part (a)

state 1 from rest ($v_{G_1}=0$, $\omega_1=0$)



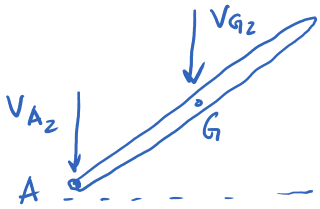
$T_1=0$

$V_1=mgh$

no non-cons. work

energy is conserved 1→2

State 2 just bf impact



$\omega_2=0$

$v_{G_2}=v_{A_2}$

$T_2=\frac{1}{2}m v_G^2$

$V_2=mg(h-1)$

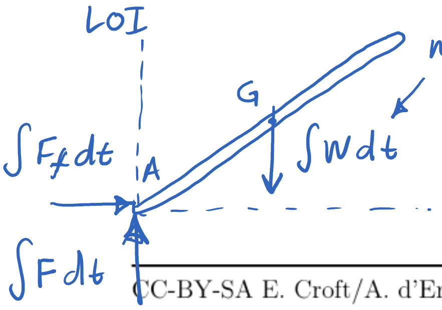
$\therefore \cancel{T_1}^{0}+V_1=T_2+V_2$

$0+mgh=\frac{1}{2}m v_{G_2}^2+mg(h-1)$

$\frac{1}{2}\cancel{m}v_{G_2}^2=\cancel{m}g$

$v_{G_2}=\sqrt{2g}=v_{A_2}$

state 3 just after impact



negligible impulse

$\int W dt << \int F dt$

rough floor plastic impact

$e=0=\frac{v_{Ay3}}{v_{Ay2}}$

$\Rightarrow v_{Ay3}=0$

$\vec{L}_1+\int \Sigma \vec{F} dt=\vec{L}_2$

doesn't help! don't know $\int \vec{F} dt$

(Example continued)

angular momentum is conserved about A ONLY (recall $\int W dt \simeq 0$)

$$\vec{H}_{A_2} = \vec{H}_{A_3}$$

before impact

$$\vec{H}_{A_2} = \cancel{I_G \vec{\omega}_2}^{\,0} + \underbrace{\vec{r}_{G/A}} \times \underbrace{\vec{L}_2} = \overbrace{\frac{l}{2}\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)}^{\vec{r}_{G/A}} \times \overbrace{\left(m v_{G_2}(-\hat{j})\right)}^{\vec{L}_2}$$

$$= \frac{\sqrt{3}}{4} l m v_{G_2} (-\hat{k})$$

set equal

after impact — A acts like a pin

$$\vec{H}_{A_3} = \underline{\underline{I_A}}\, \vec{\omega}_3 = \frac{1}{3} m l^2 \vec{\omega}_3$$

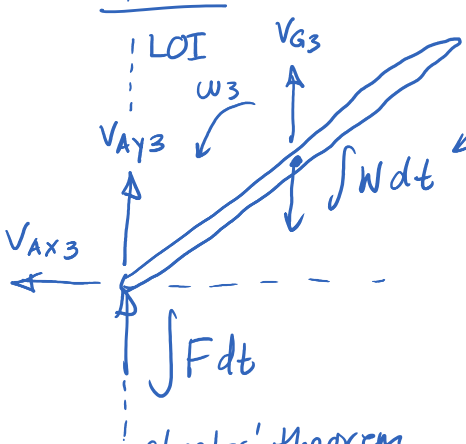
$$\frac{\sqrt{3}}{4} l m v_{G_2} (-\hat{k}) = \frac{1}{3} m l^2 \vec{\omega}_3$$

$$\Rightarrow \boxed{\vec{\omega}_3 = -5.75 \text{ rad/s } \hat{k}}$$

part b elastic impact $e = 1$ $\rightarrow$ state 1, state 2 are same as above (only change is at impact)

smooth floor $\rightarrow F_f = 0$

state 3



negligible $\int W dt << \int F dt$

along LOI:

$$e = \frac{V_{Ay_3}}{-V_{Ay_2}} = 1$$

$$\Rightarrow V_{Ay_3} = -V_{Ay_2} = -V_{G_2}$$

linear momentum is conserved in x-direction

$$\left(\Sigma \int F_x dt = 0\right) \quad m\cancel{\vec{V}_{Gx_2}}^{\,0} = m\vec{V}_{Gx_3} = 0$$

$$\vec{V}_{G_3} = V_{G_3}(\hat{j})$$

Chasles' theorem

$$\vec{V}_{G_3} = \vec{V}_{A_3} + \vec{\omega}_3 \times \vec{r}_{G/A}$$

$$V_{G_3}\hat{j} = \overbrace{+V_{G_2}\hat{j} + V_{Ax_3}\hat{i}} + \omega_3 \hat{k} \times \frac{l}{2}\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$$\hat{j}: \quad V_{G_3} = V_{G_2} + \omega_3 l \frac{\sqrt{3}}{4}$$

$$\hat{i}: \quad 0 = V_{Ax_3} - \omega_3 \frac{l}{4} \Rightarrow V_{Ax_3} = \frac{\omega_3 l}{4}$$

Angular momentum is conserved @ A:

$$\vec{H}_{A_2} = \vec{H}_{A_3}$$

$$\vec{H}_{A2} = \vec{H}_{A3}$$

as before $\vec{H}_{A2} = I_G \cancel{\omega_2}^{0} + \vec{r}_{G/A} \times \vec{L}_2 = \frac{\sqrt{3}}{4} l m v_{G2} (-\hat{k})$

now $\vec{H}_{A3} = \underset{=}{I_G} \vec{\omega}_3 + \vec{r}_{G/A} \times \vec{L}_3$

$$= \frac{1}{12} m l^2 \omega_3 \hat{k} + \underbrace{\frac{l}{2}\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)}_{\vec{r}_{G/A}} \times m \overbrace{\left(v_{G2} + \omega_3 \frac{l\sqrt{3}}{4}\right)(+\hat{j})}^{\vec{v}_{G3}}$$

set equal

$$-\frac{\sqrt{3}}{4} l m v_{G2} = \frac{13}{48} m l^2 \omega_3 + \frac{\sqrt{3}}{4} m l v_{G2}$$

$$\Rightarrow \boxed{\vec{\omega}_3 = -\frac{24}{13}\frac{\sqrt{3}}{l} v_{G2} \hat{k} = -14.2 \text{ rad/s } \hat{k}}$$

* $e = 1$, energy is conserved across impact *

can check $T_2 = T_3$