More Random Sampling Methods

This version of the document is dated 2020-08-20.

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1 Contents

1.1 Specific Distributions

Requires random real numbers. This section shows algorithms to sample several popular non-uniform distributions. The algorithms are exact unless otherwise noted, and applications should choose algorithms with either no error (including rounding error) or a user-settable error bound. See the **appendix** for more information.

1.1.1 Normal (Gaussian) Distribution

The ***normal distribution*** (also called the Gaussian distribution) takes the following two parameters:

- `mu` (μ) is the mean (average), or where the peak of the distribution's "bell curve" is.
- `sigma` (σ), the standard deviation, affects how wide the "bell curve" appears. The probability that a normally-distributed random number will be within one standard deviation from the mean is about 68.3%; within two standard deviations (2 times `sigma`), about 95.4%; and within three standard deviations, about 99.7%. (Some publications give σ^2, or variance, rather than standard deviation, as the second parameter. In this case, the standard deviation is the variance's square root.)

There are a number of methods for normal random number generation, including the following. An application can combine some or all of these.

1. The ratio-of-uniforms method (given as `NormalRatioOfUniforms` below).
2. In the *Box–Müller transformation*, `mu + radius * cos(angle)` and `mu + radius * sin(angle)`, where `angle = RNDRANGEMaxExc(0, 2 * pi)` and `radius = sqrt(Expo(0.5)) * sigma`, are two independent normally-distributed random numbers. The polar method (given as `NormalPolar` below) likewise produces two independent normal random numbers at a time.
3. Karney's algorithm to sample from the normal distribution, in a manner that minimizes approximation error and without using floating-point numbers (Karney 2014)**(1)**.
4. The following are approximations to the normal distribution:
    - The sum of twelve `RNDRANGEMaxExc(0, sigma)` numbers (see Note 13), subtracted by 6 * `sigma`. See `NormalCLT` below, which also includes an optional step to "warp" the random number for better accuracy (Kabal 2000/2019)**(2)** See also **"Irwin–Hall distribution" on Wikipedia**. D. Thomas (2014)**(3)**, describes a more general approximation called CLT_k, which combines `k` uniform random numbers as follows: `RNDU01() - RNDU01() + RNDU01() - ...`.
    - **Inversions** of the normal distribution's cumulative distribution function (CDF), including those by Wichura, by Acklam, and by Luu (Luu 2016)**(4)**. See also **"A literate program to compute the inverse of the normal CDF"**. Notice that the normal distribution's inverse CDF has no closed form.

For surveys of normal random number generators, see (Thomas et al. 2007)**(5)**, and (Malik and Hemani 2016)**(6)**.

```
METHOD NormalRatioOfUniforms(mu, sigma)
    while true
        a=RNDU01ZeroExc()
        b=RNDRANGE(0,sqrt(2.0/exp(1.0)))
        if b*b <= -a * a * 4 * ln(a)
          return (RNDINT(1) * 2 - 1) *
            (b * sigma / a) + mu
        end
    end
END METHOD

METHOD NormalPolar(mu, sigma)
  while true
    a = RNDU01ZeroExc()
    b = RNDU01ZeroExc()
    if RNDINT(1) == 0: a = 0 - a
    if RNDINT(1) == 0: b = 0 - b
    c = a * a + b * b
    if c != 0 and c <= 1
       c = sqrt(-ln(c) * 2 / c)
       return [a * sigma * c + mu, b * sigma * c + mu]
    end
  end
END METHOD

METHOD NormalCLT(mu, sigma)
  sum = 0
  for i in 0...12: sum=sum+RNDRANGEMaxExc(0, sigma)
  sum = sum - 6*sigma
  // Optional: "Warp" the sum for better accuracy
  ssq = sum * sum
  sum = ((((0.0000001141*ssq - 0.0000005102) *
           ssq + 0.00007474) *
           ssq + 0.0039439) *
```

```
           ssq + 0.98746) * sum
  return sum + mu
 end
END METHOD
```

Notes:

1. The *standard normal distribution* is implemented as `Normal(0, 1)`.
2. Methods implementing a variant of the normal distribution, the *discrete Gaussian distribution*, generate *integers* that closely follow the normal distribution. Examples include the one in (Karney 2014)[1], an improved version in (Du et al. 2020)[7], as well as so-called "constant-time" methods such as (Micciancio and Walter 2017)[8] that are used above all in *lattice-based cryptography*.

1.1.2 Gamma Distribution

The following method generates a random number that follows a *gamma distribution* and is based on Marsaglia and Tsang's method from 2000[9] (which is an approximate but simple algorithm) and (Liu et al. 2015)[10]. Usually, the number expresses either—

- the lifetime (in days, hours, or other fixed units) of a random component with an average lifetime of `meanLifetime`, or
- a random amount of time (in days, hours, or other fixed units) that passes until as many events as `meanLifetime` happen.

Here, `meanLifetime` must be an integer or noninteger greater than 0, and `scale` is a scaling parameter that is greater than 0, but usually 1 (the random gamma number is multiplied by `scale`).

```
METHOD GammaDist(meanLifetime, scale)
    // Needs to be greater than 0
    if meanLifetime <= 0 or scale <= 0: return error
    // Exponential distribution special case if
    // `meanLifetime` is 1 (see also (Devroye 1986), p. 405)
    if meanLifetime == 1: return Expo(1.0 / scale)
    if meanLifetime < 0.3 // Liu, Martin, Syring 2015
       lamda = (1.0/meanLifetime) - 1
       w = meanLifetime / (1-meanLifetime) * exp(1)
       r = 1.0/(1+w)
       while true
            z = 0
            x = RNDU01()
            if x <= r: z = -ln(x/r)
            else: z = -Expo(lamda)
            ret = exp(-z/meanLifetime)
            eta = 0
            if z>=0: eta=exp(-z)
            else: eta=w*lamda*exp(lamda*z)
            if RNDRANGE(0, eta) < exp(-ret-z): return ret * scale
       end
    end
    d = meanLifetime
    v = 0
    if meanLifetime < 1: d = d + 1
    d = d - (1.0 / 3) // NOTE: 1.0 / 3 must be a fractional number
    c = 1.0 / sqrt(9 * d)
    while true
        x = 0
```

```
        while true
           x = Normal(0, 1)
           v = c * x + 1;
           v = v * v * v
           if v > 0: break
        end
        u = RNDU01ZeroExc()
        x2 = x * x
        if u < 1 - (0.0331 * x2 * x2): break
        if ln(u) < (0.5 * x2) + (d * (1 - v + ln(v))): break
    end
    ret = d * v
    if meanLifetime < 1
       ret = ret * pow(RNDU01(), 1.0 / meanLifetime)
    end
    return ret * scale
END METHOD
```

1.1.3 Beta Distribution

The beta distribution is a bounded-domain probability distribution; its two parameters, `a` and `b`, are both greater than 0 and describe the distribution's shape. Depending on `a` and `b`, the shape can be a smooth peak or a smooth valley.

The following method generates a random number that follows a *beta distribution*, in the interval [0, 1).

```
METHOD BetaDist(a, b)
  if b==1 and a==1: return RNDU01()
  // Min-of-uniform
  if a==1: return 1.0-pow(RNDU01(),1.0/b)
  // Max-of-uniform.  Use only if a is small to
  // avoid accuracy problems, as pointed out
  // by Devroye 1986, p. 675.
  if b==1 and a < 10: return pow(RNDU01(),1.0/a)
  x=GammaDist(a,1)
  return x/(x+GammaDist(b,1))
END METHOD
```

I give an **error-bounded sampler** for the beta distribution (when `a` and `b` are both 1 or greater) in a separate page.

1.1.4 von Mises Distribution

The *von Mises distribution* describes a distribution of circular angles and uses two parameters: `mean` is the mean angle and `kappa` is a shape parameter. The distribution is uniform at `kappa = 0` and approaches a normal distribution with increasing `kappa`.

The algorithm below generates a random number from the von Mises distribution, and is based on the Best–Fisher algorithm from 1979 (as described in (Devroye 1986)**(11)** with errata incorporated).

```
METHOD VonMises(mean, kappa)
    if kappa < 0: return error
    if kappa == 0
        return RNDRANGEMinMaxExc(mean-pi, mean+pi)
    end
    r = 1.0 + sqrt(4 * kappa * kappa + 1)
    rho = (r - sqrt(2 * r)) / (kappa * 2)
    s = (1 + rho * rho) / (2 * rho)
```

```
    while true
        u = RNDRANGEMaxExc(-pi, pi)
        v = RNDU01ZeroOneExc()
        z = cos(u)
        w = (1 + s*z) / (s + z)
        y = kappa * (s - w)
        if y*(2 - y) - v >=0 or ln(y / v) + 1 - y >= 0
           if angle<-1: angle=-1
           if angle>1: angle=1
           // NOTE: Inverse cosine replaced here
           // with `atan2` equivalent
           angle = atan2(sqrt(1-w*w),w)
           if u < 0: angle = -angle
           return mean + angle
        end
    end
END METHOD
```

1.1.5 Stable Distribution

As more and more independent random numbers, generated the same way, are added together, their distribution tends to a ***stable distribution***, which resembles a curve with a single peak, but with generally "fatter" tails than the normal distribution. (Here, the stable distribution means the "alpha-stable distribution".) The pseudocode below uses the Chambers–Mallows–Stuck algorithm. The `Stable` method, implemented below, takes two parameters:

- `alpha` is a stability index in the interval (0, 2].
- `beta` is an asymmetry parameter in the interval [-1, 1]; if `beta` is 0, the curve is symmetric.

```
METHOD Stable(alpha, beta)
    if alpha <=0 or alpha > 2: return error
    if beta < -1 or beta > 1: return error
    halfpi = pi * 0.5
    unif=RNDRANGEMinMaxExc(-halfpi, halfpi)
    c=cos(unif)
    if alpha == 1
       s=sin(unif)
       if beta == 0: return s/c
       expo=Expo(1)
       return 2.0*((unif*beta+halfpi)*s/c -
         beta * ln(halfpi*expo*c/(unif*beta+halfpi)))/pi
    else
       z=-tan(alpha*halfpi)*beta
       ug=unif+atan2(-z, 1)/alpha
       cpow=pow(c, -1.0 / alpha)
       return pow(1.0+z*z, 1.0 / (2*alpha))*
          (sin(alpha*ug)*cpow)*
          pow(cos(unif-alpha*ug)/expo, (1.0 - alpha) / alpha)
    end
END METHOD
```

Methods implementing the strictly geometric stable and general geometric stable distributions are shown below (Kozubowski 2000)**(12)**. Here, `alpha` is in (0, 2], `lamda` is greater than 0, and `tau`'s absolute value is min(1, 2/`alpha` - 1).

```
METHOD GeometricStable(alpha, lamda, tau)
```

```
   // If tau is 0, this is a symmetric Linnik distribution.
   // If tau is 1 and alpha is less than 1, this is
   // a Mittag--Leffler distribution.
   rho = alpha*(1-tau)/2
   sign = -1
   if RNDINT(1)==0 or RNDU01() < tau
       rho = alpha*(1+tau)/2
       sign = 1
   end
   w = 1
   if rho != 1
      rho = rho * pi
      cotparam = RNDRANGE(0, rho)
      w = sin(rho)*cos(cotparam)/sin(cotparam)-cos(rho)
   end
   return Expo(1)*sign*pow(lamda*w, 1.0/alpha)
END METHOD

METHOD GeneralGeoStable(alpha, beta, mu, sigma)
   z = Expo(1)
   if alpha == 1: return mu*z+Stable(alpha, beta)*sigma*z+
          sigma*z*beta*2*pi*ln(sigma*z)
   else: return mu*z+
          Stable(alpha, beta)*sigma*pow(z, 1.0/alpha)
END METHOD
```

1.1.6 Multivariate Normal (Multinormal) Distribution

The following pseudocode calculates a random vector (list of numbers) that follows a ***multivariate normal (multinormal) distribution***. The method `MultivariateNormal` takes the following parameters:

- A list, `mu` (μ), which indicates the means to add to the random vector's components. `mu` can be `nothing`, in which case each component will have a mean of zero.
- A list of lists `cov`, that specifies a *covariance matrix* (Σ, a symmetric positive definite N×N matrix, where N is the number of components of the random vector).

```
METHOD Decompose(matrix)
  numrows = size(matrix)
  if size(matrix[0])!=numrows: return error
  // Does a Cholesky decomposition of a matrix
  // assuming it's positive definite and invertible
  ret=NewList()
  for i in 0...numrows
    submat = NewList()
    for j in 0...numrows: AddItem(submat, 0)
    AddItem(ret, submat)
  end
  s1 = sqrt(matrix[0][0])
  if s1==0: return ret // For robustness
  for i in 0...numrows
    ret[0][i]=matrix[0][i]*1.0/s1
  end
  for i in 0...numrows
    msum=0.0
    for j in 0...i: msum = msum + ret[j][i]*ret[j][i]
    sq=matrix[i][i]-msum
    if sq<0: sq=0 // For robustness
    ret[i][i]=math.sqrt(sq)
```

```
  end
  for j in 0...numrows
    for i in (j + 1)...numrows
      // For robustness
      if ret[j][j]==0: ret[j][i]=0
      if ret[j][j]!=0
        msum=0
        for k in 0...j: msum = msum + ret[k][i]*ret[k][j]
        ret[j][i]=(matrix[j][i]-msum)*1.0/ret[j][j]
      end
    end
  end
  return ret
END METHOD

METHOD MultivariateNormal(mu, cov)
  mulen=size(cov)
  if mu != nothing
    mulen = size(mu)
    if mulen!=size(cov): return error
    if mulen!=size(cov[0]): return error
  end
  // NOTE: If multiple random points will
  // be generated using the same covariance
  // matrix, an implementation can consider
  // precalculating the decomposed matrix
  // in advance rather than calculating it here.
  cho=Decompose(cov)
  i=0
  ret=NewList()
  vars=NewList()
  for j in 0...mulen: AddItem(vars, Normal(0, 1))
  while i<mulen
    nv=Normal(0,1)
    msum = 0
    if mu == nothing: msum=mu[i]
    for j in 0...mulen: msum=msum+vars[j]*cho[j][i]
    AddItem(ret, msum)
    i=i+1
  end
  return ret
end
```

Note: The **Python sample code** contains a variant of this method for generating multiple random vectors in one call.

Examples:

1. A **binormal distribution** (two-variable multinormal distribution) can be sampled using the following idiom: `MultivariateNormal([mu1, mu2], [[s1*s1, s1*s2*rho], [rho*s1*s2, s2*s2]])`, where `mu1` and `mu2` are the means of the two normal random numbers, `s1` and `s2` are their standard deviations, and `rho` is a *correlation coefficient* greater than -1 and less than 1 (0 means no correlation).
2. **Log-multinormal distribution**: Generate a multinormal random vector, then apply `exp(n)` to each component `n`.
3. A **Beckmann distribution**: Generate a random binormal vector `vec`, then apply `Norm(vec)` to that vector.
4. A **Rice (Rician) distribution** is a Beckmann distribution in which the binormal random pair is generated with `m1 = m2 = a / sqrt(2)`, `rho = 0`, and

`s1 = s2 = b`, where `a` and `b` are the parameters to the Rice distribution.

5. **Rice–Norton distribution**: Generate `vec = MultivariateNormal([v,v,v], [[w,0,0],[0,w,0],[0,0,w]])` (where `v = a/sqrt(m*2)`, `w = b*b/m`, and `a`, `b`, and `m` are the parameters to the Rice–Norton distribution), then apply `Norm(vec)` to that vector.
6. A **standard complex normal distribution** is a binormal distribution in which the binormal random pair is generated with `s1 = s2 = sqrt(0.5)` and `mu1 = mu2 = 0` and treated as the real and imaginary parts of a complex number.
7. **Multivariate Linnik distribution**: Generate a multinormal random vector, then multiply each component by `GeometricStable(alpha/2.0, 1, 1)`, where `alpha` is a parameter in (0, 2] (Kozubowski 2000)[12].

1.1.7 Random Real Numbers with a Given Positive Sum

Generating *n* `GammaDist(1, 1)` numbers and dividing them by `total` times their sum[13] will result in *n* uniform random numbers, in random order, that sum to `total` assuming no rounding error (see a **Wikipedia article**). For example, if `total` is 1, the numbers will (approximately) sum to 1. Note that in the exceptional case that all numbers are 0, the process should repeat.

Notes:

1. Notes 1 and 2 in the section "Random Integers with a Given Positive Sum" apply here.
2. The **Dirichlet distribution**, as defined in some places (e.g., *Mathematica*; (Devroye 1986)[11], p. 593-594), can be sampled by generating *n*+1 random **gamma-distributed** numbers, each with separate parameters, taking their sum[13], dividing them by that sum, and taking the first *n* numbers. (The *n*+1 numbers sum to 1, but the Dirichlet distribution models the first *n* of them, which will generally sum to less than 1.)

1.1.8 Gaussian and Other Copulas

A *copula* is a way to describe the dependence between random numbers.

One example is a *Gaussian copula*; this copula is sampled by sampling from a **multinormal distribution**, then converting the resulting numbers to *dependent* uniform random numbers. In the following pseudocode, which implements a Gaussian copula:

- The parameter `covar` is the covariance matrix for the multinormal distribution.
- `erf(v)` is the **error function** of the number `v` (see the appendix).

```
METHOD GaussianCopula(covar)
   mvn=MultivariateNormal(nothing, covar)
   for i in 0...size(covar)
      // Apply the normal distribution's CDF
      // to get uniform random numbers
      mvn[i] = (erf(mvn[i]/(sqrt(2)*sqrt(covar[i][i])))+1)*0.5
   end
   return mvn
END METHOD
```

Each of the resulting uniform random numbers will be in the interval [0, 1], and each one

can be further transformed to any other probability distribution (which is called a *marginal distribution* here) by taking the quantile of that uniform number for that distribution (see "**Inverse Transform Sampling**", and see also (Cario and Nelson 1997) **(14)**.)

Examples:

1. To generate two correlated uniform random numbers with a Gaussian copula, generate `GaussianCopula([[1, rho], [rho, 1]])`, where `rho` is the Pearson correlation coefficient, in the interval [-1, 1]. (Other correlation coefficients besides `rho` exist. For example, for a two-variable Gaussian copula, the **Spearman correlation coefficient** `srho` can be converted to `rho` by `rho = sin(srho * pi / 6) * 2`. Other correlation coefficients, and other measures of dependence between random numbers, are not further discussed in this document.)

2. The following example generates two random numbers that follow a Gaussian copula with exponential marginals (`rho` is the Pearson correlation coefficient, and `rate1` and `rate2` are the rates of the two exponential marginals).

```
METHOD CorrelatedExpo(rho, rate1, rate2)
   copula = GaussianCopula([[1, rho], [rho, 1]])
   // Transform to exponentials using that
   // distribution's quantile function
   return [-log1p(-copula[0]) / rate1,
     -log1p(-copula[1]) / rate2]
END METHOD
```

Note: The Gaussian copula is also known as the *normal-to-anything* method.

Other kinds of copulas describe different kinds of dependence between random numbers. Examples of other copulas are—

- the **Fréchet–Hoeffding upper bound copula** *[x, x, ..., x]* (e.g., `[x, x]`), where `x = RNDU01()`,
- the **Fréchet–Hoeffding lower bound copula** `[x, 1.0 - x]` where `x = RNDU01()`,
- the **product copula**, where each number is a separately generated `RNDU01()` (indicating no dependence between the numbers), and
- the **Archimedean copulas**, described by M. Hofert and M. Mächler (2011)**(15)**.

1.1.9 Exponential Distribution: Another Error-Bounded Algorithm

The following method samples from an exponential distribution with a λ parameter greater than 0, expressed as `lnum`/`lden` (where the sampling occurs within an error tolerance of $2^{-precision}$). For more information, see "**Partially-Sampled Random Numbers**".

```
METHOD ZeroOrOneExpMinus(x, y)
  // Generates 1 with probability exp(-x/y) (Canonne et al. 2020)
  if y <= 0 or x<0: return error
  if x==0: return 1 // exp(0) = 1
  if x > y
    xf = floor(x/y)
    x = mod(x, y)
    if x>0 and ZeroOrOneExpMinus(x, y) == 0: return 0
    for i in 0...xf
      if ZeroOrOneExpMinus(1,1) == 0: return 0
```

```
    end
    return 1
  end
  r = 1
  oy = y
  while true
    if ZeroOrOne(x, y) == 0: return r
    r=1-r
    y = y + oy
  end
END METHOD

METHOD LogisticExp(lnum, lden, prec)
    // Generates 1 with probability 1/(exp(2^-prec)+1).
    // References: Alg. 6 of Morina et al. 2019; Carinne et al. 2020.
    denom=pow(2,prec)*lden
    while true
       if RNDINT(1)==0: return 0
       if ZeroOrOneExpMinus(lnum, denom) == 1: return 1
    end
END METHOD

METHOD ExpoExact(lnum, lden, precision)
   ret=0
   for i in 1..precision
    if LogisticExp(lnum, lden, i)==1: ret=ret+pow(2,-i)
   end
   while ZeroOrOneExpMinus(lnum,lden)==1: ret=ret+1
   return ret
END METHOD
```

> **Note:** After `ExpoExact` is used to generate a random number, an application can append additional binary digits (such as `RNDINT(1)`) to the end of that number while remaining accurate to the given precision (Karney 2014)[1].

1.2 Weighted Choice with Biased Coins

This section describes a way to implement weighted choice of one or more items from coins with unknown bias. Assuming that we only have—

- a list of non-negative integer weights (that need not sum to 1), and
- a "biased coin" which returns true with *unknown* probability of heads and false otherwise,

then the solution involves turning a biased coin to a fair coin, and then turning the fair coin into a loaded die.

1. Biased coin to fair coin: This can be achieved with *randomness extraction* (see my **Note on Randomness Extraction**).
2. Fair coin to loaded die: There are many ways to solve this problem. For example, fair coins can serve as the source of random numbers for `RNDINT` (see "**Uniform Random Integers**"), and `RNDINT` can in turn be used to implement `WeightedChoice`, which implements loaded dice. Some algorithms also produce a loaded die *directly* from fair coins, such as the **Fast Loaded Dice Roller**.

If we have multiple biased coins (n of them), each with a separate unknown bias, we can choose one of them at random according to their bias via rejection sampling, also known as the *Bernoulli race* (Dughmi et al. 2017)[16]; see also (Morina et al., 2019)[17]:

1. Set *i* to `RNDINT(n - 1)`.
2. Flip coin *i* (the first coin is 0, the second is 1, etc.). If the coin returns 1 or heads, return *i*. Otherwise, go to step 1.

Coins of Known Bias: If we only have a list of probabilities (`probs`) that sum to 1, as well as `UnfairCoin(p)`, which returns 1 with a given probability `p` and zero otherwise (such as `ZeroOrOne` or `RNDU01() < p`), one of the following two algorithms chooses an integer at random according to its probability (see the ***Stack Overflow* question** by Daniel Kaplan). However, since we can treat `UnfairCoin(q)` (for any fixed value of `q` in (0, 1)) as a coin with *unknown* bias in this case, these algorithms are only given for completeness. The algorithms are error-bounded when all the probabilities in `probs` are rational numbers. The **first algorithm** uses iteration and is as follows: `cumu = 1.0; for i in 0...size(probs): if UnfairCoin(probs[i]/cumu)==1: return i; else: cumu = cumu - probs[i]`. For a proof of its correctness, see "**Darts, Dice, and Coins**" by Keith Schwarz. The **second algorithm** is the *Bernoulli race*: `while true; y=RNDINT(size(probs)-1); if UnfairCoin(probs[y])==1: return y; else continue; end`, where `UnfairCoin(0.5)` serves as the source of random numbers for `RNDINT`.

2 Notes

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(12) Tomasz J. Kozubowski, "Computer simulation of geometric stable distributions", *Journal of Computational and Applied Mathematics* 116(2), 2000.

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3 Appendix

3.1 Implementation of `erf`

The pseudocode below shows how the **error function** `erf` can be implemented, in case the programming language used doesn't include a built-in version of `erf` (such as JavaScript at the time of this writing). In the pseudocode, `EPSILON` is a very small number to end the iterative calculation.

```
METHOD erf(v)
    if v==0: return 0
    if v<0: return -erf(-v)
    if v==infinity: return 1
    // NOTE: For Java `double`, the following
    // line can be added:
    // if v>=6: return 1
    i=1
    ret=0
    zp=-(v*v)
    zval=1.0
    den=1.0
    while i < 100
        r=v*zval/den
        den=den+2
        ret=ret+r
        // NOTE: EPSILON can be pow(10,14),
        // for example.
        if abs(r)<EPSILON: break
        if i==1: zval=zp
        else: zval = zval*zp/i
        i = i + 1
    end
    return ret*2/sqrt(pi)
END METHOD
```

3.2 A Note on Integer Generation Algorithms

There are many algorithms for the `RNDINT(maxInclusive)` method, which generates uniform random integers in [0, maxInclusive]. This section deals with "optimal" `RNDINT` algorithms in terms of the number of random bits they use on average (assuming we have a "true" random generator that outputs independent unbiased random bits).

Knuth and Yao (1976)[18] showed that any algorithm that uses only random bits to generate random integers with separate probabilities can be described as a *binary tree* (also known as a *DDG tree* or *discrete distribution generating tree*). Random bits trace a path in this tree, and each leaf (terminal node) in the tree represents an outcome. They also gave lower bounds on the number of random bits an algorithm needs on average for this purpose. In the case of `RNDINT`, there are `n = maxInclusive + 1` outcomes that each occur with probability `1/n`, so any *optimal* algorithm for `RNDINT` needs at least `log2(n)` and at most `log2(n) + 2` bits on average (where `log2(x) = ln(x)/ln(2)`).[19]

As also shown by Knuth and Yao, however, any integer generating algorithm that is both optimal *and unbiased (exact)* will also run forever in the worst case, even if it uses few random bits on average. This is because in most cases, `n` will not be a power of 2, so that `n` will have an infinite binary expansion, so that the resulting DDG tree will have to either be infinitely deep, or include "rejection leaves" at the end of the tree. (If `n` is a power of 2, the binary expansion will be finite, so that the DDG tree will have a finite depth and no rejection leaves.)

Because of this, there is no general way to "fix" the worst case of running forever, while still having an unbiased (exact) algorithm. For instance, modulo reductions can be represented by a DDG tree in which rejection leaves are replaced with labeled outcomes, but the bias occurs because only some outcomes can replace rejection leaves this way. Even with rejection sampling, stopping the rejection after a fixed number of iterations will likewise lead to bias, for the same reasons.

The following are some ways to implement `RNDINT`. (The column "Unbiased?" means whether the algorithm generates random integers without bias, even if `n` is not a power of 2.)

Algorithm	Optimal?	Unbiased?	Time Complexity
Rejection sampling: Sample in a bigger range until a sampled number fits the smaller range.	Not always	Yes	Runs forever in worst case
Multiply-and-shift reduction: Generate `bignumber`, a `k`-bit random integer with many more bits than `n` has, then find `(bignumber * n) >> k` (see (Lemire 2016)[20], (Lemire 2018)[21], and the "Integer Multiplication" algorithm surveyed by M. O'Neill).	No	No	Constant
Modulo reduction: Generate `bignumber` as above, then find `rem(bignumber, n)`	No	No	Constant
Fast Dice Roller (Lumbroso 2013)[22]	Yes	Yes	Runs forever in worst case
Math Forum (2004)[23] or (Mennucci 2018)[24] (batching/recycling random bits)	Yes	Yes	Runs forever in worst case
"FP Multiply" surveyed by **M. O'Neill**	No	No	Constant
Algorithm in "Conclusion" section by O'Neill	No	Yes	Runs forever in worst case
"Debiased" and "Bitmask with Rejection" surveyed by M. O'Neill	No	Yes	Runs forever in worst case

There are various techniques that can reduce the number of bits "wasted" by an integer-generating algorithm, and bring that algorithm closer to the theoretical lower bound of Knuth and Yao, even if the algorithm isn't "optimal". These techniques, which include batching, bit recycling, and randomness extraction, are discussed, for example, in the Math Forum page and the Lumbroso and Mennucci papers referenced above, and in (Devroye and Gravel 2015)[25].

3.3 A Note on Weighted Choice Algorithms

Just like integer generation algorithms (see the previous section), weighted choice algorithms (implementations of `WeightedChoice` that sample with replacement) involve generating random integers with separate probabilities. And all of them can be described as a binary DDG tree just like integer generating algorithms.

In this case, though, the number of random bits an algorithm uses on average is bounded from below by the sum of binary entropies of all the probabilities involved. For example, say we give the four integers 1, 2, 3, 4 the following weights: 3, 15, 1, 2. The binary entropies of these weights are 0.4010... + 0.3467... + 0.2091... + 0.3230... = 1.2800... (because the sum of the weights is 21 and the binary entropy of 3/21 is `(3/21) * log2(21/3) = 0.4010...`, and so on for the other weights), so an optimal algorithm will use anywhere from 1.2800... to 3.2800... bits on average to generate a random number with these weights.[19] Another difference from integer generation algorithms is that usually a special data structure has to be built for the sampling to work, and often there is a need to make updates to the structure as items are sampled. The following are some ways to implement `WeightedChoice`; the algorithms are generally not optimal in terms of the number of bits used, unless noted.

- The Fast Loaded Dice Roller (Saad et al., 2020)[26]. This sampler comes within 6 bits, on average, of the optimal number of bits. The paper for this algorithm also reviews rejection samplers in section 4.
- The samplers described in (Saad et al., 2020)[27]. The samplers are optimal in the sense given here as long as the sum of the weights is of the form 2^k or $2^k - 2^m$.
- The data structures surveyed and mentioned in (Klundert 2019)[28].
- The Bringmann–Larsen succinct data structure (Bringmann and Larsen 2013)[29].
- A sampler that is designed to work on a sorted list of weights (Bringmann and Panagiotou 2012)[30].
- The parallel weighted random samplers described in (Hübschle-Schneider and Sanders 2019)[31].
- The two-level search, multi-level search, and flat method described in (Tang 2019)[32].

There are other weighted choice algorithms that don't necessarily take integer weights. They include:

- The algorithms given in "Weighted Choice with Biased Coins" in this appendix.
- The *alias method* by Walker (1977)[33]. Michael Vose's version of the alias method (Vose 1991)[34] is described in "**Darts, Dice, and Coins: Sampling from a Discrete Distribution**". The alias method ought to be implemented using rational-valued weights and rational-number arithmetic, since this method is hard to apply to integer weights if the sum of the weights is not divisible by the number of weights.
- The Knuth and Yao algorithm that generates a DDG tree from the binary expansions of the probabilities, an algorithm that is optimal, or at least nearly so. This is suggested in exercise 3.4.2 of chapter 15 of (Devroye 1986, p. 1-2)[11], implemented

in *randomgen.py* as the `discretegen` method, and also described in (Roy et al. 2013)[35]. `discretegen` can work with probabilities that are irrational numbers (which have infinite binary expansions) as long as there is a way to calculate the binary expansion "on the fly".

- The Han and Hoshi algorithm (Han and Hoshi 1997)[36] that uses the cumulative probabilities as input and is described in (Devroye and Gravel 2015)[25]. This algorithm comes within 3 bits, on average, of the optimal number of bits.

For all weighted-choice algorithms in this section (except the last two algorithms), floating-point arithmetic and floating-point random number generation (such as `RNDRANGE()`) ought to be avoided, since they often introduce bias in real-world implementations.

3.4 A Note on Error-Bounded Algorithms

There are three kinds of randomization algorithms:

1. An *error-bounded algorithm* is an algorithm that samples a distribution in a manner that minimizes approximation error. This means the algorithm samples from a continuous distribution that is close to the ideal distribution within a user-specified error tolerance (see below for details), or samples exactly from a discrete distribution (one that takes on a countable number of values). Thus, the algorithm gives every representable number the expected probability of occurring. In general, the only random numbers the algorithm uses are random bits (binary digits). An application should use error-bounded algorithms whenever possible.
2. An *exact algorithm* is an algorithm that samples from the exact distribution requested, assuming that computers can store and operate on real numbers of any precision and can generate independent uniform random real numbers of any precision (Devroye 1986, p. 1-2)[11]. Without more, however, an exact algorithm implemented on real-life computers can incur rounding and other errors, especially when floating-point arithmetic is used or when irrational numbers or transcendental functions are involved. An exact algorithm can achieve a guaranteed bound on accuracy (and thus be an *error-bounded algorithm*) using either arbitrary-precision or interval arithmetic (see also Devroye 1986, p. 2)[11]. In this page, all methods given here are exact unless otherwise noted. Note that `RNDU01` or `RNDRANGE` are exact in theory, but have no required implementation.
3. An *inexact*, *approximate*, or *biased algorithm* is neither exact nor error-bounded; it uses "a mathematical approximation of sorts" to generate a random number that is close to the desired distribution (Devroye 1986, p. 2)[11]. An application should use this kind of algorithm only if it's willing to trade accuracy for speed.

Most algorithms on this page, though, are not *error-bounded*, but even so, they may still be useful to an application willing to trade accuracy for speed.

On the other hand, if an algorithm returns results that are accurate to a given number of digits after the point (for example, 53 bits after the point), it can generate any number of digits uniformly at random and append those digits to the result's digit expansion while remaining accurate to that many digits. For example, after it generates a normally-distributed random number, an algorithm can fill it with enough uniform random bits, as necessary, to give the number 100 bits after the point (Karney 2014)[1], see also (Oberhoff 2018)[37] (example: `for i in 54..100: ret = ret + RNDINT(1) * pow(2,-i)`).

There are many ways to describe closeness between two distributions. As one suggestion found in (Devroye and Gravel 2015)[25], an algorithm has accuracy ε (the user-specified

error tolerance) if it samples random numbers whose distribution is close to the ideal distribution by a Wasserstein L_∞ distance of not more than ε.

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