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Object: CQF Module 4 Exam
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1) In this problem we use continuous compounding.

   a) We have: $Z(t=0; T=3) = 82 = 100e^{-3y}$
   Solving for y give us:

$$\begin{aligned} y &= -\frac{1}{3}\log(\frac{82}{100}) \\ &= 6.615\% \end{aligned}$$

   Duration:

$$\begin{aligned} D &= -\frac{1}{Z}\frac{\partial Z}{\partial y} \\ &= -\frac{1}{100e^{-3y}} \times -300e^{-3y} \\ &= 3 \end{aligned}$$

   Convexity:

$$\begin{aligned} C &= \frac{1}{Z}\frac{\partial^2 Z}{\partial y^2} \\ &= \frac{1}{Z}\frac{\partial Z}{\partial y}[-300e^{-3y}] \\ &= \frac{1}{100e^{-3y}}900e^{-3y} \\ &= 9 \end{aligned}$$

   b) Calculation for this question is provided in the attached C++ code file **bondTest.cpp**.
   $V = 3 \times [e^{-y} + e^{-2y} + e^{-3y} + e^{-4y} + e^{-5y}] + 100e^{-5y} = 90.$
   Solving for y, we obtain $y = 5.2\%$.

$$\begin{aligned} D &= -\frac{1}{V}\frac{\partial V}{\partial y} \\ &= -\frac{1}{V} \times 3 \times \big[-e^{-y} - 2e^{-2y} - 3e^{-3y} - 4e^{-4y} - 5e^{-5y}\big] - 5 \times 100e^{-5y} \\ &= 4.7 \\ C &= \frac{1}{V}\frac{\partial^2 V}{\partial y^2} \\ &= \frac{1}{V} \times 3 \times \big[e^{-y} + 2e^{-2y} + 9e^{-3y} + 16e^{-4y} + 25e^{-5y}\big] + 25 \times 100e^{-5y} \\ &= 22.91 \end{aligned}$$

2) Consider the Black-Derman & Toy (BDT) short-rate model given by:

$$d(\log r) = \left[\theta(t) - \frac{d[\log(\sigma(t)]}{dt}\log(r)\right]dt + \sigma(t)dX$$

Consider the function $f(X) = \exp(X) = r$ and applying $It\hat{o}$ lemma to this function:

$$\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2 \\
&= 0 + \exp(X)dX + \frac{1}{2}\exp(X)(dX)^2 \\
&= rd[\log(r)] + \frac{1}{2}r\left[d[\log(r)]\right]^2 \\
&= rd[\log(r)] + \frac{1}{2}r\left[\left(\theta(t) - \frac{d[\log(\sigma(t)]}{dt}\log(r)\right)dt + \sigma(t)dX\right]^2 \\
&= rd[\log(r)] + \frac{1}{2}r\sigma^2 dt \\
&= r\left[\theta(t) - \frac{d[\log(\sigma(t)]}{dt}\log(r) + \frac{1}{2}\sigma^2\right]dt + r\sigma(t)dX \\
&= A(r,t)dt + B(r,t)dX
\end{aligned}$$

with:

$$\begin{aligned}
A(r,t) &= r\left[\theta(t) - \frac{d[\log(\sigma(t)]}{dt}\log(r) + \frac{1}{2}\sigma^2\right] \\
B(r,t) &= r\sigma(t)
\end{aligned}$$

3) Consider the spot rate r, which evolves according to the SDE:

$$dr(t) = u(r,t)dt + w(r,t)dX$$

The extended Hull and White model has drift and diffusion:

$$\begin{aligned}
dr(t) &= u(r,t)dt + w(r,t)dX \\
&= [\eta_t - \gamma r]dt + cdX
\end{aligned}$$

Thus the pricing equation is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}c^2\frac{\partial^2 V}{\partial r^2} + [\eta_t - \gamma r]\frac{\partial V}{\partial r} - rV = 0$$

And the final condition for a zero-coupon bond is $Z(r,T;T) = 1$, and look for a solution of the form $Z = \exp[A(t) - rB(t)]$.

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\exp[A(t) - rB(t)]\right] &= [\dot{A}(t) - r\dot{B}(t)]Z \\
\frac{\partial}{\partial r}\left[\exp[A(t) - rB(t)]\right] &= -B(t)Z \\
\frac{\partial^2}{\partial r^2}\left[\exp[A(t) - rB(t)]\right] &= B(t)^2 Z
\end{aligned}$$

Substituting these expressions into the pricing equation:

$$\begin{aligned}(\dot{A}(t) - r\dot{B}(t))Z + \frac{1}{2}c^2B(t)^2Z + (\eta_t - \gamma r)B(t)Z - rZ &= 0\\ (\dot{A}(t) - r\dot{B}(t)) + \frac{1}{2}c^2B(t)^2 - (\eta_t - \gamma r)B(t) - r &= 0\end{aligned}$$

Grouping the terms, we have an expression that is linear in r:

$$[\dot{A}(t) + \frac{1}{2}c^2B(t)^2 - \eta_t B(t)] + r[-\dot{B}(t) + \gamma B(t) - 1] = 0$$

Both of the expressions in parentheses must be zero.

We have two first order ordinary differential equations for $A(t)$ and $B(t)$:

$$\begin{cases}\dot{B}(t) - \gamma B(t) = -1\\ \dot{A}(t) + \frac{1}{2}c^2B(t)^2 - \eta_t B(t) = 0\end{cases}$$

In order for the final condition at $t = T$ to be satisfied we need

$$\begin{cases}\exp[A(T) - rB(T] = 1\\ A(T) - rB(T) = 0 \ \forall r\end{cases}$$

and so $A(T) = B(T) = 0$.
Solving for $B(t)$:

$$\begin{aligned}\dot{B}(t) - \gamma B(t) &= -1\\ \exp[-\gamma t]\dot{B}(t) - \exp[-\gamma t]\gamma B(t) &= -1 \times \exp[-\gamma t]\\ \frac{\partial}{\partial t}\Big[\exp[-\gamma t]B(t)\Big] &= -1 \times \exp[-\gamma t]\\ \int_t^T \Big[\exp[-\gamma t]B(t)\Big] &= -\int_t^T \exp[-\gamma t]\\ \exp[-\gamma T]B(T) - \exp[-\gamma t]B(t) &= \frac{1}{\gamma}\big[\exp[-\gamma T] - \exp[-\gamma t]\big]\end{aligned}$$

The solution is:

$$B(t;T) = \frac{1}{\gamma}[1 - \exp[-\gamma(T - t)]]$$

Solving for $A(t;T)$:

$$
\begin{aligned}
\dot{A}(t)+\frac{1}{2}c^2B(t)^2-\eta_tB(t) &= 0\\
\frac{dA}{dt} &= -\frac{1}{2}c^2B(t)^2+\eta_tB(t)\\
A(T;T)-A(t;T) &= -\int_t^T\left[\frac{1}{2}c^2B(s)^2+\eta_sB(s)\right]ds\\
A(t;T) &= \int_t^T\frac{1}{2}c^2\left[\frac{1}{\gamma}\left[1-e^{-\gamma(T-s)}\right]\right]^2ds-\int_t^T\eta_sB(s)ds\\
A(t;T) &= \frac{c^2}{2\gamma^2}\int_t^T\left[1-e^{-\gamma(T-s)}\right]^2ds-\int_t^T\eta_sB(s)ds\\
A(t;T) &= \frac{c^2}{2\gamma^2}\int_t^T\left[1-2e^{-\gamma(T-s)}+e^{-2\gamma(T-s)}\right]ds-\int_t^T\eta_sB(s)ds\\
A(t;T) &= \frac{c^2}{2\gamma^2}\left[(T-t)-2\int_t^T e^{-\gamma(T-s)]}ds+\int_t^T e^{-2\gamma(T-s)}ds-\int_t^T\eta_sB(s)ds\right.\\
A(t;T) &= \frac{c^2}{2\gamma^2}\left[(T-t)-\frac{2}{\gamma}\left[1-e^{-\gamma(T-t)}\right]+\frac{1}{2\gamma}\left[1-e^{-2\gamma(T-t)}\right]-\int_t^T\eta_sB(s)ds\right.\\
A(t;T) &= \frac{c^2}{2\gamma^2}\left[(T-t)-\frac{2}{\gamma}+\frac{2}{\gamma}e^{-\gamma(T-t)}+\frac{1}{2\gamma}-\frac{1}{2\gamma}e^{-2\gamma(T-t)}\right]-\int_t^T\eta_sB(s)ds\\
A(t;T) &= \frac{c^2}{2\gamma^2}\left[(T-t)+\frac{2}{\gamma}e^{-\gamma(T-t)}-\frac{1}{2\gamma}e^{-2\gamma(T-t)}-\frac{4}{2\gamma}+\frac{1}{2\gamma}\right]-\int_t^T\eta_sB(s)ds\\
A(t;T) &= \frac{c^2}{2\gamma^2}\left[(T-t)+\frac{2}{\gamma}e^{-\gamma(T-t)}-\frac{1}{2\gamma}e-2\gamma(T-t)-\frac{3}{2\gamma}\right]-\int_t^T\eta_sB(s)ds\\
A(t;T) &= \frac{c^2}{2\gamma^2}\left[(T-t)+\frac{2}{\gamma}e^{-\gamma(T-t)}-\frac{1}{2\gamma}e^{-2\gamma(T-t)}-\frac{3}{2\gamma}\right]-\int_t^T\eta(s)B(s;T)ds
\end{aligned}
$$

4) Consider the process given by:

$$
dU_t \;=\; -\gamma U_tdt+\sigma dX_t,\ U_0=u
$$

Consider the function: $Y(t)=U_t\exp(\gamma t)$

$$
\begin{aligned}
dY(t) &= dU_t\exp(\gamma t)+\gamma U_t\exp(\gamma t)dt\\
&= [-\gamma U_tdt+\sigma dX_t]\exp(\gamma t)+\gamma U_t\exp(\gamma t)dt\\
&= -\gamma U_t\exp(\gamma t)dt+\sigma\exp(\gamma t)dX_t+\gamma U_t\exp(\gamma t)dt\\
&= \sigma\exp(\gamma t)dX_t
\end{aligned}
$$

Intergrating $Y(t)$ between 0 and t give us:

$$
Y(t)-Y(0) \;=\; U_t\exp(\gamma t)-U_0=\sigma\int_0^t\exp(\gamma u)dX(u)
$$

Multiplying by both sides by $\exp(-\gamma t)$ and rearranging give us:

$$\begin{aligned}
U_t &= U_0 \exp(-\gamma t) + \sigma \exp(-\gamma t) \int_0^t \exp(\gamma u) dX(u) \\
\mathbb{E}(U_t) &= \mathbb{E}\Big[U_0 \exp[-\gamma t] + \sigma \exp(-\gamma t) \int_0^t \exp(\gamma u) dX(u)\Big] \\
\mathbb{E}(U_t) &= \mathbb{E}\big[U_0 \exp[-\gamma t]\big] + \mathbb{E}\big[\sigma \exp(-\gamma t) \int_0^t \exp(\gamma u) dX(u)\big] \\
\mathbb{E}(U_t) &= \mathbb{E}\big[U_0 \exp[-\gamma t]\big] + 0 \\
\mathbb{E}(U_t) &= U_0 \exp[-\gamma t]
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}(U_t) &= \mathbb{V}\Big[U_0 \exp(-\gamma t) + \sigma \exp(-\gamma t) \int_0^t \exp(\gamma u) dX(u)\Big] \\
&= \mathbb{V}\big[\sigma \exp(\gamma t) \int_0^t \exp(\gamma u) dX(u))\big] \\
&= \sigma^2 \exp[-2\gamma t] \mathbb{V}\Big[\int_0^t \exp(\gamma u) dX(u)\Big] \\
&= \sigma^2 \exp[-2\gamma t] \Big[\int_0^t \mathbb{E}[\exp(2\gamma u)] du\Big] \\
&= \sigma^2 \exp[-2\gamma t] \Big[\int_0^t \exp(2\gamma u) du\Big] \\
&= \sigma^2 \exp[-2\gamma t] \frac{1}{2\gamma} \Big[\int_0^{2\gamma t} \exp(u) du\Big] \\
&= \sigma^2 \exp[-2\gamma t] \frac{1}{2\gamma} \Big[\exp[2\gamma t] - 1\Big] \\
&= \frac{1}{2\gamma} \sigma^2 \Big[1 - \exp[-2\gamma t]\Big]
\end{aligned}$$

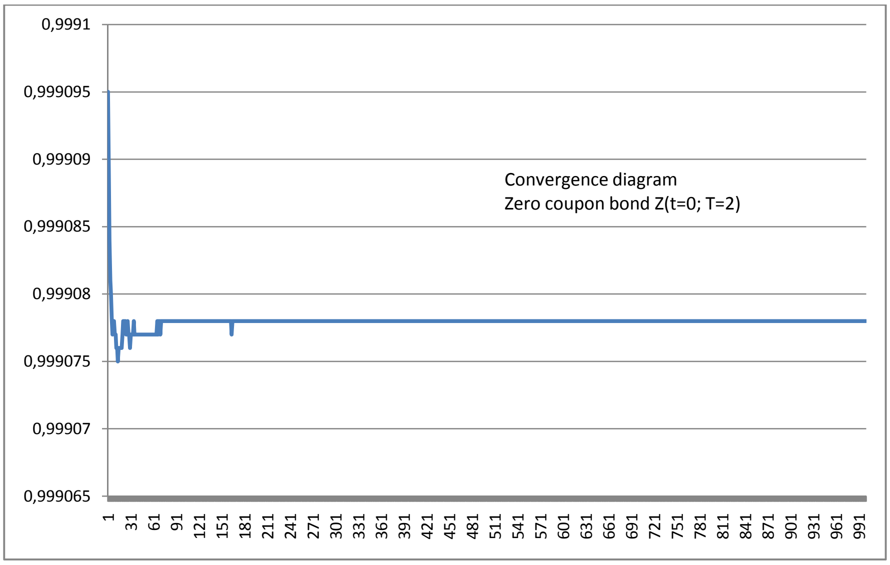
5)

$$\begin{aligned}
dZ &= r(t) Z dt \\
\frac{dZ}{Z} &= r(t) dt \\
d \log(Z) &= r(t) dt \\
\int_t^T d \log(Z) &= \int_t^T r(s) ds \\
\log(Z(r, T; T)) - \log(Z(r, t; T)) &= \int_t^T r(s) ds \\
0 - \log(Z(r, t; T)) &= \int_t^T r(s) ds \\
Z(r, t; T) &= \exp\left(-\int_t^T r(s) ds\right)
\end{aligned}$$

Taking the expectation under the risk-neutral measure, we have:
$Z(r,t;T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right]$

My focus was on the calculation from the provided file **HJM model - MC.xlsm**, re-coded en C++. I obtained the following zero coupon bond price with $t = 0$ and maturity $T = 2$ years: $Z(0;2) = 0.999078$

The convergence diagram for the zero coupon bond $Z(0;2)$, on the next page, was generated from the file **HJM_MC_convergence_diagram.csv** produced by the test in **HJMTest.cpp**.


Convergence diagram
Zero coupon bond Z(t=0; T=2)
0,9991
0,999095
0,99909
0,999085
0,99908
0,999075
0,99907
0,999065
1
31
61
91
121
151
181
211
241
271
301
331
361
391
421
451
481
511
541
571
601
631
661
691
721
751
781
811
841
871
901
931
961
991