A Butterfly Subdivision Scheme for Surface Interpolation with Tension Control

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A new interpolatory subdivision scheme for surface design is presented. The new scheme is designed for a general triangulation of control points and has a tension parameter that provides design flexibility. The resulting limit surface is C^1 for a specified range of the tension parameter, with a few exceptions. Application of the butterfly scheme and the role of the tension parameter are demonstrated by several examples.

Categories and Subject Descriptors: G.1.1 [**Numerical Analysis**]: Interpolation; I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling—*surface representation*; J.6 [**Computer Applications**]: Computer-Aided Engineering—computer-aided design (*CAD*)

General Terms: Algorithms, Design

Additional Key Words and Phrases: General triangulation, subdivision scheme, surface interpolation, tension control

INTRODUCTION

The basic approach to the design of curves and surfaces in CAGD consists of using control points which define control polygons or control polyhedrons, together with a smoothing scheme. The scheme defines a smooth curve out of a control polygon, or a smooth surface out of a control polyhedron, and the desired shape is achieved by maneuvering the control points. In the case of a curve, we are given control points $\{\bar{p}_i^0\}_1^n$, $\bar{p}_i^0 \in \mathbb{R}^3$. A B-spline curve is then defined as

$$\bar{p}(t) = \sum_{i=1}^{n} \bar{p}_i^0 B_i(t), \qquad 1 \le t \le n, \tag{1}$$

This work was supported by the Israel Academy of Science and Humanities grant 548-86 and the U.S.A.-Israel Binational Science Foundation grant 86-00243.
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where $\{B_i\}_1^n$ is a proper B-spline basis. Another type of scheme for defining curves uses a binary subdivision process that computes recursively from the given set of control points $\{\bar{p}_i^0\}_{i=0}^n$ new sets $\{\bar{p}_i^k\}_{i=0}^{2^k n}$, $k = 1, 2, \ldots$. The points at level k define a polygon $\bar{p}^k(t)$, $t \in [0, n]$, and the sequence $\{\bar{p}^k(t)\}_{k=0}^\infty$ converges to a smooth curve if the recursive process is properly devised. All the B-spline curves can be defined by recursive subdivision. Two examples that fit into this class are

(1) Chaikin's algorithm [5]

$$\begin{cases} \bar{p}_{2i}^{k+1} = \frac{3}{4}\,\bar{p}_i^k + \frac{1}{4}\,\bar{p}_{i+1}^k, \\ \bar{p}_{2i+1}^{k+1} = \frac{1}{4}\,\bar{p}_i^k + \frac{3}{4}\,\bar{p}_{i+1}^k, \end{cases} \tag{2}$$

which converges to the quadratic B-spline curve and is C^1 continuous; and

(2) the cubic spline algorithm [12]

$$\begin{cases} \bar{p}_{2i}^{k+1} = \frac{1}{2}\,\bar{p}_i^k + \frac{1}{2}\,\bar{p}_{i+1}^k, \\ \bar{p}_{2i+1}^{k+1} = \frac{1}{8}\,\bar{p}_i^k + \frac{3}{4}\,\bar{p}_{i+1}^k + \frac{1}{8}\,\bar{p}_{i+2}^k, \end{cases} \tag{3}$$

which converges to the cubic B-spline curve and is C^2 continuous.

Catmull and Clark [4] presented a version of (3) for nonregular polyhedrons of control points, and Chaikin's algorithm was generalized to this setting by Doo and Sabin [8].

The above schemes are based on chopping corners of the control polygon. Hence, they are not interpolatory, and this may be a drawback in some applications. Nasri [16] suggested a method for achieving interpolation by applying the Doo–Sabin scheme to a modified set of control points with the same topology. Computation of the modified points involves solution of a sparse linear system, and the resulting method is nonlocal. In this work we present an explicit, local, interpolatory subdivision scheme for surfaces that is based on the local four-point interpolatory subdivision scheme for curves studied in [9, 10].

Given control points $\{\bar{p}_i^0\}_{i=-2}^{n+2}$, the four-point interpolatory scheme defines points at level $k+1$ of the recursion by

$$\begin{cases} \bar{p}_{2i}^{k+1} = \bar{p}_i^k & -1 \le i \le 2^k n + 1, \\ \bar{p}_{2i+1}^{k+1} = (\frac{1}{2} + w)(\bar{p}_i^k + \bar{p}_{i+1}^k) - w(\bar{p}_{i-1}^k + \bar{p}_{i+2}^k) & -1 \le i \le 2^k n. \end{cases} \tag{4}$$

Obviously the scheme is interpolatory, $\bar{p}_{2^k i}^k = \bar{p}_i^0$, $0 \le i \le n$, since at each stage we keep all the old points and insert new points "in between" the old ones.

Let us associate the point $\bar{p}_i^k$ with the parameter value $t_i^k = 2^{-k} i$, $0 \le i \le n$, and denote by $\bar{p}^k(t)$, $t \in [0, n]$, the polygonal line connecting the points $\{\bar{p}_i^k\}_{i=0}^{2^k n}$. The convergence analysis deals with the limit curve

$$\bar{p}(t) = \lim_{k \to \infty} \bar{p}^k(t), \qquad t \in [0, n], \tag{5}$$

investigating existence and smoothness of $\bar{p}(t)$ for different values of the parameter w.

The convergence analysis of the four-point scheme and the properties of the limit curve are presented in [9] and [10], and the results are as follows:

(1) For any $|w| < \frac{1}{2}$, the points produced by scheme (4) lie on a continuous curve $\bar{p}_w(t)$.
(2) For any $0 < w < (\sqrt{5} - 1)/8$, the curve $\bar{p}_w(t)$ is a C^1 curve; that is, it has a continuous tangent.
(3) For general sets of initial control points, there is no value of w for which the curve $\bar{p}_w(t)$ is C^2 continuous.
(4) The parameter w serves as a tension parameter; that is, as $w \to 0$ the curve is tightened toward the control polygon.

A C^2 limit curve is obtained by the six-point interpolatory scheme:

$$\begin{cases} \bar{p}_{2i}^{k+1} = \bar{p}_i^k, \\ \bar{p}_{2i+1}^{k+1} = (\frac{9}{16} + 2\theta)(\bar{p}_i^k + \bar{p}_{i+1}^k) - (\frac{1}{16} + 3\theta)(\bar{p}_{i-1}^k + \bar{p}_{i+2}^k) \\ \qquad\qquad + \theta(\bar{p}_{i-2}^k + \bar{p}_{i+3}^k). \end{cases} \tag{6}$$

For $\theta = 0$, this scheme reduces to the four-point scheme with $w = \frac{1}{16}$, and for $0 < \theta < 0.02$, the limit curve is C^2, that is, it has continuous curvature [17]. For necessary conditions on a general interpolatory $2r$-point scheme to produce a C^k curve, see [10].

The simplest generalization of the univariate subdivision processes to surfaces is when the control points form a regular squarelike grid. In this case one can define a tensor-product version of the univariate schemes. The tensor-product form of the interpolatory four-point scheme is the following:

Starting from the set of control points $\{\bar{p}_{i,j}^0\}$, define

$$\begin{cases} \bar{p}_{2i,2j}^{k+1} = \bar{p}_{i,j}^k, \\ \bar{p}_{2i+1,2j}^{k+1} = (\frac{1}{2} + w)(\bar{p}_{i,j}^k + \bar{p}_{i+1,j}^k) - w(\bar{p}_{i-1,j}^k + \bar{p}_{i+2,j}^k), \\ \bar{p}_{i,2j+1}^{k+1} = (\frac{1}{2} + w)(\bar{p}_{i,2j}^{k+1} + \bar{p}_{i,2j+2}^{k+1}) - w(\bar{p}_{i,2j-2}^{k+1} + \bar{p}_{i,2j+4}^{k+1}). \end{cases} \tag{7}$$

Further information on subdivision methods can be found in the works of Boehm [1, 2], Boehm, Farin, and Kahmann [3], Cohen, Lyche, and Riesenfeld [6, 7], and Micchelli and Prautzsch [13–15].

1. INTERPOLATORY SUBDIVISION SCHEMES FOR SURFACES

We propose a generalization of the four-point interpolatory scheme to a general triangulation of control points $\{\bar{p}_i^0\}$. The scheme transforms recursively each triangular face of the control polyhedron into a patch consisting of four triangular faces interpolating the old control points. Thus, the refined triangulation retains the vertices of the coarser triangulation, and new vertices are added corresponding to the edges of the old triangulation. The rule for inserting new points is an eight-point rule we call a "butterfly scheme" because it is based on the configuration shown in Figure 1. Using the butterfly configuration in Figure 1, we examine rules for inserting a new point $\bar{q}^{k+1}$, corresponding to the edge $(\bar{p}_1^k, \bar{p}_2^k)$, of the symmetric form

$$\bar{q}^{k+1} = u(\bar{p}_1^k + \bar{p}_2^k) + v(\bar{p}_3^k + \bar{p}_4^k) - w(\bar{p}_5^k + \bar{p}_6^k + \bar{p}_7^k + \bar{p}_8^k). \tag{8}$$

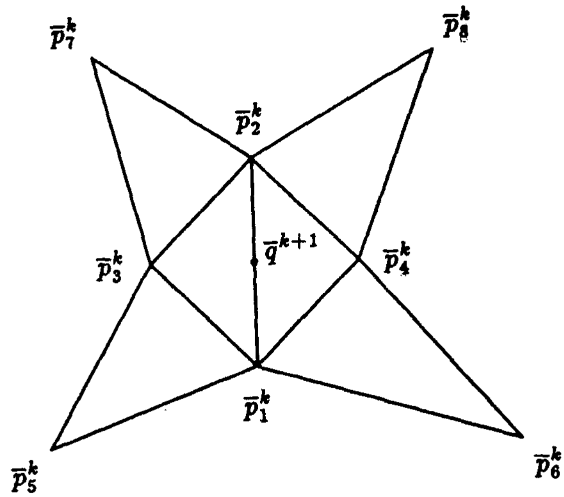

Fig. 1. Configuration of points in the butterfly scheme.

After inserting the new points, each corresponding to an edge of the given triangulation, a refined triangulation is formed, and the process is repeated recursively. The refined triangulation consists of all the edges connecting each new point $\bar{q}^{k+1}$ with the old points $\bar{p}_1^k$, $\bar{p}_2^k$ and the four new points corresponding to the edges $(\bar{p}_i^k, \bar{p}_j^k)$, $i = 1, 2, j = 3, 4$. Thus, each old triangle is replaced by four new triangles. In (8) u, v, w are parameters to be chosen so that in the limit the process will produce a C^1 surface (C^1).

The convergence analysis of subdivision schemes for surfaces was treated by Doo and Sabin [8] and by Micchelli and Prautzsch [14]. In [14] necessary conditions and sufficient conditions are given for the convergence and the smoothness of the limit surface produced by a general uniform subdivision scheme defined on a uniform rectangular grid. In [8] necessary conditions for the regularity of the surface near a given point are stated. The analysis of a butterfly scheme for a general triangulation should combine both the approach of [8] and that of [14]. The uniform grid analysis treats the case where all the vertices in the triangulation are regular vertices, namely, of degree six (the degree of a vertex being the number of edges meeting at the vertex). Note that all the new vertices generated by the scheme are regular vertices. Therefore, the uniform analysis of [14] applies to most of the surface, excluding neighborhoods of the initial irregular vertices, where the analysis in [8] applies.

By considering the necessary conditions of [8] near a regular vertex, it follows that for a C^0 surface we must have

$$2u + 2v - 4w = 1 \tag{9}$$

and for C^1 we should take $u = \frac{1}{2} + h(w)$, $h(0) = 0$. Choosing $h = 0$ and combining both conditions, we have that $v = 2w$, and the scheme is

$$\bar{q}^{k+1} = \frac{1}{2}(\bar{p}_1^k + \bar{p}_2^k) + 2w(\bar{p}_3^k + \bar{p}_4^k) - w(\bar{p}_5^k + \bar{p}_6^k + \bar{p}_7^k + \bar{p}_8^k). \tag{10}$$

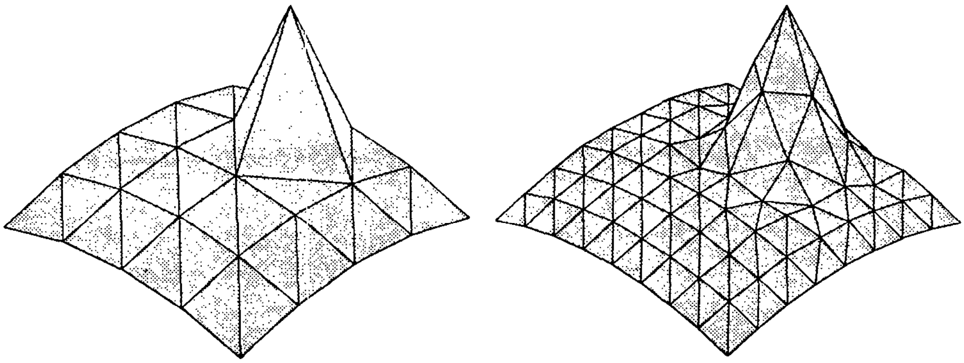

Fig. 2. One step of subdivision with $\omega = \frac{1}{16}$.

In Figure 2 we give the original control polyhedron and the resulting piecewise linear surface obtained after one subdivision iteration using scheme (10) with $w = \frac{1}{16}$.

That scheme (10) is an extension of the four-point scheme (4) becomes apparent when the control points describe function values over a three-direction mesh; if these values are constant along one of the directions, then all the new values will be constant along this direction, and the scheme reduces to the univariate scheme (4) along the other two directions.

The smoothness properties of the surface produced by the butterfly scheme depend on the degrees of the vertices in the triangulation. It was found that if there is no vertex of degree three in the triangulation then scheme (10) with $0 < w < w_0$, $w_0 > \frac{1}{16}$, satisfies the necessary conditions of [8] and the sufficient conditions for C^1 derived from [11] and [14]. It is conjectured that if the necessary conditions of [8] for C^1 hold at irregular points and if the surface is C^1 at all other points, then the surface is globally C^1. The full analysis near irregular points, and the exact range of w and its local dependence on the degree of the vertex are still under investigation. A detailed analysis of the regular case for $w_0 > 0$ small, is given in [8]. In a neighborhood of a vertex of degree three, the surface is certainly not C^1 since the necessary conditions for C^1 do not hold. A modification of scheme (10) for points near such a vertex is also under investigation.

In scheme (10) w serves as a tension parameter in the sense that as w tends to zero the limit surface is tightened toward the piecewise linear control polyhedron. For design flexibility in manipulating both curves and surfaces, one would like to have different tension in different segments and different tension in different directions.

Local tension is achieved by preassigning a tension value w_i^0 to each control point $\bar{p}_i^0$ and by assigning recursively a tension value w_i^k to a new point $\bar{p}_i^k$ by linear interpolation. That is, the definition of a new point by scheme (10) is preceded by defining the parameter w there as

$$w = \frac{1}{2}\,(w_1^k + w_2^k), \tag{11}$$

and this tension value is assigned to the new point.

A generalization of the scalar tension w to a directed tension is obtained by replacing the scalar w in schemes (4), (7), and (10) by a tension matrix,

$$w = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}. \tag{12}$$

To explain the geometric meaning of directed tension, let us rewrite scheme (10) as

$$\bar{q}^{k+1} = \frac{1}{2}(\bar{p}_1^k + \bar{p}_2^k) + w\bar{s} \tag{13}$$

where $\bar{s} = 2(\bar{p}_3^k + \bar{p}_4^k) - (\bar{p}_5^k + \bar{p}_6^k + \bar{p}_7^k + \bar{p}_8^k)$. Hence, $\bar{q}^{k+1}$ is the midpoint of the segment $(\bar{p}_1^k, \bar{p}_2^k)$ "corrected" by $w\bar{s}$. By letting w be a matrix, we provide flexibility in the direction of the correction as well as in its magnitude. For example, in the case of a diagonal matrix w the parameter w_{ii} corresponds to the tension of the ith component of the curve or the surface.

2. IMPLEMENTATION AND EXAMPLES

A software package for implementing and testing the new butterfly schemes has been developed on a SUN 3/50 workstation. It accepts any set of control points in $\mathbb{R}^3$, which together with a proper triangulation form a control polyhedron with piecewise linear triangular faces. The software has the following options:

(1) reading the control polyhedron from an input file;
(2) a global definition of a scalar tension parameter;
(3) local definition of a diagonal matrix tension;
(4) global and local updating of tension parameters;
(5) translation of the control points in space;
(6) flipping edges of the triangulation;
(7) iterated application of the butterfly scheme;
(8) a variety of graphic viewing and displaying options; and
(9) saving the designed shape, that is, its control polyhedron and tension parameters.

In each iteration of the subdivision process, the number of triangles is multiplied by four. Special attention has therefore been given to the design of an economical data structure. The initial control polyhedron is kept in a data structure that allows general triangulation: Each triangle is a record containing its vertices and pointers to neighboring triangles. In order to reduce memory requirements, however, finer triangulation within any of the initial triangles is kept as a two-dimensional array of points in $\mathbb{R}^3$.

The following examples exhibit the main features and the performance of the butterfly scheme for general triangulation. The control polyhedron for the first example is the fish-like shape presented in Figure 3. Figure 4 presents the resulting surface after four subdivision iterations with the butterfly scheme, with a global tension parameter $w = \frac{1}{16}$, in wireframe form and as a shaded surface. The sharp points at the front and the rear are due to the fact that the vertices at the tips of the mouth and the tail are of degree three. The next two figures

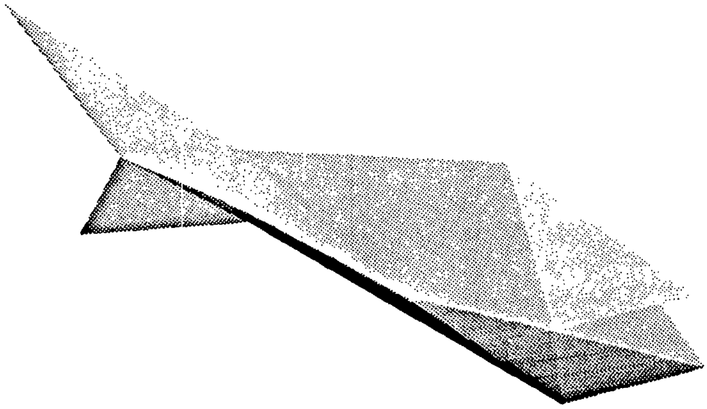

Fig. 3. Control polyhedron of a fish-like shape.

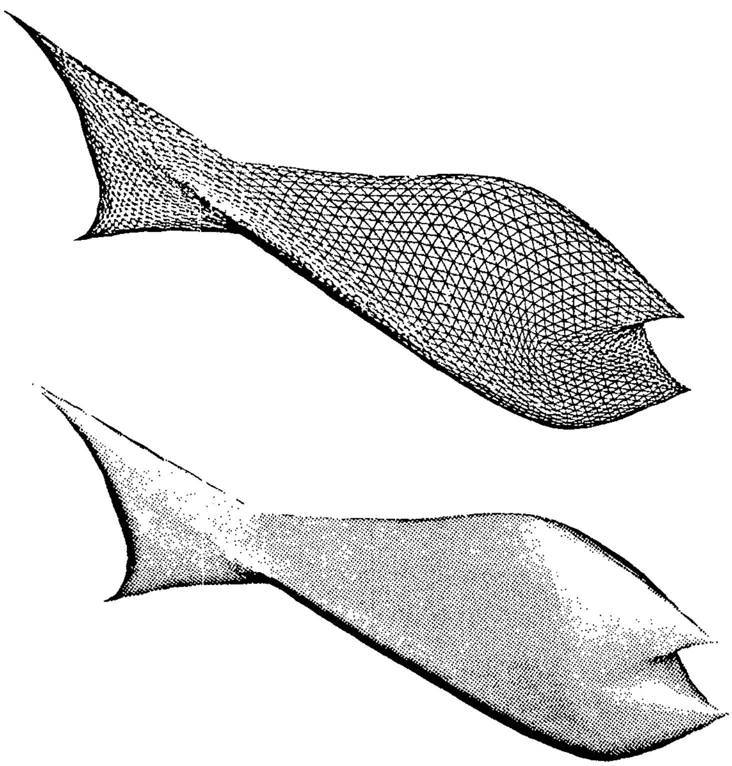

Fig. 4. A fish-like shape after four iterations.

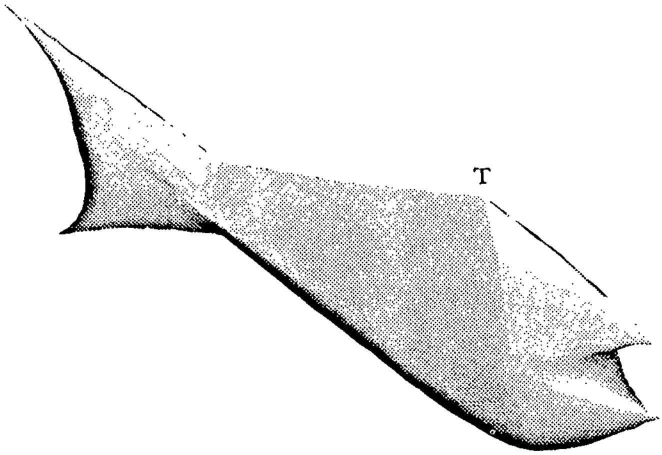

Fig. 5. Effect of local tension $\omega = 0$ at T.

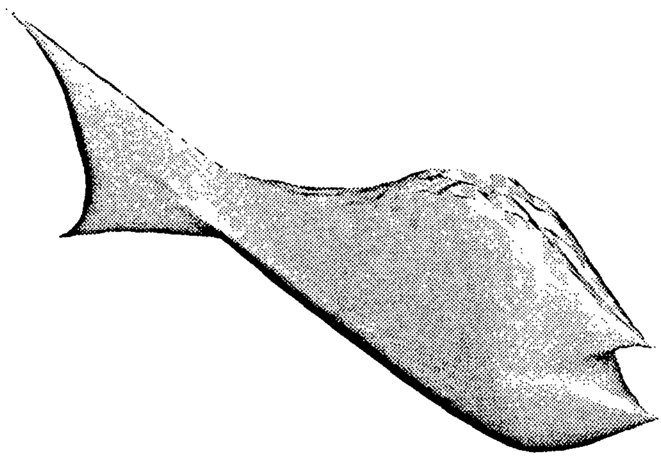

Fig. 6. Effect of directional tension at T outside the allowed range.

exhibit the effects of local tension changes. In Figure 5, we set $w = 0$ at the point marked by T. The result is a surface that is locally tense near T, with a sharp corner. In Figure 6, we set $w = \mathrm{diag}(\frac{1}{16}, \frac{1}{16}, \frac{1}{4})$ at the point T. This tension is outside the C^1 range, and the effect is a fractal-like behavior in a neighborhood of T.

The next example in Figure 7 is of a head-like control polyhedra and the resulting surfaces after two and after four iterations with $\omega = \frac{1}{16}$.

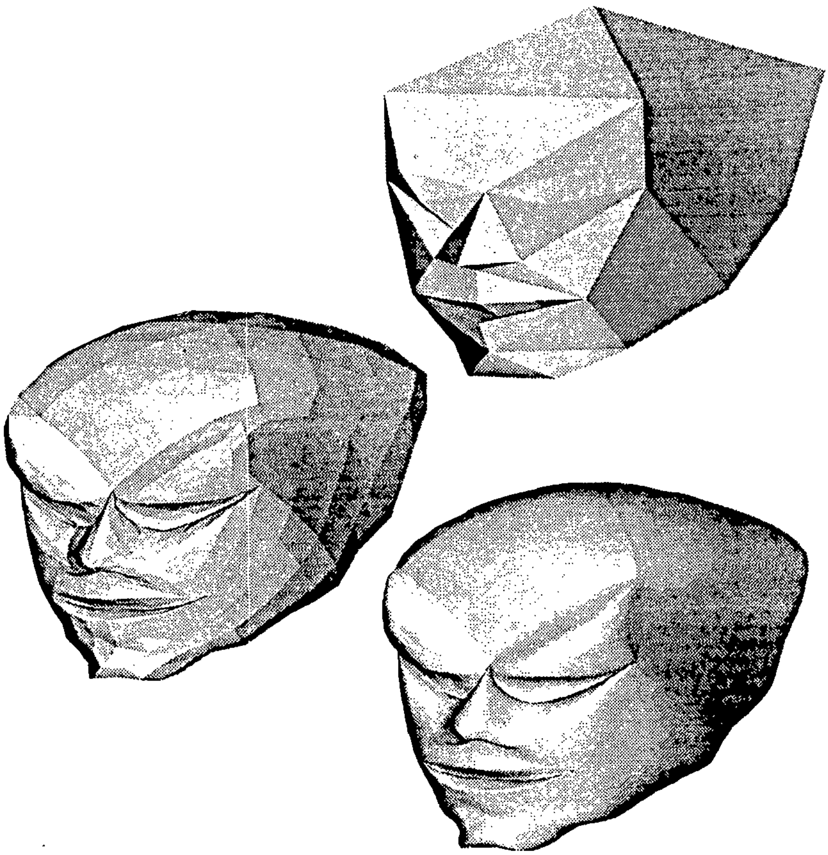

Fig. 7. From control polyhedron of a face-like shape to its smoothed versions after two and four iterations.

ACKNOWLEDGMENTS

The authors are grateful to Aharon Betashvilli, who wrote the computer programs.

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Received September 1988; revised February 1988; accepted May 1989

Editors: R. H. Bartels and R. N. Goldman