# DYN L12 13 14

September 24, 2019 9:18 AM



# 9 L12/L13/L14: General Plane Motion and Kinematic Constraints

Readings

## 9.1 Objective

To apply Newton's Second Law to **planar** rigid bodies under a number of different conditions. To recognize and use **kinematic constraints** in order to solve these problems. Of specific consideration are the problems involving wheels where it is not known whether the body is rolling or slipping, and this fact must be determined as part of the problem.

## 9.2 Review of General Methodology

1. Draw the free body diagram for each separate body
2. Write Equations of Motion
   - $\sum \boldsymbol{F} = m\boldsymbol{a}$ (for planar, two equations - in plane - such as $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$)
   - $\sum \boldsymbol{M_G} = I_G \boldsymbol{\alpha}$ (for planar, one equation – out of the plane $\hat{\boldsymbol{k}}$ coordinate)
     (can use a different moment equation if appropriate – i.e. for pinned rotation)
3. Apply KINEMATIC CONSTRAINTS (use your brain)
4. Solve

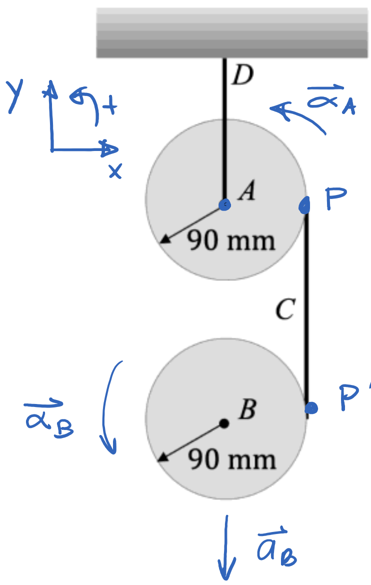
SOME IMPORTANT NOTES TO REMEMBER:

- Forces must be EQUAL AND OPPOSITE on connecting links
- Forces due to strings or cables are along the string or cable
- Forces due to frictionless roller contact on a surface are perpendicular (normal) to the surface.
- For surfaces with friction, friction is along the surface and opposes the CONTACT motion. The Normal force is perpendicular to the surface.
- In some texts (e.g. Hibbeler), a "rough surface" indicates friction present. A "smooth surface" indicates you can neglect friction.

### 9.2.1 Example

A cord, $C$, is wrapped around both $10\,kg$ disks. If they are released from rest, determine the tension in the fixed cord, $D$. Neglect the mass of the cord.




$\vec a_A = 0$ $\qquad \vec\alpha_A \neq 0$

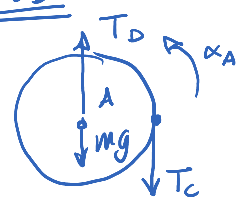
$\vec a_B \neq 0$ $\qquad \vec\alpha_B \neq 0$

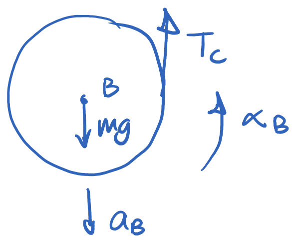
assume

$\vec\alpha_A = \alpha_A \hat k$

$\vec\alpha_B = \alpha_B \hat k$

FBD




$I_A = I_B = \frac{1}{2} m r^2$

$r_A = r_B = r$

$m_A = m_B = m$

EOM (both bodies)

A: $\Sigma F_x: 0 = 0$

$\Sigma F_y: T_D - mg - T_C = 0$ (1)

$\Sigma M_A: -T_C r = I_A \alpha_A$ (2)

B: $\Sigma F_x: 0 = 0$

$\Sigma F_y: T_C - mg = m a_B$ (3)

$\Sigma M_B: T_C r = I_B \alpha_B$ (4)

4 eqns, unknowns: $T_C, T_D, a_B, \alpha_A, \alpha_B$ (5) – need another eqn.

KIN CONSTRAINT

Rope acts like rigid body, $a_P = a_{P'} = a_C$ (↑ dir)

$\vec a_P = \vec a_C = \alpha_A r \hat j$ ($\vec\omega_A = 0$ starts from rest ; $\vec a_A = 0$)

$\vec a_B = \underbrace{\vec a_{P'}}_{=\vec a_P = \vec a_C} - \cancel{\omega_B^2 r}^{\,0} - \alpha_B r \hat j \Rightarrow \vec a_B = \alpha_A r \hat j - \alpha_B r \hat j$

$= (\alpha_A - \alpha_B) r \hat j$ (5)

5th eqn relating $\alpha_A, \alpha_B, a_B$

SOLVE:



(2)+(4): $\alpha_B = -\alpha_A$ $\quad$ (3) $\Rightarrow T_C = m(g + \overbrace{(-\alpha_B - \alpha_B) r}^{a_{By}})$ $\quad$ (4) $\Rightarrow \overbrace{m(g - 2\alpha_B r)}^{T_C} r = I_B \alpha_B$

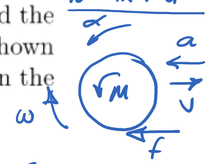
$\Rightarrow$ $\boxed{\alpha_B = \frac{2}{5}\frac{g}{r} \quad \alpha_A = -\frac{2}{5}\frac{g}{r}}$ $\quad T_C = -\frac{1}{\cancel{2}} m r^{\cancel{2}} \left(-\frac{\cancel{2}}{5}\frac{g}{\cancel{r}}\right)$ $\boxed{T_C = \frac{mg}{5}}$ $\quad \boxed{T_D = mg + T_C = \frac{6}{5} mg}$

## 9.3 Type of Kinematic Constraints

This is a class of problem ~~were~~ where the type of **kinematic constraint** (in this case slipping or non-slipping) must be determined as part of the solution.

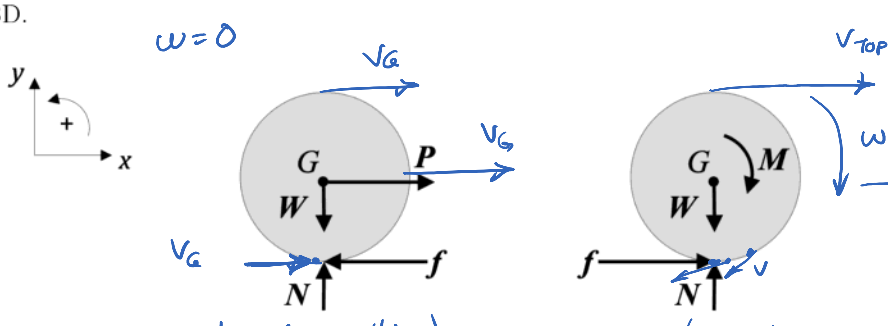
The first step is to draw the free body diagram, including friction. It is helpful to keep in mind the direction in which friction is expected to act in considering the motion that is expected. As shown in this figure, this can be different depending on the rest of the forces and moments involved in the FBD.

BRAKING




$\Sigma F_x: -f = -ma$

$\Sigma M_G: M - fr = I\alpha$




sliding (no rolling) rolling (w/o slipping, or w. slipping)

The next step is to write the equations of motion related to the free body diagram. Considering the free body diagram on the right, we can write the equations of motion as:

$$\Sigma F_x: \quad f = m a_{Gx} \qquad ①$$

$$\Sigma F_y: \quad N - W = 0 \qquad ②$$

$$\Sigma M_G: \quad -M + f \cdot r = I_G \alpha \qquad ③$$

Given W and M, these three equations have four unknowns:

$f,\ a_{Gx},\ N,\ \alpha$

In the case that the wheel is rolling without slipping the force of friction,

$$f \le \mu_s N = \mu_s W = \mu_s mg$$

Where $\mu_s$ is the static coefficient of friction. For rolling without slipping, we know that the velocity of the bottom of the wheel is the same as the velocity of contact surface (relative velocity is zero). We can use this kinematic constraint to specify the relationship between the acceleration of the center of the wheel and the angular acceleration of the wheel.



rolling w/o slipping 4th eqn:

④

rolling w/o slipping 4th eqn:

$$a_{G_x} = \boldsymbol{\alpha} \times \boldsymbol{r}$$ $\vec{a}_{Gx} = \alpha \hat{k} \times r \hat{j} = -\alpha r \hat{i}$ ④

This fourth equation allows the problem to be solved provided that $f \leq \mu_s N$. check this is true

①+④: $f = -m\alpha r$

③: $-M - m\alpha r^2 = I_G \alpha \Rightarrow \alpha = \frac{-M}{I_G + mr^2}$

④: $a_{Gx} = \frac{Mr}{I_G + mr^2}$

CHECK: $f = \frac{\cancel{m} M r}{I_G + mr^2} \leq \mu_s \cancel{m} g$ $\quad \mu_s \geq \frac{Mr}{g(I_G + mr^2)}$ OTHERWISE ...

What if the solutions yields $f > \mu_s N$? Then slipping occurs, and our kinematic constraint does not hold. In this case we use $f = \mu_k N$ as our fourth equation and then we can solve the problem.

4th eqn

① $\Rightarrow f = m a_{Gx} = \mu_k N = \mu_k mg$

③ $\Rightarrow -M + (\mu_k mg) r = I_G \alpha \Rightarrow \alpha = \frac{-M + \mu_k mg}{I_G}$

## 9.4 Method

1. Draw the free body diagram for each separate body
2. Write Equations of Motion
   - $\sum \boldsymbol{F} = m\boldsymbol{a}$ (for planar, two equations - in plane - such as $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$)
   - $\sum \boldsymbol{M_G} = I_G \boldsymbol{\alpha}$ (for planar, one equation – out of the plane $\hat{\boldsymbol{k}}$ coordinate) (can use a different moment equation if appropriate – i.e. for pinned rotation)
3. Apply KINEMATIC CONSTRAINTS. First assume no slipping $f \leq \mu_s N$ and solve. Check the no-slipping assumption. If this yields $f > \mu_s N$ then use $f = \mu_k N$ as the extra relationship and solve the problem.

Can I skip a step and use $f = \mu_k N$ right away? NO! You cannot make the sliding case assumption without checking the non-sliding condition first.

NOTE: Typically $\mu_k < \mu_s$

The determination of the type of kinematic constraint (sliding/not sliding) can also occur in problems such as this one:





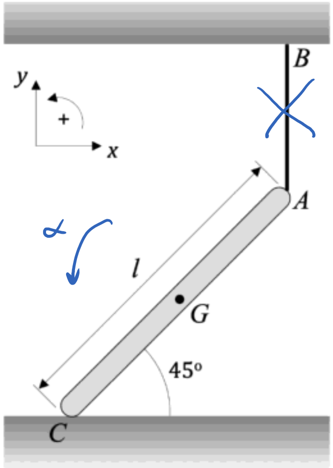
### 9.4.1 Example

$\Sigma M_c = I_G \alpha + m \vec{r}_{G/c} \times \vec{a}_G$
$= I_c \alpha + m \vec{r}_{G/c} \times \vec{a}_c$

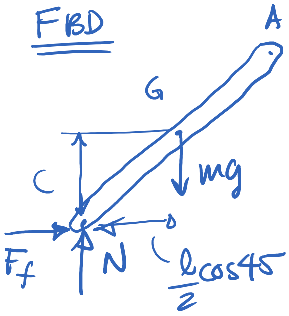
Find the angular acceleration of the bar and the linear acceleration of point $G$, immediately after

$$= I_C\alpha + m\,\underbrace{r_{G/C}\times a_C}_{=0}$$

Find the angular acceleration of the bar and the linear acceleration of point $G$, immediately after line $AB$ is cut. Take $m = 10\,kg$, $l = 2\,m$, $\mu_s = 0.5$, $\mu_k = 0.4$, $g = 9.81\,m/s^2$, and $\theta = 45°$.




FBD




$\vec{\omega} = 0$

EOM

$\Sigma F_x$: $F_f = m a_{Gx}$ ①

$\Sigma F_y$: $N - mg = m a_{Gy}$ ②

$\Sigma M_G$: $F_f\left(\frac{l}{2\sqrt{2}}\right) - N\left(\frac{l}{2\sqrt{2}}\right) = I_G\alpha$ ③

unknowns: $\alpha, a_{Gx}, a_{Gy}, F_f, N$

KIN CONSTRAINTS – ASSUME NO SLIP $F_f \le \mu_s N$

$\vec{a}_C = 0$ (acts like a pin)

$$\therefore\ \vec{a}_G = \cancel{\vec{a}_C}^{\,0} + \vec{\alpha}\times\vec{r}_{G/C} - \cancel{\omega^2}^{\,0}\vec{r}_{G/C}$$ ④

$$\vec{a}_G = \alpha\hat{k}\times\frac{l}{2}\left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right) = \underbrace{-\frac{\alpha l}{2\sqrt{2}}}_{=a_{Gx}}\hat{\imath} + \underbrace{\frac{\alpha l}{2\sqrt{2}}}_{=a_{Gy}}\hat{\jmath}$$ 2 more eqns

SOLVE

①+④: $F_f = -\frac{m\alpha l}{2\sqrt{2}}$

②+④: $N = m\left(g + \frac{\alpha l}{2\sqrt{2}}\right)$

③ $$\underbrace{-\frac{m\alpha l}{2\sqrt{2}}}_{F_f}\cdot\frac{l}{2\sqrt{2}} - \underbrace{m\left(g + \frac{\alpha l}{2\sqrt{2}}\right)}_{N}\cdot\frac{l}{2\sqrt{2}} = \underbrace{\frac{1}{12}ml^2}_{I_G}\alpha$$

$$\Rightarrow \alpha = -\frac{3g}{2\sqrt{2}\,l}\qquad \boxed{\vec{\alpha} = -\frac{3g}{2\sqrt{2}\,l}\hat{k}}$$

(Alternately: revise $\Sigma M \Rightarrow \Sigma M_C = I_C\alpha$ (no slipping, acts like pin))

(Example continued)

CHECK ASSUMPTION (NO SLIPPING):

① + ④ + $\alpha$ ⇒ $F_f = \frac{3}{8}mg$ friction needed to keep bar from slipping

② + ④ + $\alpha$ ⇒ $N = m\left(g - \frac{3g}{8}\right) = \frac{5\,mg}{8}$

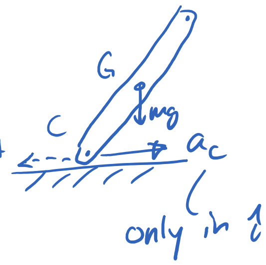
Find $F_{f_s max} = \mu_s N = 0.5\left(\frac{5\,mg}{8}\right) = \frac{2.5\,mg}{8}$

$F_f \geq \mu_s N$ RATS! slipping occurs

RE-SOLVE

$\vec{a}_c \neq 0$ ; $\boxed{F_f = \mu_k N} = m a_{Gx}$ ← 1 eqn

KIN CONSTRAINT: $\vec{a}_G = \underbrace{\vec{a}_c}_{\vec{a}_c = a_c \hat{i}} + \vec{\alpha} \times \vec{r}_{G/c} - \cancel{\omega^2 \vec{r}_{G/c}}$ 0 starts from rest




$\vec{a}_G = a_c \hat{i} + \frac{\alpha l}{2\sqrt{2}}(-\hat{i} + \hat{j})$

$a_{Gx} = a_c - \frac{\alpha l}{2\sqrt{2}}$ $a_{Gy} = \frac{\alpha l}{2\sqrt{2}}$ ← 2 more eqns

+ 1 more unknown ($a_c$)

(total: 6 eqns, 6 unknowns)

SOLVE

② ⇒ $N = m\left(g + \frac{\alpha l}{2\sqrt{2}}\right)$

③ ⇒ $\underbrace{\mu_k N}_{F_f} \frac{l}{2\sqrt{2}} - \frac{N l}{2\sqrt{2}} = \frac{1}{12} m l^2 \alpha$ ⇒ $N = \frac{\sqrt{2}}{6} \frac{m l \alpha}{(\mu_k - 1)}$

equate ⇒ $\cancel{m}\left(g + \frac{\alpha l}{2\sqrt{2}}\right) = \frac{\sqrt{2}}{6} \frac{\cancel{m} l \alpha}{(\mu_k - 1)}$ ⇒ $\alpha\left(\frac{\sqrt{2}\,l}{6(\mu_k - 1)} - \frac{l}{2\sqrt{2}}\right) = g$

$\boxed{\vec{\alpha} = -6.64\,\text{rad/s}^2\,\hat{k}}$ $N = 10\text{kg}\left(9.81\,\text{m/s}^2 - 6.64\,\text{rad/s}^2 \frac{(2\text{m})}{2\sqrt{2}}\right)$ ⇒ $\vec{N} = 51.1\,N\hat{j}$



$\vec{F}_f = \mu_k N \hat{i} = 20.5\,N\,\hat{i}$ $\vec{a}_{Gy} = \frac{\alpha l}{2\sqrt{2}}\hat{j} = -4.7\,\text{m/s}^2\,\hat{j}$

$\vec{a}_{Gx} = \vec{F}_f/m = 2.0\,\text{m/s}^2\,\hat{i}$ $\boxed{\vec{a}_G = (2.0\hat{i} - 4.7\hat{j})\,\text{m/s}}$

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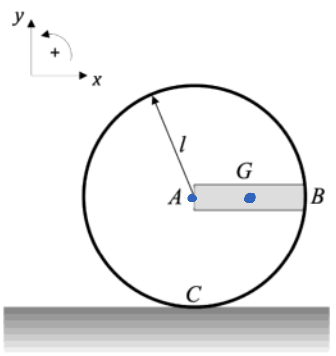
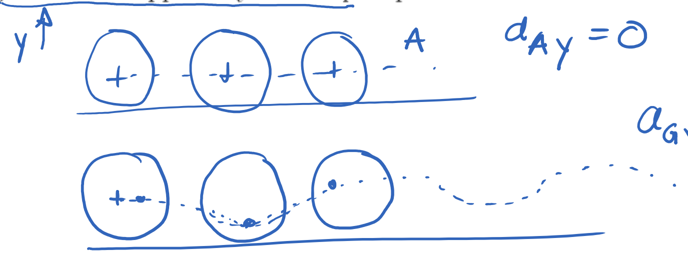
## 9.5 Review Question

The uniform bar $AB$, mass $m$ is attached to the hoop of negligible mass. The hoop is released from rest in the position shown. What is the minimum coefficient of friction for the hoop to roll without slipping – in this case give the force of friction acting on the hoop, as well as the normal force at $C$. Also find the moment acting on the bar applied by the hoop at point $B$.

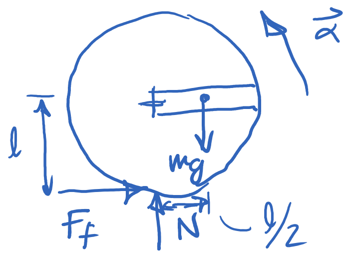



$d_{Ay} = 0 \quad (a_{Ax} \neq 0)$

$a_{Gy} \neq 0$

FBD




EOM

$\Sigma F_x: \; F_f = m a_{Gx}$ ①

$\Sigma F_y: \; N - mg = m a_{Gy}$ ② $(a_{Gy} \neq 0)$

$\Sigma M_G: \; -N\frac{l}{2} + F_f l = I_G \alpha = \underbrace{\frac{1}{12} m l^2}_{I_G} \alpha$ ③

unknowns: $F_f, a_{Gx}, a_{Gy}, \alpha, N$ (5)

3 eqns

KINEMATIC CONSTRAINTS

at this instant [ $\vec{\omega} = 0$ released fr. rest $\therefore a_{Cy} = 0 \; (\omega^2 r)$ ; $a_{Cx} = 0$ rolling w/o slipping ] $\vec{a}_C = 0$

$\vec{a}_G = \cancel{\vec{a}_C}^{\,0} + \vec{\alpha} \times \vec{r}_{G/C} - \cancel{\omega^2}^{\,0} \vec{r}_{G/C}$

$= \alpha \hat{k} \times \left(\frac{l}{2}\hat{i} + l\hat{j}\right) = \underbrace{\frac{\alpha l}{2}}_{a_{Gy}} \hat{j} - \underbrace{\alpha l}_{a_{Gx}} \hat{i}$ 2 more equations ④

SOLVE

① + ④ : $F_f = -m\overbrace{\alpha l}^{a_{Gx}}$

② + ④ : $N = m\left(g + \overbrace{\frac{\alpha l}{2}}^{a_{Gy}}\right)$

with ③ : $-\underbrace{m\left(g+\frac{\alpha l}{2}\right)}_{N}\frac{l}{2} + \underbrace{(-m\alpha l)}_{F_f}\, l = \frac{1}{12}ml^2\alpha$

$$\Rightarrow \cancel{m}\alpha l^{\cancel{2}}\left(\frac{1}{12}+\frac{1}{4}+1\right) = -\cancel{m}g\frac{\cancel{l}}{2}$$

$$\alpha l\left(\frac{4}{3}\right) = \frac{-g}{2} \Rightarrow \alpha = -\frac{3}{8}\frac{g}{l}, \quad \vec{\alpha} = -\frac{3}{8}\frac{g}{l}\hat{k}$$

$$F_f = -m\left(-\frac{3}{8}\frac{g}{l}\right)l = \frac{3}{8}mg \qquad \vec{F} = \frac{3}{8}mg\,\hat{i}$$

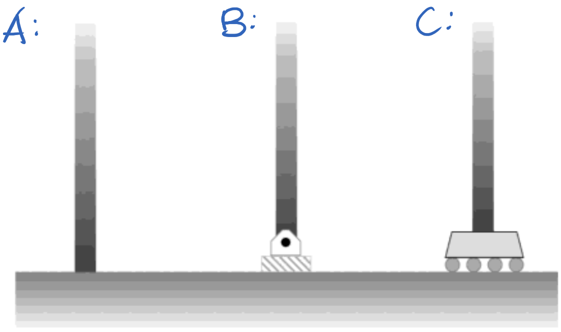
$$N = m\left(g - \frac{3}{8}\frac{g}{\cancel{l}}\cdot\frac{\cancel{l}}{2}\right) = \frac{13}{16}mg \qquad \vec{N} = \frac{13}{16}mg\,\hat{j}$$

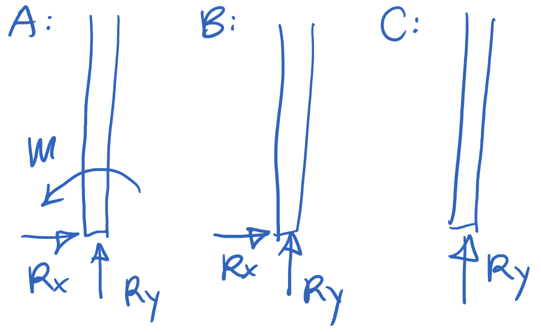
For rolling w/o slipping:

$F_f \le \mu_s N$ for minimum $\mu_s$, $F_f = \mu_{s_{min}} N$

$$\mu_{s_{min}} = \frac{F_f}{N} = \frac{\frac{3}{8}\cancel{mg}}{\frac{13}{16}\cancel{mg}} = \frac{6}{13} \qquad \boxed{\mu_{s_{min}} = \frac{6}{13}}$$

REVIEW: Types of contacts related to the forces and moments they can support.



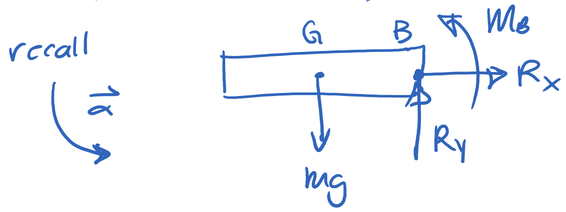



(Example continued)

FIND MOMENT ACTING ON BAR @B

or $\underbrace{\underline{I_B}\vec{\alpha} + m\vec{r}_{G/B}\times\underline{\vec{a}_B}}$

FBD (BAR ONLY)




$\Sigma M_B$: $M_B\hat{k} + mg\frac{l}{2}\hat{k} = \underline{I_G}\vec{\alpha} + m\vec{r}_{G/B}\times\underline{\vec{a}_G}$

$\Rightarrow M_B\hat{k} = -mg\frac{l}{2}\hat{k} + \underbrace{\frac{1}{12}ml^2}_{I_G}\alpha\hat{k}$

$+ m\left(\underbrace{\frac{-l}{2}\hat{i}}_{\vec{r}_{G/B}}\times\underbrace{\left(-\alpha l\hat{i} + \frac{\alpha l}{2}\hat{j}\right)}_{\vec{a}_G}\right)$

recall: $-\hat{i}\times -\hat{i} = 0$
$-\hat{i}\times\hat{j} = -\hat{k}$

$$M_B = ml\left[-\frac{g}{2} - \frac{\alpha l}{4} + \frac{\alpha l}{12}\right]$$

$$= ml\left[-\frac{g}{2} - \frac{\alpha l}{6}\right]$$

$$= ml\left[-\frac{g}{2} - \frac{l}{6}\underbrace{\left[\frac{-3g}{8l}\right]}_{\alpha}\right] = \frac{-7}{16}mgl \Rightarrow \boxed{\vec{M}_B = \frac{-7}{16}mgl\hat{k}}$$