Homework 2

1) $P(A) = 0.3$
$P(B) = 0.4$
$P(A \cap B) = 0.2$

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.5$

b) If A & B independant, $P(A) \cdot P(B) = P(A \cap B)$
$0.3 \cdot 0.4 = 0.2$ ✓

A & B are independent

c) $P(A^c \cap B) = P(B) - P(A \cap B)$
$= 0.2$

2) $f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

a) $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^1 2x^2\, dx$

$= \frac{2}{3} x^3 \Big|_0^1$

$= \boxed{\frac{2}{3}}$

$Var(x) = E(x^2) - E(x)^2$
$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

2 continued)

a continued) $= \int_0^1 2x^3 \, dx$

$= \frac{1}{2} x^4 \Big|_0^1$

$= \frac{1}{2}$

$Var(x) = \frac{1}{2} - \left(\frac{2}{3}\right)^2$

$= \frac{1}{2} - \frac{4}{9}$

$= \boxed{\frac{1}{18}}$

b) $P(X=x) = \phi^x (1-\phi)^{1-x}$

$P(X=1) = \phi^1 (1-\phi)^0$

$= \phi$

$P(X=0) = \phi^0 (1-\phi)^1$

$= (1-\phi)$

$E(x) = \sum_{x=0}^{1} x \cdot P(X=x)$

$= 0(1-\phi) + 1(\phi)$

$= \phi$

$Var(x) = E(x^2) - E(x)^2$

$E(x^2) = \sum_{x=0}^{1} x^2 P(X=x)$

$= 0(1-\phi) + 1(\phi)$

$= \phi$

$Var(x) = \phi - \phi^2$

$= \phi(1-\phi)$

3) $f_{xy}(x,y) = \begin{cases} 6xy & 0 \le x \le 1, \quad 0 \le y \le 1 \\ \end{cases}$

a) $f_x(x) = \int_0^1 6xy \, dy$

$= 6x \left(\frac{y^2}{2} \Big|_0^1 \right)$

$\boxed{f_x(x) = 3x}$

Similarly,

$\boxed{f_y(y) = 3y}$

b) $f_x(x) \cdot f_y(y) = 9xy \neq 6xy$

Not independent

4) Event A: $P(5 \text{ or } 6) = 2/6$

Event B: $P((3,6)\ (4,6)\ (5,6)\ (6,6)\ (6,5)\ (6,4)\ (6,3)\ (5,4)\ (5,5)\ (4,5)$

$= 10/36$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

$|A \cap B| = |(5,4)\ (5,5)\ (5,6)\ (6,3)\ (6,4)\ (6,5)\ (6,6)|$

$= 7$

$\frac{7/36}{2/6} = \frac{7}{12}$

5) rv X w/ pdf $f(x) = 3(1-x)^2$
$Y = (1-X)^3$

$$Y^{1/3} = 1-x$$

$$X = 1 - Y^{1/3}$$

$$\frac{dx}{dy} = -\frac{1}{3} Y^{-2/3}$$

$$f_y(Y) = f_x(1 - Y^{1/3}) \left| \frac{dy}{dx} \right|$$

$$= 3(1 - (1 - Y^{1/3}))^3 \left| \frac{dy}{dx} \right|$$

$$= 3 (y^{1/3})^2 \cdot \frac{1}{3} (y^{-2/3})$$

$$= \underline{1}$$

$$f_y(y) = \begin{cases} \underline{1} & 0 < y < 1 \\ 0 & 0, \text{ otherwise} \end{cases}$$

6) $Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} a_j X_j\right) = \sum_1^{+} \sum_1^{+} a_i a_j Cov(X_i, X_j) = a^T S a$

$$Y = \sum_{i=1}^{n} a_i X_i \qquad Z = \sum_{j=1}^{n} a_j X_j$$

$$E(Y) = E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

$$E(Z) = \sum_1^{+} a_j E(X_j)$$

$$E(YZ) = E\left(\left(\sum_1^{+} a_i X_i\right)\left(\sum_1^{+} a_j X_j\right)\right)$$

6 continued)

$$= E\left(\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j X_i X_j\right)\right)$$

$$= \sum \sum a_i a_j E(X_i X_j)$$

$$\mathrm{Cov}(Y, Z) = E(YZ) - E(Y)E(Z)$$

$$= \sum\sum a_i a_j E(X_i X_j) - \sum a_i E(X_i) \sum a_j E(X_j)$$

$$= \sum\sum a_i a_j E(X_i X_j) - \sum\sum a_i a_j E(X_i) E(X_j)$$

$$= \sum\sum a_i a_j \left(E(X_i X_j) - E(X_i)E(X_j)\right)$$

$$= \sum\sum a_i a_j \mathrm{Cov}(X_i, X_j)$$

$$a^T S a = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i S_{ij} a_j = \sum\sum a_i a_j \mathrm{Cov}(X_i, X_j)$$

7) $f(x) = \log(1+e^x)$ $\qquad \sigma(x) = \dfrac{e^x}{1+e^x}$

① $$\frac{d f(x)}{dx} = \frac{1}{1+e^x} \cdot \frac{d}{dx}(1+e^x) = \frac{e^x}{1+e^x}$$

② $$\frac{d\sigma(x)}{dx} = \frac{\frac{d}{dx} e^x \cdot (1+e^x) - e^x \cdot \frac{d}{dx}(1+e^x)}{(1+e^x)^2}$$

$$= \frac{e^x(1+e^x) - e^{2x}}{(e^x+1)^2}$$

$$= \frac{e^x}{(1+e^x)^2}$$

$$= \sigma(x) \cdot \frac{1}{1+e^x}$$

$$= \sigma(x) \cdot \left(\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x} \right)$$

$$= \sigma(x)(1-\sigma(x))$$

③ $\log(\sigma(x)) = -f(-x)$

$$\log\left(\frac{1}{1+e^{x}}\right)$$

$$= \log(e^{x}) - \log(1+e^{x})$$

$$= x - \log(1+e^{x})$$

$$= x - f(x)$$

$$= -f(-x)$$

④ $1-\sigma(x) = 1 - \dfrac{e^{x}}{e^{x}+1} = \dfrac{e^{x}+1}{e^{x}+1} - \dfrac{e^{x}}{e^{x}+1} = \dfrac{1}{e^{x}+1}$

$\sigma(-x) = \dfrac{e^{-x}}{1+e^{-x}} = \dfrac{1}{1+e^{x}}$ $\quad \sigma(-x) = 1-\sigma(x)$

⑤ $\sigma^{-1}(x) = \log\left(\dfrac{x}{1-x}\right)$

$$\frac{e^{y}}{1+e^{y}} = x$$

$$e^{y} = x(1+e^{y})$$

$$x = e^{y} - xe^{y}$$

$$e^{y}(1-x) = x$$

$$e^{y} = \frac{x}{1-x}$$

$$y = \log\left(\frac{x}{1-x}\right)$$

⑥ $\forall x > 0 \quad f^{-1}(x) = \log(\exp(x) - 1)$

$$y = \log(1+e^x)$$

$$e^y = 1 + e^x$$
$$e^x = e^y - 1$$
$$x = \log(e^y - 1)$$

$$f^{-1}(x) = \log(e^x - 1)$$

⑦ $f(x) = \int_{-\infty}^{x} \sigma(y)\, dy$

$$\int_{-\infty}^{x} \frac{e^y}{1+e^y}\, dy$$

$$u = 1 + e^y$$
$$du = e^y$$

$$\int \frac{1}{u}\, du$$

$$= \ln(1+e^y)\Big|_{-\infty}^{x}$$

$$= \ln(1+e^x) - \ln(1+e^{-\infty})$$

$$= \ln(1+e^x) - \ln(1)$$

$$= \ln(1+e^x)$$

⑧ $f(x) - f(-x) = x$

$$f(x) - f(-x) = \log(1+e^x) - \log(1+e^{-x})$$

$$= \log(1+e^x) - \log\left(\frac{e^x+1}{e^x}\right)$$

$$= \log(1+e^x) - (\log(e^x+1) - x)$$

$$= x$$