Determining the path traced by an orbiting satellite relative to the Earth's surface

Satellites have many useful functions in the modern day. Many of these, such as meteorology, communications, navigation, and cartography, depend largely on the satellites' positions relative to points on Earth's surface, because the area on the Earth that the satellite is capable of monitoring depends on where in the sky the satellite is. This means that knowing the geographical location of the satellite (i.e. the satellite's coordinates on a map) is extremely important.

Unfortunately, satellite's don't usually trace simple paths on a map: they follow elliptical paths in orbit around the globe, constantly in freefall at very high speeds. To make matters worse, the altitudes and velocities of these satellites are constantly changing, and the Earth itself is rotating in place (Montenbruck and Gill, 2012). In this investigation, I will devise equations for calculating and predicting the geographic coordinates of the point on the Earth's surface that is radially below a satellite. The equations for satellites in elliptical orbits are very complex, because in an elliptical orbit the satellite's speed is not constant, which requires advanced physics to properly model. In fact, they result in a differential equation that can not be solved algebraically. Due to this complexity, as well as length restrictions, I will only address circular orbits: orbits whose altitude and velocity never change – a condition that makes the equations solvable. There are many real-life satellites that have near-circular orbits, meaning that my equations can accurately predict their motion by making the assumption that they are perfectly circular. To demonstrate this, I'll use the equations to model the path traced over the Earth's surface by the International Space Station, a real satellite with a near-circular orbit.

Before I begin, I will briefly define the system of geographical coordinates that will be used throughout this investigation. These coordinates are expressed in terms of two angles, latitude and longitude:

- Latitude is the "North-South" angle – it is zero at the equator, positive in the Northern Hemisphere, and negative in the Southern Hemisphere. At the North Pole latitude is 90° (π/2 radians), and at the South Pole it is -90° (-π/2).
- Longitude is the "East-West" angle – it is zero at the prime meridian, positive in the Eastern Hemisphere, and negative in the Western Hemisphere. The longitude at the antimeridian (the line on the opposite side of the map from the prime meridian) is both 180° and -180° (π and -π radians).

In the interest of simplicity, I will express these angles exclusively in radians for the rest of the investigation.

Arguably the simplest possible orbit a satellite can have is a circular equatorial orbit. This is a special case of a circular orbit where the satellite never strays north or south of the equator – that is, its latitude is constant at 0. A circular equatorial orbit's longitude changes at a rate equal to the angular speed of the satellite relative to the Earth's surface. This can be calculated as the difference between the angular speed of the satellite and the angular speed of the Earth's rotation. The equations for this satellite's coordinates over time are therefore:

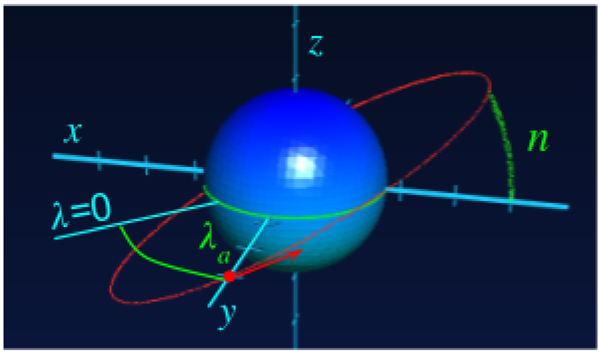
$$\phi = 0$$

$$\lambda = \lambda_0 + (\omega_s - \omega_E)t$$

I will consistently use λ to represent longitude, ϕ to represent latitude, and t to represent time. There are a few other quantities worth discussing:

- λ_0 is the satellite's longitude at $t = 0$
- ω_s is the angular speed of the satellite
- ω_E is the angular speed of the Earth. Since the Earth rotates once per 24 hours, its angular speed is $2\pi \div (24 \times 60 \times 60) = 7.27 \times 10^{-5}$ radians per second.

Though these equations are simple, they suggest an interesting phenomenon: when $\omega_s = \omega_E$, both latitude and longitude are constant functions, $\phi = 0$ and $\lambda = \lambda_0$. This specific type of circular equatorial orbit is called a geostationary orbit, and it is useful for "assigning" a satellite to constantly hover above a location near the equator (Montenbruck and Gill, 2012). However, this can only be done with an equatorial orbit: most cases are more complex. For instance, take a satellite orbiting in a circle at an angle n to the equator:



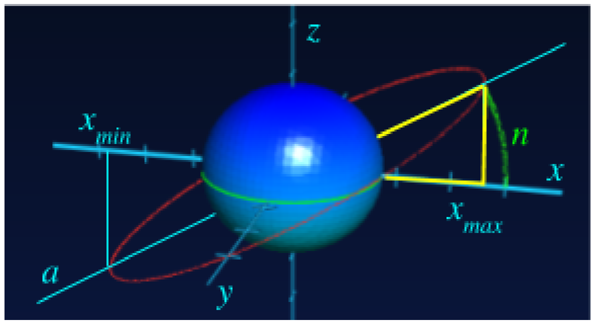
I'll take some time to describe the elements of the diagram. First, the red circle is the path taken by the satellite in its orbit, and the plane it lies in is called the *orbital* plane. The red dot is the satellite's position at $t = 0$, and the red arrow shows its direction. The green line on the planet is the equator, and the x-y plane is called the *equatorial* plane. The z-axis is the axis of the planet's rotation: 'up', or positive z, is North and 'down', or negative z, is south.

The line labelled $\lambda = 0$ points towards the intersection of the equator and the prime meridian: it's the point on the equator where longitude is zero. The angle λ_a is the longitude of the point where the satellite crosses the equatorial plane from negative z to positive z; this point is known as the ascending node, and λ_a is the longitude of the ascending node at $t = 0$. For simplicity of calculations, I've defined the y-axis so that the ascending node lies on it (and therefore y is the line of intersection between the orbital and equatorial planes). The angle n (shown in green) is the *inclination* of the orbital plane relative to the equatorial plane.

We'll calculate the satellite's coordinates as a function of θ, the angle between the satellite and the ascending node, and r, the orbital radius. Since this is a circular orbit, θ increases at a constant speed ω_s and r is constant. Also, θ is zero at $t = 0$, since we are measuring time starting at the ascending node for simplicity.

The first step to finding the latitude and longitude of the satellite is finding its Cartesian (x, y, z) coordinates. The easiest of the three is y: since the y-axis lies in the orbital plane, the y-coordinate is just the cartesian coordinate of a point on a circle, meaning y is a sinusoidal function of θ. As the satellite travels π radians around its circular orbit, y varies from r to $-r$: this is a cosine relationship, so $y = r\cos(\theta)$. Regardless of the orbit's inclination n, this will be true as the y-axis will still lie in the orbital plane.

However, the x and z positions depend on the inclination (n) of the orbit. When $n = 0$, the circle lies in the equatorial x-y plane, so the x-coordinate is $r\sin(\theta)$, and the z-coordinate is constant at zero. The general case for other values of n is more complicated, and I'll illustrate it using a diagram:



I've drawn on the diagram a blue line labelled a – call it the a-axis. The a-axis is perpendicular to the y-axis and lies in the orbital plane, so that the angle between the a-axis and the x-axis is n (shown in green on the diagram). As θ changes, the satellite moves in a circle in the a-y plane, where $y = r\cos(\theta)$ and $a = r\sin(\theta)$. There is a right-angled triangle between the a-coordinate, the x-coordinate, and the origin, shown in yellow. The hypotenuse of this triangle has length a, so $x = a\cos(n)$ and $z = a\sin(n)$. Substituting the value of a gives us $x = r\sin(\theta)\cos(n)$ and $z = r\sin(\theta)\sin(n)$.

As time passes, θ increases at a rate of ω_s radians per second, so $\theta = \omega_s \times t$. Thus, the cartesian coordinates of the satellite are:

$$x = r\sin(\omega_s t)\cos(n)$$
$$y = r\cos(\omega_s t)$$
$$z = r\sin(\omega_s t)\sin(n)$$

We must next find equations relating x, y, and z to latitude and longitude. Below is a diagram showing the latitude, longitude, and Cartesian coordinates of a point P at altitude r above the center of the Earth. The ascending node is located at $\lambda = \lambda_a,\ \phi = 0$, or, alternatively, on the y-axis at $(0,\ r,\ 0)$ in Cartesian coordinates.

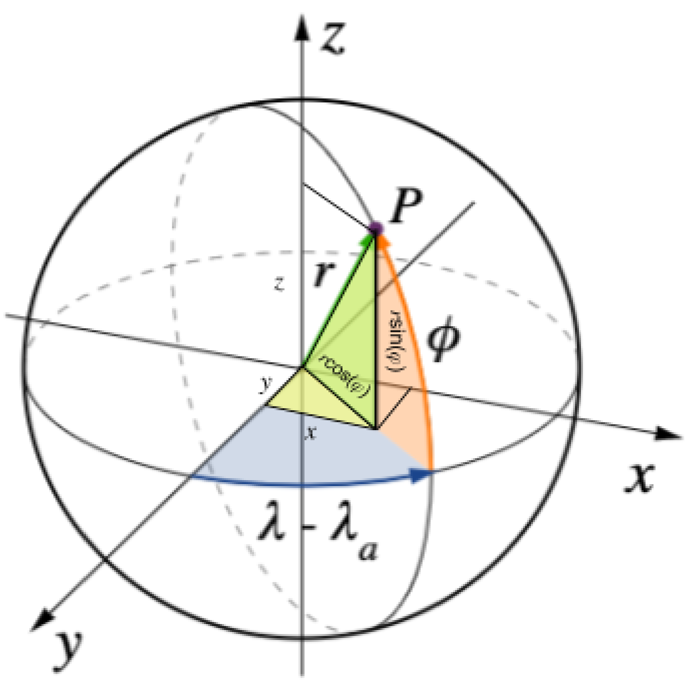


Image modified from work of wikimedia user Andeggs (who released it into the public domain)

From the diagram, the green right-angled triangle makes it clear that $\sin(\phi) = \frac{z}{r}$, so:

$$z = r\sin(\phi)$$

The shorter side of the green triangle is therefore $r\cos(\phi)$. This line segment is also the hypotenuse of the yellow triangle. This means that we can derive:

$$x = r\cos(\phi)\sin(\lambda - \lambda_a)$$
$$y = r\cos(\phi)\cos(\lambda - \lambda_a)$$

from right-angle trigonometric ratios. We can equate these to our previous equations for the satellite's position:

$$x = r\sin(\omega_s t)\cos(n) = r\cos(\phi)\sin(\lambda - \lambda_a)$$
$$y = r\cos(\omega_s t) = r\cos(\phi)\cos(\lambda - \lambda_a)$$
$$z = r\sin(\omega_s t)\sin(n) = r\sin(\phi)$$

We can divide everything by r, and then solve for λ and ϕ. The z equation gives us ϕ easily:

$$\tfrac{z}{r} = \sin(\omega_s t)\sin(n) = \sin(\phi)$$
$$\phi = \arcsin(\sin(\omega_s t)\sin(n))$$

And the other two equations give us λ:

$$\tfrac{y}{r} = \cos(\omega_s t) = \cos(\phi)\cos(\lambda - \lambda_a)$$
$$\tfrac{x}{r} = \sin(\omega_s t)\cos(n) = \cos(\phi)\sin(\lambda - \lambda_a)$$
$$\cos(\phi) = \frac{\cos(\omega_s t)}{\cos(\lambda - \lambda_a)} = \frac{\sin(\omega_s t)\cos(n)}{\sin(\lambda - \lambda_a)}$$
$$\frac{\sin(\lambda - \lambda_a)}{\cos(\lambda - \lambda_a)} = \frac{\sin(\omega_s t)\cos(n)}{\cos(\omega_s t)}$$
$$\tan(\lambda - \lambda_a) = \tan(\omega_s t)\cos(n)$$
$$\lambda = \lambda_a + \arctan(\tan(\omega_s t)\cos(n))$$

This equation gives us the satellite's longitude relative to the Earth's center, but the Earth's surface (which we want to use as our frame of reference for calculating longitude) is rotating at a constant angular speed. To account for this, we can add a longitude shift of $-\omega_E t$:

$$\lambda = \lambda_a + \arctan(\tan(\omega_s t)\cos(n)) - \omega_E t$$

These equations are central to this paper, so I will denote them as equations 1 and 2:

Equation 1 : $\phi = \arcsin(\sin(\omega_s t)\sin(n))$

Equation 2 : $\lambda = \lambda_a + \arctan(\tan(\omega_s t)\cos(n)) - \omega_E t$

This shows that if the satellite were not to move at all, its longitude would still decrease at a constant rate as the Earth's surface moves below it (ω_E is positive, because a point on the Earth's surface moves eastward – that's why at morning the sunlight appears in the east, and why at nightfall the darkness spreads from the east).

We can illustrate the behavior of these equations by graphing them. The equations depend on multiple parameters: n, ω_E, ω_s, and λ_a. For the sake of simplicity, we can set λ_a to zero, since it only represents a constant shift in longitude. The angular speed of the satellite is also rather arbitrary for graphing purposes, so we will just set $\omega_s = 1$. We will assign a meaningful value to ω_s when we find the equations for a real-life satellite.

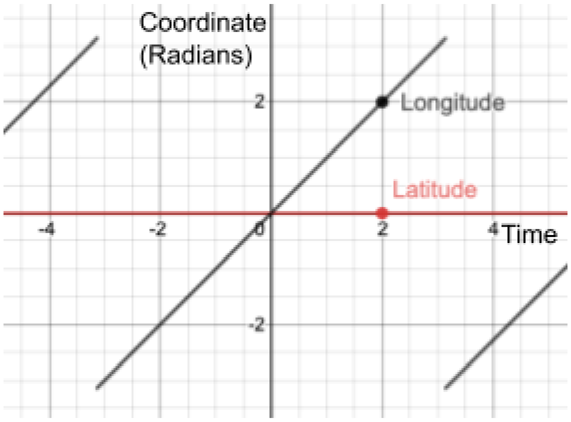
The parameters n and ω_E have a large impact on the model's behavior. We'll start by considering different values of n where $\omega_E = 0$ – that is, the Earth is not rotating. From equation 1:

$$\phi = \arcsin(\sin(1 \times t)\sin(0))$$
$$\phi = \arcsin(0) = 0$$

And equation 2:

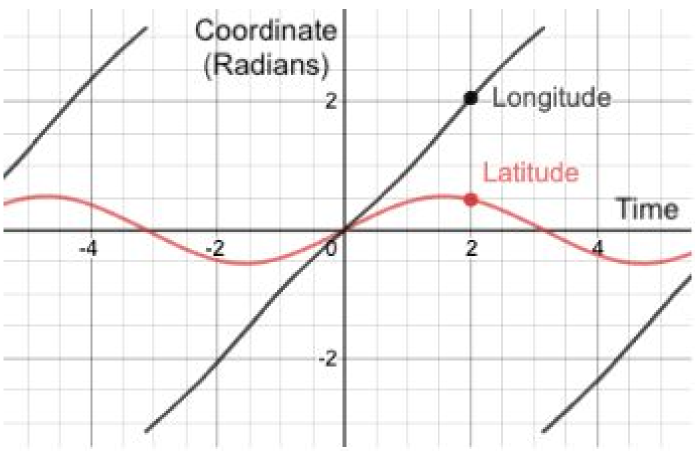
$$\lambda = 0 + \arctan(\tan(1 \times t)\cos(0)) - 0 \times t$$
$$\lambda = t$$

These are actually just a specific case of the circular equatorial orbit I discussed earlier. When graphed:



As shown in the graph, latitude stays constant at zero, and longitude increases linearly. Since $\omega_s = 1$, the satellite orbits at 1 radian per time unit, it will travel from 0 to π radians of longitude in π time units, as shown above. Since a turn of π radians represents a half rotation around the world, the graph shows a discontinuous jump, analogous to the satellite disappearing off the eastern side of a map, and reappearing on the western side. This means that the range of possible longitudes is $-\pi$ to π, as we established when defining longitude.

If we tilt the orbit to $n = \pi/6$ under the same conditions, the equations become:



$$\phi = \arcsin(\sin(t)\sin(\tfrac{\pi}{6})) = \arcsin(\tfrac{1}{2}\sin(t))$$

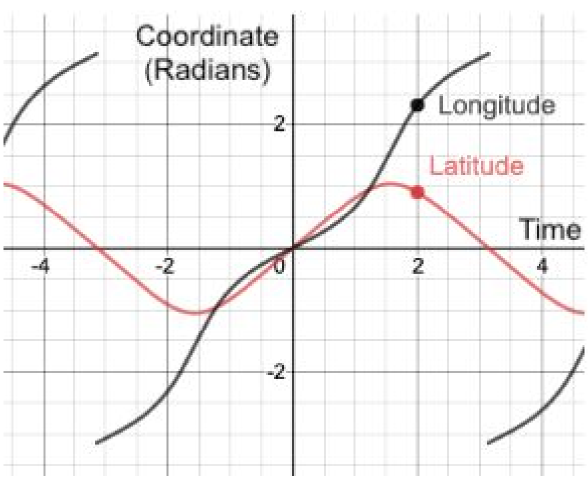
$$\lambda = \arctan(\tan(t)\cos(\tfrac{\pi}{6})) = \arctan(\tfrac{\sqrt{3}}{2}\tan(t))$$

Latitude now varies in a manner similar to a sinusoidal wave as the satellite's orbit brings it above and below the equator.

Since we defined the satellite's starting point to be at the ascending node (where it crosses the equatorial plane from below to above), the graph shows the satellite directly at the equator at $t = 0$, and moving northwards.

The longitude is still roughly a straight line, but has some deformations due to the fact that at greater latitudes, the longitude lines on the globe are closer together, and thus crossed faster.

Tilting to $n = \pi/3$:

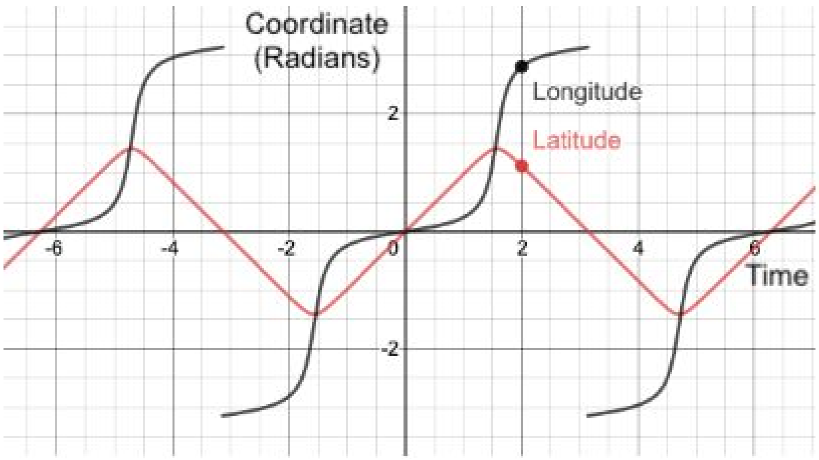


$$\phi = \arcsin(\sin(t)\sin(\tfrac{\pi}{3})) = \arcsin(\tfrac{\sqrt{3}}{2}\sin(t))$$

$$\lambda = \arctan(\tan(t)\cos(\tfrac{\pi}{3})) = \arctan(\tfrac{1}{2}\tan(t))$$

Here, the latitude wave has a greater amplitude, and looks decidedly less sinusoidal, with sharper peaks and straighter segments between them. Longitude is clearly no longer linear. Its gradient appears to be greater at high latitudes and lower at low latitudes.

As we approach a vertically north-south orbit at $n = \pi/2$, these deformations increase, as shown below for $n = 0.45\pi$:



$$\phi = \arcsin(\sin(t)\sin(0.45\pi))$$
$$\lambda = \arctan(\tan(t)\cos(0.45\pi))$$

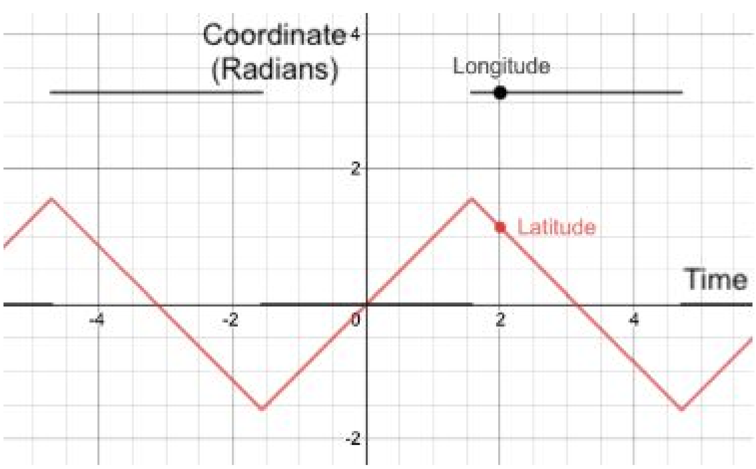
Latitude is almost linear, as we would expect from a vertical orbit: the satellite travels vertically northward at a nearly constant rate, and then vertically southward at the same rate. Longitude changes very slowly when the satellite is near the equator, and then extremely quickly near the poles at $\phi = \frac{\pi}{2}$. This can be best visually understood by looking at a globe with latitude and longitude lines shown:

At greater latitudes, the lines of longitude are closer together near the poles than near the equator. So a satellite moving at a constant angular speed over the Earth will cross more longitude lines in a given amount of time, meaning the rate of change of longitude is highest when the satellite is farthest from the equator.

Image credit: Hellerick (licensed under CC BY-SA 3.0)

This also means that the range of possible latitudes is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, since it is impossible to travel more than a quarter-circle rotation north of the equator.

At $n = \pi/2$, latitude is perfectly linear, and longitude jumps between 0 (in this case zero) and $\pm\pi$:



$$\phi = \arcsin(\sin(t)\sin(\tfrac{\pi}{2})) = \arcsin(\sin(t))$$
$$\lambda = \arctan(\tan(t)\cos(\tfrac{\pi}{2})) = \arctan(0) = 0 \pm \pi$$

On the graph, the peak longitude is only shown at π, but since $\lambda = \pi$ and $\lambda = -\pi$ are equivalent, it doesn't really matter. The only reasons both values aren't shown is because my graphing engine has trouble graphing relations that fail the vertical line test, and because it would take unnecessary vertical space on the page.

We can also see that latitude oscillates through its full possible range of values, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

An interesting application of these equations is to determine the path traced by a satellite over a map. A map is essentially a two-dimensional representation of latitude and longitude, the simplest form being a graph with latitude on the y-axis and longitude on the x-axis. To graph the latitude and longitude of our satellite on an x-y graph, we must find a function that outputs the satellite's latitude at a given longitude. Essentially, we must express ϕ in terms of λ. Since we know ϕ in terms of t, we can find $\phi(\lambda)$ by finding an expression for t in terms of λ and substituting that expression into $\phi(t)$. Unfortunately, it is not possible to rearrange equation 2 to solve for t:

$$\lambda = \lambda_a + \arctan(\tan(\omega_s t)\cos(n)) - \omega_E t$$

Fortunately, there are two cases that make finding the inverse possible. The first is the narrow case $n = 0$, which is the equatorial orbit I discussed earlier. The second is $\omega_E = 0$, where we model the earth as standing still (which is clearly false, but is a valid approximation if the satellite's angular speed is much greater than the Earth's angular speed – which is true for low-altitude satellites like the International Space Station), the equation for λ becomes:

$$\lambda = \lambda_a + \arctan(\tan(\omega_s t)\cos(n))$$

At this point the inverse is easily determined:

$$\tan(\lambda - \lambda_a) = \tan(\omega_s t)\cos(n)$$

$$\arctan\left(\frac{\tan(\lambda-\lambda_a)}{\cos(n)}\right) = \omega_s t$$

$$t = \frac{\arctan\left(\frac{\tan(\lambda-\lambda_a)}{\cos(n)}\right)}{\omega_s}$$

Substituting this into $\phi = \arcsin(\sin(\omega_s t)\sin(n))$ gives us:

$$\phi = \arcsin\left(\sin\left[\omega_s \frac{\arctan\left(\frac{\tan(\lambda-\lambda_a)}{\cos(n)}\right)}{\omega_s}\right]\sin(n)\right)$$

Which simplifies to:

$$\phi = \arcsin\left(\sin\left[\arctan\left(\frac{\tan(\lambda-\lambda_a)}{\cos(n)}\right)\right]\sin(n)\right)$$

While this function is quite complex, we can see that for small values of n, the equation can be approximated by the fact that $\sin(\theta) \approx \arcsin(\theta) \approx \theta$ for small values of θ. In these cases, $\cos(n) \approx 1$ and $\sin(n) \approx n$:

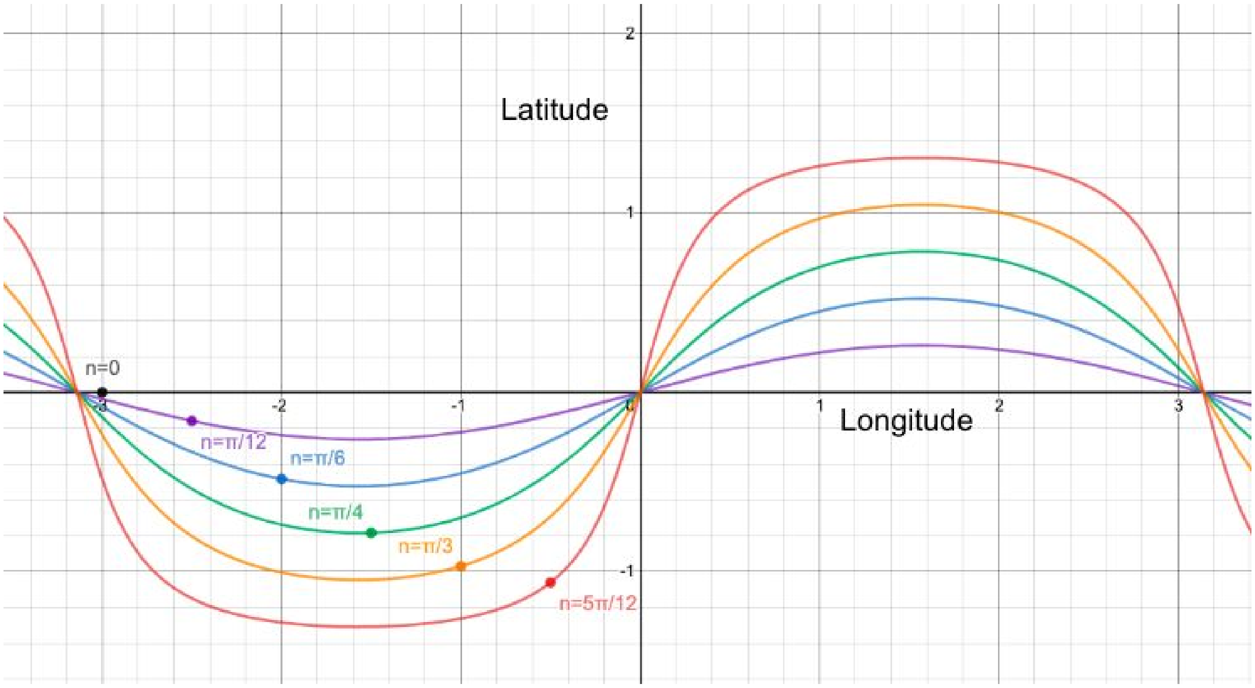
$$\phi \approx \arcsin(n \sin[\arctan(\tan(\lambda - \lambda_a)]])$$

$$\phi \approx \arcsin(n \sin(\lambda - \lambda_a))$$

Since $n \sin(\theta) < n$, $\arcsin(n \sin(\theta)) \approx n \sin(\theta)$ for small n. So when n is relatively small:

$$\phi \approx n \sin(\lambda - \lambda_a)$$

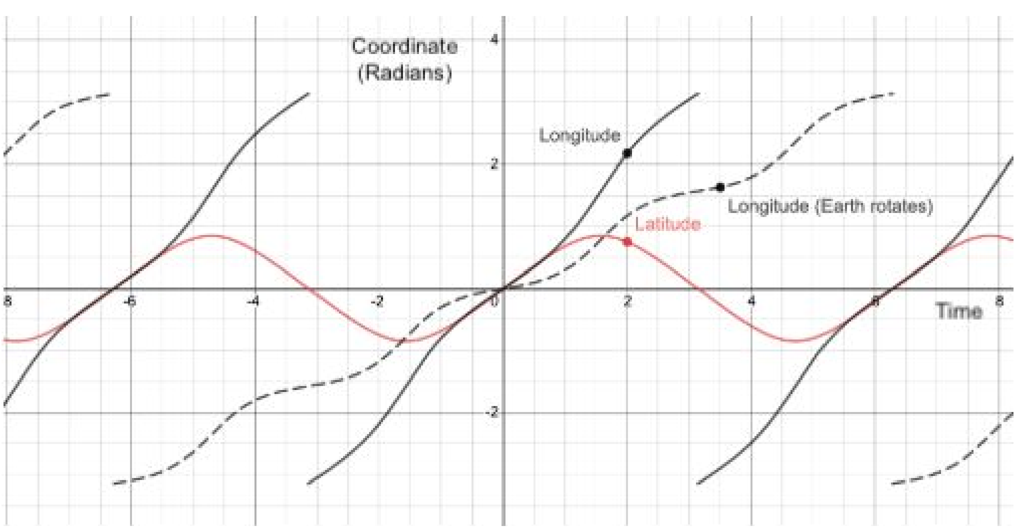
Indeed, this is visible when ϕ is graphed as a function of λ (with values of $n = 0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$):



Above, graphs showing the path traced over a stationary Earth's surface is shown for identical satellites with different values of n. While the wave produced appears to be sine-like for small values of n, such as $\pi/4$ or lower, it quickly becomes more square for more tilted orbits. Naturally, the apparent speed with which the satellite traces this path is not constant; as discussed earlier, the satellite appears to move faster over the map when it is farther from the equator..

I will now look at a more realistic situation, where the angular speed of the satellite is close or equal to the angular speed of the Earth, and therefore ω_E is not negligible. If we begin with a simple example, where the Earth rotates at half the angular speed of the satellite, we can see that the longitude over time graph appears stretched out horizontally. Substituting $\omega_E = 0.5,\ \omega_s = 1$ gives:

$$\phi = \arcsin(\sin(t)\sin(n))$$
$$\lambda = \arctan(\tan(t)\cos(\tfrac{\pi}{3})) - \omega_E t$$
$$\lambda = \arctan(\tan(t)\cos(\tfrac{\pi}{3})) - \tfrac{t}{2}$$



On the graph, the solid black line is the longitude if we model the Earth as stationary, and the dashed black line is the longitude if we model the Earth rotating.This is significant because the latitude is not affected, meaning that the latitude and longitude no longer oscillate with the same period. As a result, the satellite won't trace the same path every time it orbits: each orbit will be out of sync with the previous one. As time passes, the path traced by the satellite will appear to 'drift' with every full north-south oscillation.

To understand the longitude drift phenomenon, look at the satellite's net change in longitude over one full orbit. The time taken to orbit 2π radians around the planet is $2\pi/\omega_s$. If we substitute $t = 2\pi/\omega_s$ into the longitude equation:

$$\lambda = \lambda_a + \arctan(\tan(\omega_s t)\cos(n)) - \omega_E t$$

$$\lambda = \lambda_a + \arctan(\tan(2\pi)\cos(n)) - 2\pi\frac{\omega_E}{\omega_s}$$

$$\lambda = \lambda_a - 2\pi\frac{\omega_E}{\omega_s}$$

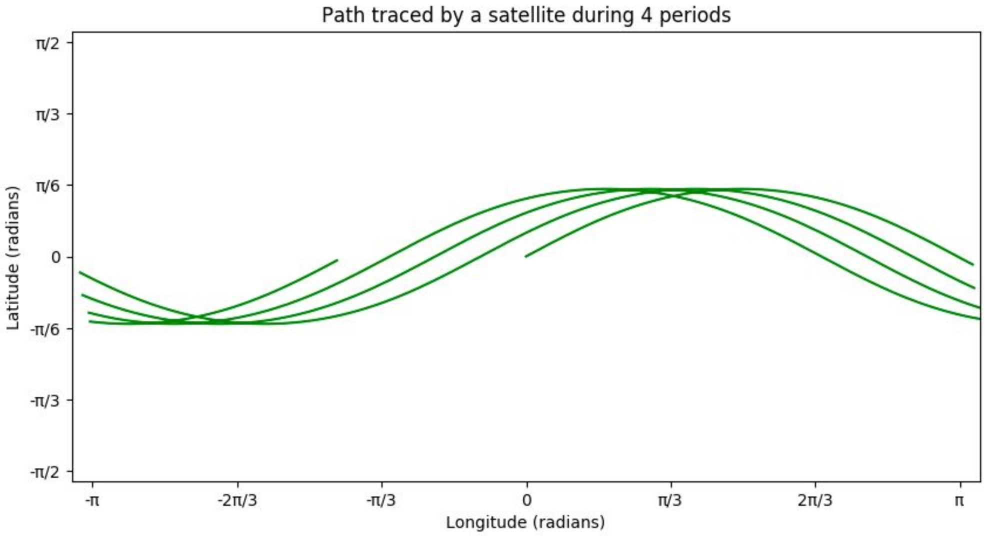
$$\lambda = \lambda_a - \omega_d$$

Where $\omega_d = 2\pi\frac{\omega_E}{\omega_s}$. ω_d, the drift rate, represents how much the satellite's longitude drifts each orbital period, and is determined by the ratio of ω_E to ω_s. When the two are equal, $\omega_d = 2\pi$. If the satellite moves faster than the Earth, $\omega_d < 2\pi$; if it moves slower, $\omega_d > 2\pi$ After one orbit, the longitude of the satellite has changed by $-\omega_d$ radians. If the satellite rotates in the same direction as the Earth, $-\omega_d$ will be negative, so it will drift to the west. If the satellite rotates in the opposite direction, it will drift to the East. For satellites orbiting faster than the Earth (satellites travel faster the closer they are to the planet, so low-altitude satellites orbit faster than the Earth rotates), the drift rate will be less than one full rotation per orbit: $|\omega_d| < 2\pi$. At a specific altitude, 35,786 km above sea level (Howell, 2020), the satellite orbits at the same angular speed as the Earth's rotation, orbiting every 24 hours, so it doesn't appear to drift at all. At altitudes beyond 35,786 km, the satellite lags behind the Earth's rotation, meaning that $|\omega_d| > 2\pi$.

If ω_d is a rational number multiple of 2π, the satellite will eventually return to its starting position relative to the Earth's surface. For example, if $\omega_d = 2\pi/5$, the satellite will return to its initial coordinates after 5 full orbits, since the drifts accumulate to one full rotation. If $\omega_d = 10\pi/8 = 2\pi \times 5/8$, the satellite will return after 8 orbits, with the cumulative drift summing to 5 full rotations. Following this pattern, if ω_d is written as a fully simplified fraction of 2π, it will return to its starting coordinates after *denominator* orbits, having drifted by *numerator* full rotations. But if $\omega_d \div \pi$ is irrational, it can't be written as a fraction with a finite denominator – therefore, it will not return to its initial state in finite time. A satellite with an irrational $\omega_d \div \pi$ will pass over every reachable point between its minimum and maximum latitudes and never return to its starting state.

In reality, it's unclear whether the ratio of speeds would be rational or irrational. Even if in real-life cases where it is rational, the numerator and denominator are both likely to be very large, meaning that even if the satellite returns to its initial state, it will take a very long time to do so ($41^{50} \div 43^{50}$ is rational, but a satellite with that ratio takes 43^{50} orbits to return to its initial state). It's safe to assume that any real satellite will not return to the same state within the scope of this investigation (pass over exactly the same coordinates traveling in the same direction) more than once.

On a map, this drift phenomenon can look something like this:



Satellite path for $n = 0.15,\ \omega_s = 1,\ \omega_E = 0.05,\ \lambda_a = 0$. This was created by plotting $x = \lambda(t)$ and $y = \phi(t)$ parametrically as functions of time, using computer graphing software. $\omega_E \div \omega_s = 1 \div 20$, so this hypothetical satellite will retrace its path after 20 full orbits.

Using the equations I've formulated in this investigation, it is now possible for us to model the orbit of any real-life satellite with a near-circular orbit. One good example of this is the International Space Station (ISS), whose altitude varies between 415 and 422 kilometers above sea level. The earth's radius is close to 6,371 km, so the distance between the center of the orbit and the ISS itself varies between 6,786 km and 6,793 km – a variation of about 0.1%, meaning it's fairly safe to model its orbit as a perfect circle. The space station has an orbital inclination (n) of 51.6° and makes 15.5 orbits each day (meaning they see between 15 and 16 sunrises and sunsets every 24 hours). From this we can calculate the orbital period:

$$\frac{24\times60\times60}{15.49241305} = 5576.92 \text{ seconds.}$$

5,576.92 seconds is about 93 minutes to complete one full rotation around the Earth – which shows how unimaginably fast satellites must travel to stay in Low-Earth Orbit. From this, we can calculate the angular speed of the satellite:

$$\omega_s = \frac{2\pi}{5576.92} = 0.00113 \text{ radians per second.}$$

We can also find the orbital inclination in radians:

$$n = 51.6429 \times \pi \div 180 = 0.901 \text{ radians.}$$

We know that the Earth makes one rotation every 24 hours, so:

$$\omega_E = \frac{2\pi}{24\times60\times60} = 0.0000727 \text{ radians per second.}$$

If we interpret these angular speeds as exact values, we can calculate that $\omega_E \div \omega_s = 727 \div 11300$. Since 727 is a prime number, this fraction can't be simplified, so the ISS would exactly retrace its path after 11,300 orbits. But these values are really just approximations themselves, truncated to 3 significant figures to simplify calculation. In reality, the ISS will never perfectly retrace its path.

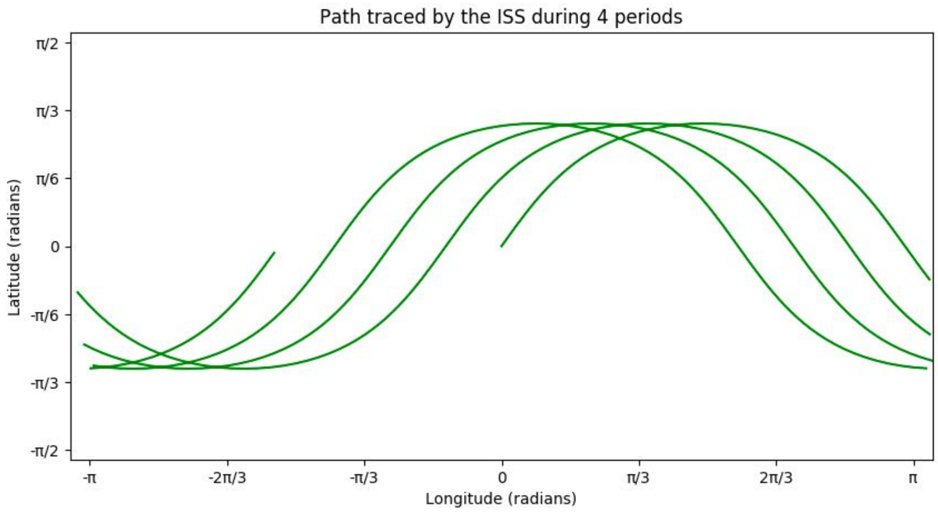
The only missing parameter is λ_a, the longitude of the ascending node, at $t = 0$. As the Earth rotates, the longitude of the ascending node changes, so the actual value of λ_a depends on the time our model starts, and the value we choose is somewhat arbitrary. To simplify things, I'll pick $\lambda_a = 0$. Now that we have all of our information, we can create the parametric equations for the path traced by the ISS. Latitude:

$$\phi = \arcsin(\sin(\omega_s t)\sin(n))$$
$$\phi = \arcsin(\sin(0.00113 \times t)\sin(0.901))$$
$$\phi = \arcsin(\sin(0.00113 \times t) \times 0.784)$$

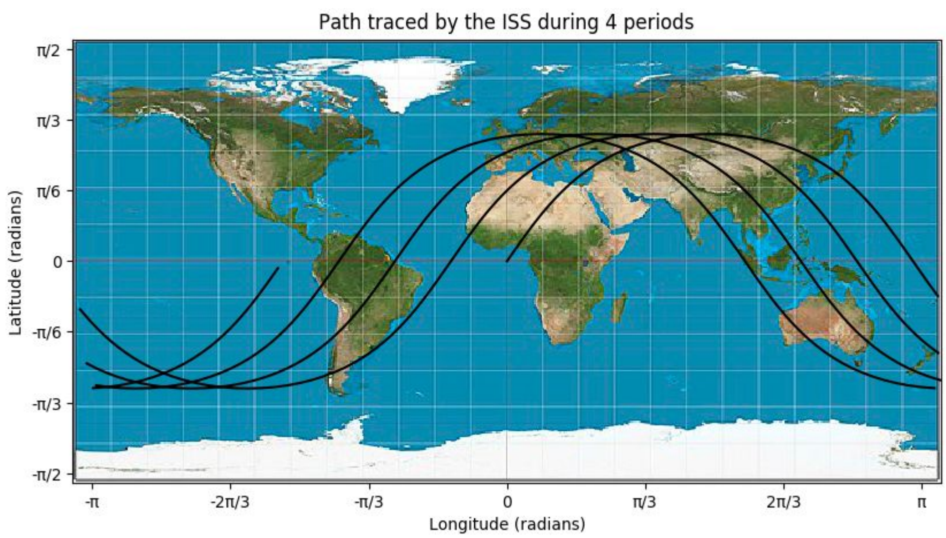
And longitude:

$$\lambda = \lambda_a + \arctan(\tan(\omega_s t)\cos(n)) - \omega_E t$$
$$\lambda = 0 + \arctan(\tan(0.00113 \times t)\cos(0.901)) - 0.0000727 \times t$$
$$\lambda = \arctan(\tan(0.00113 \times t) \times 0.621) - 0.0000727 \times t$$

Finally, we graph this as a parametric curve:



We can also overlay this on a map:



Map background image credit: Strebe (licensed under CC BY-SA 3.0)

This example shows how, using mathematical calculations, we can determine the location of a satellite relative to the Earth by modelling its orbit as a perfect circle. Despite the fact that this is a model, it successfully determined the coordinates of the ISS during a 20 hour period, during which I repeatedly retrieved the true ISS coordinates from the European Space Agency and compared them to my model's predictions. The output of the model was correct to within at least 3 significant figures, the maximum precision given on ESA's website. For example, at time 13:02:05, the ISS was over the point $\lambda = -2.93,\ \phi = -0.56$, and at 09:10:25 on the following day, it was at $\lambda = 1.57,\ \phi = 0.77$. My model predicted that at 09:10:25 it would be over $\lambda = 1.57284,\ \phi = 0.76600$, which is impressive considering that the model was derived entirely from theory. Numerous assumption were made: I addressed the fact that the satellite's orbit is in fact never perfectly circular, but in addition, this model assumes that:

- The density (and thus the gravitational field) of the Earth is uniform, and the Earth is a perfect sphere. In fact, the Earth is not uniformly dense, and due to its rotation it bulges slightly near the equator, causing gravitational perturbations to the orbit.
- There are no other objects in space (the moon or sun, for example) that have any effect on the satellite. While the influence of the sun is limited, moons can have a major impact on the orbits of satellites because they are nearby and exert a strong gravitational pull.
- The satellite never fires thrusters to change its orbital trajectory, and that it does not rendezvous and dock with any other spacecraft. Naturally any of these events would cause the satellite to deviate from its orbit.

But since these conditions are very uncommon for most man-made satellites orbiting Earth, this is a useful model for predicting the orbits of satellites in circular orbits. Another use of these equations is to predict the variations in satellite phone coverage over time at a specific location. Iridium Communications, a prominent satellite phone service, operates a network of satellites in circular orbits with an inclination of 86.4° (almost perpendicular to the equatorial plane) and an orbital period of 100 minutes. Knowing this information, these equations can predict the movement of the satellite over the Earth's surface over time, and therefore determine when a specific location will have improved or reduced signal strength.

In this investigation, I have devised a set of equations for calculating and predicting the latitude and longitude of a satellite in circular orbit around a planet. I've investigated specific cases, such as the equatorial orbit, geostationary orbit, and the simplified model where the planet doesn't rotate. I also explored general relationships, such as the effects of inclination and angular velocity on the satellite's path, and phenomena such as the satellite's tendency to drift East or West due to the planet's rotation. These equations are versatile and useful for predicting the path over the Earth taken by any of the multitude of artificial satellites in circular orbits around the planet.

Bibliography:

1. Montenbruck, O. and Gill, E., 2012. Satellite Orbits. Berlin: Springer.
2. Howell, E., 2020. What Is A Geosynchronous Orbit?. [online] Space.com. Available at: <https://www.space.com/29222-geosynchronous-orbit.html> [Accessed 5 April 2020].