Lesson 8:
Sources, Sinks, Swirls and Singularities

Try It2

The cell below will be filled in by the grader:

Correctness/Completeness:/10

Quality of Explanations: /10

Total Score: /20

Comments:

2) Singularity sources, sinks and swirls

2.a) Singularity sources and sinks

Given a vector field

Field[x, y] = {m[x, y], n[x, y]},

you can calculate

divField[x, y] = D[m[x, y], x] + D[n[x, y], y] = $\frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}$

to look for sources and sinks.

If you locate a source at {x, y} and {x, y} is not a singularity, you can expect new fluid to be slowly oozing into the flow at {x, y}.

If you locate a sink at {x, y} and {x, y} is not a singularity, you can expect old fluid to be slowly soaking out of the flow at {x, y}.

The vivid sources and sinks are often found at singularities; in fact if you envision a sink at a singularity to be a black hole, you are thinking correctly.

To detect a source or a sink at a singularity, you center a small circle C[r] of radius r at the singularity and calculate

$$\oint_{C[r]} -n[x, y]\,dx + m[x, y]\,dy \ .$$

→ If

$$\lim_{r \to 0} \oint_{C[r]} -n[x, y]\,dx + m[x, y]\,dy > 0 \ ,$$

you have located a singularity source (a gusher) at the singularity.

→ If

$$\lim_{r \to 0} \oint_{C[r]} -n[x, y]\,dx + m[x, y]\,dy < 0 \ ,$$

you have located a singularity sink (a black hole) at the singularity.

→ If

$$\lim_{r \to 0} \oint_{C[r]} -n[x, y]\,dx + m[x, y]\,dy = 0 \ ,$$

you have located a singularity that is neither a source nor a sink.

Try it out:

The point {a, b} is a singularity of the 2D electric field coming from a point charge of strength 2 placed at {a, b}:

```
Clear[ElectricField, a, b, m, n, q, x, y];

{m[x_, y_], n[x_, y_]} = (q {x - a, y - b}) / ((x - a)^2 + (y - b)^2);
ElectricField[x_, y_] = {m[x, y], n[x, y]}
```

$$\left\{\frac{q\,(-a+x)}{(-a+x)^2+(-b+y)^2},\ \frac{q\,(-b+y)}{(-a+x)^2+(-b+y)^2}\right\}$$

Center a circle C[r] of radius r at {a, b} and calculate

$$\oint_{C[r]} -n[x, y]\,dx + m[x, y]\,dy \ :$$

```
singularity = {a, b};
```

```
Clear[xr, yr, r, t];
{xr[t_], yr[t_]} = singularity + r {Cos[t], Sin[t]};
Integrate[(-n[xr[t], yr[t]]) xr'[t] + m[xr[t], yr[t]] yr'[t], {t, 0, 2 Pi}]
```

$2\pi q$

This tells you that the singularity at {a, b} is a source of new juice if q > 0 (positive charge at {a, b}) and is a sink for old juice if q < 0 (negative charge at {a, b}).

2.a.i)

Does the electric field above have sources or sinks other than at the singularity?

```
Simplify[divF[x_, y_] = D[m[x, y], x] + D[n[x, y], y]]
```

0

Student Response

There is no divergence at any other points other than the singularity.

2.a.ii)

Here's a vector field related to the electric field:

```
Clear[Field, a, b, m, n, x, y];

{m[x_, y_], n[x_, y_]} = (8 {x - a, y - b}) / Sqrt[(x - a)^2 + (y - b)^2];

Field[x_, y_] = {m[x, y], n[x, y]}
```

$$\left\{\frac{8(-a+x)}{\sqrt{(-a+x)^2+(-b+y)^2}}, \frac{8(-b+y)}{\sqrt{(-a+x)^2+(-b+y)^2}}\right\}$$

Note the singularity at {a, b}.
Now look at the following information:

```
singularity = {a, b};


Clear[xr, yr, r, t];
{xr[t_], yr[t_]} = singularity + r {Cos[t], Sin[t]};

Integrate[(-n[xr[t], yr[t]]) xr'[t] + m[xr[t], yr[t]] yr'[t], {t, 0, 2 Pi}]
```

$16\pi\sqrt{r^2}$

How does this help to tell you that the singularity at {a, b} is neither a singularity source nor a singularity sink?

Student Response

This relies on r, as there is no a and b. As r-> 0 then divergence -> 0.

2.a.iii)

Stay with the same vector field as in part ii) above and look at divField[x, y] together with
$(x-a)^2+(y-b)^2$:

```
Clear[Field, a, b, m, n, x, y];

{m[x_, y_], n[x_, y_]} = (8 {x - a, y - b}) / Sqrt[(x - a)^2 + (y - b)^2];

Field[x_, y_] = {m[x, y], n[x, y]};
```

```
{Together[D[m[x, y], x] + D[n[x, y], y]], Expand[(x - a)^2 + (y - b)^2]}
```

$$\left\{\frac{8}{\sqrt{(-a+x)^2+(-b+y)^2}},\ a^2+b^2-2\,a\,x+x^2-2\,b\,y+y^2\right\}$$

How does this help to tell you that all points {x, y} other than the singularity at {a, b} are sources?
How does this tell you that the big-time sources of new fluid for this field are packed near the singularity?

Student Response

This tells us that there are other points that are sources, since its always positive. As you stray away from the singularity the divergence decreases.

2.a.iv)

Here's another vector field related to the electric field:

```
Clear[Field, a, b, q, m, n, x, y];

{m[x_, y_], n[x_, y_]} = (5 {x - a, y - b}) / ((x - a)^2 + (y - b)^2)^(3/2);

Field[x_, y_] = {m[x, y], n[x, y]}
```

$$\left\{\frac{5\,(-a+x)}{((-a+x)^2+(-b+y)^2)^{3/2}},\ \frac{5\,(-b+y)}{((-a+x)^2+(-b+y)^2)^{3/2}}\right\}$$

Determine whether {a, b} is a singularity source, a singularity sink, or neither.
Also determine whether there are sources or sinks other than at the singularity.

```
singularity = {a, b};
Clear[xr, yr, r, t];
{xr[t_], yr[t_]} = singularity + r {Cos[t], Sin[t]};
Limit[Integrate[(-n[xr[t], yr[t]]) xr'[t] + m[xr[t], yr[t]] yr'[t], {t, 0, 2 Pi}], r -> 0]

Simplify[D[m[x, y], x] + D[n[x, y], y]]
```

$$\infty$$

$$-\frac{5}{((a-x)^2+(b-y)^2)^{3/2}}$$

Student Response

Its a singularity a source. There are other points with Sources/ Sinks since it will always be pos/ neg.

2.b.i) Singularity swirls

When you pull the plug in a bathtub, you see a good example of a singularity swirl (and a singularity sink).
To detect a singularity swirl, you center a small circle C[r] of radius r at the singularity and calculate a limit of a certain path integral. If this limit is positive, then you have located a counterclockwise singularity swirl. If this limit is negative, then you have located a clockwise singularity swirl.
If this limit is 0, then you have located singularity that has no swirling effect at all.
What limit do you look at?

Student Response

Limit[Integrate[(m[xr[t], yr[t]]) xr'[t] + n[xr[t], yr[t]] yr'[t], {t, 0, 2 Pi}], r -> 0]

2.b.ii)

Here's a vector field with a singularity at {0, 0}:

```
Clear[Field, m, n, x, y];
```

```
{m[x_, y_], n[x_, y_]} = 3 ({y, -x} / (x^2 + y^2));

Field[x_, y_] = {m[x, y], n[x, y]}
```

$$\left\{\frac{3\,y}{x^2+y^2},\ -\frac{3\,x}{x^2+y^2}\right\}$$

Here is rotField[x, y]:

```
rotField[x, y] = Together[D[n[x, y], x] - D[m[x, y], y]]
```

This field has no swirl around any point other than possibly the singularity at {0, 0}.
Look at a plot of this vector field.

```
scalefactor = 0.5;

fieldplot = Table[Vector[Field[x, y], Tail -> {x, y}, ScaleFactor -> scalefactor, HeadSize -> 0.5],
   {x, -4.5, 4.5, 9 / 7}, {y, -4.5, 4.5, 9 / 7}];

Show[fieldplot, Axes -> True, AxesLabel -> {"x", "y"}]
```

Big-time clockwise singularity swirl around the singularity at {0, 0}.
Test your answer to part b.i) immediately above to see whether your limit agrees with reality.