Decision Making and Reinforcement Learning

Module 1: Decision Making and Utility Theory

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Topics

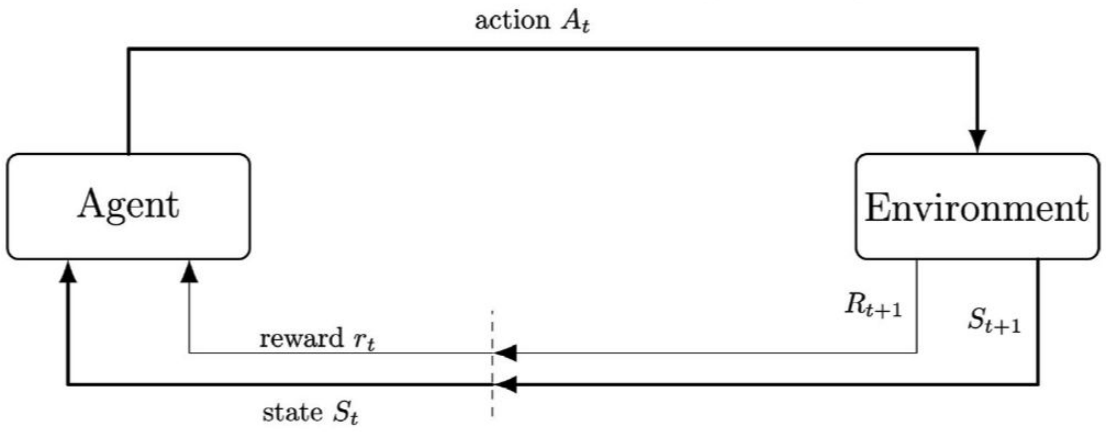
- Rational agents and environments
- Utilities and maximization of expected utility
- Preferences and axioms of utility theory
- Uncertain and multi-attribute utilities
- Value of perfect information

Learning Objectives

- **Describe** rational decision-making as maximization of expected utility
- **List** and **understand** the axioms that govern rational preferences and existence of utility functions
- **Understand** properties of uncertain and multi-attribute utility functions
- **Compute** the value of perfect information in an information-gathering problem

Agent-Environment Interface

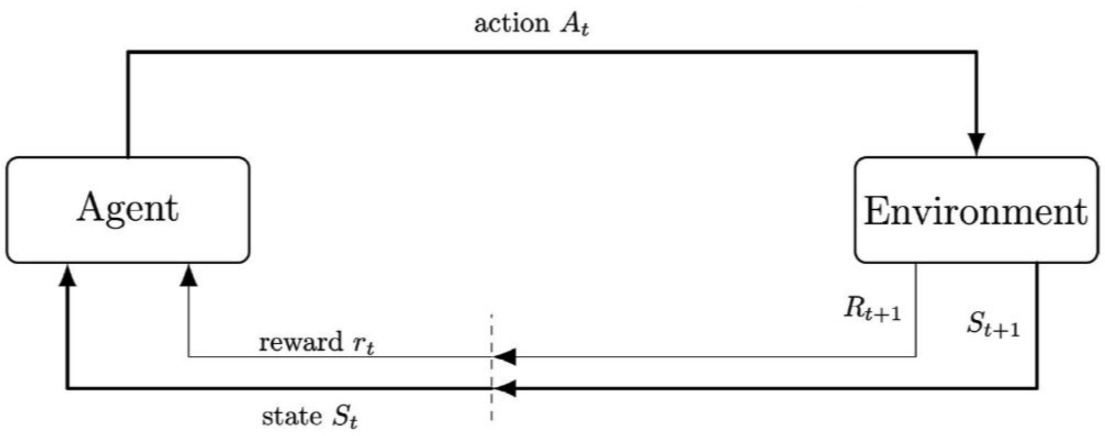
- *Decision-making* is the process in which an **agent** performs an *action* or set of actions in an **environment**



Agent-Environment Interface

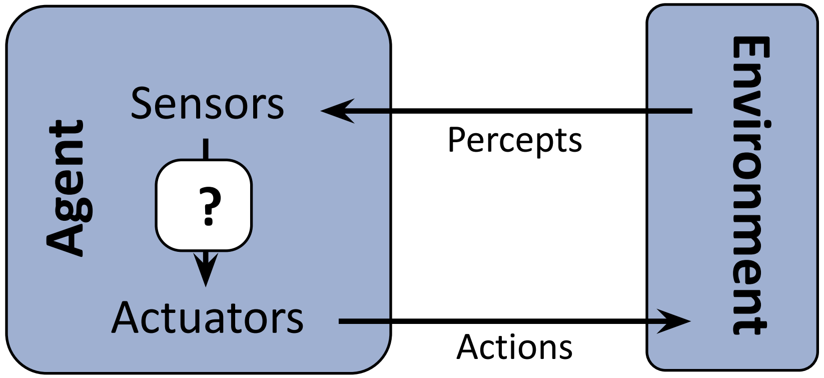
- *Decision-making* is the process in which an **agent** performs an *action* or set of actions in an **environment**

- The action may change the **state** of the agent and environment
- The agent can receive percepts from the environment, e.g., *rewards*



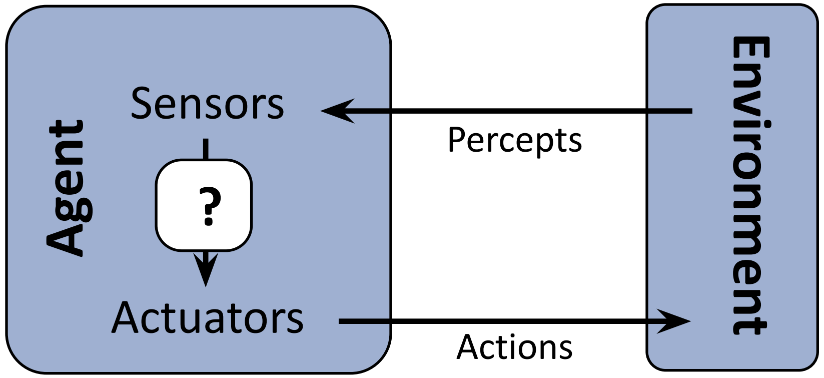
Agent Functions

- What do we need to know about an agent to consider decision-making?
- An agent may have **sensors** and **actuators**



Agent Functions

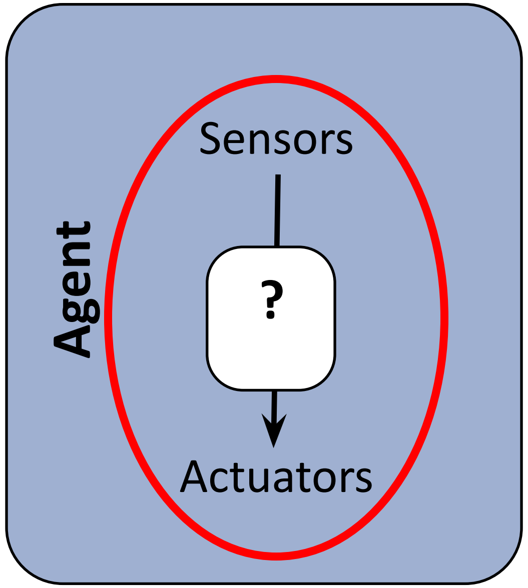
- What do we need to know about an agent to consider decision-making?
- An agent may have **sensors** and **actuators**
- Agent's actions depend on its percepts
- May even store an entire *percept sequence*
- An **agent function** maps percept sequences to action



Agent Programs

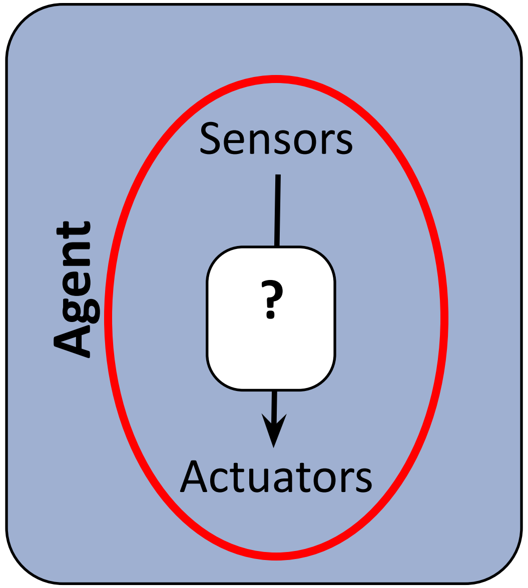
- **Agent programs** (percept to action) *implement* agent functions (percept sequence to action)

- One idea: Lookup table with all possible percept sequences



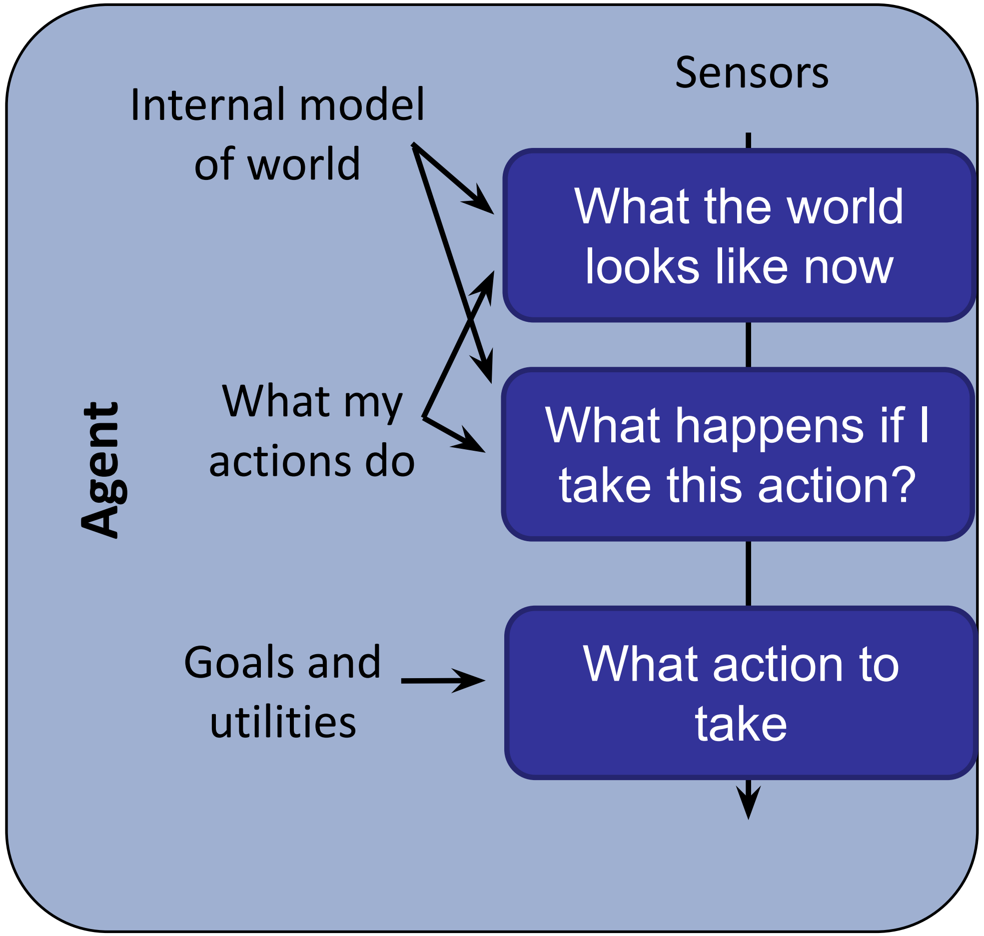
Agent Programs

- **Agent programs** (percept to action) *implement* agent functions (percept sequence to action)
- One idea: Lookup table with all possible percept sequences
- Program usefulness depends on hardware, limitations
- E.g., may be impossible to implement a program to solve chess on a slow PC



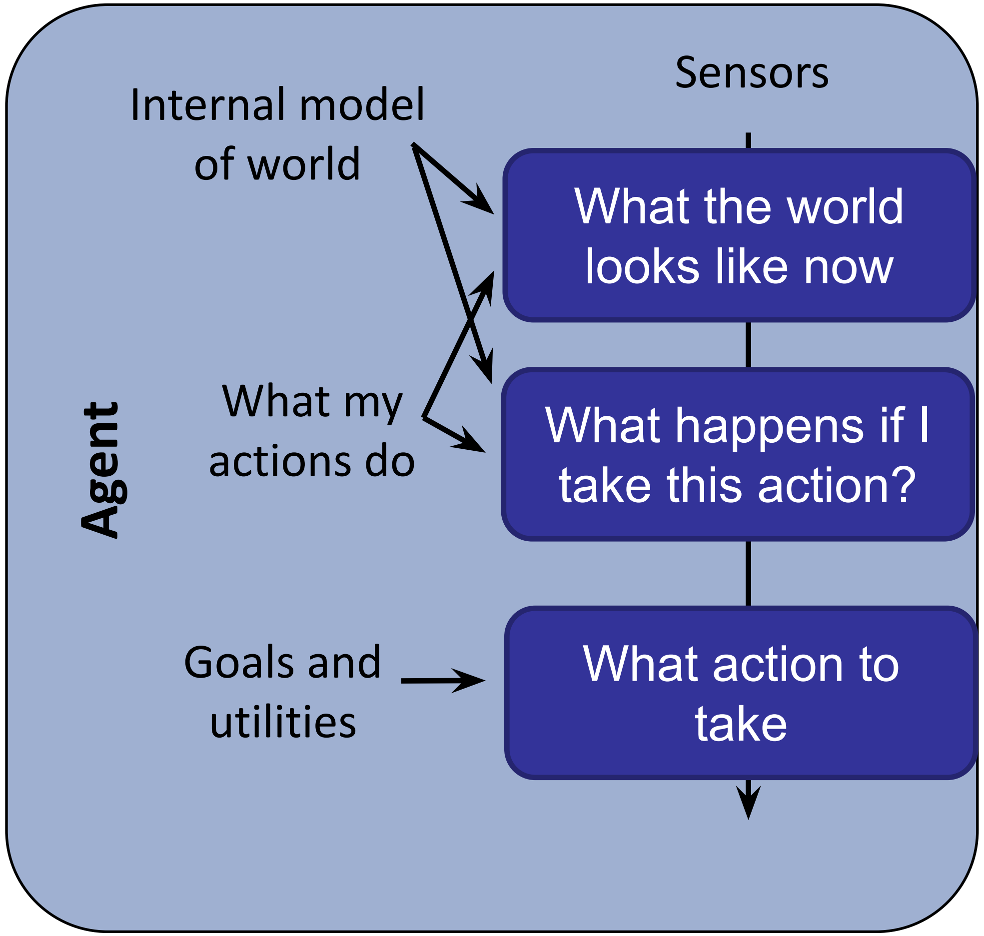
Goals and Utilities

- An agent program may need to store and use internal models of the world
- These models can help the agent update its state and consider action consequences



Goals and Utilities

- An agent program may need to store and use internal models of the world
- These models can help the agent update its state and consider action consequences
- Finally, we may have goals and utilities
- Decision-making is performed so as to achieve goals or maximize utilities



Utilities

- **Utility function** $U: S \to \mathbb{R}$: Mapping from state to real numbers

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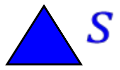
- Principle of **maximum expected utility**: A rational agent chooses actions so as to maximize *expected* utility, given its knowledge

Maximizing Expected Utility

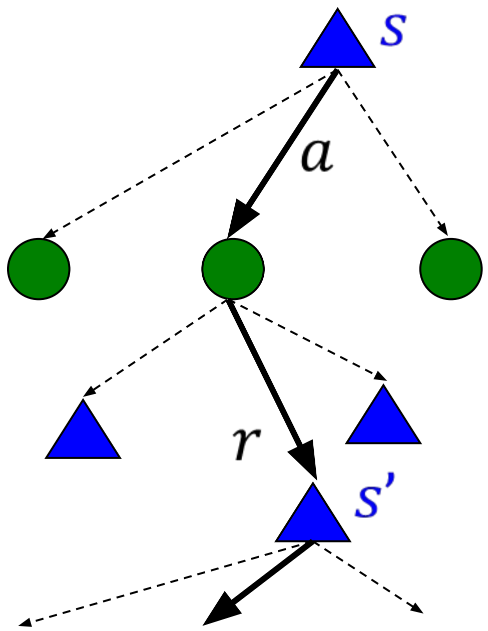
- MEU tells what an agent *should* do, but it doesn't solve the problem ☹

- Suppose our agent is currently in a state *s*



Maximizing Expected Utility

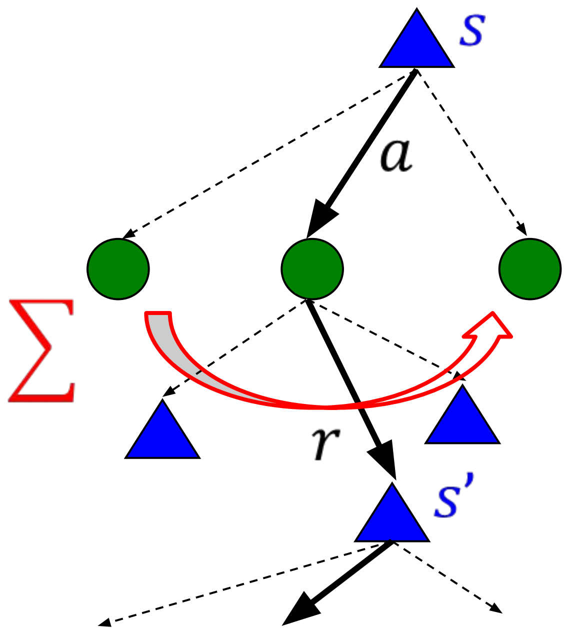
- MEU tells what an agent *should* do, but it doesn't solve the problem ☹
- Suppose our agent is currently in a state s
- If it takes action a, it may end up in one of several possible *successor states* s' according to a *transition model*



Maximizing Expected Utility

- The *expected utility of an action* a is the weighted average utility over all s':

$$EU(a) = \sum_{s'} \Pr(s')\, U(s')$$



Maximizing Expected Utility

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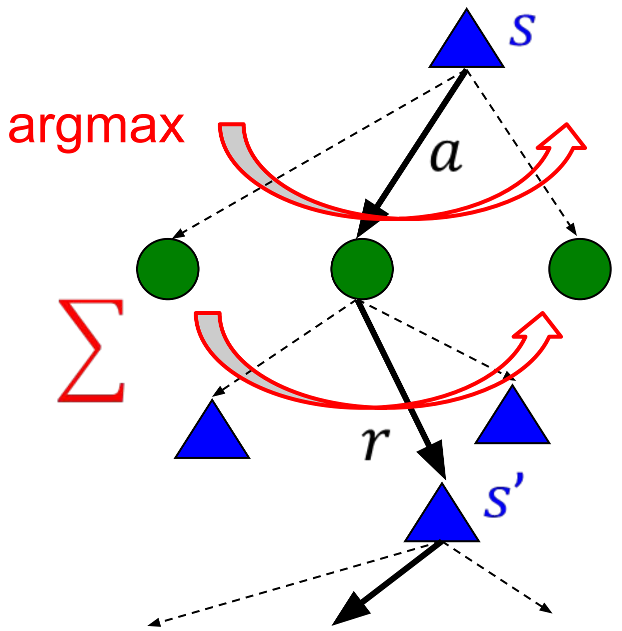
- To *maximize EU*, we choose the "best" action—easier said than done!

$$a^* = \underset{a}{\operatorname{argmax}}\, EU(a)$$



Outcomes and Lotteries

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- We can also have uncertainty among multiple states or outcomes

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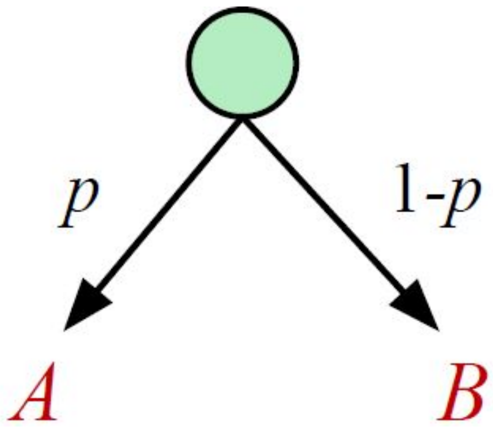
- We can also have uncertainty among multiple states or outcomes

- A **lottery** is a set of possible outcomes with associated probabilities
- An agent has preferences over both definite outcomes and lotteries



$$L = [p, A; 1 - p, B]$$

Rational Preferences

- We will use the following notation to express preferences:

Rational Preferences

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- *Rational* preferences must satisfy certain axioms!

- Orderability: $A \succ B$ or $B \succ A$ or $A \sim B$
- Transitivity: $A \succ B$ and $B \succ C$ implies $A \succ C$
- Continuity: $A \succ B \succ C$ implies $\exists p \; [p, A; 1-p, C] \sim B$

Axioms of Utility Theory

- *Rational* preferences must satisfy certain axioms!

- Substitutability: $A \sim B$ implies $[p, A; 1-p, C] \sim [p, B; 1-p, C]$
- Monotonicity: $A \succ B$ implies ($p > q$ iff $[p, A; 1-p, B] \succ [q, A; 1-q, B]$)

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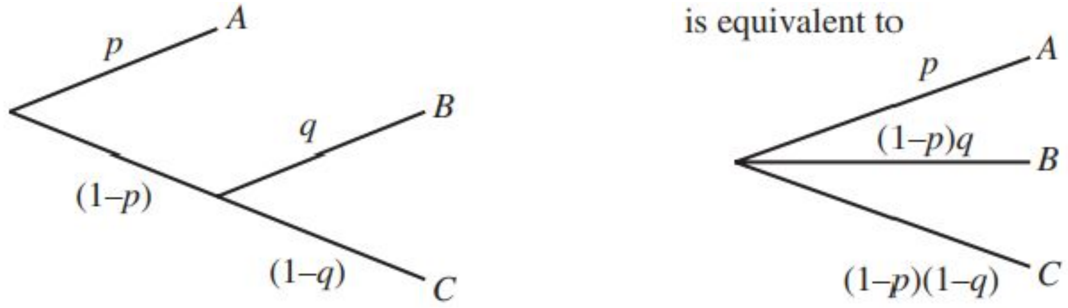
- Decomposability: $[p, A; 1-p, [q, B; 1-q, C]] \sim [p, A; (1-p)q, B; (1-p)(1-q), C]$



Irrational Preferences

- Preferences that do not preserve all the previous axioms may yield irrational behavior

- Suppose that an agent has preferences among three goods:
- $A \succ B, B \succ C, C \succ A$

- These are *intransitive* preferences
- Each one is incompatible with the other two

Irrational Preferences

- If the agent has good C, it would trade it away for good B for $X
- Now suppose it is offered good A; it would trade B away for A for $X
- It would do the same to retrieve good C
- The agent is back to where it started, less $3X!

Existence of Utilities

- von Neumann and Morgenstern, 1944: Given a set of outcomes $S_1, \ldots, S_n$ satisfying the preceding axioms, there exists a *utility function U* such that

$$U(S_i) \geq U(S_j) \;\Leftrightarrow S_i \gtrsim S_j$$

$$U[p_1, S_1; \ldots; p_n, S_n] = \sum_i p_i U(S_i)$$

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- Values assigned by U preserve preferences over prizes and lotteries

- U is not unique! Agent behaviors do not change if we replace U with a *positive affine transformation* of it: $U'(S) = aU(S) + b, a > 0$

Preference Elicitation

- Utility functions are guaranteed to exist, but how to come up with one?

- Suppose we have a *standard lottery* with *normalized utilities*:

$$[p, u_{\top}; 1 - p, u_{\bot}]$$

Preference Elicitation

- Utility functions are guaranteed to exist, but how to come up with one?

- Suppose we have a *standard lottery* with *normalized utilities*:

$$[p, u_{\top}; 1-p, u_{\perp}]$$

- $u_{\top} = 1$ corresponds to "best possible prize", $u_{\perp} = 0$ to "worst possible outcome"

- The utility of a prize S is the value p s.t. $S \sim [p, u_{\top}; 1-p, u_{\perp}]$

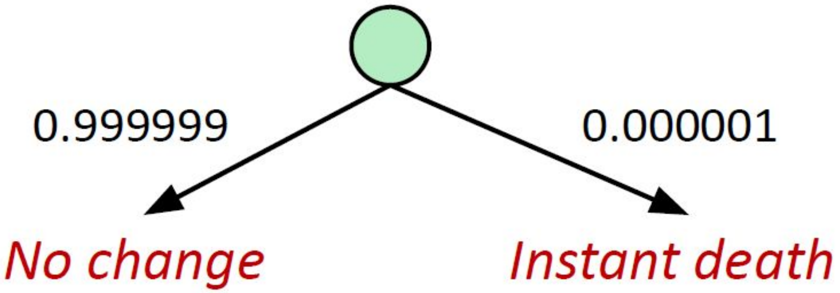
Preference Elicitation

$$S \sim [p, u_\top; 1 - p, u_\bot]$$

- Ex: Research has shown that people value a 1-in-a-million chance of death (a *micromort*) at about $50

- Many activities have an associated micromort assignment

$$Pay\ \$50 \qquad \sim$$



Utility of Money

- Money typically does not behave exactly like a utility function
- Consider the lotteries $L_1 = [0.5, \$2.1M; 0.5, \$0]$ vs $L_2 = [1, \$1M]$
- L_1 has a higher expected monetary value than L_2, but which would you choose?

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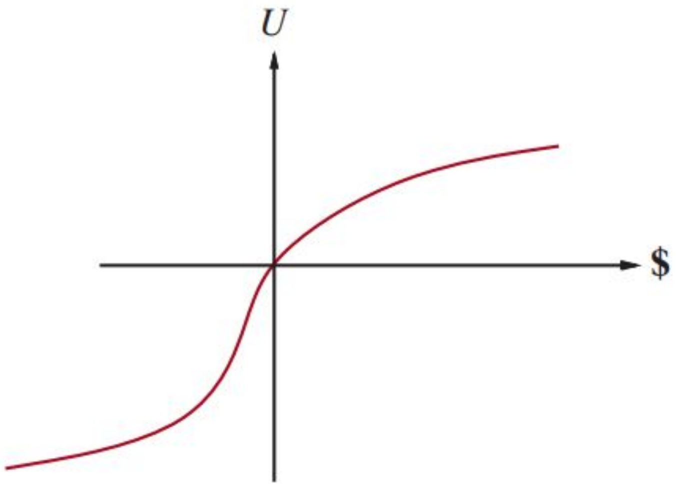
- L_1 has a higher expected monetary value than L_2, but which would you choose?
- Most people would choose L_2 because they are *risk-averse*
- Utilities increase more slowly than dollar amounts

Utility of Money

- A wealthier person may have a linear utility curve on a larger range
- Differences in risk acceptance give rise to insurance premiums



Utility of Money

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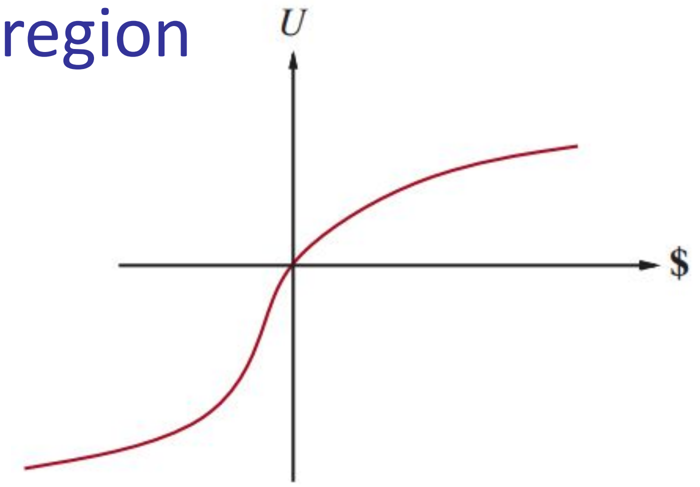
- Differences in risk acceptance give rise to insurance premiums
- People have concave utility curves for expensive products
- Curves for insurance companies are linear in the same region



Uncertain Utilities

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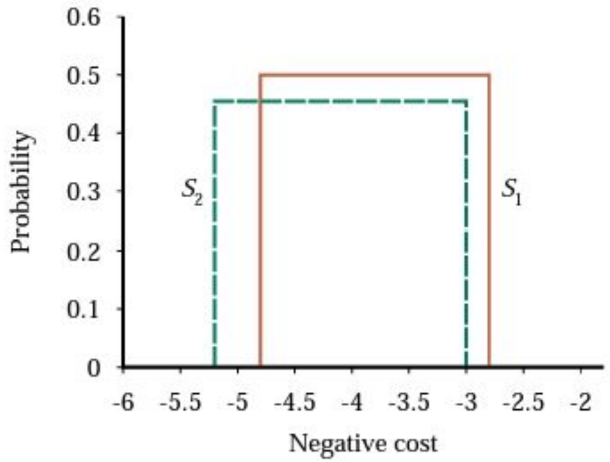
- We say that outcome S_1 **strictly dominates** outcome S_2 if $U(S_1) > U(S_2)$

Uncertain Utilities

- Probability density function (pdf)



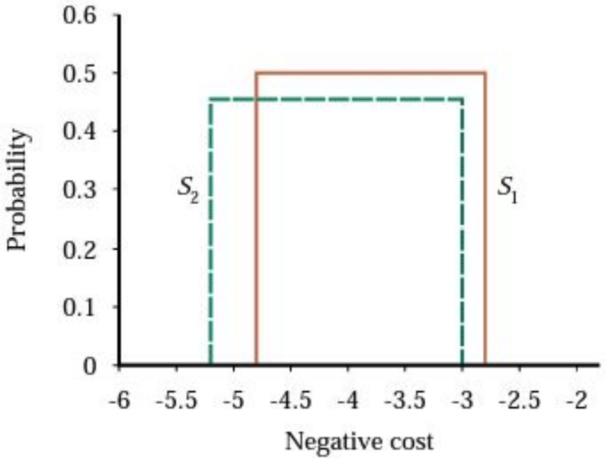
- We say that outcome S_1 **strictly dominates** outcome S_2 if $U(S_1) > U(S_2)$

- What if the utilities are uncertain and described as probability distributions $p_1(x)$ and $p_2(x)$ over an attribute X?

Uncertain Utilities

-

Probability density function (pdf)



- S_1 **stochastically dominates** S_2 if $\Pr(S_1 \geq x) \geq \Pr(S_2 \geq x)$ for all x

Uncertain Utilities

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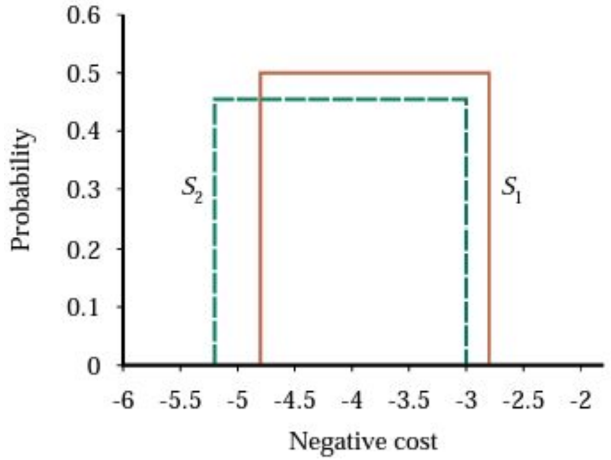

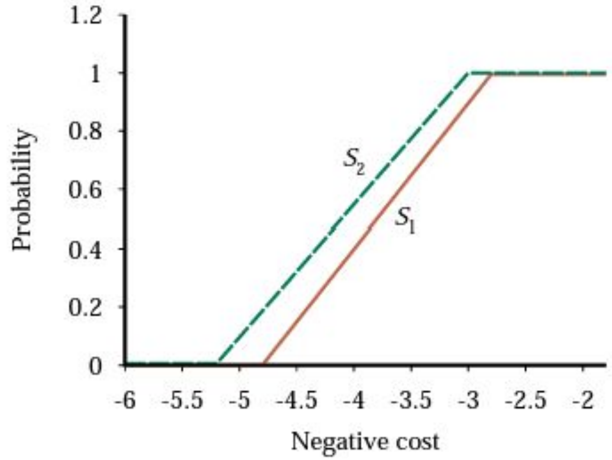

Cumulative distribution function (cdf)

- S_1 **stochastically dominates** S_2 if $\Pr(S_1 \geq x) \geq \Pr(S_2 \geq x)$ for all x

- The *cumulative distribution function* of S_1 is smaller than or equal to that of S_2 for all x

Multi-attribute Utilities

- An outcome may be described by *multiple* attributes $\boldsymbol{X} = X_1, \dots, X_n$

- E.g., job A: \$150k salary, 2 wks vacation; job B: \$130k salary, 4 wks vacation

Multi-attribute Utilities

- An outcome may be described by *multiple* attributes $\boldsymbol{X} = X_1, \dots, X_n$
- E.g., job A: \$150k salary, 2 wks vacation; job B: \$130k salary, 4 wks vacation
- An outcome is preferable to another if it is stochastically dominant across all attributes
- Otherwise, we may need a **multi-attribute utility function** $U(x_1, \dots, x_n)$

Multi-attribute Utilities

- The size of multi-attribute utility functions can grow *exponentially*
- If we have n attributes with d values each, this function must be defined for d^n values!

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- Special case: If attributes are **additive independent**, then we can write

$$U(x_1, \dots, x_n) = \sum_{i=1}^{n} k_i U_i(x_i)$$

- Uncertain attributes with weaker forms of independence may lead to *multiplicative* utility functions

Value of Information

- Suppose current best action for a problem is α: $EU(\alpha) = \max_a EU(a)$

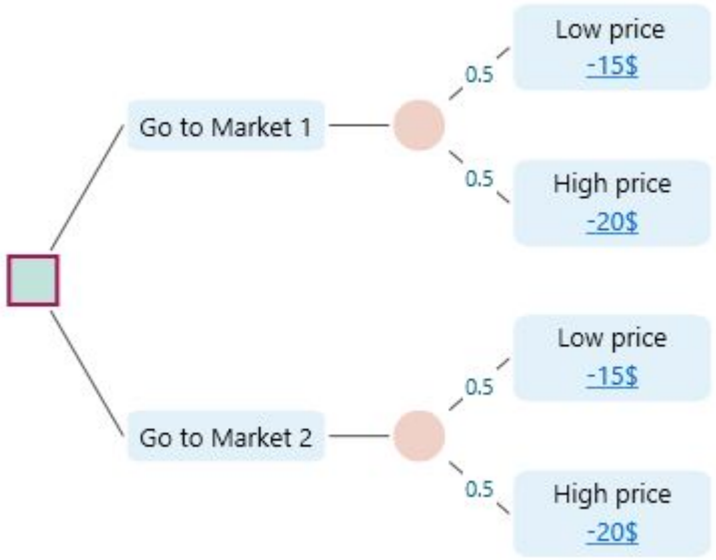
- Suppose we can learn new information that may change α

Value of Information

- Suppose current best action for a problem is α: $EU(\alpha) = \max_a EU(a)$
- Suppose we can learn new information that may change α
- We might learn something about a *random variable* E with possible outcomes e_i
- We may change our action and thus utility depending on e_i

Value of Information

- For each value $E = e_i$, the (possibly new) best action is α_i: $EU(\alpha_i|e_i) = \max_a EU(a|e_i)$



Value of Information

- For each value $E = e_i$, the (possibly new) best action is α_i: $EU(\alpha_i | e_i) = \max_a EU(a | e_i)$

- **Value of perfect information (VPI)** is the *expected* improvement in expected utility:

$$VPI(E) = \left(\sum_e \Pr(E = e) EU(\alpha_e | e) \right) - EU(\alpha)$$

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Example: VPI

- An oil company is trying to choose one of n possible drilling sites
- Each site may contain oil with probability $\frac{1}{n}$
- If the net profit of finding oil is C, then the EU of drilling in any site is C/n

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- An oil company is trying to choose one of n possible drilling sites
- Each site may contain oil with probability $\frac{1}{n}$
- If the net profit of finding oil is C, then the EU of drilling in any site is C/n
- A seismologist offers to survey one site and definitively find out if it contains oil
- What are the possible outcomes?

Example: VPI

- If oil is found ($p = 1/n$), best action is to drill there to obtain utility C
- If oil is *not* found ($p = \frac{n-1}{n}$), best action is to *not* drill there to obtain expected utility $\frac{C}{n-1}$

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- The new EU is thus $\frac{1}{n} \times C + \frac{n-1}{n} \times \frac{C}{n-1} = \frac{2C}{n}$

- The VPI is the *difference* between new and old EU: $VPI = \frac{2C}{n} - \frac{C}{n} = \frac{C}{n}$

Properties of VPI

- Similar analysis can be applied in any information gathering scenario

- Ex: Should a doctor order more tests to be done on a patient?
- Ex: Should an investment firm hire a consultant to better understand the market?

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- Theorem: VPI is non-negative; it is never disadvantageous to acquire more information
- Maximization of VPI can be used to design an *information-gathering agent*

Summary

- An agent's underlying preferences must satisfy certain axioms in order to be considered rational
- Rational preferences lead to utility functions; we maximize expected utility
- Utilities may be uncertain or may be described by multiple attributes
- Value of information can be quantified by expected gain in expected utility