

Demonstrating Quantum Speed-Up with a Two-Transmon Quantum Processor.

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Chapter 1

Introduction & Summary

1.1 Quantum Computing with Superconducting Circuits

This thesis presents experiments performed with a superconducting two-qubit quantum processor. The main goal of this work was to demonstrate a possible quantum computing architecture based on superconducting qubits that follows the canonical blueprint of a quantum processor as sketched in fig. 1.1, in accordance with the five criteria formulated by DiVincenzo [17]. By this definition, a universal quantum computer is a register of well-defined quantum bits (1) with long coherence times (2), on which one can implement any unitary evolution using a universal set of quantum gates (3), fitted with individual high fidelity readout of the qubits (4) and with the possibility to reset them to their ground state (5). Implementing this allegedly simple list of requirements in a system of superconducting qubits has been a major research challenge during the last decade, and is part of more general line of research on superconducting quantum circuits briefly summarized below.

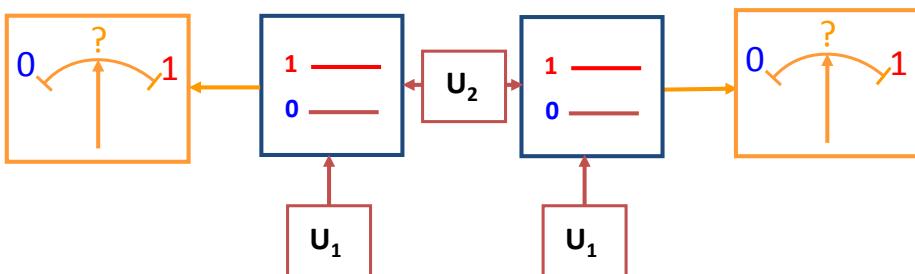


Figure 1.1: The blueprint of a “canonical” two-qubit quantum processor. The two qubits can be individually manipulated (U_1) and a universal two-qubit gate U_2 can be applied to them. Each of the qubits can be read out individually.

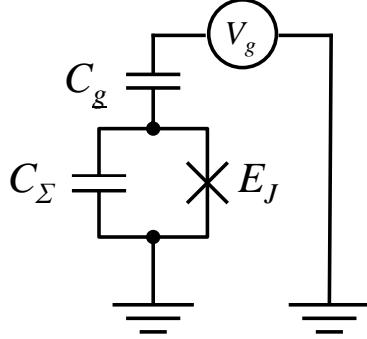


Figure 1.2: The schematic of the simple Cooper pair box (CPB) circuit, consisting of a capacitor in parallel with a Josephson junction capacitively coupled to a voltage source.

1.1.1 Context of this thesis work: 25 years of superconducting quantum circuits

The observation of quantum tunneling in a current-biased Josephson junction switching out of its zero-voltage state by Devoret *et. al.* [14, 35], first demonstrated that a collective electrical variable such as the superconducting phase difference across a Josephson junction (or the conjugated variable, i.e. the number of Cooper pairs that crossed it) can exhibit quantum properties. Then, the observation of microwave induced transitions between the quantum states of the junction by Martinis *et. al.* [35] further confirmed the quantum nature of this degree of freedom (See also [35, 36, 8]). A somewhat simpler quantum electrical circuit called the single Cooper Pair Box (CPB), made of a Josephson junction in series with a gate capacitor and a voltage source as shown in fig. 1.2, was later developed in the Quantronics group during the 1990s [6], and its ground state was characterized. With this electrical circuit, Nakamura *et. al.* [40] performed the first superconducting qubit experiment, demonstrating coherent oscillations between its ground and first excited eigenstates. Although the achieved coherence time was quite short, in the 5-10 ns range, this result attracted a huge interest and triggered the active development of research on superconducting quantum bits.

In the years after, several types of superconducting qubits were proposed using Josephson junctions in different configurations. Different regimes, in which the quantum state of the junctions ranges from almost Cooper pair number states to phase states, were realized. Let us cite here the flux qubit [38, 7] and the phase qubit [37], which were very successful qubits in many aspects. On the side of Cooper Pair Boxes, the Quantronics group contributed to the progress by operating a new circuit called the *Quantronium* (Vion *et. al.* [52]), fitted with a strategy for fighting dephasing due to the noise of the electrical control parameters, and with a single-shot readout (although with limited fidelity). The robustness of the quantronium arises from its operation at a so-called *sweet spot* where the qubit frequency is stationary in respect to variations in charge and phase control parameters. The improved coherence of the quantronium allowed to perform all the basic manipulations possible on spins and more generally on

two level systems [10]. Shortly after, another CPB design, inspired from cavity-QED, was developed at Yale by Wallraff *et. al.* [53]. In this so-called circuit-QED (CQED) design, the CPB, embedded in a microwave resonator, can be thought of as an artificial atom in a resonant cavity. The qubit readout is performed dispersively, i.e. through the cavity pull of the resonator frequency controlled by the qubit state. This small frequency change results in a small phase change of a resonant microwave pulse, which yields –after sufficient repetition– the probability of the two qubit states [5].

Another great bonus of CQED is that the electromagnetic environment in which the qubit relaxes its energy consists of a microwave resonator with a well-controlled impedance. The modern version of the Cooper Pair Box called *Transmon*, follows this design with an extra feature that makes it insensitive to the charge noise which plagues single electron and single Cooper Pair devices. This feature consists in placing the Cooper Pair Box in the phase regime by adding an extra capacitance in parallel with the junction: the qubit frequency is then totally insensitive to the gate charge, and hence to the charge noise to all orders in charge, thus transforming the sweet spot into a “sweet line” in the charge-phase parameter space. This new design, that still leaves sufficient anharmonicity to operate the device as a qubit and allows to drive it, yielded a sizable improvement in coherence times, qubit robustness and usability. The CQED concept was thus rapidly extended to flux and phase qubits [28].

In 2010, a new type of CQED architecture has been developed by Paik *et. al.* [42] that combines Transmon qubits with 3D cavities instead of CPW resonators, resulting again in an impressive increase of qubit coherence times by a large factor, with reported qubit relaxation times approaching $100\ \mu\text{s}$ and dephasing times of the same order. Very recently, these drastically improved coherence times have made possible the realization of elemental quantum feedback and error correction schemes with these systems [51].

The progress achieved during the last decade on the Cooper Pair Box and on the phase qubits has benefited to quantum processors. So far, superconducting CQED processors with up to three qubits have been realized and two- and three-qubit quantum gates [20], multi-qubit entanglement [16, 1] and simple quantum algorithms [15, 34] as well as quantum error correction [45] have recently been demonstrated.

1.1.2 This Thesis Work

At the beginning of this thesis work, CQED processors having demonstrated quantum algorithms did not comply with the rules established by DiVincenzo [17]. In other words, they did not follow the canonical blueprint able to demonstrate quantum speed-up: they were all fitted with a joint readout, which allows to measure the average value of a collective variable of the qubit register, but not each qubit individually. By repeating a given

sequence of gates a large number of times, one can nevertheless determine the quantum state of the qubit register at different steps of the algorithm being run. Since the whole interest of quantum computing is precisely to perform computational tasks more efficiently than with a classical processor, it was essential, in our mind, to demonstrate the quantum speed-up expected from quantum algorithms with a CQED quantum processor fitted with an individual single shot and non-demolishing (QND) readout for each qubit. Such a high fidelity single-shot readout had been developed for a single transmon during a previous thesis in the Quantronics group [33, 43], and it was natural to use it in the present work.

This thesis discusses therefore the realization of a superconducting two-qubit processor based on Transmon qubits, fitted with individual single-shot readouts. In chapter 2, we present the theoretical building blocks of this work. Chapter 3 outlines the design of the two-qubit processor, whereas chapter 4 presents the most relevant measurement techniques used here. With the two-qubit processor, we implement elementary one- and two-qubit quantum operations, as detailed in chapter 5. We use it to run a simple quantum algorithm that demonstrates probabilistic quantum speed-up: the Grover search algorithm, as explained in chapter 6. Finally, we discuss in chapter 7 the realization of a four-qubit quantum processor using a more scalable approach that could possibly be extended to an even larger number of qubits.

Note that during this thesis work, quantum speed-up was also demonstrated for the Deutsch-Josza algorithm with a phase qubit processor using individual single-shot and destructive readouts [54].

1.2 Realizing a Two-Qubit Quantum Processor

The quantum processor implemented in this work is shown in fig. 1.3. It consists of two superconducting quantum bits of the Transmon type, each equipped with its own drive and readout circuit. In order to obtain a high fidelity single-shot readout of the qubit register, we use the Cavity Josephson Bifurcation Amplifier (CJBA) readout method, which is based on the Josephson Bifurcation Amplifier (JBA) readout first developed in the team of Michel Devoret at Yale for the quantronium qubit [48, 50, 47]. This method had indeed already been successfully adapted to the transmon, and yielded a 93 % readout fidelity [33].

Each qubit can be manipulated by driving it with microwave pulses through its readout resonator, allowing robust and fast single-qubit operations. The qubit frequencies can be tuned individually using fast flux lines, allowing us to change the frequency of each qubit over a range of several GHz. The coupling between the two qubits is realized through a fixed capacitance that connects the two top-electrodes of the Transmons

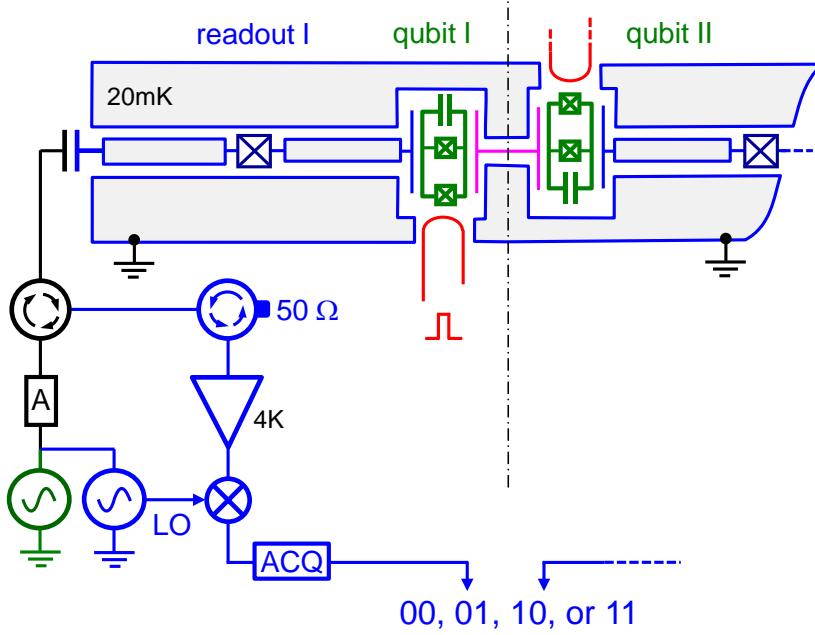


Figure 1.3: Circuit schematic of the two-qubit processor realized in this thesis work, showing the two qubits (in green) coupled by a fixed capacitor (in purple), as well as the fast flux lines (in red) used to tune the qubit frequencies and the qubit readouts (in blue). Each qubit is embedded in its own nonlinear readout resonator and can be driven and read out by microwave reflectometry through an individual microwave line.

and implements a fixed σ_{xx} -type qubit-qubit coupling. This coupling allows us to generate entangled two-qubit states and to implement a two-qubit gate. We use this simple processor to generate entangled two-qubit states, test the Bell inequality, implement a universal two-qubit gate and perform a simple quantum algorithm that demonstrates quantum speed-up, as discussed in the following sections.

1.3 Demonstrating Simultaneous Single-Shot Readout

For readout, each qubit is capacitively coupled to a coplanar waveguide resonator made nonlinear by placing a Josephson junction in its central conductor. We exploit the frequency pull of the bifurcation transition that occurs in this resonator when driven at a suitable frequency and power to map the qubit states on the bifurcated and non bifurcated cavity states, which are then discriminated by reflectometry. Here, the hysteretic character of the bifurcation transition allows to reduce the measuring power, to latch the cavity state, and to measure it without being affected by subsequent qubit relaxation. The state of the resonator can thus be determined reliably without being limited by qubit relaxation, thereby providing a high-fidelity single-shot qubit readout. Contrary to previous CQED processors, our processor is fitted with individual readouts, and a

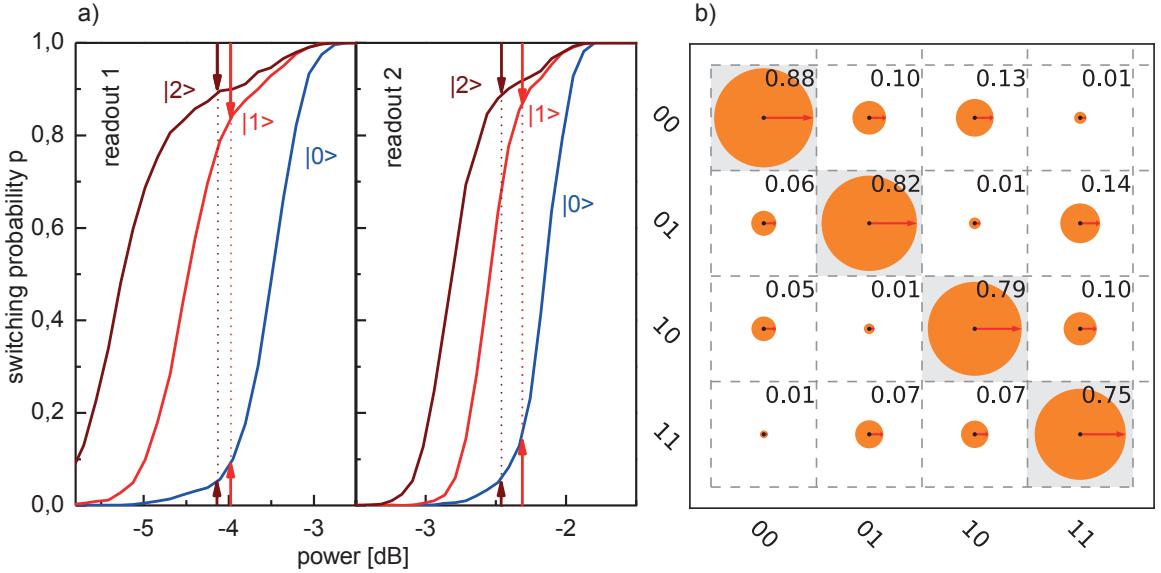


Figure 1.4: a) Switching probabilities of the two readout resonators as a function of the readout drive power at a fixed driving frequency. The measurement is performed after preparing the qubits in the states $|0\rangle$, $|1\rangle$ and $|2\rangle$. The readout contrast is given as the difference in probability between the curves corresponding to the states $|0\rangle$ and $|1\rangle$ or $|2\rangle$, respectively. The highest contrasts of 88 and 89 % are achieved when the qubit is shelved from state $|1\rangle$ to $|2\rangle$. b) Readout matrix of the two-qubit system, giving the probabilities to obtain the different outcomes ij after having prepared the register in the different computational basis states $|kl\rangle$.

simultaneous readout of the full two-qubit register is possible, as requested by the Di-Vincenzo criteria. For a single-qubit CJBA readout, fidelities up to 93 % have been achieved [33] by shelving the first excited state of the transmon $|1\rangle$ to the higher excited state $|2\rangle$. However, due to the higher complexity and design constraints of our system, only 83-89 % fidelity has been achieved for the processor presented here. The full characterization of the readout of our processor is shown in fig. 1.4. Panel a shows the switching probabilities of each individual readout as a function of the drive amplitude, measured at a fixed drive frequency. Individual curves correspond to the qubit being prepared (or shelved) in different states $|0\rangle$, $|1\rangle$ or $|2\rangle$, the difference between either two curves giving the readout contrast between those qubit states. Shelving the qubit from state $|1\rangle$ to state $|2\rangle$ before readout can increase the readout fidelity by more than 10 % and is therefore often used in the experiments presented in this thesis. Panel b shows the full readout matrix of the two-qubit register that relates measured readout switching probabilities with real qubit state occupation probabilities and allows us to correct readout errors when performing quantum state tomography. In chapter 5 we discuss all relevant readout fidelities and errors in detail and analyze different error sources limiting the readout performance in our experiments.

1.4 Generating and Characterizing Entanglement

The capacitive coupling between the two qubits provides a σ_{xx} -type interaction that can be used to generate entangled two-qubit states. Conveniently, this coupling is only effective when the qubit frequencies are near-resonant and can therefore be effectively switched on and off by tuning the qubit frequencies in and out of resonance. For the processor realized in this work, the effective coupling constant g of the two qubits has been measured as $2g/2\pi = 8.2$ MHz. When the two qubits are in resonance, the effective evolution operator of the two-qubit system is:

$$U(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi t g & i \sin 2\pi t g & 0 \\ 0 & i \sin 2\pi t g & \cos 2\pi t g & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}} \quad (1.1)$$

By using fast flux pulses to non-adiabatically tune the qubits in and out of resonance, we can switch on this interaction for a well-defined time. We first characterize the effect of the coupling on the qubit register by preparing the state $|10\rangle$, tuning the qubits in resonance for a given time and measuring the qubit state afterward. The resulting curve is shown in fig. 1.5a and shows swapping oscillations between the two qubits. Analyzing this curve allows us to extract the effective coupling strength between them. Leaving the interaction between the qubits on for a well-defined time allows us to generate entangled Bell states that we characterize by performing quantum state tomography. The experimental reconstruction of the density matrix of such a Bell-state of the type $|\psi\rangle = (|01\rangle + i|10\rangle)/\sqrt{2}$ is shown in fig. 1.5b. The measured fidelity of the prepared state of 91 % and the concurrence of 85 % confirm that entanglement is present in the system. We also characterize the entanglement between the two qubits by measuring the average value of the so-called *Clauser-Horne-Shimony-Holt* operator (CHSH) [9], which combines measurements of the state of the two qubits along different axes on the Bloch sphere and provides a test that can distinguish between classical correlations and quantum entanglement in a two-qubit system.

For classical states, the maximum value of the CHSH operator is bound by 2 but for entangled states it can reach a maximum of $2\sqrt{2}$. Figure 1.6 shows the result of such a CHSH-type measurement performed on a state created by the method described above, showing the value of $\langle \text{CHSH} \rangle$ as a function of the angle ϕ of the measurement basis. We observe a violation of the classical boundary 2 of the operator by 22 standard deviations when correcting the readout errors that are present in our system. The raw, uncorrected data fails to exceed the classical threshold because of readout errors mainly caused by qubit relaxation during the readout. Nevertheless, the observed violation in

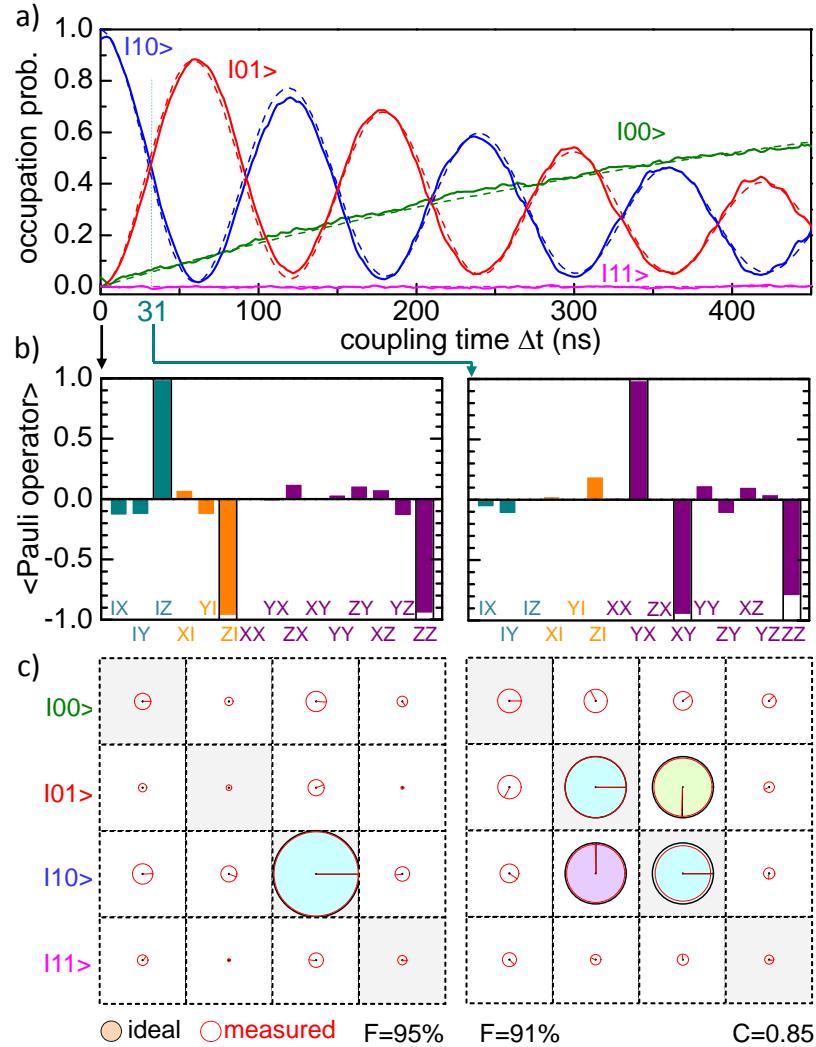


Figure 1.5: Coherent exchange of a single quantum of excitation (swapping oscillations) between the two qubits initially prepared the register state $|10\rangle$, obtained from the resonant interaction between them. a) Register state probabilities as a function of the swapping time Δt . The frequency of the oscillations corresponds to $2g/2\pi = 8.7$ MHz. b) Measured average values of the Pauli operators products $\{I, \sigma_x, \sigma_y, \sigma_z\} \otimes \{I, \sigma_x, \sigma_y, \sigma_z\}$ (Pauli set) for the register states obtained at times 0 ns and 31 ns. c) Corresponding reconstructed density matrices. The area of each circle corresponds to the absolute value of each matrix element and the color and direction of the arrow to the phase of the element. The black circles correspond to the density matrices of the ideal states $|10\rangle$ and $(|10\rangle + i|01\rangle)/\sqrt{2}$, respectively.

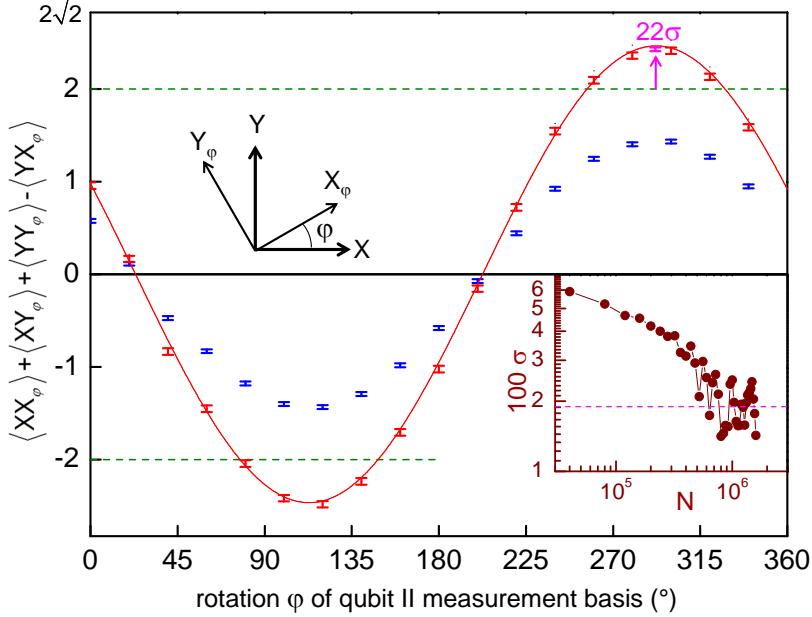


Figure 1.6: Measured average value of CHSH operator for a prepared Bell state. After readout error corrections, the CHSH expectation value (red points) exceeds the classical boundary of 2. The raw measurement data (blue points) lies below this critical threshold. The inset shows the standard deviation σ at the highest point of the curve as a function of the measurement sample size. For the highest sample count, the classical boundary is exceeded by 22 standard deviations.

the calibrated data is a strong indication of entanglement in the system. A more detailed overview of this experiment can be found in chapter 5.

1.5 Realizing a Universal Two-Qubit Quantum Gate

The swapping evolution given by eq. (1.1) allows not only to prepare entangled two-qubit states but also to implement a universal two-qubit gate: When switching on the interaction for a time $t_{\pi/2} = 1/8g$ one realizes the so-called $\sqrt{i\text{SWAP}}$ gate, represented by the evolution operator

$$U(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}} \quad (1.2)$$

which forms together with single qubit gates a universal set of gates, on which any algorithm can be decomposed. We characterize the operation and errors of our implementation of this gate by performing quantum process tomography, obtaining a gate fidelity of 90 %. The 10 % error in gate fidelity is caused mainly by qubit relaxation and

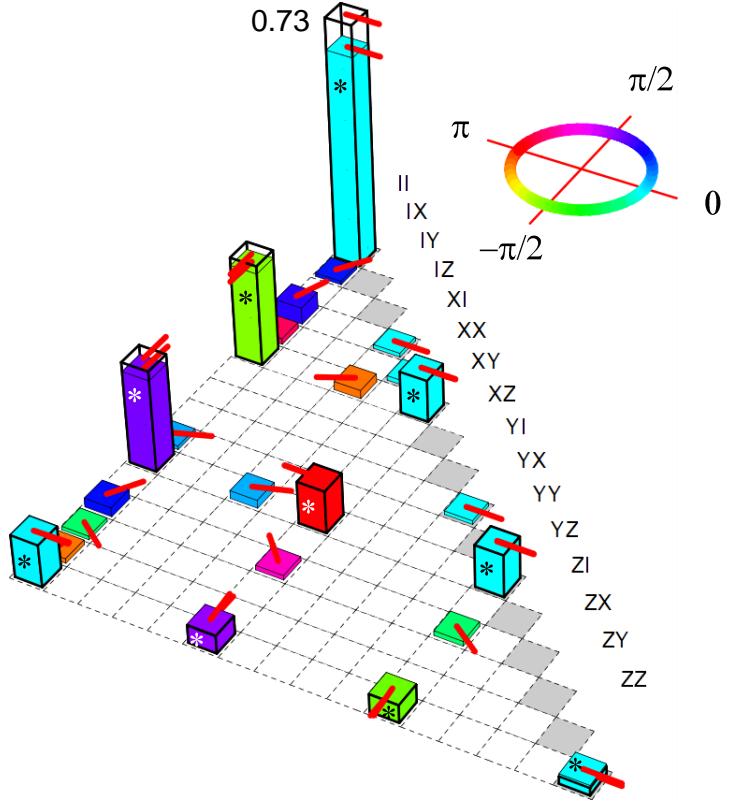


Figure 1.7: Measured χ -matrix of the implemented $\sqrt{i\text{SWAP}}$ gate. The row labels correspond to the indices of the E_i Pauli operators, the height of each bar to the absolute value of the corresponding matrix element, and the color and red arrow direction to the argument of the element. The ideal χ -matrix of the $i\sqrt{\text{SWAP}}$ gate is given by the outlined bars. The upper half of the positive-hermitian matrix is not shown.

dephasing during the gate operation and only marginally by deterministic preparation errors, as will be discussed in chapter 5. Figure 1.7 shows the measured χ matrix of the gate, that describes its effect in the Pauli basis of two-qubit operators. The χ matrix provides the full information on the unitary and non-unitary action of the gate. The achieved fidelity of the gate operation is sufficient to allow the implementation of simple quantum algorithms with our processor.

1.6 Running a Quantum Search Algorithm

Using a two-qubit quantum gate related to the one described above, we run a simple quantum algorithm on our processor, the so called *Grover search algorithm* [25]. The version of this algorithm that we implement operates on the two-qubit basis $x_i \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and can distinguish between four different *Oracle functions* $\mathcal{C}_j(x)$ with $x \in x_i$ that give $\mathcal{C}_j(x = x_j) = 1$ and $\mathcal{C}_j(x \neq x_j) = 0$. In the two-qubit case, this algorithm requires only one evaluation of the Oracle function $\mathcal{C}_j(x)$, implemented as a unitary operator, to determine which state among the four possible ones it tags. This case thus provides a simple benchmark of the operation of the quantum processor, and a simple and illustrative example of quantum speed-up in comparison with classical algorithms, as discussed in chapter 6. The diagram of the Grover search algorithm implemented

in our processor is shown in fig. 1.8a and involves two *i*SWAP gate operations and six single-qubit operations along with a single-shot qubit readout at the end of the algorithm. We measure the success probability of the algorithm from the obtained outcomes, and complete the analysis of its operation by performing the tomography of the quantum state at different steps of the algorithm. We first discuss this evolution that sheds light on how quantum speed-up is achieved.

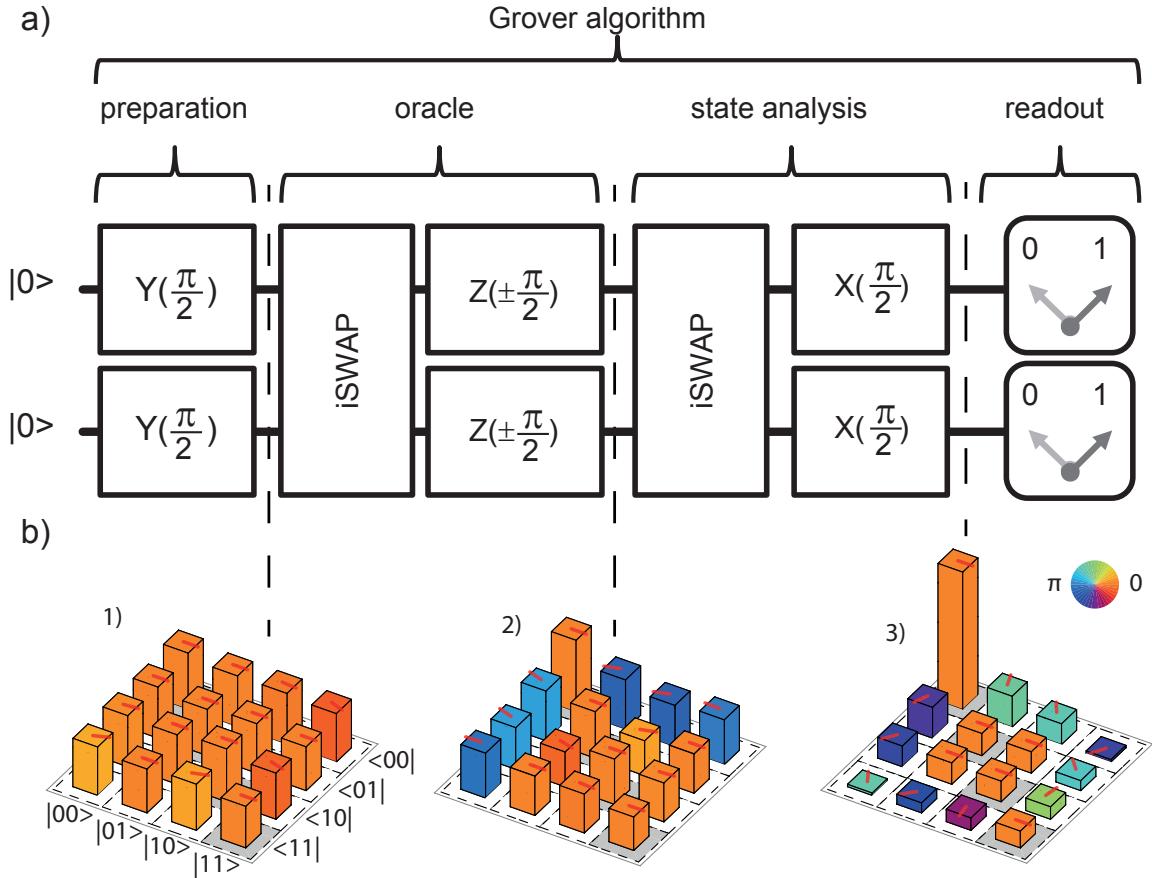


Figure 1.8: a) A two-qubit version of the Grover search algorithm that we implemented on our quantum processor. The algorithm consists in preparing a fully superposed state, applying a given Oracle operator to it only once, and analyzing the resulting output to determine the quantum state tagged by this Oracle operator. b) Measured density matrices when running the Grover search algorithm with a search oracle marking the state $|00\rangle$. 1) shows the state after the generalized Hadamard transform, 2) after applying the quantum oracle and 3) after the final step of the algorithm.

Fig. 1.8b shows the density matrices determined experimentally when running the Grover search algorithm with the Oracle operator tagging the state $|00\rangle$. State tomography is first shown after preparation with a generalized Hadamard transform applied to the initial state $|00\rangle$. It clearly corresponds to a superposition of all the computational basis states. The quantum state after having applied the quantum Oracle is $-|00\rangle + |01\rangle + |10\rangle + |11\rangle$ and the information on the tagged state is encoded in the

phase of the state $|00\rangle$. After extracting this phase information, the tomography displays a large peak on state $|00\rangle$ at the end of the algorithm, just as expected. The fidelity of the final quantum state of the algorithm is 68%, 61%, 64% and 65% for the four different Oracle operators, respectively. These fidelities, corrected for readout errors, do not quantify the quantum speed-up achieved when running the algorithm. For this, it is necessary to analyze the results obtained after a single run, which does not allow for any corrections of the readout outcomes.

1.7 Demonstrating Quantum Speed-Up

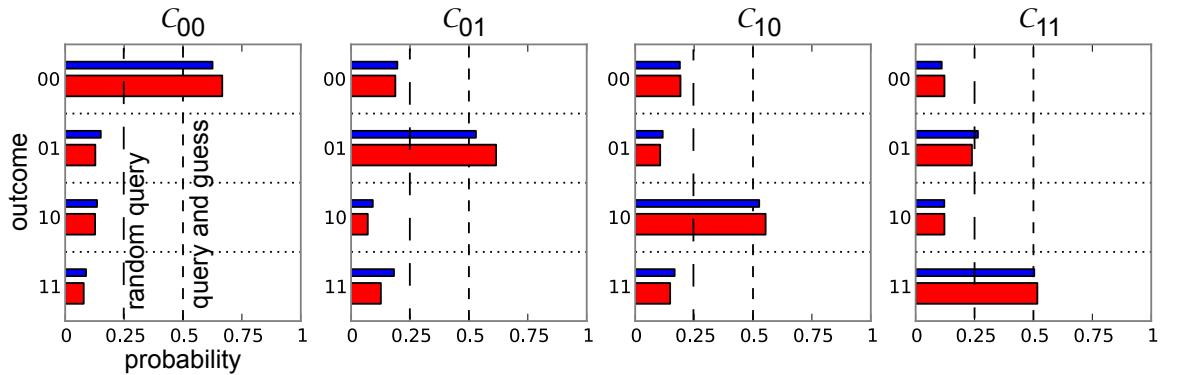


Figure 1.9: Single-run results when running the Grover search algorithm on our two-qubit quantum processor. Shown are the probabilities of measuring the qubit register in state $|i\rangle$ for an Oracle function \mathcal{C}_j marking the state $|j\rangle$ provided to the algorithm. In all four cases, the success probability of the algorithm is $> 50\%$, thus outperforming any classical “random query” or “query and guess” algorithm using a single Oracle call.

The main interest of running a quantum algorithm is to obtain an advantage in the run-time in comparison to a classical algorithm, the so-called *quantum speed-up*. To characterize this speed-up as obtained with our processor, we run the Grover algorithm for all four possible Oracle functions and directly read out the state of the qubit register after the last step of the algorithm instead of performing quantum state tomography, thus not correcting any readout errors. By averaging the outcomes of many such individual runs of the algorithm with different Oracle functions we obtain the so-called *single-run fidelities*, which –for the four different Oracle functions– have been measured as 66%, 55%, 61% and 52%. The full probability distributions for the four possible cases are shown in fig. 1.9. The achieved success probability is always lower than the theoretically possible value of 100 %, mainly because of relaxation and decoherence of the qubit state during the run time of the algorithm and –to a small degree– errors in the pulse sequence. The measured success probabilities are however larger than the 50% success probability of a classical query-and-guess algorithm using the outcome of a sin-

gle query. The algorithm thus demonstrates quantum speed-up, as explained in greater detail in chapter 6.

1.8 Towards a Scalable Multi-Qubit Architecture

The approach to superconducting quantum computing outlined in the previous sections is well suited for the implementation of simple quantum processors with a few qubits. However, due to several design limitations it is not suitable for implementing a large scale quantum computer. As an example, the direct qubit-qubit coupling employed in this thesis work is not suitable for coupling a large number of qubits since it becomes increasingly difficult to deterministically switch on and off the coupling between individual qubits as the number of qubits increases, a problem sometimes referred to as “frequency crowding”. Also, fitting each qubit of the processor with individual drive and readout circuitry –as done in this work– is usually not extensible to a large number of qubits due to topological and space constraints on the chip, as well as financial and practical constraints for a cryogenic experimental setup.

Recently, several research groups have started to address these issues by devising new architectures for superconducting quantum processors that can –in theory– be scaled to a larger number of qubits. Here we will mention only the so-called “Rez-Qu” architecture [22] and the surface-code approach [18] pursued e.g. by IBM. In this thesis work we discuss our own approach towards scalable quantum computing, where we have developed a revised version of our qubit chip that provides a way to implement a system with a larger –albeit still small– number of qubits. Key elements of this architecture are a quantum bus in form of a high-Q microwave resonator that is used for coupling the qubits and a multiplexed drive and readout circuit that allows us to measure

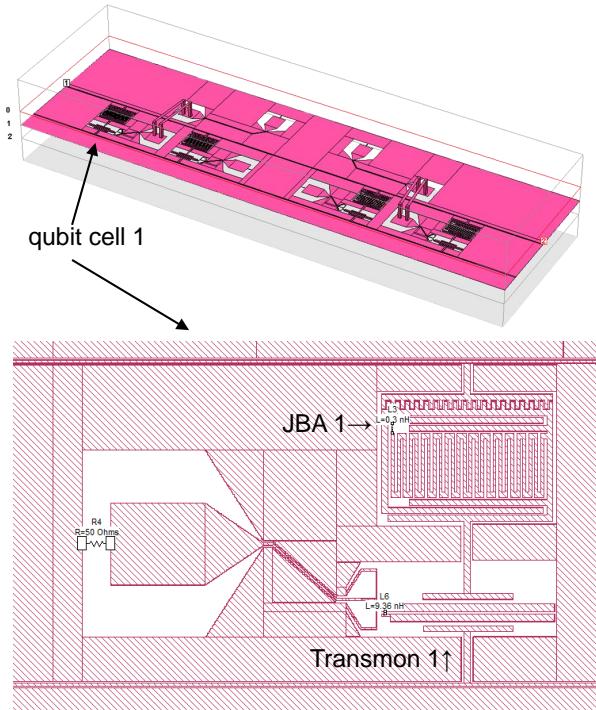


Figure 1.10: Schematic of the four-qubit chip realized in this thesis work. Shown on top is the whole chip with connectors for the drive and readout transmission line and for the four fast flux lines. Below we show a single qubit cell with a Transmon qubit coupled to the quantum bus, a flux line and a JBA readout.

and manipulate all qubits through one single microwave transmission line.

b Figure 1.10 shows the schematic of the first version of this architecture. Our chip contains four Transmon qubits which are capacitively coupled to a distributed high-Q resonator acting as a quantum bus. In addition, each qubit is coupled to a low-Q non-linear lumped element resonator acting as a CJBA, which is used for reading out the qubit state. Each of these resonators is in turn coupled capacitively to the input transmission line. The resonance frequencies of the readout resonators are arranged in ascending order with a frequency spacing of ≈ 50 MHz between adjacent resonators. This frequency spacing allows us to address each resonator individually and to read out the state of the full qubit register in parallel using only one single transmission line. Figure 1.11

shows the measured $|S_{12}|$ matrix element of such a chip with four CJBA resonators at frequencies between 6.78 GHz and 6.95 GHz, plotted as a function of the incident microwave power. The “knee” in each of the four resonance curves appearing between -45 and -40 dBm corresponds to the bifurcation point of the resonators.

Still, this approach suffers from a relatively bad ON/OFF ratio in the qubit-qubit coupling. To alleviate this problem, the Transmon qubit used in the current version of this architecture can be replaced by a qubit with a tunable coupling [49]. Alternatively, it is possible to use a fixed-frequency coupling scheme for the qubits, thereby altogether eliminating the need for frequency tuning of individual qubits and also reducing the number of input transmission lines from $n + 1$ to 1, with n the number of qubits.

A detailed discussion of the scalable architecture and the first preliminary measurements performed with a four-qubit chip are presented in chapter 7 of this thesis.

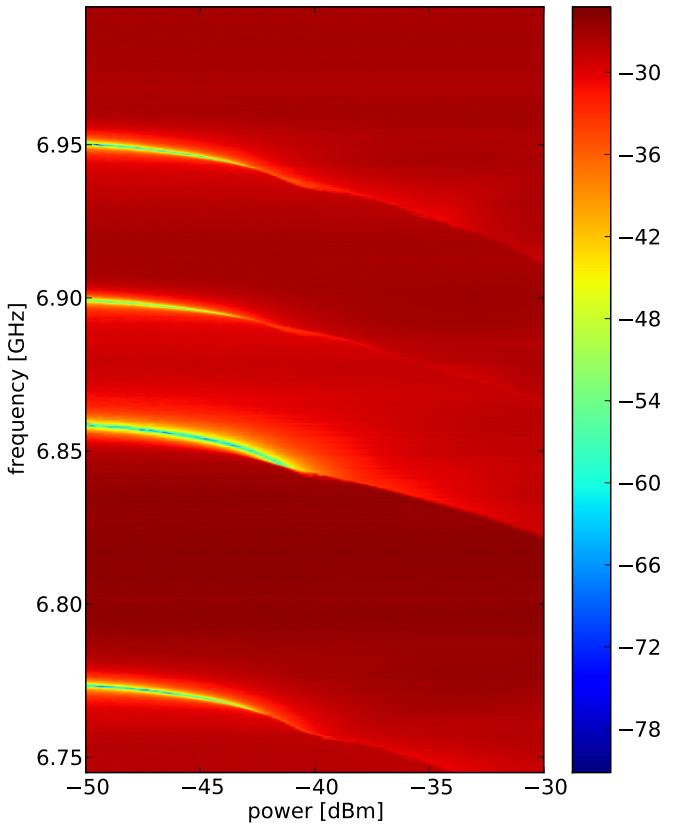


Figure 1.11: The measured $|S_{12}|$ transmission coefficient for the input transmission line of our four-qubit chip. Clearly visible are four resonances of the CJBA resonators and the bifurcation of each resonator at high input power.

Chapter 2

Theoretical Foundations

The goal of this chapter is to provide the theoretical foundations needed to interpret and analyze the experiments discussed in the following chapters. We will therefore briefly introduce some basic concepts of quantum mechanics and quantum computing, discuss Transmon qubits and circuit quantum electrodynamics (CQED) and introduce the reader to the *cavity bifurcation amplifier (CBA)* that we use in our experiments. More detailed discussions of the properties of all the circuit components discussed here will be provided in the relevant sections of the “Experiments” chapter.

2.1 Clasical & Quantum Information Processing

By definition, computing designates the activity of using computer hardware and software to process information, or *data*. Classical information processing can be divided in so-called *analog and digital information processing*, the former being based on continuously changeable physical quantities whereas the latter is based on incrementally changeable quantities. The fundamental unit of digital information processing is the so-called *bit*, which represents a boolean (true/false) information. The discipline of theoretical computer science has been created to investigate the fundamental limits and properties of classical information processing. One of the main foundational theorems of theoretical computer science is the so-called *Church-Turing thesis* which provides a universal computing model by saying (basically) that everything which is computable can be efficiently computed using a *Turing machine*. Such a Turing machine, in turn, is a simple theoretical devices which is able to run programs that operates on a discrete set of data using a well-specified set of operations. The Turing machine is universal in the sense that any other classical computing device can be efficiently simulated using a Turing machine with the appropriate program and data.

Surprisingly, Richard Feynman discovered in the early 1980 that a classical Turing machine as described above would be unable to efficiently simulate a quantum-

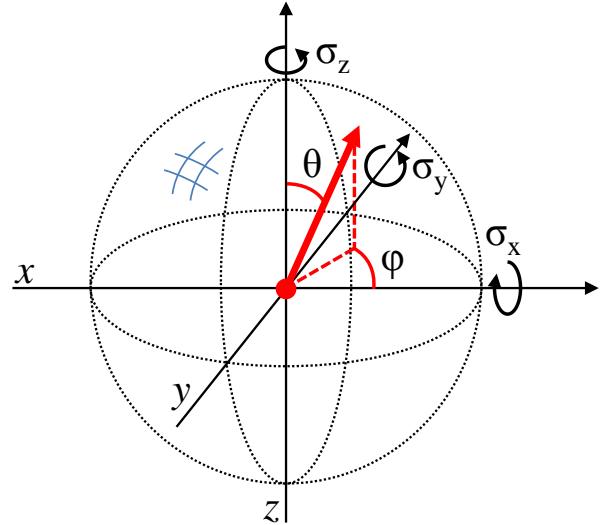


Figure 2.1: The Bloch sphere representation of a qubit state $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$. The state $|\psi\rangle$ is fully characterized by specifying its “altitude” and “azimuth” angles θ and ϕ . Pure quantum states will always lie on the surface of the Bloch sphere, whereas mixed quantum states can also lie anywhere inside the sphere.

mechanical system [21]. A few years later, David Deutsch took up Feynman’s idea and developed an information processing framework based on quantum mechanics. He showed that by using this so-called *quantum computing* or *quantum information processing* framework one could solve certain problems faster than would be possible with any classical Turing machine [12]. The work by Feynman and Deutsch created a large interest in the physics community and led to a huge experimental and theoretical effort aimed at realizing a working quantum computer.

2.2 Principles of Quantum Computing

In this section we will discuss the basic principles of quantum computing, including quantum bits and quantum gates. We will also briefly discuss some examples of quantum algorithms that are relevant to this work.

2.2.1 Quantum Bits

Similar to classical computing, in quantum computing one can define a fundamental unit of information, the so called *quantum bit* or *qubit*. Such a qubit is a quantum-mechanical two-level system described by the wavefunction

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \quad (2.1)$$

As can be seen, the state of such a qubit can be described by a pair of real numbers θ and ϕ that characterize the occupation probability of each of the two basis states $|0\rangle$ and $|1\rangle$ and the phase between them. A useful and intuitive representation of such a single-qubit state is the so-called *Bloch sphere representation* of a quantum state, which

is shown in fig. 2.1. In this representation, the state $|\psi\rangle$ is located on a unit sphere. The north and south poles of this sphere correspond to the qubit states $|0\rangle$ and $|1\rangle$. All states lying between those two correspond to superposition states, which are characterized by their “altitude” and “azimuth” angles θ and ϕ .

2.2.2 Quantum Gates

Analogously to classical information processing it is necessary to define *quantum gates* which act on individual or multiple qubits and allow us to process information with them. In the most general sense, a quantum gate is a unitary quantum operator that acts on one or several qubits. Theoretically there is an infinite number of possible quantum gates, however in order to describe all possible quantum operations that can be performed on a register of qubits it is sufficient to define an *universal set of quantum gates*. Such a universal gate set that will be especially relevant to this work consists of the three single-qubit rotation matrices

$$R_x(\theta) = \begin{pmatrix} \cos \theta/2 & -i \sin \theta/2 \\ -i \sin \theta/2 & \cos \theta/2 \end{pmatrix} \quad (2.2)$$

$$R_y(\theta) = \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ +\sin \theta/2 & \cos \theta/2 \end{pmatrix} \quad (2.3)$$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{+i\theta/2} \end{pmatrix} \quad (2.4)$$

together with the so-called *iSWAP* two-qubit operator, which has the representation

$$i\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.5)$$

in the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$. Often we will represent these gates schematically as shown in fig. ??.

2.2.3 Quantum Algorithms

The interest in quantum computing is mainly due to the fact that certain problems can be solved faster on a quantum computer than on a classical computer. By faster we mean here that the order \mathcal{O} of the runtime of the algorithm increases faster on a classical computer than on a quantum computer as a function of the problem size. Up to this

day it has not been shown that a quantum computer can perform all tasks faster than a classical computer, however certain real-world problems have been found that can be solved exponentially to polynomially faster on a quantum computer. Here we will briefly discuss some of them:

1. **Grover's Search Algorithm**: Discovered by Lev Grover in 1996 [24], this search algorithm can find a single well-defined state in an unsorted database of size N in $\mathcal{O}(\sqrt{N})$ steps, being hence quadratically faster than a classical search algorithm.
2. **Shor's Factorization Algorithm** Discovered by Peter Shor in 1994 [10], this algorithm can factorize a binary number of length N into its prime factors in $\mathcal{O}(\log^3 N)$ steps, therefore exponentially outperforming any known classical factorization algorithm. There is large interest in this algorithm since products of large prime numbers are routinely used in asymmetric cryptography.

2.2.4 Quantum Simulation

Another domain of interest for quantum computers is so called *quantum simulation*. Here the goal is to simulate the behaviour of an arbitrary quantum system using a quantum computer by either engineering the quantum computer in direct analogy with the system being modeled (so called *analog quantum simulation*) or by numerically simulating the Hamiltonian of the quantum system on a general-purpose quantum computer (so-called *digital quantum simulation*). Since no classical computer can simulate a quantum system efficiently, there is a large interest in quantum simulation, especially in the fields of biology and chemistry (e.g. for protein folding[] [!1!](#))

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2.2.5 Realization of a Quantum Computer

To realize a working quantum computer it is necessary to implement highly coherent qubits that can be manipulated, read out and coupled with high fidelity. So far, no fully working quantum computer has been experimentally demonstrated. However, larger progress towards the realization of such a quantum computer has been achieved in the last decade. Promising approaches for the realization of a quantum computer include –among others– ions trapped in magnetic and electric fields [], cold atomic gases [], photonic circuits [], semiconductor structures [] and, last but not least, superconducting structures. Since this work treats only superconducting qubits of the Transmon type we will focus our attention on them in the following sections. We will explain how we can realize a reliable qubit using superconducting structures and how we can implement circuits to manipulate, couple and read out the qubit state.

2.3 Superconduting Quantum Circuits

In this section we will discuss several types of superconducting circuit elements that are most relevant to this work. We will introduce the reader to the Josephson junction, discuss the properties of transmission lines and transmission line resonators and introduce a general method for the quantization of arbitrary circuits. Afterwards we will discuss the Cooper pair box and the Transmon qubit, give a short overview of circuit quantum electrodynamics and finally introduce the reader to the Josephson and cavity bifurcation amplifier.

2.3.1 The Josephson junction

The core element used to construct quantum circuits is the so-called *Josephson junction*, being equivalent to the transistor in classical circuits in significance. A Josephson junction is based on a discovery of Brian Josephson, which published a now-classical paper on quantum tunneling between weakly coupled superconductors [29]. He found, that such a *weak link* between two superconductors could support a supercurrent I described by the simple formula

$$I = I_c \sin \phi \quad (2.6)$$

where $\phi = \phi_2 - \phi_1$ and ϕ_1 and ϕ_2 are the superconducting phases at each side of the link. This simple equation, together with the current-phase relation of a Josephson junction,

$$U = \frac{\hbar}{2e} \frac{\partial \phi}{\partial t} \quad (2.7)$$

yields a system exhibiting a wealth of interesting physical phenomena that are used today in various applications in physics. The energy associated with the Josephson junction is given as

$$E = E_J(1 - \cos \phi) \quad (2.8)$$

where $E_J = I_c \Phi_0 / 2\pi$ is the so-called *Josephson energy*. In addition to this Josephson energy, the junction usually has an energy associated to the capacitance formed by the two separated electrodes of the junction and given as $E_c = Q^2 / 2C$.

For currents $I \ll I_c$, a Josephson junction behaves approximatively like a nonlinear inductor with inductance

$$L_J = \frac{\Phi_0}{2\pi I_c \cos \phi} \quad (2.9)$$

where $\Phi_0 = h/2e \approx 2.05 \times 10^{-15}$ Wb is the so-called *magnetic flux quantum*. Later we will show how to make use of these properties of the Josephson junction to construct a qubit with it.

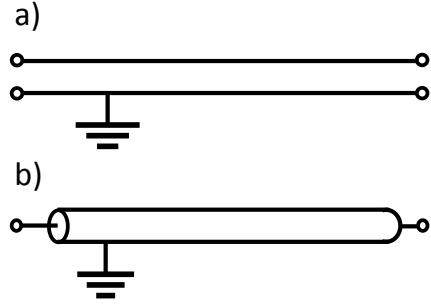


Figure 2.2: a) The circuit diagram of a grounded transmission line. b) The circuit diagram of a grounded coaxial transmission line.

2.3.2 Transmission Lines

Another circuit element that we will encounter many times in this work is the so-called *transmission line*. In the most general way, a transmission line is a structure with a large extension in one direction which is capable of transmitting electromagnetic waves along itself. Two possible symbols by which one usually designates transmission lines in a circuit schematic are shown in fig. 2.2. A detailed treatment of the physics of transmission lines can be found e.g. [44]. For this introduction we will skip these basics and start directly with the equation that describes the propagation of an electromagnetic wave along the extended dimension z of the transmission line and which is given as

$$V(z, t) = \exp(i\omega t) \cdot (V^+ \exp(-i\gamma z) + V^- \exp(i\gamma z)) \quad (2.10)$$

$$I(z, t) = \frac{1}{Z_0} \exp(i\omega t) \cdot (V^+ \exp(-i\gamma z) - V^- \exp(i\gamma z)) \quad (2.11)$$

Here, $\gamma = \alpha + i\beta = \sqrt{(R + i\omega L)(G + i\omega C)}$ is the so-called *propagation constant* which describes the dispersion and damping of electromagnetic waves along the transmission line and ω the circular frequency of the electromagnetic wave. The voltages V^+ and V^- correspond to waves traveling in different directions along the waveguide.

If we regard a CPW of fixed length l , we can model the voltages and currents at its end by using the formula[44]

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \cos \gamma l & iZ_r \cos \gamma l \\ iY_r \sin \gamma l & \cos \gamma l \end{pmatrix} \cdot \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} \quad (2.12)$$

2.3.3 Transmission Line Resonators

One can easily create a resonator using a coplanar waveguide as described in the last section. As an example, we will discuss an open-ended $\lambda/2$ CPW resonator that is used in our experiments as well and thus highly relevant to this work. To create a resonator out of a CPW line, we terminate the line by an open end and connect it to a drive line through an input capacitance C . To calculate the voltages and currents going into the

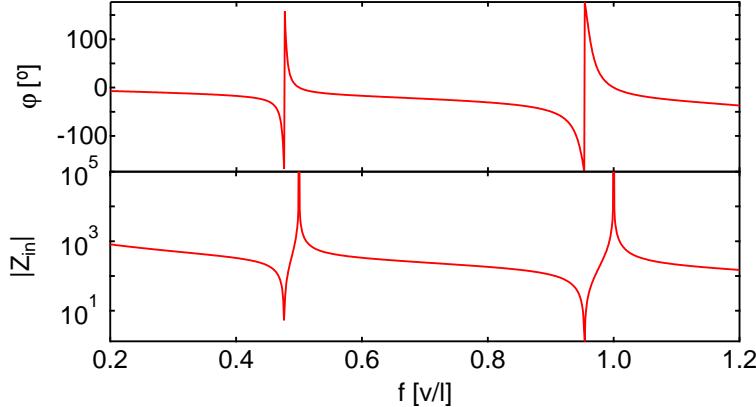


Figure 2.3: This is the side caption which is maybe already to small to support such long text..

CPW resonator, we can use eq. (2.12). Since the far end of the resonator is open we demand that $I_2 = 0$ and thus obtain for the voltage V_1 and current I_2 the relation

$$V_1 = \cos \gamma l V_2 \quad (2.13)$$

$$I_1 = i Y_r \sin \gamma l V_2 \quad (2.14)$$

Hence, the impedance of the resonator is given as $Z_{res} = V_1/I_1 = i Z_r \cot \gamma l$, where Z_r is the impedance of the transmission line of which the resonator is made of. We couple this resonator to an input line through a gate capacitance C_g , thus the impedance seen from the input of the circuit is given as

$$Z_{in} = i Z_r \cot \gamma l - \frac{i}{\omega C_g} \quad (2.15)$$

It is straightforward to calculate the S_{11} reflection coefficient of the resonator when coupling it to an input line with impedance Z_0 as

$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_r \cot \gamma l - / \omega C_g - Z_0}{Z_r \cot \gamma l - / \omega C_g + Z_0} \quad (2.16)$$

Now, when measuring the reflection of an incoming signal with voltage V^+ at frequency $f = 2\pi\omega$ and phase ϕ_0 , the phase of the reflected signal ϕ_{ref} will be simply given as $\phi_{ref} = \text{Arg}[V^-/V^+] - \phi_0 = \text{Arg}[S_{11}] - \phi_0$. Fig 2.3 shows this phase for an exemplary resonator, plotted for $f = 2\pi\omega$ in reduced units of $[l/v]$, with impedances $Z_r, Z_0 = 50 \Omega$, $\alpha = 0$ and with the resonator coupled to the input line through a normalized capacitance $C_g = 10^{-3}/\omega$ [Hz · F].

We can calculate the quality factor of such an open-ended CPW resonator by modeling it as a series-LC circuit. Indeed, if we regard the input impedance of the resonator in the vicinity of ω_0 such that $\Delta\omega = \omega - \omega_0$ with $\Delta\omega \ll \omega_0$ and $\beta l = \pi + \pi\Delta\omega/\omega_0$, we

obtain an effective impedance

$$Z_{in} = \frac{Z_r}{\alpha l + i(\Delta\omega\pi/\omega_0)} \quad (2.17)$$

We can identify the quantities in this equation with the input impedance of a parallel LCR-resonator, which at $\omega \approx \omega_0$ is approximatively given as

$$Z_{in} = \frac{R}{1 + 2iQ\Delta\omega/\omega_0} \quad (2.18)$$

with $W = \omega_0 RC$. This yields an effective resistance, inductance and capacitance for the transmission line resonator of

$$R_r = \frac{Z_r}{\alpha l} \quad (2.19)$$

$$L_r = \frac{1}{\omega_0^2 C} \quad (2.20)$$

$$C_r = \frac{\pi}{2\omega_0 Z_r} \quad (2.21)$$

When coupling this resonator to an input transmission line of impedance Z_0 through a gate capacitance C_g as before, the quality factor of the coupled (or *loaded*) resonator will be given as [23]

$$Q_L = \omega_0^* \frac{C + C^*}{1/R_r + 1/R^*} \quad (2.22)$$

where we have introduced an effective resistance, capacitance and resonance frequency given as

$$R^* = \frac{1 + \omega_0^2 C_g^2 Z_0^2}{\omega_0^2 C_g^2 Z_0} \quad (2.23)$$

$$C^* = \frac{C_g}{1 + \omega_0^2 C_g^2 Z_0^2} \quad (2.24)$$

$$\omega_0^* = \frac{1}{\sqrt{L_r(C_r + C_g)}} \quad (2.25)$$

Thus, we can effectively tune the quality factor of the resonator by changing the gate capacitance by which we couple it to the input transmission line.

2.3.4 Quantization of Electrical Circuits

In this section we will outline a general method to treat arbitrary electrical circuits as the ones discussed before within the framework of quantum-mechanics, hence *quantizing* them. This introduction on circuit quantization presented in this chapter is based on an

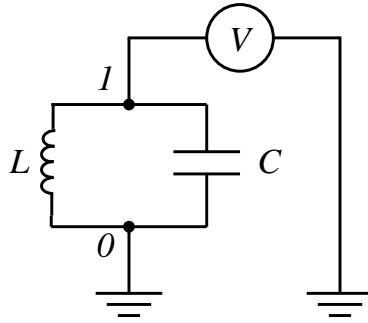


Figure 2.4: An exemplary superconducting circuit made of a Josephson junction, a capacitor and a voltage source. The circuit topology can be described by one node (plus ground) and one branch.

article by [13].

Fig. 2.4 shows an exemplary circuit made of an inductance, a capacitor and a voltage source. A circuit as this one is fully characterized by the parameters of its elements and its topology. The latter can be described – following the laws of Kirchhoff – as a set of nodes connected by a number of branches. In classical circuit theory, each branch i is described by a voltage V_i between its ends and a current I_i flowing through it. The Kirchhoff laws demand that the sum of the branch voltages V_i along any closed path must be zero, i.e. $\sum \oint V_i = 0$. Equivalently one may demand that the sum of currents flowing in and out of each node must be zero. For the quantization of electrical circuits it is usually more convenient to replace voltages and currents with branch charges and fluxes that are defined as

$$\Phi_i(t) = \int_{-\infty}^t V_i(t') dt' \quad (2.26)$$

$$Q_i(t) = \int_{-\infty}^t I_i(t') dt' \quad (2.27)$$

In analogy with the Kirchhoff laws for the sums of currents and voltages along a closed branch, we can formulate a Kirchhoff law for the charges Q_i at each node of the circuit, given as

$$\sum_i Q_i = Q_0 \quad (2.28)$$

where Q_0 is constant. To quantize such a circuit made up of non-dissipative elements we can follow the method given in [55], writing the Lagrangian of the circuit as

$$\mathcal{L} = \sum_i V_i - \sum_i T_i \quad (2.29)$$

where V_i and T_i are the potential and kinetic energies associated to each circuit element. **!2!** . For a circuit composed entirely of capacitors and inductor, this Lagrangian is given

To Do 2: Clarify why kinetic energy is mapped to capacitive energy and potential energy to inductive energy

as

$$\mathcal{L} = \frac{1}{2} \sum_i \frac{Q_i^2}{C_i} - \frac{1}{2} \sum_i L \left(\frac{dQ_i}{dt} \right)^2 \quad (2.30)$$

If needed, resistors can be described within the Lagrangian formalism by modeling them as transmission lines with a characteristic impedance matching their resistance [55]. From the Lagrangian as given in eq. (2.30) we obtain then the equations of motion of the system by variation of the action

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial (\partial_t Q_i)} \right) - \frac{\partial \mathcal{L}}{\partial Q_i} = 0 \quad (2.31)$$

Finally, by imposing the charge-conservation equations as given by eq. (2.28) we obtain then a complete description of the underlying circuit. From the variable Q_i we obtain the canonically conjugate momentum Φ_i by solving the equation

$$\Phi_i = \frac{\partial \mathcal{L}}{\partial (\partial_t Q_i)} \quad (2.32)$$

First Quantization of the circuit variables can then be done by imposing commutation relations between the set of canonical variables Q_i and Φ_i such that

$$[Q_i(t), Q_j(t')] = 0 \quad (2.33)$$

$$[\Phi_i(t), \Phi_j(t')] = 0 \quad (2.34)$$

$$[Q_i(t), \Phi_j(t')] = i\hbar\delta_{ij}\delta(t - t') \quad (2.35)$$

Having obtained Φ_i and Q_i , it is also trivial to obtain the Hamiltonian \mathcal{H} of the system by applying the transformation

$$\mathcal{H} = \sum_j \Phi_i \dot{Q}_i - \mathcal{L} \quad (2.36)$$

In the next section we will use this method to quantize the Cooper pair box circuit.

2.3.5 The Cooper Pair Box

The *Cooper pair box (CPB)* is a device containing a (split) Josephson junction coupled to an input voltage source through a gate capacitance C_g , as shown in fig. 2.5. Following the circuit quantization method outlined in the last section, we obtain the following Hamiltonian for the CPB [11]

$$\hat{H} = 4E_C (\hat{n} - n_g)^2 - E_J \cos \hat{\theta} \quad (2.37)$$

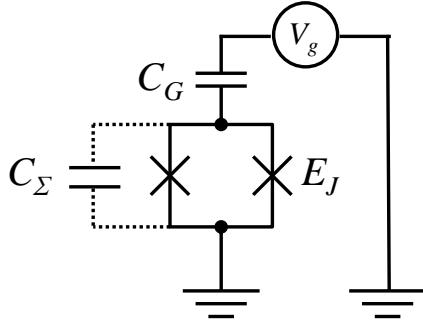


Figure 2.5: The circuit schematic of a Split Cooper Pair Box. The device consists of two Josephson junctions arranged in a loop, where the capacitance of the junctions is modeled as the extra capacitor indicated on the left. By putting the two junctions in series with a gate capacitance one creates an island one which charges can accumulate. The gate voltage can be controlled by an external voltage source.

Here we have defined $E_C = e^2/C_\Sigma$ as the charging energy of the system, with $C_\Sigma = C_J + C_B + C_g$ the total gate capacitance, $\hat{n} = \hat{Q}/2e$ the number of Cooper pairs transferred between the islands, $n_g = V_g C_g / 2e$ the gate charge, E_J the Josephson energy of the junction and $\hat{\phi}$ the quantum phase across the junction.

\hat{n} and θ are conjugate variables such that $[\theta, \hat{n}] = -i$, the corresponding wavefunction $\Psi_k(\theta) = \langle \theta, k \rangle$ will therefore satisfy a Schrödinger equation of the form

$$E_k \Psi_k(\theta) = E_C \left(\frac{1}{i} \frac{\partial}{\partial \theta} - n_g \right)^2 \Psi_k(\theta) - E_J \cos(\theta) \Psi_k(\theta) \quad (2.38)$$

Since the potential $E_J \cos(\theta)$ is periodic, we may demand that the solution also possesses a periodicity of the form

$$\Psi_k(\theta) = \Psi_k(\theta + 2\pi) \quad (2.39)$$

Using this assumption it is possible to map eq. (2.38) to a similar and well-known differential equation, the so-called *Mathieu equation*, that is given as

$$\frac{d^2y}{dx^2} + [a - 2q \cos(2x)] y = 0 \quad (2.40)$$

The *Floquet theorem* states that all solutions to this equation can be written in the form

$$F(a, q, x) = \exp(i\mu x) P(a, q, x) \quad (2.41)$$

Using such a Floquet ansatz for eq. (??), the most general solutions that one obtains are given as [11]

$$\Psi_k(r, q, \theta) = \mathcal{C}_1 \exp(in_g \theta) \mathcal{M}_C \left(\frac{4E_k}{E_C}, -\frac{2E_J}{E_C}, \frac{\theta}{2} \right) + \mathcal{C}_2 \exp(in_g \theta) \mathcal{M}_S \left(\frac{4E_k}{E_C}, -\frac{2E_J}{E_C}, \frac{\theta}{2} \right) \quad (2.42)$$

with

$$E_k = \frac{E_C}{4} \mathcal{M}_A \left(r_k, -\frac{2E_J}{E_C} \right) \quad (2.43)$$

Here, \mathcal{M}_C , \mathcal{M}_S are the so-called *Mathieu functions* and \mathcal{M}_A corresponds to the eigenvalue corresponding to each solution. Following the convention in [11] we order the E_k such that the energy increases with increasing k , yielding

$$r_k = k + 1 - [(k + 1)\text{mod}2] + 2n_g(-1)^k \quad (2.44)$$

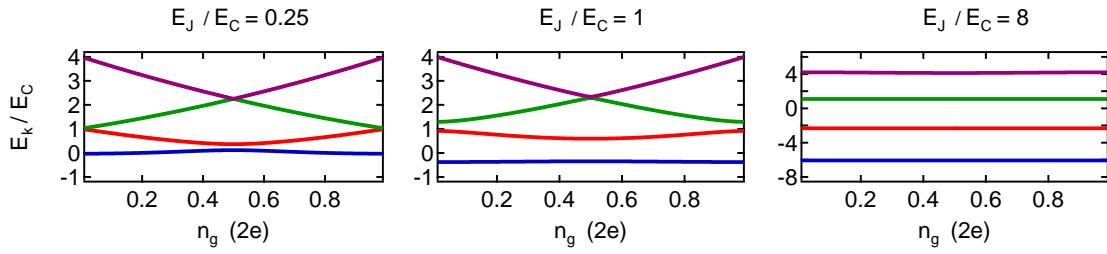


Figure 2.6: Energies of the first four energy level of the Cooper pair box for different ratios E_J/E_C , plotted as a function of the gate charge n_g . As can be seen, for $E_J \ll E_C$, the charge-dispersion curve becomes almost completely flat.

We denote the energy differences between individual energy level by $E_{ij} = E_j - E_i$. We also define the absolute and relative anharmonicities of the first two energy levels as $\alpha_{12} \equiv E_{12} - E_{01}$ and $\alpha_r \equiv \alpha/E_{01}$. In the limit $E_J \gg E_C$ these anharmonicities are well approximated by $\alpha \simeq -E_C$ and $\alpha_r \simeq -(8E_J/E_C)^{-1/2}$. An in-depth treatment of the Cooper pair box can be found e.g. in [11] [!3!](#).

To Do 3: cite original paper by Zorin?

2.4 The Transmon Qubit

The Transmon qubit is a Cooper pair box operated in the regime where $E_J \gg E_C$. As shown above, in this regime the charge dispersion of the energy levels of the Cooper pair box becomes flat, thus rendering the transition frequency E_{01} almost insensitive to the value of the gate charge n_g . This reduced sensitivity to charge noise is highly advantageous in experiments. However, when increasing the ratio E_J/E_C , we also reduce the anharmonicity α_r of the qubit, therefore limiting the speed of gate operations that can be realized with this system (driving errors related to weak anharmonicity will be discussed more thoroughly in the Experiments section of this thesis). Remarkably though, α_r decreases only geometrically with E_J/E_C , whereas the sensitivity of the qubit to charge noise decreases exponentially with the ratio of Josephson and charging energy. Fig ?? shows the relaxation rate and relative anharmonicity of a Cooper Pair box for different values of E_J/E_C . As can be seen,

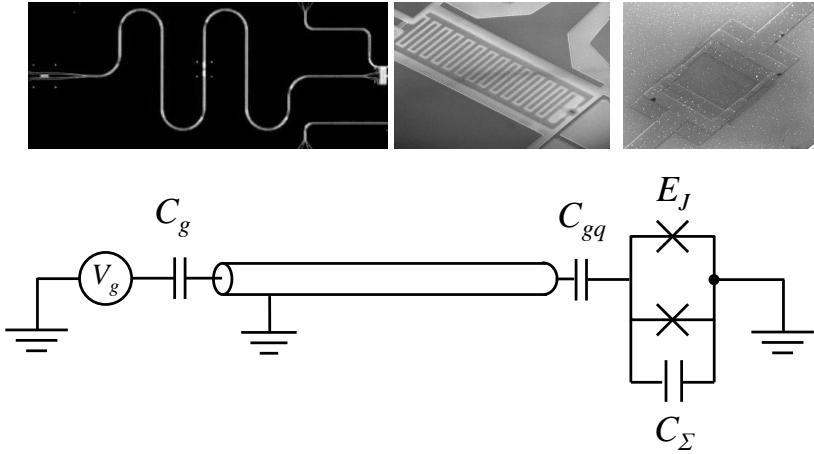


Figure 2.7: A typical circuit QED setup consisting of a Transmon qubit capacitively coupled to a $\lambda/2$ resonator.

2.5 Circuit Quantum Electrodynamics

Cavity quantum electrodynamics is a research domain where one investigates the physics of an atom (or a large number of atoms) coupled to a high-Q microwave cavity. Analogously, *circuit quantum electrodynamics* investigates the coupling of an artificial atom (e.g. a Transmon qubit) coupled to a microwave resonator that is realized on the surface of a chip. Usually, such a qubit-resonator system can be represented as in fig. 2.7. There, a Transmon qubit is capacitively coupled to a $\lambda/2$ resonator which itself is capacitively coupled to an input transmission line. Due to the capacitance between the qubit and the resonator, a coupling energy of the form

$$E_{rq} = \frac{1}{2}C_g\hat{V}_g^2 = \frac{1}{2}C_g \left(V_{rms}^0(a^\dagger + a) - \hat{V} \right)^2 \quad (2.45)$$

arises, where V_g is the voltage across the coupling capacitance C_g , $\hat{V} = 2e/C_\Sigma \cdot (n_g - \hat{n})$ is the voltage across the Transmon electrodes and $V_{rms}^0 = \sqrt{\hbar\omega_r/2C_r}$ is the root mean square voltage corresponding to one photon in the resonator. Hence the coupling energy can be rewritten as

$$\begin{aligned} E_{rq} &= \frac{1}{2}C_g \left(V_{rms}^0(a^\dagger + a) - \frac{2e}{C_\Sigma} [n_g - \hat{n}] \right)^2 \\ &= 2e\beta V_{rms}^0 \hat{n}(a^\dagger + a) + \dots \end{aligned} \quad (2.46)$$

where we defined $\beta = C_g/C_\Sigma$. The terms omitted in eq. (2.46) correspond to energy shifts of the qubit and the resonator which are not directly relevant for the coupling between them. In the limit where the resonator capacity $C_r \gg C_\Sigma$ we can write the effective Hamiltonian of the system using the uncoupled basis states $|i\rangle$ as

$$\hat{H} = \hbar \sum_j \omega_j |j\rangle \langle j| + \hbar\omega_r \hat{a}^\dagger \hat{a} + \hbar \sum_{i,j} g_{ij} |i\rangle \langle j| (\hat{a} + \hat{a}^\dagger) \quad (2.47)$$

Here, $\omega_r = 1/\sqrt{L_r C_r}$ gives the resonator frequency of the resonator. The coupling energies g_{ij} are given as

$$\hbar g_{ij} = 2\beta e V_{rms}^0 \langle i | \hat{n} | j \rangle = \hbar g_{ji}^* \quad (2.48)$$

When the coupling between the resonator and the Transmon is weak, such that $g_{ij} \ll \omega_r, E_{01}/\hbar$ we can ignore the terms in eq. (2.47) that describe simultaneous excitation or deexcitation of the Transmon and the resonator and obtain the so-called *rotating wave approximation*, which is given as

$$\hat{H} = \hbar \sum_j \omega_j |j\rangle \langle j| + \hbar \omega_r \hat{a}^\dagger \hat{a} + \hbar \sum_i (g_{i,i+1} |i\rangle \langle i+1| \hat{a}^\dagger + H.c.) \quad (2.49)$$

This Hamiltonian describes a multi-level quantum system coupled to a resonator through a capacitive interaction. The first two terms correspond to the energies of the n-level system and the resonator, respectively. The term $|i\rangle \langle i+1| \hat{a}^\dagger$ describes the creation of a photon in the resonator accompanied by the deexcitation of the n-level system by one energy level.

2.5.1 Dispersive Limit & Qubit Readout

When the qubit frequency is far detuned from the resonator frequency such that $|\omega_{ij} - \omega_r| \gg g_{ij}$, direct qubit-resonator interactions are almost completely suppressed and the only a dispersive shift of the transition frequency of both systems remains as an effect of the coupling between them. This effect has been discussed e.g. in [30] and yields an effective Hamiltonian of the form

$$\hat{H}_{eff} = \frac{\hbar \omega'_{01}}{2} \hat{\sigma}_z + (\hbar \omega'_r + \hbar \chi \hat{\sigma}_z) \hat{a}^\dagger \hat{a} \quad (2.50)$$

Here, the resonance frequencies of the qubit and the resonator are shifted as $\omega'_{01} = \omega_{01} + \chi_{01}$ and $\omega'_r = \omega_r - \chi_{12}/2$ and the dispersive shift is given as $\chi = \chi_{01} - \chi_{12}/2$, where $\chi_{ij} = g_{ij}^2 / (\omega_{ij} - \omega_r)$. As can be seen, for a state with n photons, the energy difference between the two qubit levels is given as

$$\omega_{01}^n = \omega'_{01} + 2\chi n \quad (2.51)$$

Thus, there is a dispersive shift of the qubit transition frequency that is proportional to the number of photons in the resonator. Likewise, the resonance frequency of the resonator gets also shifted by 2χ depending on the state of the qubit. The latter effect is very useful since it allows us to read out the state of the qubit by measuring the state-dependent frequency displacement of the resonator, as will be explained later.

2.5.2 Qubit-Qubit Interaction

In this section we will discuss possible qubit-qubit coupling schemes. We will regard a direct coupling scheme involving a capacitive coupling between two qubits and an indirect scheme involving the coupling of multiple qubits to a resonator which acts as a “quantum bus”.

Coupling Bus

[4] showed that extending the single-qubit rotating-wave Hamiltonian as given in eq. (2.49) to the case of two qubits yields an effective qubit-qubit coupling Hamiltonian of the form

$$H_{2q} = \hbar \frac{g_1 g_2 (\Delta_1 + \Delta_2)}{2\Delta_1 \Delta_2} (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) \quad (2.52)$$

This approximation is valid in the limit of large qubit-resonator detuning where $\Delta_1 \gg g_1, \Delta_2 \gg g_2$. Here $\Delta_{1,2} = \omega_{01}^{1,2} - \omega_r$ is the detuning of the $|0\rangle \rightarrow |1\rangle$ transition frequency of each qubit to the bus resonator. Full energy-exchange between the qubits is achieved when the qubit frequencies are in resonance. By detuning the qubits from the resonator, the effective coupling constant can be varied, which is advantageous in many settings.

Direct Capacitive Coupling

A direct capacitive coupling C_{qq} between two qubits yields a coupling Hamiltonian of the form

$$\hat{H}_{qq} = \frac{1}{2} C_{qq} \hat{V}_{qq}^2 = \frac{1}{2} C_{qq} \left(\frac{2e}{C_{\Sigma 1}} (n_{g1} - \hat{n}_1) - \frac{2e}{C_{\Sigma 2}} (n_{g2} - \hat{n}_2) \right)^2 \quad (2.53)$$

$$= \frac{4e^2 C_{qq}}{C_{\Sigma 1} C_{\Sigma 2}} \hat{n}_1 \hat{n}_2 + \dots \quad (2.54)$$

This equation is valid in the limit where $C_{qq} \ll C_{\Sigma 1}, C_{\Sigma 2}$, otherwise the coupling gets renormalized by a factor $\alpha = 1/(1 - C_{qq}^2/[C_{\Sigma 1} C_{\Sigma 2}])$ [?]. Rewriting it in the basis of uncoupled qubit states yields the effective Hamiltonian

$$\hat{H}_{qq} = \hbar 2 g_{qq} (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) \quad (2.55)$$

where we have defined the effective qubit-qubit coupling as $\hbar g_{qq} = 2e^2 C_{qq} / C_{\Sigma 1} C_{\Sigma 2}$. As before, full energy exchange between the qubits is achieved when the qubit frequencies are in resonance. Directly coupling two qubits can be advantageous since it simplifies the circuit layout and does not require an auxiliary quantum system. However, since the

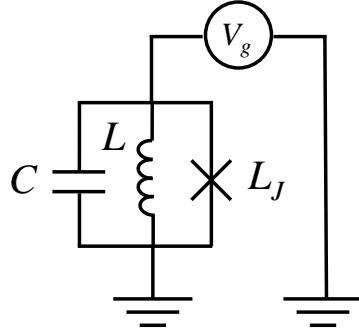


Figure 2.8: The circuit model of the JBA.

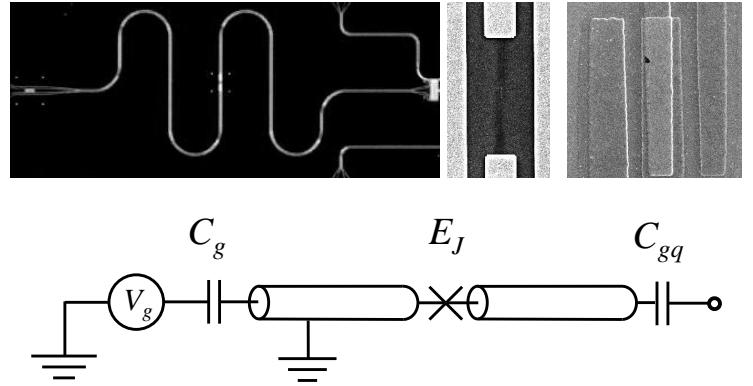


Figure 2.9: The Cavity Bifurcation Amplifier used in this work.

coupling is always turned on it is difficult to achieve a sufficiently good ON/OFF ratio that is needed for many applications (e.g. to realize a quantum gate).

2.6 The Josephson Bifurcation Amplifier

In this section we will briefly discuss the physics of superconducting nonlinear bifurcation amplifiers. The two devices that we will discuss are the so-called *Josephson bifurcation amplifier (JBA)* as shown in fig. 2.8 and the so-called *cavity bifurcation amplifier (CBA)* as shown in fig. 2.9. The JBA typically consists of a Josephson junction in parallel with a capacitor and (optionally) an inductor. The CBA, on the other hand, consists of a transmission line resonator with a Josephson junction embedded in its central conductor. As shown in one of the last sections, a transmission line resonator can be treated mathematically as a lumped elements resonator, hence the physics of the CBA can be mapped to that of the JBA. Hence we will restrict our discussion in this section to the JBA. A more detailed comparision between the JBA and CBA can be found e.g. in [43].

The circuit in fig. 2.8 can be modeled as

$$[L_e + L_J(i)]\ddot{q} + R_e\dot{q} + \frac{q}{C_e} = V_e \cos(\omega_m t) \quad (2.56)$$

Here, L_J the Josephson inductance as given by eq. (2.9) and L_e , C_e and R_e are the

resistance, inductance and capacitance parallel to the Josephson junction in the circuit. V_e is the amplitude of the driving voltage, which oscillates at a frequency ω_m . Expanding the Josephson inductance in this equation to second order leads to the equation

$$\left(L_e + L_J \left[1 + \frac{\dot{q}^2}{2I_0^2} \right] \right) \ddot{q} + R_e \dot{q} + \frac{q}{C_e} = V_e \cos(\omega_m t) \quad (2.57)$$

Defining the total inductance $L_t = L_e + L_J$, the participation ratio $p = L_J/L_t$, the resonance frequency $\omega_r = 1/\sqrt{L_t C_e}$ and the quality factor $Q = \omega_r L_t / R_e$ we can rewrite this as

$$\ddot{q} + \frac{\omega_r}{Q} \dot{q} + \omega_r^2 q + \frac{p \dot{q}^2 \ddot{q}}{2I_0} = \frac{V_e}{L_t} \cos(\omega_m t) \quad (2.58)$$

Introducing the reduced variables $\beta = (V_e/\phi_0\omega_m)^2(pQ/2\Omega)^3$, $\Delta_m = \omega_r - \omega_m$, $\tau = t\Delta_m$, $\Omega = 2Q\Delta_m/\omega_r$ and $u(t) = \sqrt{pQ/2\Omega} \cdot q(t)\omega_m/I_0$ we can further rewrite this equation to obtain

$$\begin{aligned} & \frac{\Delta_m}{\omega_m} \frac{d^2 u}{d\tau^2} + \left(\frac{1}{Q\omega_m} + 2i \right) \frac{du}{d\tau} \\ & + \left[2 \left(\frac{\omega_r^2 - \omega_m^2}{2\omega_m \Delta_m} \right) + \frac{1}{Q\Delta_m} - 2|u|^2 \right] u = 2\sqrt{\beta} \end{aligned} \quad (2.59)$$

In the limit where $Q \gg 1$, $\Delta_m \omega_m \ll 1$ such that $\omega_m + \omega_r \approx 2\omega_m$ and $\ddot{u} \ll \omega_m \dot{u}$ we can simplify this equation to obtain

$$\frac{du}{d\tau} = -\frac{u}{\Omega} - iu(|u|^2 - 1) - i\sqrt{\beta} \quad (2.60)$$

The stationary solutions of this equation describe stable oscillator states and have the form

$$\frac{|u|^2}{\Omega^2} + |u|^2 \left(|u|^2 - 1 \right)^2 = \beta(\Omega) \quad (2.61)$$

This equation can have one or two physical solutions depending on the parameter $\beta(\Omega)$. The region where multiple solutions exist is usually called the *bistability region* and is limited by the the two parameter boundaries $\beta^\pm(\Omega)$ that are given as

$$\beta^\pm(\Omega) = \frac{2}{27} \left[1 + \left(\frac{3}{\Omega} \right)^2 \pm \left(1 - \frac{3}{\Omega^2} \right)^{3/2} \right] \quad (2.62)$$

Fig. 2.10 shows the values of $\beta^\pm(\Omega)$ as well as the phase $\phi \simeq \arg(u(t))$ and the amplitude $\sqrt{B} \propto A \simeq |u(t)|$ of different stable solutions of eq. (2.60) for a JBA with $\alpha = 0.05$ and $\gamma = 0.01$, where $\alpha = V_e p^{3/2} / \phi_0 \omega_r$ and $\gamma = R_e / L_t \omega_r$ characterize the drive strength and dissipation in the JBA, respectively.

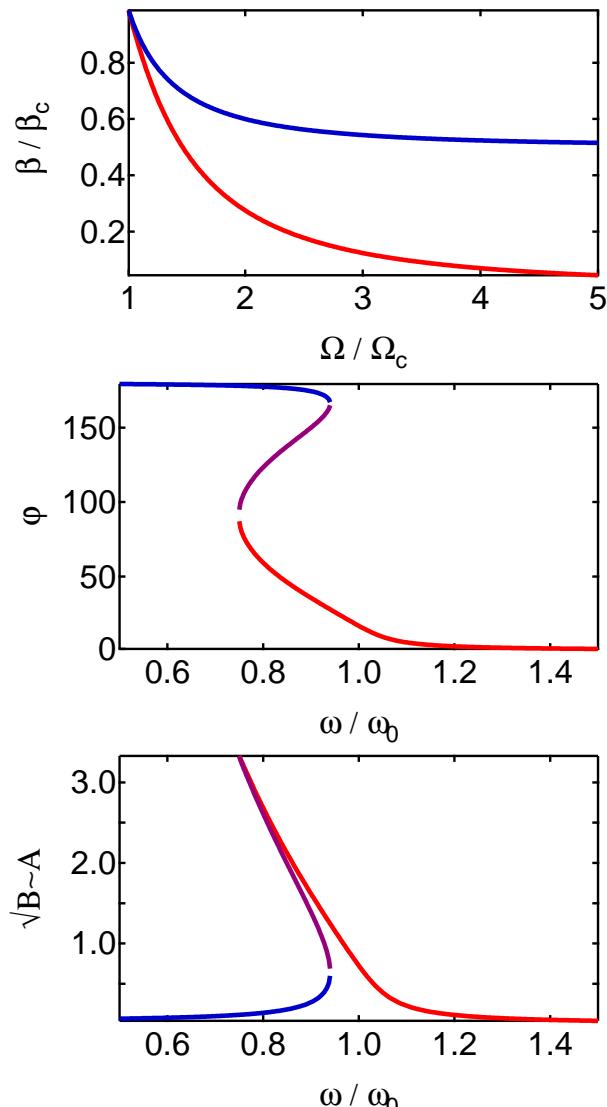


Figure 2.10: a)The bistability boundaries β^\pm of a JBA, plotted as a function of Ω/Ω_c . b/c)The phase $\phi = \arg(u(t))$ and amplitude $A \simeq |u(t)|$ of different solutions of eq. (2.60) for the JBA parameters $\alpha = 0.05$ and $\gamma = 0.01$. In the hysteretic region, three solutions exists, two of which are stable. The solution realized at a given moment depends thus on the history of the system, allowing for hysteretic behaviour.

Chapter 3

Realizing a Two-Qubit Processor

This chapter discusses in depth the design process for the realization of the 2-qubit processor which was used in this work. We will start by introducing the general constraints we face when designing a two-qubit processor, followed by a component-wise discussion of the individual parts of the processor and the associated parameters we need to choose.

3.1 Introduction & Motivation

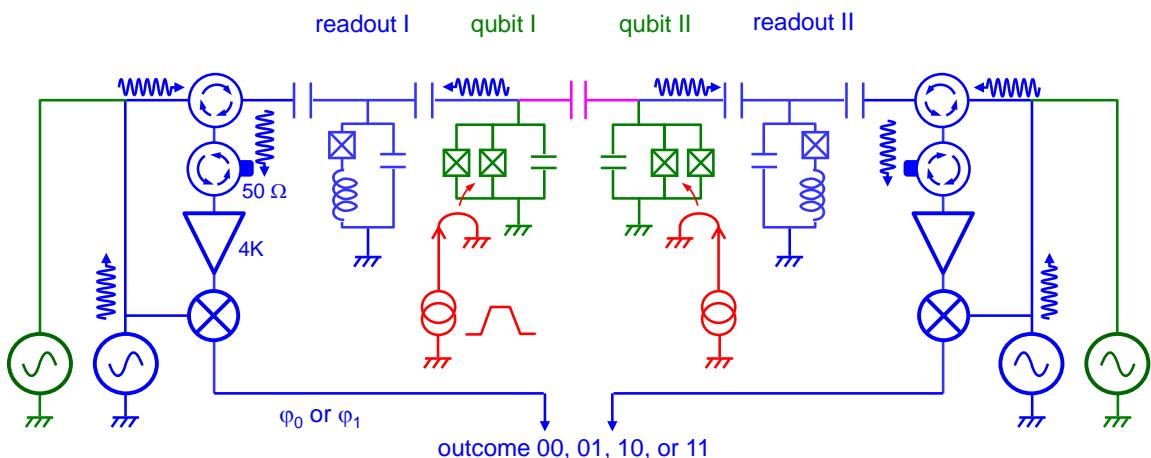


Figure 3.1: The circuit schematic of the two-qubit processor used in this work. Shown are the two Transmon qubits in green, the drive and readout circuit in blue, the fast flux lines in red and the coupling capacitance in magenta.

As discussed in the introduction, the most simple, usable quantum processor contains two qubits that can be manipulated and read out individually and between which one can realize a universal 2-qubit gate. We implement such a two-qubit processor using two Transmon qubits that are coupled by a fixed capacitance and that can be read

out out individually by a pair of Josephson bifurcation amplifiers (JBAs). The circuit diagram of our processor is shown in fig. 3.1, it shows the two qubits in green, the drive and readout circuit in red, the coupling capacitance between them in magenta and the fast flux lines in red. Also shown is the basic microwave setup used for manipulating and reading out the qubits. In the following sections we will discuss each of these components and the parameters we have chosen for them individually.

3.2 Qubit Design

For the qubits, the main design goal is high coherence time, good frequency tunability and the possibility of fast qubit driving. The coherence time of the qubits is limited by relaxation to the ground state and dephasing due to coupling to external noise sources. The dephasing of the qubit state is caused mainly by noise-induced fluctuations of the qubit frequency, which we will discuss in the next section. The relaxation is caused mainly by the coupling of the Transmon to its environment and by internal losses inside the Josephson junctions and the capacitor of the qubit.

Frequency tunability is important since it will allow us to implement two-qubit gates and to position the qubit ideally for readout or manipulation, as will be explained later. However, strongly coupling the qubit to a fast flux line can induce relaxation and dephasing as well. In the following sections we discuss therefore which coupling strength is adequate and how much the flux coupling contributes to the decoherence time of the qubit.

The drive speed of the qubit –i.e. the frequency at which we can drive qubit transitions– is ultimately limited by its anharmonicity. When the drive frequency becomes comparable to the anharmonicity we start to induce transitions to higher Transmon levels, therefore “leaking” out of the computational basis and producing unitary errors. As we will see, the anharmonicity of the Transmon cannot be chosen arbitrarily high since the sensitivity of the Transmon to charge noise increases exponentially with its anharmonicity, it is therefore necessary to find a compromise.

We will start our discussion of the qubit design by discussing the choice of the following qubit parameters:

1. The maximum qubit transition frequency f_{01}
2. The qubit anharmonicity α
3. The qubit critical current asymmetry d
4. The size of the qubit flux loop and its coupling to the fast flux line

3.2.1 Qubit Transition Frequency and Anharmonicity

The maximum transition frequency $f_{01}^0 = \sqrt{E_J E_C / 2\hbar}$ [4] of the qubit should be chosen in function of the readout resonator frequency such that it is possible to tune the two of them sufficiently close, thereby achieving the optimal readout fidelity. Chosing a qubit frequency above the cavity frequency should therefore be avoided, since it implies a stronger flux dependence of the qubit frequency at frequencies below the resonator frequency, thus increasing qubit dephasing when operating the qubit there. Hence, ideally the maximum qubit frequency should be chosen such that it corresponds to the optimal frequency for the qubit readout, as will be discussed later.

To Do 4: verify constants!

The anharmonicity $\alpha_r = -(8E_J/E_C)^{-1/2}$ of the qubit can be tuned by changing the ratio of charging and Josephson energy E_J/E_C . In general, one aims to have as much anharmonicity as possible in order to avoid inducing leakage to higher level of the quantum system when driving the qubit at high transition frequencies. On the other hand, increasing the anharmonicity (and thus decreasing E_J/E_C) will also lead to more charge-induced dephasing of the qubit, thus limiting the coherence time. For our setup, we chose an anharmonicity of the order of -240 MHz which allows us to perform single qubit gates at ≈ 100 MHz transition frequency without inducing too much leakage to the higher qubit states (a detailed analysis of this leakage will be given in the main section of this thesis).

Given f_{01}^0 and α_r (or equivalently α), we can calculate the required charging and Josephson energies of our qubit easily as

$$E_C = -\alpha \quad (3.1)$$

$$E_J = \frac{(4\pi\hbar f_{01}^0)^2}{|\alpha|} \quad (3.2)$$

For our processor, we chose values $f_{01}^0 = 7.0$ GHz and $\alpha = -240$ MHz, respectively, which yields a Josephson energy $E_J =$ and a charging energy $E_C =$.

We can estimate the dephasing time of the qubit as a function of these energies by using an equation given in [30]:

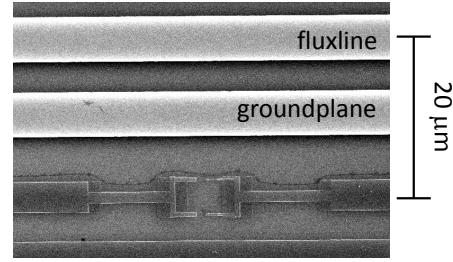
$$T_2 \simeq \frac{\hbar}{A} \left| \frac{\partial E_{01}}{\partial n_g} \right|^{-1} \simeq \frac{\hbar}{A\pi|\epsilon_1|} \quad (3.3)$$

where A is an empirical parameter and ϵ_m depends exponentially on E_J/E_C and is given as

$$\epsilon_m \simeq (-1)^m E_C \frac{2^{4m+5}}{m!} \sqrt{\frac{2}{\pi}} \left(\frac{E_J}{2E_C} \right)^{m/2+3/4} e^{-\sqrt{8E_J/E_C}} \quad (3.4)$$

The dephasing time of the qubit is thus increasing exponentially with the ratio E_J/E_C if

Figure 3.2: Image of the fast flux line used in our 2-qubit processor. The fluxline is separated from the qubit SQUID loop through a stretch of ground plane, the separation between the fluxline and the qubit is $\approx 20 \mu\text{m}$. The flux coupling coefficient is given as $1\Phi_0/20 \text{ mA}$.



$E_J \gg E_C$, whereas the anharmonicity decreases only geometrically with E_J/E_C . For our values of E_J and E_C and for a value $A =$, we obtain a dephasing time $T_2 =$.

3.2.2 Qubit Critical Current Asymmetry

The critical current asymmetry of the qubit SQUID determines the modulation depth of the transition frequency as a function of the flux applied to the SQUID loop. The interest of choosing a non-zero d (and thus a reduced frequency modulation depth) is to reduce the flux-induced dephasing of the qubit state, which is given as [30]

$$T_2 = \frac{\hbar}{A} \left| \frac{\partial E_{01}}{\partial \Phi} \right|^{-1} \quad (3.5)$$

where A is an emperical parameter. Hence, the coherence rate increases proportional to the derivative of the qubit energy as a function of the applied flux. On the other hand, cricital current asymmetries can open additional relaxation channels [] such that the overall effect of increasing d is not necessarily monotonous. For our qubits, we chose values of the asymmetry of $d_I = 0.2$ and $d_{II} = 0.35$. However, as we will show later this choice did not led to increased dephasing times of the qubits and should generally be avoided. !5!

To Do 5: cite relevant sections of the Koch paper!

3.2.3 Coupling of the Qubit to the Fast Flux Line

Another critical parameter to chose is the coupling of the qubit the fast flux line. This coupling must be sufficiently strong to allow large frequency displacements of the qubit during operation of the processor but not so strong as to induce additional qubit relaxation or dephasing. A comprehensive summary of fluxline-induced dephasing and relaxation mechanism can be found in [30]. For our qubit chip, we chose a geometry as depicted in fig. 3.2. Here, the distance between the qubit SQUID loop and the flux line is approx. $20 \mu\text{m}$, thus the flux coupling to the qubit is given as approx. $1\Phi_0/20 \text{ mA}$. The capacitive coupling between the qubit electrodes and the fluxline is $C_{qf} = ?? \text{ fF}$. This capacitive coupling yields an additional contribution to the qubit relaxation rate of

the order of $\Gamma_{qf} \approx$!6! .

To Do 6: add more info on the relevant relaxation

3.2.4 Qubit-Qubit Coupling

To realize universal two-qubit gate operations it is necessary to provide a kind of coupling between the two qubits of our processor. As discussed before, we use direct capacitive coupling to achieve this for our processor. The coupling strength g_{qq} between the two qubits can be calculated by using eq. (2.54). This coupling strength must be chosen such that the interaction between the qubits is sufficiently fast to realize two-qubit gate operations with adequate fidelity but not too strong in order to still allow us to controllably switch on and off the coupling by detuning the qubit frequencies. In general, by diagonalizing the Hamiltonian given in eq. (2.55) we find for the swapping frequency f_{qq} and the swap amplitude a_{qq} of the two coupled qubits as a function of the qubit-qubit detuning $\Delta = f_{01}^I - f_{01}^{II}$ the values

$$\begin{aligned} f_{qq} &= \frac{1}{2} \sqrt{16g^2 + \Delta^2} \\ a_{qq} &= \frac{4g}{\sqrt{16g^2 + \Delta^2}} \end{aligned} \quad (3.6)$$

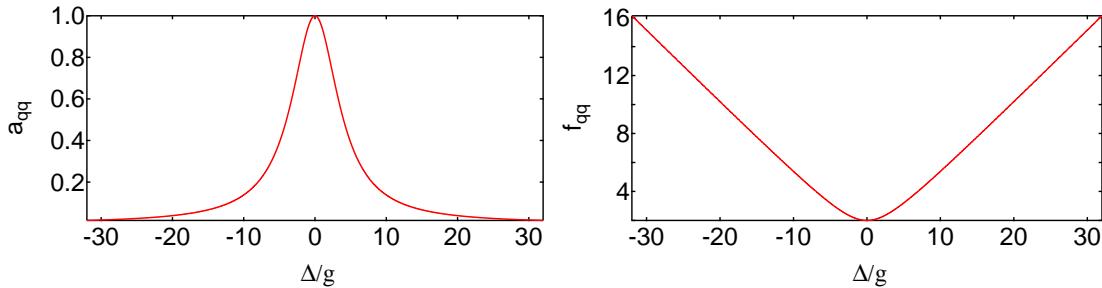


Figure 3.3: The swapping amplitude a_{qq} and frequency f_{qq} as given by eq. (3.6). For $\Delta \gg g$, the amplitude of the swap decreases $\propto 1/\Delta$ and the frequency increases $\propto \Delta$. To effectively switch off the qubit-qubit level below the 1 % error level, a detuning of $\Delta \approx 40g$ is required.

Fig. 3.3 shows this amplitude and frequency as a function of the normalized qubit-qubit detuning Δ/g . As can be seen, for $\Delta \gg g$ the swap amplitude decreases $\propto 1/\Delta$ whereas the swap frequency increases $\propto \Delta$. To turn off the qubit-qubit coupling below the 1 % level it is necessary to detune the qubits by $\Delta \approx 40g$. It is therefore important to choose g such that it is possible to tune the qubits in and out of resonance sufficiently fast in order to realize a reliable two-qubit gate and switch off the qubit-qubit interaction if wanted. For our processor, we chose $2g = 10$ MHz, hence we need to detune the qubits by only 200 MHz to switch off the coupling between them, which is easily achievable using our fast flux lines. In resonance, the swap frequency of 100 MHz allows us to

realize an $\sqrt{i\text{SWAP}}$ gate in 25 ns and an $i\text{SWAP}$ gate in 50 ns, which is sufficiently fast compared to the estimated relaxation and dephasing times of the qubits.

3.3 Readout Design

In the following section we discuss the design of the qubit readout. The main parameters that we need to choose for the readout are

1. The frequency ω_r , quality factor Q and nonlinearity K of the readout resonator
2. The coupling g_{qr} between the qubit and the readout resonator

The frequency f_r of the resonator is chosen high enough to have a negligibly low thermal photon occupation probability at the operating temperature $T = 20$ mK of our experiment. Also, the choice of frequency is limited by the availability of sufficiently good cryogenic microwave components (especially circulators) in the chosen frequency band. For our qubit processor, we chose resonator frequencies of $f_r^I = 6.7$ GHz and $f_r^{II} = 7.85$ GHz. The resonance frequencies of the two resonators are detuned by 150 MHz in order to reduce the effect of unwanted microwave crosstalk between them.

The other relevant parameters of the readout resonator are its quality factor Q and its Kerr constant K . The choice of the quality factor influences the relaxation rate of the qubit through the Purcell effect as well as the rate at which photons can be created in or extracted from the resonator, which in turn also determines the speed of the qubit readout. A comprehensive review of the interplay between qubit relaxation, readout speed and readout fidelity can be found in [33]. To quantify the Purcell relaxation of the qubit through the readout resonator we can use an equation given in [30]:

$$\gamma_k^{(i,i+1)} = \kappa \frac{g_{i,i+1}^2}{(\omega_{i,i+1} - \omega_r)^2} \quad (3.7)$$

Here, ω_r is the frequency of the readout resonator, $\omega_{i,i+1}$ is the $|i\rangle \rightarrow |i+1\rangle$ transition frequency of the Transmon and $g_{i,i+1}$ is the qubit-resonator coupling strength for this transition. As can be seen, the Purcell relaxation rate decreases quadratically with the qubit-resonator detuning and increases quadratically with the qubit-resonator coupling strength $g_{i,i+1}$!?!. For our resonators, we chose $Q = 800$ which yields $\kappa_I = 52.6$ MHz and $\kappa_{II} = 53.8$ MHz. Thus, for the above quality factors, a qubit-resonator coupling $2g_{qr}^{01} = 2\pi \cdot 50$ MHz and a detuning of $\omega_{01} - \omega_r = 2\pi \cdot 500$ MHz we obtain a Purcell relaxation time of $T_1^{01} \approx 7.5$ μ s, which is bigger than the intrinsic relaxation time of a typical Transmon qubit by a large factor (however, certain new types of Transmon qubits can surpass this relaxation time).

To Do 7: choose consistent notation, e.g. $g_{qr}^{i,i+1}$!

The choice of the Kerr nonlinearity K determines the number of photons in the low- and high-amplitude oscillation state of the resonator and the size of the “bifurcation region” of the JBA. It is important to choose K sufficiently high in order to be able to achieve reliable readout operation. On the other hand, the photons present in the resonator during the readout can induce decoherence in the qubit and will shift its resonance frequency proportional to the number of photons. This frequency shift can create unwanted effects such as recoupling of the two qubits of our processor and should therefore be minimized. Theoretically, the maximum readout fidelity is achieved when the bifurcation point of the resonator is shifted by more than its linewidth, as a function of the qubit state. The linewidth of the switching probability distribution of the JBA as a function of the incident microwave power can be calculated numerically but usually we use experimental values to characterize it. In our case, this measured linewidth in dB was given as $\Delta p = xx$ dB. !8!

To Do 8: add more detail about the readout contrast at a given detuning for the chosen set of parameters.

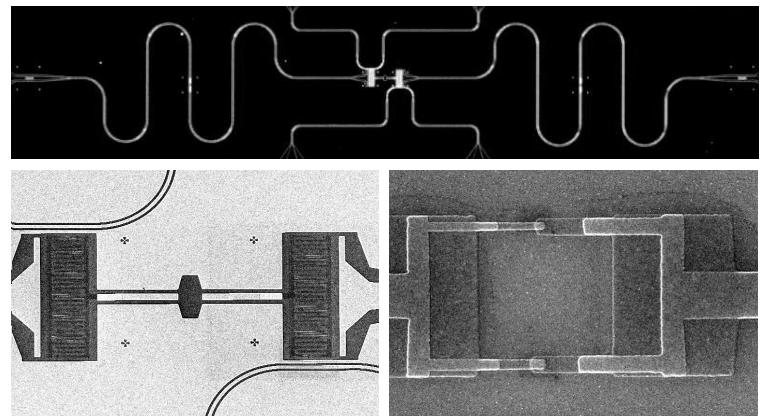
3.4 Processor Fabrication

We fabricate the processor on a silicon substrate with a 50 nm thermal oxide layer. First, we deposit a 150 nm layer of Niobium by magnetron sputtering. Afterwards, we spin a photoresist and define an etch mask through optical lithography. Then we dry-etch in aSF_6 plasma, defining the readout resonators, transmission lines and qubit flux lines on the chip. This optical patterning is performed for the wafer as a whole. Afterwards, we spin a bilayer of MAA/PMMA electron beam resist (with typically 1050 nm of MMA and 115 nm of PMMA thickness). Then the wafer gets diced and the qubits and JBA junctions are patterned per chip using electron beam lithography, using a double-angle shadow evaporation technique to define the Josephson junctions and capacitances on the chip. The e-beam resist is then lifted off chemically in an Acetone bath. We characterize the chip optically afterwards. In addition, we place “twin” structures of the Transmon qubits and the JBAs on each chip whose normal state resistance we measure at room temperature. Giving the normal-state resistance of a Josephson junction we can calculate the Josephson energy by using the Ambegaokar-Baratoff relation

$$E_J = \frac{2\pi^2\Delta}{R_n h} \quad (3.8)$$

, where $E_J = hI_c/4\pi e$. Furthermore, we perform numerical microwave simulations to extract the values of all relevant capacitances and inductances on the chip, which allows us to calculate all relevant parameters of our qubit chip.

Figure 3.4: Optical and electron microscope photos of the two-qubit processor realized in this work. a) shows the full processor with the two coupled qubits, fluxlines and readout resonator. b) shows an enlarged version of the central region of the chip with the two qubits and the coupling capacitance. c) Shows a single Transmon qubit.



Chapter 4

Measurement Setup & Techniques

In this chapter we will discuss the measurement setup and techniques that we use in our experiments. We will discuss all individual parts of our signal generation and measurement chain, putting emphasis on the generation and measurement of high-frequency signals. Afterwards we will briefly discuss the calibration and compensation techniques that we use to correct signal imperfections.

Finally we will introduce the reader to the different measurement techniques that we use in this work, including techniques used for qubit readout and driving as well as more advanced measurement methods that we use to characterize the qubits and their decoherence times.

4.1 Sample Holder & PCB

The qubit chip is first glued to a high-frequency PCB. The coplanar transmission lines present on the chip are bond-wired to their counterparts on the microwave PCB, where they are terminated by a set of Mini-SMA !1! connectors. Additional bond wires are used to connect ground planes of the chip to the PCB. Also, bond wires are used to connect different parts of the on-chip ground plane that are isolated from each other due to the circuit topology. As [46] showed, non-connected on-chip ground planes can induce parasitic resonances on the qubit chip and should therefore be avoided. The mounted chip on the PCB is then placed in a Copper or Aluminium sample holder that fully encloses the PCB and provides electromagnetic shielding of the environment. The inner dimensions of the sample holder are chosen such that no spurious box resonances in the frequency range relevant to our experiment can arise.

Comment
SMP???

1:

Figure 4.1: The sample holder with the mounted PCB carrying the qubit chip. Bond wires are used to connect the on-chip transmission lines to the PCB, which in turn uses a Mini-SMP connector to connect the bond-wire to a set of external coaxial SMP cables. The top part of the sample holder is screwed to the bottom part, forming a closed box with cavities only for the transmission lines and the qubit chip. The whole sample holder is screwed to the 20 mK stage of the dilution cryostat.



4.2 Cryogenic Wiring

Microwave signals that are generated at room temperature are sent to the qubit chip through a series of transmission lines. Fig. 4.2 shows the wiring of our experiment from room temperature down to the 20 mK state of the dilution cryostat. Superconducting cables are used where adequate to minimize signal attenuation, in addition lossy cables made from special compounds (e.g. CuNi) are used to minimize heat transfer into the dilution cryostat, which is especially critical between the 4K and 300 mK and the 300 mK and 20 mK states of the cryostat. In addition to high frequency transmission lines we also use a pair of bifilar cables to power a superconducting Nb coil which is used for flux biasing the qubits.

4.3 Signal Generation & Acquisition

For the experiments presented below we need to generate pulsed high-frequency signals with a well-defined frequency, amplitude and phase. In addition we need to characterize microwave signals measured using microwave reflectometry using standard microwave demodulation techniques. In the following sections we discuss in detail the signal generation and measurement chain used for our experiments.

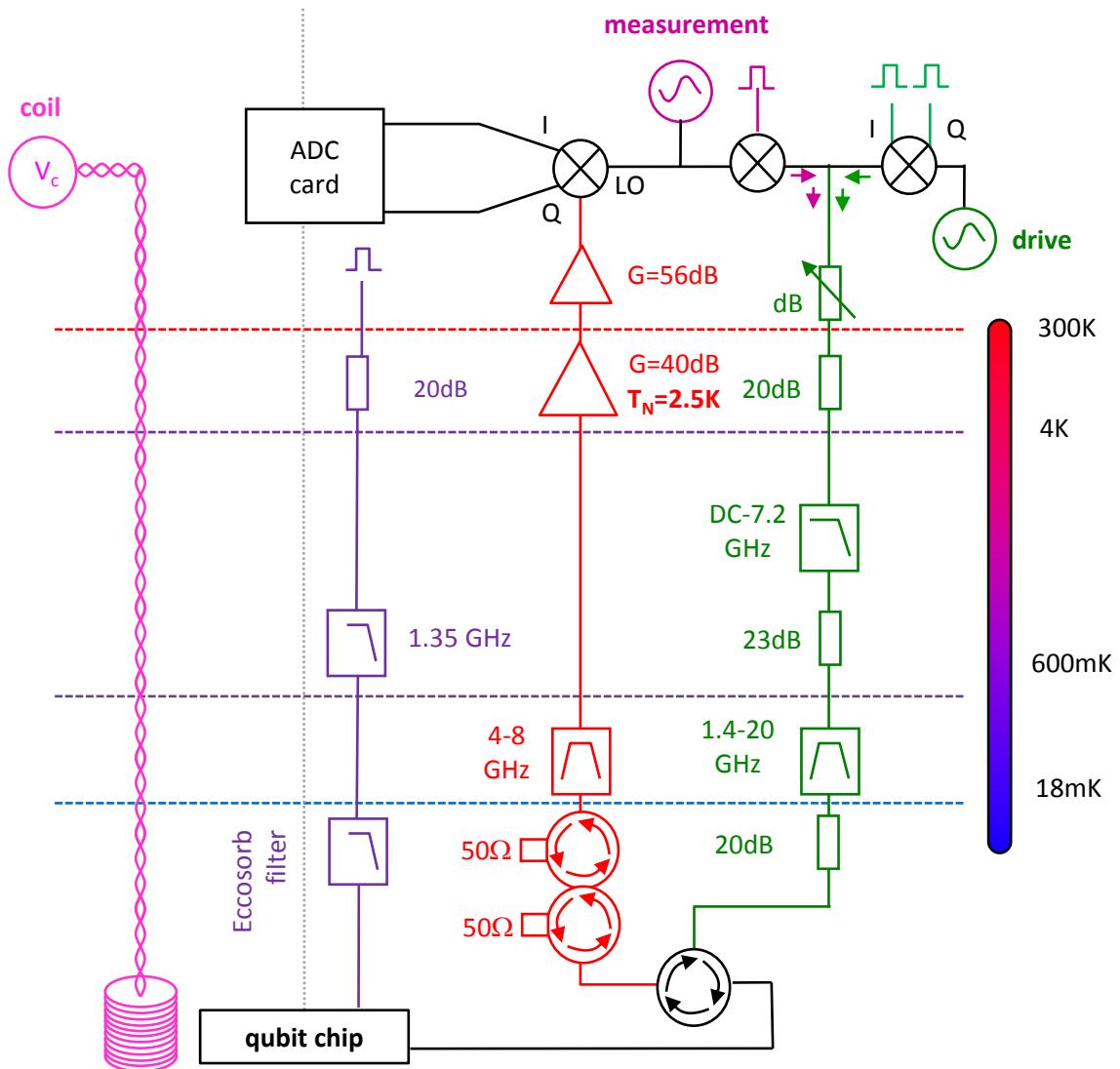


Figure 4.2: The measurement setup used for the two-qubit experiments. Exactly the same drive and readout scheme is used for both qubits with phase-locked microwave sources and arbitrary waveform generators.

4.3.1 Driving and Measurement of the Qubit

Each of the qubits together with the corresponding readout resonator on our chip is fitted with an individual drive and readout circuit. At room temperature we generate qubit and resonator drive waveforms using phase-locked single-tone microwave sources whose continuous output is mixed with fast control pulses generated by two arbitrary waveform generators (the details of this microwave mixing will be discussed in the following paragraph). The drive and readout signals are then combined and sent to the qubit chip through a series of (cryogenic) attenuators and filters. A cryogenic circulator at the 20 mK sample stage of the dilution cryostat routes the incoming pulses to the qubit

chip where they are sent to the qubit readout resonator and finally reflected by it. The reflected signal passes again through the input microwave circulator and gets routed through a double isolator and a band-pass filter to a cryogenic HEMT amplifier with a gain of 40 dB. The amplified signal gets transmitted to the room temperature electronics, where it gets filtered and amplified further. Finally, the signal is demodulated with a continuous microwave reference tone and fed to an ADC board through a pair of low-noise amplifiers.

In addition to this, each qubit is equipped with a pair of fast flux lines. High-frequency and DC flux pulses are generated using an arbitrary waveform generator at room temperature. The flux signal is then sent to the qubit chip through 20 dB of attenuation, a conventional Microtronics low-pass filter as well as a custom-made high-frequency powder filter that uses an absorptive material (Eccosorb) to attenuate high-frequency noise. After passing through the transmission line on the qubit chip, the outgoing flux signal gets routed to room temperature through a transmission line identical to the input line. There, the signal can be measured, which is useful for correcting possible signal imperfections caused by the non-ideal character of the transmission line (we will detail in one of the following sections how to numerically compensate the imperfect frequency response of the flux line).

4.3.2 Microwave Sideband Mixing

To generate the qubit drive pulses we use single-sideband mixing techniques. We use a pair of IQ mixers (Hittite !9!) that we drive with a continuous single-frequency microwave tone and two synchronized fast control signals generated by an arbitrary waveform generator (Tektronix AWG5014b). In general, when feeding a signal $LO(t) = i_0 \cos(\omega_{rft})$ to the LO port of an IQ mixer and two signals $I(t)$, $Q(t)$ to the I and Q ports of the mixer, we obtain a signal

$$RF(t) = I(t) \cos(\omega_{rft}) + Q(t) \sin(\omega_{rft}) \quad (4.1)$$

at the RF port of the mixer. Since the IQ mixer that we use is a passive, reciprocal device one can as well feed two input signals to the LO and RF ports and obtain the demodulated signal quadratures at the I and Q ports, a technique that we make use in our qubit readout scheme, as will be detailed later in this chapter.

Typically we use sideband mixing to generate drive pulses that are displaced in frequency in respect to the original LO (carrier) waveform. This is often advantageous since it eliminates microwave leakage at the signal frequency for zero voltage on the IQ inputs, which is often a big problem for commercially available IQ mixer, as we will discuss below.

To Do 9: Add exact type number

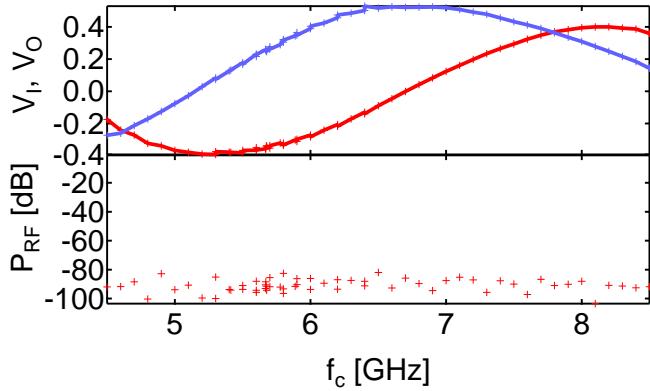


Figure 4.3: a) The offset voltages that we need to apply to the IQ ports of the mixer to eliminate unwanted leakage from the LO to the RF port of the device, plotted as a function of the carrier frequency f_c . b) The measured remaining signal power at the RF port of the mixer at the optimal IQ bias point.

Commercially available IQ mixers often deviate from the ideal behavior as given by eq. (4.1). Typical imperfections include large insertion losses –i.e. loss of signal power between the different ports of the mixer–, RF signal leakage at zero IQ-input and frequency-dependent phase and amplitude errors of the mixed sideband signals. In order to achieve reliable single-qubit operations we need to correct the signal leakage and quadrature-specific amplitude and phase errors. The signal leakage causes a small part of the LO signal to leak through to the RF port even when the IQ inputs are zeroed. This leakage can be compensated by adding center-frequency ω_c dependent DC offset voltages to the IQ ports. The appropriate offset voltages can be determined by applying a continuous input signal at a frequency ω_c to the LO port of the mixer and minimizing the measured signal power at the RF port by varying the IQ offset voltages. To correct the sideband amplitude and phase errors we apply another correction procedure that we outline here. First, for the signals at the IQ inputs of the mixer we introduce the notation

$$A(t) = I(t) + iQ(t) = a(t) \exp(-i\phi(t)) \quad (4.2)$$

We consider an IQ signal at a single sideband frequency ω_{sb} and at fixed complex amplitude $a(t) = a = a_0 \exp(i\phi_0)$ such that $A(t) = a \exp(-i\omega_{sb}t)$. The effect of the gain and phase imperfections of the IQ mixers can then be modeled by assuming that the mixer adds another IQ signal $\epsilon(\omega_{sb}, \omega_c) A^*(t)$ at the mirrored sideband frequency $-\omega_{sb}$. We can correct this unwanted signal by adding a small correction $c(\omega_{sb}, \omega_c) A^*(t)$ to our IQ input signal. The correction coefficient $c(\omega_{sb}, \omega_c)$ usually depends both on the carrier frequency ω_c and the sideband frequency ω_{sb} . We determine the correction coefficients by generating a continuous waveform at a given center and sideband frequency, measuring the amplitude of the unwanted sideband signal with a fast spectrum analyzer and minimizing its amplitude by varying the correction coefficient $c(\omega_{sb}, \omega_c)$.

Both the offset and the sideband-amplitude and -phase corrections have been automated using our data acquisition software, the resulting correction coefficients are summarized in fig. 4.3. By using the optimization techniques described above we can

achieve -80 dBm residual power at the RF port of the mixer when no input IQ signal is present and a suppression of the unwanted mirror sideband in heterodyne modulation > 70 dB !10! .

To Do 10: verify this number!

4.3.3 Fast Magnetic Flux Pulses

For the fast flux lines we use superconducting transmission lines which are attenuated by 20 dB and filtered at the 4K and 20 mK stages of the cryostat. The filtering at the 20 mK stage is realized using custom-built, highly absorptive high-frequency microwave filters. Fig. ?? shows an image of such a filter and the attenuation characteristic obtained for it. As can be seen, the filter shows an exponential attenuation and filters very effectively at high frequencies. This heavy filtering of the flux line is helpful since it greatly reduces high-frequency noise seen by the qubit but it also distorts all deterministic signals sent through the flux line. This distortion is unwanted especially at high frequencies and needs to be corrected. To do this we need to measure and compensate the frequency response of the flux line. In order to do this, we make use of the return line of the flux line and feed back the flux signal sent to the sample to room temperature. This allows us to measure the returning signal and – assuming symmetric distortion in the input and return line – to calculate the response function of the input line. Fig. 4.4 shows the different parts of the response function of the flux line as measured in our experiment. We can obtain the response function of the input part of the fluxline by sending a step-pulse through the flux line, measuring the Fourier spectrum of the returning signal and solving

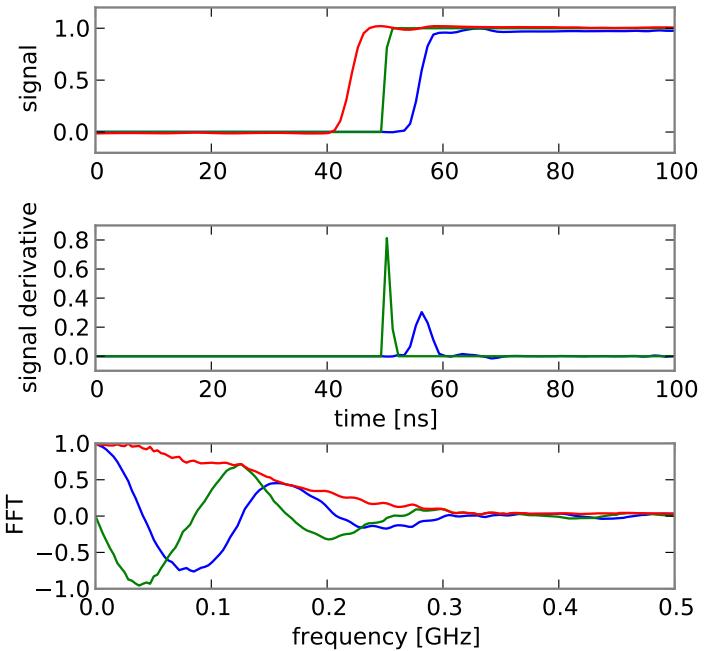


Figure 4.4: (response function filtered with a Gaussian filter with a cut-off at 0.4 GHz)

$$\chi_{out}(\omega) = \chi_{ideal}(\omega) \cdot \chi_{DAC} \cdot \chi_{in} \cdot \chi_{out} \cdot \chi_{ADC} \quad (4.3)$$

Here, χ_{ADC} and χ_{DAC} describe the response functions of the DAC and ADC, χ_{in} corresponds to the Fourier spectrum of the ideal input waveform and χ_{out} corresponds to the Fourier transform of the digitized return signal. We assume that $\chi_{in} \approx \chi_{out}$. By measuring χ_{out} and correcting the measured Fourier spectrum for the response function of the ADC χ_{ADC} we obtain the input line response $\chi_{DAC}\chi_{in}$ including the DAC response. We can then correct our digital input waveforms by applying

$$\chi_{in}^{corr} = \chi_{in} \cdot (\chi_{DAC} \cdot \chi_{in})^{-1} \cdot G(f_0) \quad (4.4)$$

Here, $G(f_0)$ is a Gaussian filter that we apply to the measured response function to attenuate possible signal distortion that is caused by the fact that we are not able to accurately measure the response function of the system above a certain frequency. Usually, we set the cutoff frequency to $f_0 = 300$ MHz which allows us to correct most signal distortion effects in the frequency band relevant to us.

4.3.4 Pulse Synchronization

We use a 10 MHz chain to synchronize all relevant signal generator and acquisition cards of our setup. The chain topology is shown in detail in fig. ???. In addition, we take great care to synchronize the frequencies of the microwave generators with the repetition interval of our arbitrary waveform generator to avoid phase-jitter which is catastrophic when generating IQ drive pulses for qubit control. In addition, we use a 1 GHz synchronization chain to phase-lock the microwave generators that produce the drive pulses for both qubits. Delays between qubit drive and fluxline signals that are caused by differences in electrical length of the respective signal lines are corrected for by using the qubit itself as a probe of the applied flux: For this, we use a step-like flux signal which shifts the qubit out of resonance with a pre-chosen Rabi pulse that performs a π rotation of the qubit state when in resonance. By varying the position of the step in respect to the π -pulse we can determine the exact timing between the two and correct for possible delays.

4.4 Measurement Techniques

In this section we will discuss the techniques used to characterize and manipulate our two-qubit processor. We will cover the qubit readout and manipulation and will describe how we can determine all relevant qubit parameters using microwave reflectometry measurements.

4.5 Qubit Readout

We use the nonlinear resonator as a Josephson bifurcation amplifier to read out the state of the qubit. This works by sending a drive pulse shaped as in fig. 4.5 to the sample. The frequency of this pulse is chosen such that it is inside the bistability region as given by eq. (2.62). Thus, there exists an amplitude for which the resonator will transit from a low-amplitude state to a high-amplitude one when continuously increasing the drive power during the first part of the pulse. The value of this amplitude depends on the resonance frequency of the resonator, which itself depends on the state of the qubit through the dispersive coupling. If we choose the peak power of our measurement pulse such that it is very close to this transition amplitude, a small qubit-induced frequency shift of the resonator frequency will be able to significantly change the transition probability of the resonator when ramping up the power during the measurement pulse, thus making possible a measurement of the qubit state. After ramping up the microwave power to its peak value and holding it there for a certain amount of time, we reduce the power slightly to put the resonator within the bistability regime where its state is no longer sensitive to a shift of the resonator frequency, thus effectively decoupling the evolution of the resonator state from that of the qubit. We can then hold this state for an arbitrarily long time and measure the phase and amplitude of the reflected microwave signal during this “latching” period, thereby obtaining a precise measurement of the resonator state. The low- and high-amplitude resonator states are easily distinguishable when demodulating the reflected signal during the latching part of the pulse and averaging their respective IQ quadratures. When plotting many such averaged IQ quadratures for a measurement pulse which induces around 50 % switching to the high-amplitude state, one obtains a family of points as shown schematically in fig. 4.5. To distinguish between the two states H and L we perform a principal axes transformation of the data, effectively obtaining the separator plane P that distinguishes ideally between the two families. Projecting the IQ data along this discrimination line yields an effective probability distribution in one dimension from which we can directly calculate the switching

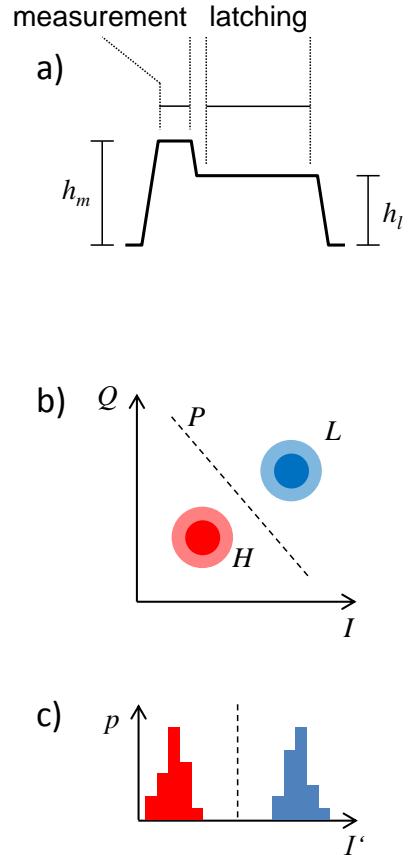


Figure 4.5: ...

probability for a given family of measurements.

4.6 Qubit Manipulation

To drive the qubits, we need to generate fast microwave pulses with a well defined frequency and phase. As described in one of the previous paragraphs we can use IQ sideband mixing to shape arbitrary drive pulses of the form

$$u(t) = I(t) \cos \omega_{rf} t + Q(t) \sin \omega_{rf} t \quad (4.5)$$

We can rewrite this as a product of two complex quantities

$$u(t) = \Re [A(t) \cdot \exp(-i\omega_{rf}t)] \quad (4.6)$$

where we defined, as before, $A(t) = I(t) + iQ(t)$. The reference phase defining an x-pulse can be chosen arbitrarily but must be conserved during one single experimental run. Thus, to realize a rotation of the qubit state in the xy-plane of the Bloch sphere around an axis defined through an angle ϕ we can use a Gaussian-shaped pulse of the form

$$A(t) = A_0 \cdot \exp\left(-\frac{(t-t_0)^2}{2\sigma_t^2}\right) \cdot \exp(-i\phi) \quad (4.7)$$

In the following chapter we will discuss more in detail the calibration of these drive pulses and possible errors induced when driving the qubit at frequencies comparable to its anharmonicity.

4.6.1 Spectroscopic Measurements of the Qubit State

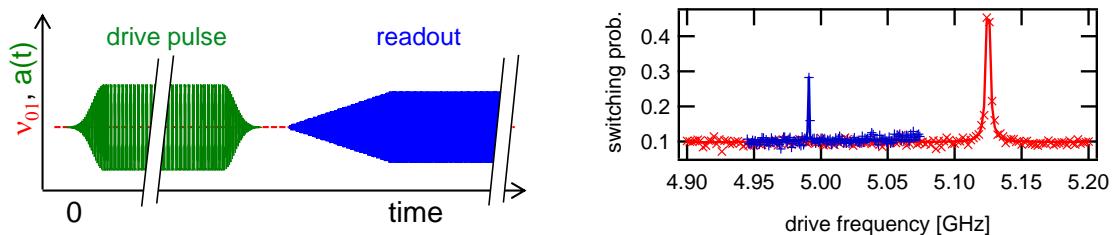


Figure 4.6: Example of measured qubit spectroscopy. Shown is the switching probability of the qubit readout when driving the qubit with a very long drive pulse (typically $1 \mu\text{s}$) at a given drive frequency. The resonance to the right corresponds to the $|0\rangle \rightarrow |1\rangle$ (at frequency f_{01}) transition of the qubit, the resonance on the left to the 2-photon $|0\rangle \rightarrow |2\rangle$ (at frequency $f_{02}/2$) transition. We perform a Lorentzian fit of the two resonances to obtain the $|0\rangle \rightarrow |1\rangle$ and $|0\rangle \rightarrow |2\rangle/2$ resonance frequencies, from which we can calculate all other qubit transition frequencies.

In order to characterize the transition frequency and anharmonicity of the qubit it is useful to perform spectroscopic measurements of the qubit state. For this we drive the qubit with a very long Rabi pulse (usually > 500 ns at a well-defined frequency f_r). When the drive frequency f_r corresponds to the f_{01} frequency of the qubit the drive pulse will induce a strong Rabi oscillation of the quantum state of the qubit. Since the decoherence time of the qubit is of the order of the length of the drive pulse, the qubit state will decay and dephase during the driven evolution, effectively yielding an equal probability to measure the qubit in either of the states $|0\rangle$ or $|1\rangle$ at the end. When the drive frequency is detuned from the qubit transition frequency, no oscillation will be induced and hence the qubit will remain in the state $|0\rangle$. The width of the resonance in frequency space is inversely proportional to the dephasing time of the qubit, following the equation

$$\dots \quad (4.8)$$

Fig. 4.6 shown an exemplary qubit spectroscopy. Plotted is the probability of measuring the qubit in state $|1\rangle$ after applying a $1\ \mu\text{s}$ rabi pulse at a given drive frequency to it. The blue curve has been measured at 10 dB higher power than the red curve and shows the $|0\rangle \rightarrow |2\rangle$ transition of the qubit. By fitting the resonance curves with a Lorentzian model we obtain the qubit frequencies f_{01} and $f_{02}/2$, which allow us to calculate the effective Josephson and charging energies at the given working point.

4.6.2 Rabi Oscillations

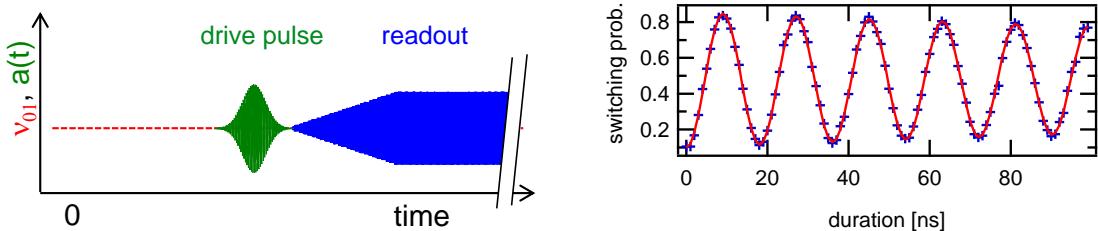


Figure 4.7: Example of a measured qubit Rabi experiment. Shown is the switching probability of the qubit readout when driving the qubit at f_{01} with a Gaussian drive pulse of varying duration. The measurement results are not corrected for readout errors.

After having obtained the proper qubit transition frequency f_{01} using the technique described above, we can perform a Rabi oscillation experiment by driving the qubit for a well-defined time with a drive pulse at the f_{01} transition frequency and measuring the state of the qubit directly afterwards. Fig. 4.7 shows an exemplary Rabi measurement. The blue dots correspond to measured data points whereas the continuous red line corresponds to a fit of a model of the form $p(t) = p_0 + a \cos \Omega t \exp -\Gamma_1 t$ to the experimental data. As can be seen, the amplitude of the Rabi oscillations gets damped the longer

the drive pulse becomes, which is due to relaxation and dephasing during the driven evolution of the qubit. The maximum probability contrast is limited due to readout errors, as we will explain in more detail in the following chapter. From the fit of the Rabi data we obtain the Rabi frequency Ω , which we can then use to perform precise single-qubit rotations, as will be explained later. Due to the finite anharmonicity of the qubit, there will always be a leakage to the second excited state $|2\rangle$ of the qubit which gets stronger, the faster we drive the system. This leakage mechanism is an important source of errors and very relevant to the experiments that will be discussed later, so we will quantify it in detail in the following chapter as well.

4.7 Dephasing Time Measurement

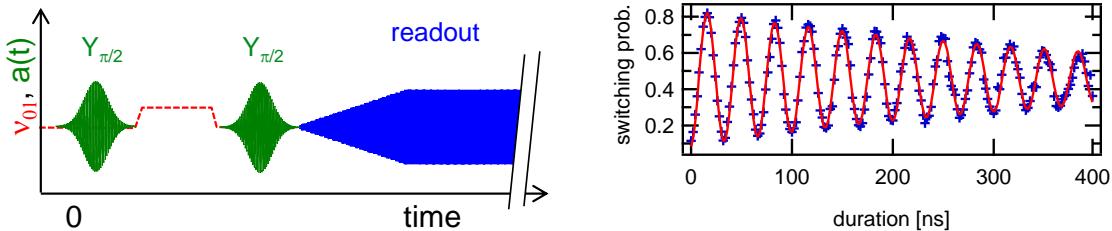


Figure 4.8: Example of a measured qubit Ramsey experiment. Shown is the switching probability of the qubit readout after performing a $X_{\pi/2}$ -wait- $X_{\pi/2}$ drive sequence at a frequency $f_{01} - \delta f$. Fitting the resulting curve with an attenuated sine-wave model allows us to determine the f_{01} frequency of the Qubit with high accuracy.

After having obtained the transition frequency f_{01} of the qubit and the Rabi frequency Ω , we can characterize the dephasing of the qubit by performing a so-called *Ramsey fringe experiment*[]. In this experiment, we perform a $Y_{\pi/2}$ rotation of the qubit from the state $|0\rangle$, obtaining thus a superposed qubit state of the form $1/\sqrt{2}(|0\rangle + |1\rangle)$. Then we displace the qubit frequency by an amount Δf by using e.g. a fast magnetic flux pulse and let the qubit state evolve freely during a certain amount of time Δt . Finally, we apply another $Y_{\pi/2}$ pulse to the qubit and measure the state of the qubit directly afterwards. Since the qubit frequency has been displaced during the free evolution, the qubit will acquire a phase $\Delta\phi = 2\pi\Delta f\Delta t$. The final state of the qubit after applying the second $Y_{\pi/2}$ pulse will therefore be given as

$$|\phi_f\rangle = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e^{i\Delta\phi} \end{pmatrix} = \begin{pmatrix} i \sin \Delta\phi/2 \\ -\cos \Delta\phi/2 \end{pmatrix} \quad (4.9)$$

Hence, the resulting state $|\phi_f\rangle$ will oscillate between the state $|0\rangle$ and $|1\rangle$ with a frequency $\Delta f/2$. As before, due to dephasing and relaxation during the free evolution

of the qubit state, the amplitude of these oscillations will decay. If the system dephasing time is limited by qubit relaxation, the decay will follow a Gaussian decay of the form $\exp(-\Gamma t^2)$, otherwise it will also exhibit an exponential decay $\simeq \exp(-\Gamma t)$ [1]. In the Ramsey sequence, instead of detuning the qubit frequency during the free evolution phase we can also detune the qubit drive frequency instead. If this detuning is small in comparison to the Rabi frequency Ω , we will induce only a negligible error when applying the first $Y_{\pi/2}$ pulse. But since the drive frequency is detuned, the qubit will also acquire a phase $\Delta f \Delta t$ during the free evolution phase. Fitting the experimental data obtained for such an experiment to a model of the form $p(|1\rangle) = p_0 + a \cos(\Delta f \Delta t + \phi_0) \exp -\Gamma_2 t^2 / 2$ we can obtain an estimate of Δf . Since we know the frequency detuning of the drive during the free evolution of the qubit we can subtract it from the fitted value in order to obtain the remaining detuning of the qubit from the drive frequency at zero drive detuning. This method allows us thus to make a precise fit of the qubit frequency and to correct drive frequency errors with an accuracy of typically 100 kHz.

4.8 Relaxation Time Measurement

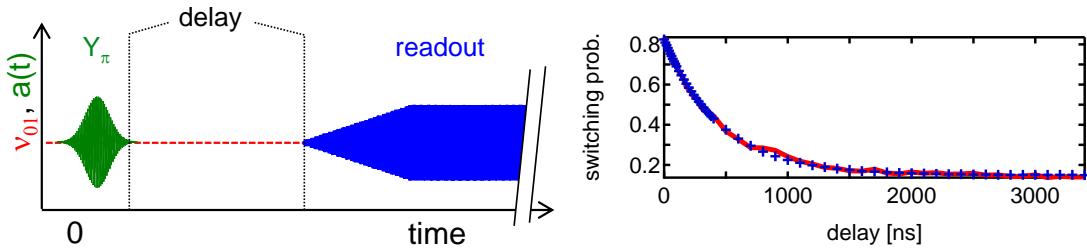


Figure 4.9: Example of a qubit relaxation time measurement. Shown is the probability of measuring the qubit in state $|1\rangle$ as a function of the delay time between the preparation of the state $|1\rangle$ and the actual measurement of the qubit state. The decay of this probability follows an exponential law of the form $p(|1\rangle) \simeq \exp(-\Gamma_1 t)$

We can characterize the relaxation time of the qubit by performing a simple experiment where we put the qubit in state $|1\rangle$ by applying a X_π pulse and let the qubit evolve freely for a given time before measuring its state. The resulting curve when performing such an experiment is shown in fig. 4.9. It shows the probability of measuring the qubit in state $|1\rangle$, plotted as a function of the delay between the initial state preparation and measurement. As can be seen, this probability decreases exponentially as a function of time. As before, in the curve the blue markers correspond to experimental data and the red line corresponds to a fit of this data to a model of the form $p(|1\rangle) = p_0 + p_a \exp(-\Gamma_1 t)$. From this fit we can then easily extract the relaxation rate Γ_1 of the qubit at the given

working point. In the following chapter we will look more in detail at the relaxation time of both qubits as a function of their transition frequency and their detuning from the readout resonator.

Chapter 5

Characterizing the Two-Qubit Processor

This section discusses the detailed characterization of individual circuit parts that will be used later to realize two-qubit gate and to run a quantum algorithm on the processor. The discussion will focus on the readout and microwave manipulation of the qubits as well as on the reconstruction of quantum states from measurement data, which will be used later for characterizing gate and processor operation.

5.1 Qubit & Readout Characterization

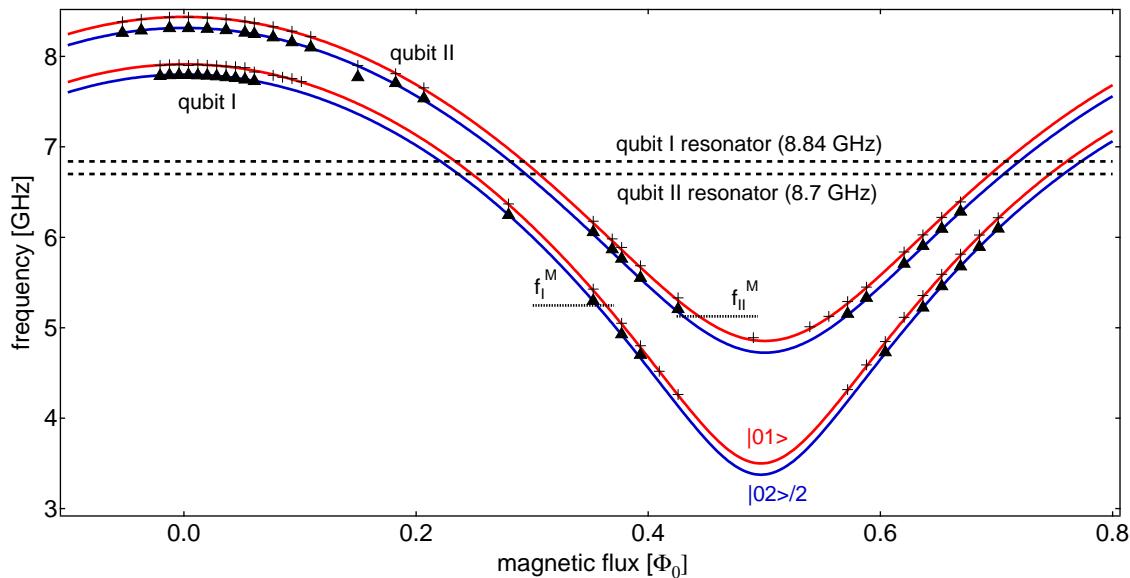


Figure 5.1: Spectroscopy of the realized two-qubit processor. $|0\rangle \rightarrow |1\rangle$ and $(|0\rangle \rightarrow |2\rangle)/2$ transition frequencies of the two qubits with fitted dependence and cavity frequencies.

The following section discusses the parameters of our two-qubit processor that have been obtained by various measurements.

5.1.1 Qubit Parameters

To obtain all the relevant parameters of our two-qubit processor, we perform a set of measurements from which we obtain the qubit frequencies, anharmonicities, junction asymmetries, the inter-qubit coupling, the coupling to the microwave drive lines, the coupling of each qubit to its readout and the relaxation and dephasing times of the qubits. The drive and readout couplings as well as the relaxation and dephasing times are measured for a range of qubit frequencies, which will allow us later to pick an ideal working point for our two-qubit experiments. The qubit parameters obtained from spectroscopic measurements are as follows:

- *Qubits*: Spectroscopic measurement of the qubit transitions yielded parameter values of $E_J^I/h = 36.2$ GHz, $E_c^I/h = 0.98$ GHz and $E_J^{II}/h = 43.1$ GHz, $E_C^{II}/h = 0.87$ GHz for the Josephson and charging energies of the two qubits and values of $d^I = 0.2$, $d^{II} = 0.35$ for the qubit junction asymmetries.
- *Readout resonator*: The frequencies of the readout resonators have been measured as $\nu_R^I = 6.84$ GHz and $\nu_R^{II} = 6.70$ GHz with quality factors $Q^I \simeq Q^{II} = 730$, independent measurements of the Kerr nonlinearities yielded $K^I/\nu_R^I \simeq K^{II}/\nu_R^{II} = -2.3 \pm 0.5 \times 10^{-5}$!11! .
- *Qubit-Resonator coupling*: The coupling of the qubits to the readout resonators has been spectroscopically determined as $g_0^I \simeq g_0^{II} = 50$ MHz

To Do 11: add inferred parameters from the bare resonator frequencies

Readout Parameters

To measure the qubit-resonator coupling, we measure the S-parameter of the resonator using microwave reflectometry while tuning the qubit transition frequency through the resonance frequency of the resonator. The results of such a measurement are shown in fig. 5.2 for the two qubits of our processor. From these measurements we can extract the qubit-resonator coupling strengths, which for both qubits are given as $g_{qr} \approx 50$ MHz.

Qubit Parameter Survey

In order to determine the optimal working point for our processor we need to characterize the properties of the qubits and readouts at many different qubit frequencies. In order to do this we perform an automated survey of qubit spectroscopies, qubit readout characterizations and T_1 measurements at different qubit frequencies. The results of

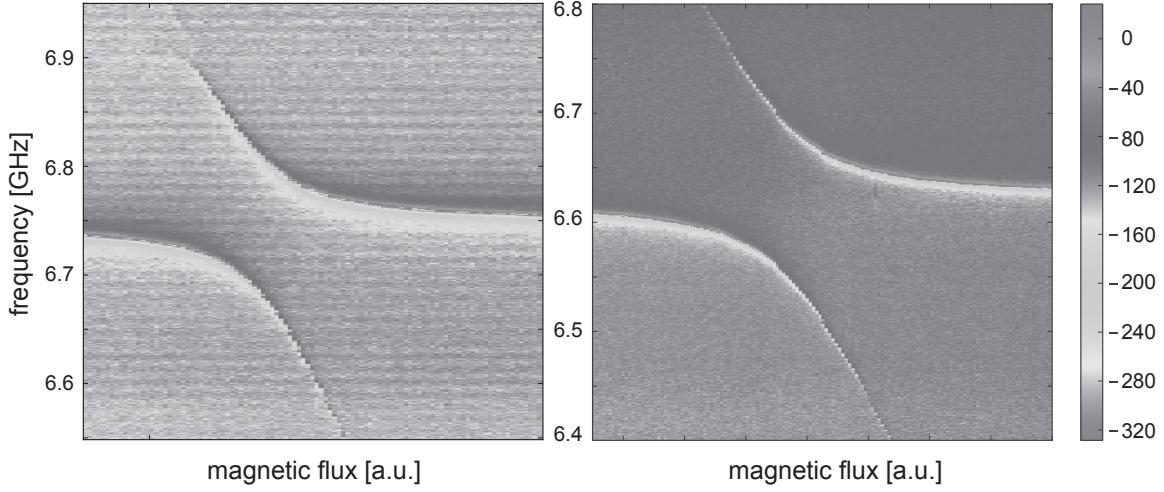


Figure 5.2: Spectroscopic measurement of the avoided level crossing between the two qubits and their corresponding readout resonator, corresponding to an effective qubit-resonator coupling strength $2g \approx 50$ MHz.

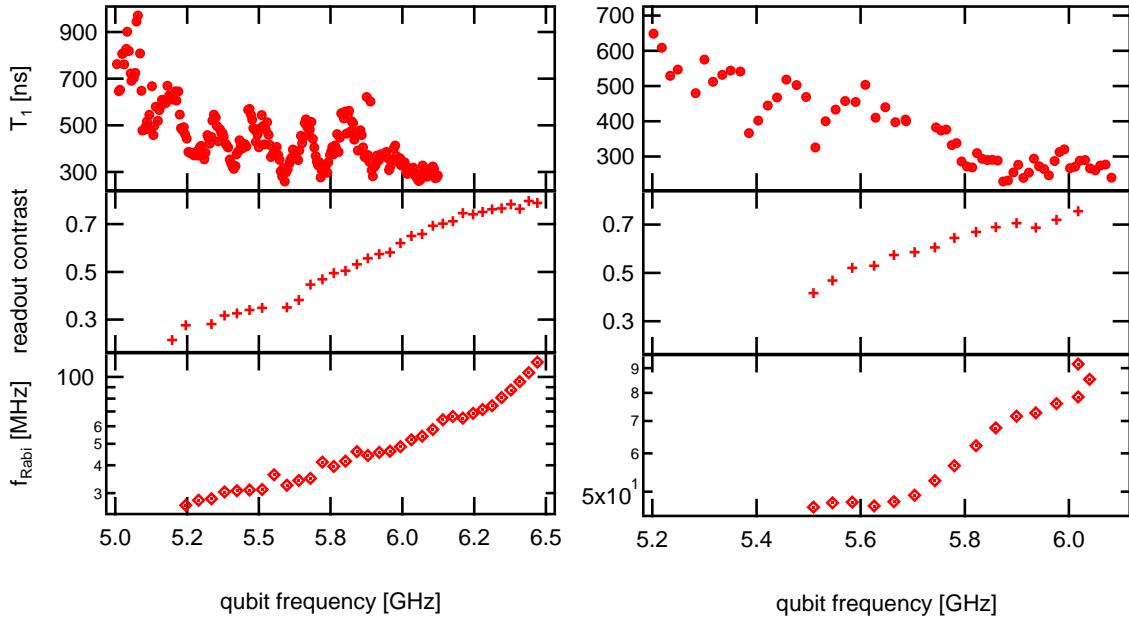


Figure 5.3: A qubit parameter survey showing the relaxation time T_1 , the readout contrast and the Rabi frequency at a fixed drive amplitude for the two qubits over a large range of qubit frequencies.

such a parameter survey are summarized in fig 5.3, showing the relaxation time T_1 , the readout contrast c_{10} and the Rabi frequency f_{Rabi} for a fixed drive amplitude for the two qubits in a frequency range between 5.2 and 6.5 GHz. As can be seen, the relaxation time of the qubits tends to increase the farther detuned each qubit is from its readout resonator. Not surprisingly, the drive frequency of the qubit also decreases when

the qubit-resonator detuning increases as expected from the Purcell effect, which filters incoming microwave signals that are far-detuned from the resonator frequency. The inverse is true for the readout contrast, which increases near-linearly when reducing the qubit-resonator detuning due to the increase of the dispersive resonator frequency shift induced by the qubit that gets stronger the less the qubit is detuned from the readout resonator.

It is interesting to note the non-monotuous characteristic of the qubit relaxation time T_1 shown in fig. 5.3, which cannot be explained by Purcell-filtering through the readout resonator and hints at a different qubit relaxation process present in the system. A possible explanation would be the coupling of the qubit to a spurious low-Q resonance in the environment. Coupling to volumetric resonance modes of the sample holder or non-CPW resonance modes of the readout resonator can be possible explanations for the data. Also, the overall dependency of the relaxation time T_1 on the qubit-resonator detuning –ignoring the “fine-structure” present in the system– is not quadratic as would be expected from the Purcell theory but rather linear. Also, by comparing the qubit relaxation time to the Rabi drive frequency reveals that the increase in T_1 is cleary not proportional to the Purcell factor that determines the qubit relaxation rate through the readout resonator. However, the observed T_1 dependency can be partially explained by taking into account the qubit relaxation through the fast fluxline, which might be to strongly-coupled to the qubit on our chip, hence inducing additional qubit relaxation beyond the Purcell and intrinsic qubit relaxation rates. This effect will therefore be studied in more detail in the following sections.

5.2 Single-Qubit Operations

To perform arbitrary single-qubit operations – as needed e.g. for implementing a quantum algorithm or performing quantum state tomography – we need to implement a universal set of X , Y and Z qubit gates with our processor. Qubit rotations in the XY -plane are implemented through microwave drive pulses, where the phase of the drive pulse in reference to an arbitrary reference determines the rotation axis and the amplitude of the drive pulse the Rabi frequency of the gate. To characterize the drive pulses, we perform an experiment where we initialize a single-qubit in the states $|1\rangle$, $1/\sqrt{2}(|0\rangle + |1\rangle)$ and $\sqrt{2}(|0\rangle + i|1\rangle)$ and subject it afterwards to a single microwave pulse of the form $a(t) = V_I \cdot \cos \omega_{rf} t + V_Q \cdot \sin \omega_{rf} t$, which we tune by changing the input voltages V_I and V_Q to the IQ -mixer that generates the pulse from a continuous input microwave-tone at frequency ω_{rf} . We measure the qubit state at different values of V_I , V_Q , obtaining the graph shown in fig. 5.4. The qubit which was prepared in state $|1\rangle$ shows a perfectly cylinder-symmetric switching probability pattern when subjecting it to an IQ-pulse of a

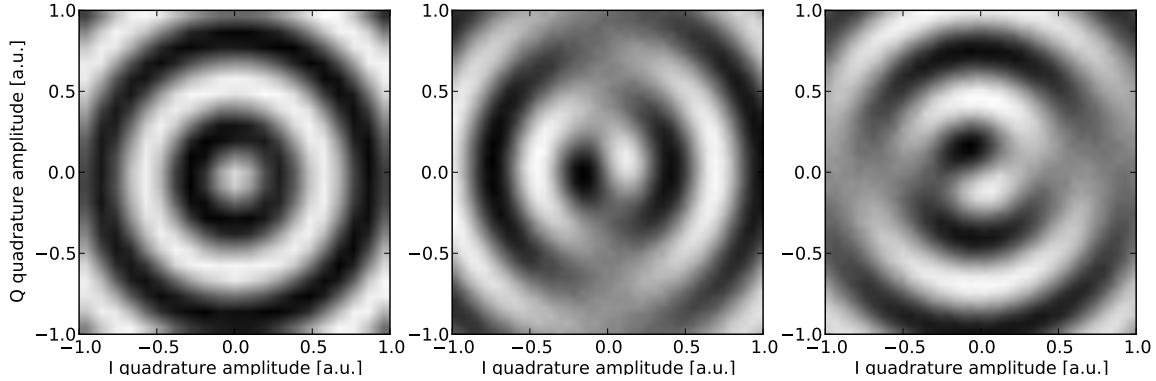


Figure 5.4: Demonstration of single-qubit IQ control. The figures show the state probability of a single qubit when preparing it in one of the states $|1\rangle$, $1/\sqrt{2}(|0\rangle+|1\rangle)$ or $1/\sqrt{2}(|0\rangle+i|1\rangle)$ and subjecting the qubit to a microwave drive pulse of the form $a(t) = V_I \cdot \cos \omega_{rf} t + V_Q \cdot \sin \omega_{rf} t$.

given phase, which is what one would expect for a qubit being prepared in either the $|0\rangle$ or $|1\rangle$ state. On the contrary, the switching probability distributions of the measured qubits prepared in the states $1/\sqrt{2}(|0\rangle+|1\rangle)$ and $1/\sqrt{2}(|0\rangle+i|1\rangle)$ are mirror-symmetric, where the switching probability does not vary at all along the drive axis which corresponds to the axis along which the qubit has been prepared. These measurements demonstrate therefore our ability to prepare and drive the qubit along arbitrary axes of the Bloch sphere. In the following sections we will analyze more in detail the drive errors inherent to our system and quantitatively analyze different error sources.

5.2.1 Estimation of drive errors

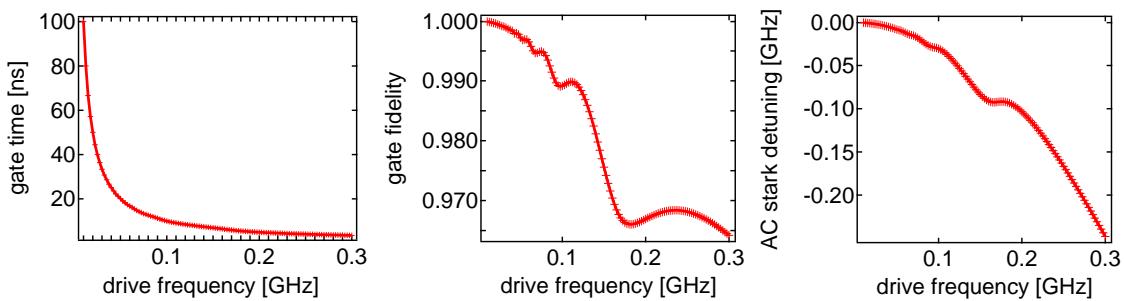


Figure 5.5: The Single-qubit π -pulse gate time, gate fidelity and AC stark detuning, plotted as a function of the drive strength ϵ applied to the three-level Transmon. As the drive strength increases, the gate time decreases as $1/\epsilon$ whereas the gate fidelity decreases approximately quadratically.

Since the Transmon is a weakly anharmonic multi-level system and thus no real qubit, driving the $|0\rangle \rightarrow |1\rangle$ transition with high power can induce transitions to higher

Transmon levels. It is important to estimate and reduce these errors when performing fast qubit gates e.g. for state preparation or tomography. To model the driving of a Transmon, we use the simple drive model in the rotating-frame approximation and as used e.g. in [39]:

$$\hat{H} = \begin{pmatrix} 0 & \epsilon^*(t) & 0 \\ \epsilon(t) & \delta & \sqrt{2}\epsilon^*(t) \\ 0 & \sqrt{2}\epsilon(t) & 2\delta + \alpha \end{pmatrix} \quad (5.1)$$

Here, $\epsilon(t) = \epsilon_x(t) + i\epsilon_y(t)$ is the complex drive IQ amplitude in the rotating qubit frame, δ is the detuning of the microwave drive from the Transmon ω_{01} transition frequency and α is the Transmon anharmonicity. To estimate the leakage to higher Transmon levels, we calculate the eigenvalues and eigenvectors of the Hamiltonian given in eq. (5.1). We then decompose an initial state $|0\rangle$ in the basis of eigenstates and let it evolve under a given constant drive amplitude ϵ_0 . By maximizing the occupation probability of the state $|1\rangle$ as a function of the evolution time and drive detuning δ we obtain the gate time, state fidelity and AC-Stark shift for the applied drive strength. These quantities are shown in fig. 5.5, which shows the numerically obtained gate time, gate fidelity and AC-stark shift of the qubit frequency as a function of the applied drive frequency. As can be seen, the gate error increases almost quadratically with the drive frequency. Typically, in our experiments we use drive frequencies between 100 - 200 MHz, inducing thus single-qubit drive errors between 1-3 %. In general, it is possible to correct these errors using optimized drive pulses [], thereby eliminating leakage to the third qubit level. In this work, we did not use such techniques but we will include possible errors arising due to this leakage to higher Transmon levels in our error models when discussing our experimental data.

5.3 Two Qubit Operations

We use the capacitive coupling between the qubits to implement 2-qubit gates and create entangled states between the two qubits. To characterize the qubit-qubit interaction, we first perform a spectroscopic measurement of the system. For this, we measure a spectroscopy of one of the qubits for a range of qubit frequencies such that the frequency of qubit I gets tuned through the frequency of qubit II. When these two frequencies are sufficiently close we observe an avoided level crossing between the qubits. At resonance, the gap between the two spectroscopic lines that appear corresponds to the qubit-qubit coupling strength $2g$. Fig 5.6 shows the results of such a measurements. Here, qubit II was positioned at a 5.125 GHz and the frequency of qubit II was tuned from 5.1 – 5.15 GHz. At resonance we measure a coupling strength of $2g = 8.9$ MHz.

If the two qubit frequencies $\omega_{01}^I, \omega_{01}^{II}$ are far detuned such that $|\omega_{01}^I - \omega_{01}^{II}| \gg 2g$, the

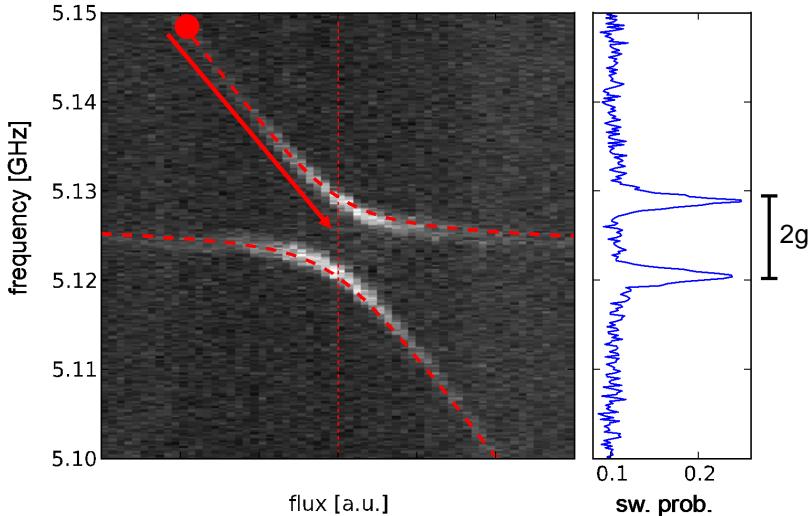


Figure 5.6: a) Spectroscopy measurement of the avoided level crossing between the two qubits of our processor. b) Single spectroscopy of qubit I when in resonance with qubit II. From this spectroscopic measurement, a qubit-qubit strength of $2g = 8.9$ MHz can be deduced.

eigenstates of the coupled two-qubit Hamiltonian correspond to a very good accuracy to the independent qubit states $|01\rangle$ and $|10\rangle$. Therefore, far from the resonance we observe only one spectroscopic line corresponding to the transition $|00\rangle \rightarrow |10\rangle$ (we drive the first qubit). However, close to resonance the eigenstates approach the Bell states $|\phi\pm\rangle = 1/\sqrt{2}(|10\rangle \pm |01\rangle)$, which can both be excited by driving qubit I. Hence we observe two spectroscopic lines corresponding to the two transitions $|00\rangle \rightarrow 1/\sqrt{2}(|01\rangle + |10\rangle)$ and $|00\rangle \rightarrow 1/\sqrt{2}(|01\rangle - |10\rangle)$.

5.3.1 Creation of Entanglement

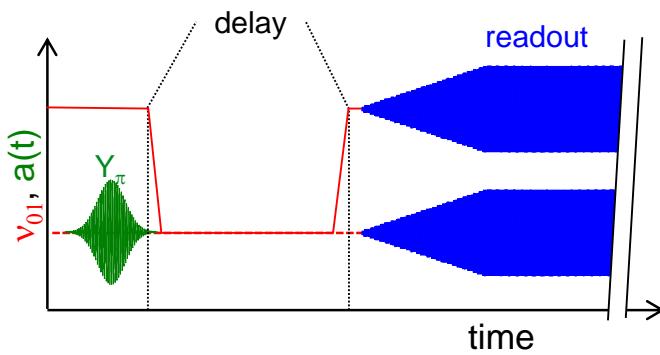


Figure 5.7: Pulse sequence used to create an entangled two-qubit state using a non-adiabatic transfer from an off-resonant to a resonant two-qubit state.

After having characterized the qubit-qubit interaction using a spectroscopic measurement we perform a coherent swapping operation between the qubits. We implement this operation by non-adiabatically changing the resonance frequency of the qubits from

an off-resonance to an on-resonance condition. When doing this after preparing the qubits in one of the states $|01\rangle$ or $|10\rangle$ we induce a coherent energy swap between the qubits. We then terminate the interaction after a well-defined time and measure the state of the two-qubit register directly afterwards. Fig. 5.7 summarizes the frequency, drive and measurement pulse sequence that we use for this experiment. When varying the time during which the qubits are in resonance we can observe the coherent swapping of energy between them by measuring the state directly after the swap. Fig. 5.8 shows these probabilities as a function of the interaction time Δt . As can be seen, as expected the frequency of the energy swap $2g = 8.9$ MHz agrees with the value obtained from the spectroscopic measurement. Fig. 5.8c shows, in addition to the measured and corrected state occupation probabilities a master equation simulation of the swapping experiment. The simulation takes into account relaxation and dephasing of the qubits. The relaxation rates put into the simulation correspond to the experimentally measured Γ_1 rates at the operating frequency of the qubits, whereas the dephasing rates $\Gamma_\phi \approx 2 \mu\text{s}$ are fitting parameters. We cannot use the measured single-qubit dephasing times in our simulation since the combined dephasing rate of the qubits in resonance vanishes at first order. This is so because the transition frequency of the two eigenstates $|\phi_\pm\rangle = 1/\sqrt{2}(|01\rangle \pm |10\rangle)$ at resonance is given as $\omega_{01}^{\pm} = \omega_{01}^{III} \pm \sqrt{4g^2 + \Delta^2}/2 \approx \omega_{01}^{III} \pm g + \Delta^2/8g$ and is therefore insensitive to variations of Δ at first order. To reproduce the experimentally measured dephasing, we use effective dephasing rates $\Gamma_\phi^{I,II} = 2 \mu\text{s}$ in our simulation. As can be seen, the simulation agrees very well with the experimental data.

5.3.2 Violation of the Bell Inequality

In a famous 1935 paper [19], A. Einstein, B. Podolsky and N. Rosen (EPR) pointed out that quantum mechanics can be considered “incomplete” if one assumes that a set of simple physical principles holds true. These two principles are *locality* and *realism*, together often referred to as *local realism*. The principle of locality states that an object is influenced directly only by its immediate surroundings, or stated differently, that no interaction between any two objects may propagate faster than the speed of light. The principle of realism states that any physical observable has, at all times, a pre-existing measurement value which exists independent of any actual measurement performed of the observable. In their paper, Einstein, Podolsky and Rosen showed that within standard quantum mechanics it is possible to write down quantum states that, when performing certain measurements on them, clearly violate at least one of the principles of locality or realism. As an example, take the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (| \uparrow\downarrow \rangle + | \downarrow\uparrow \rangle) \quad (5.2)$$

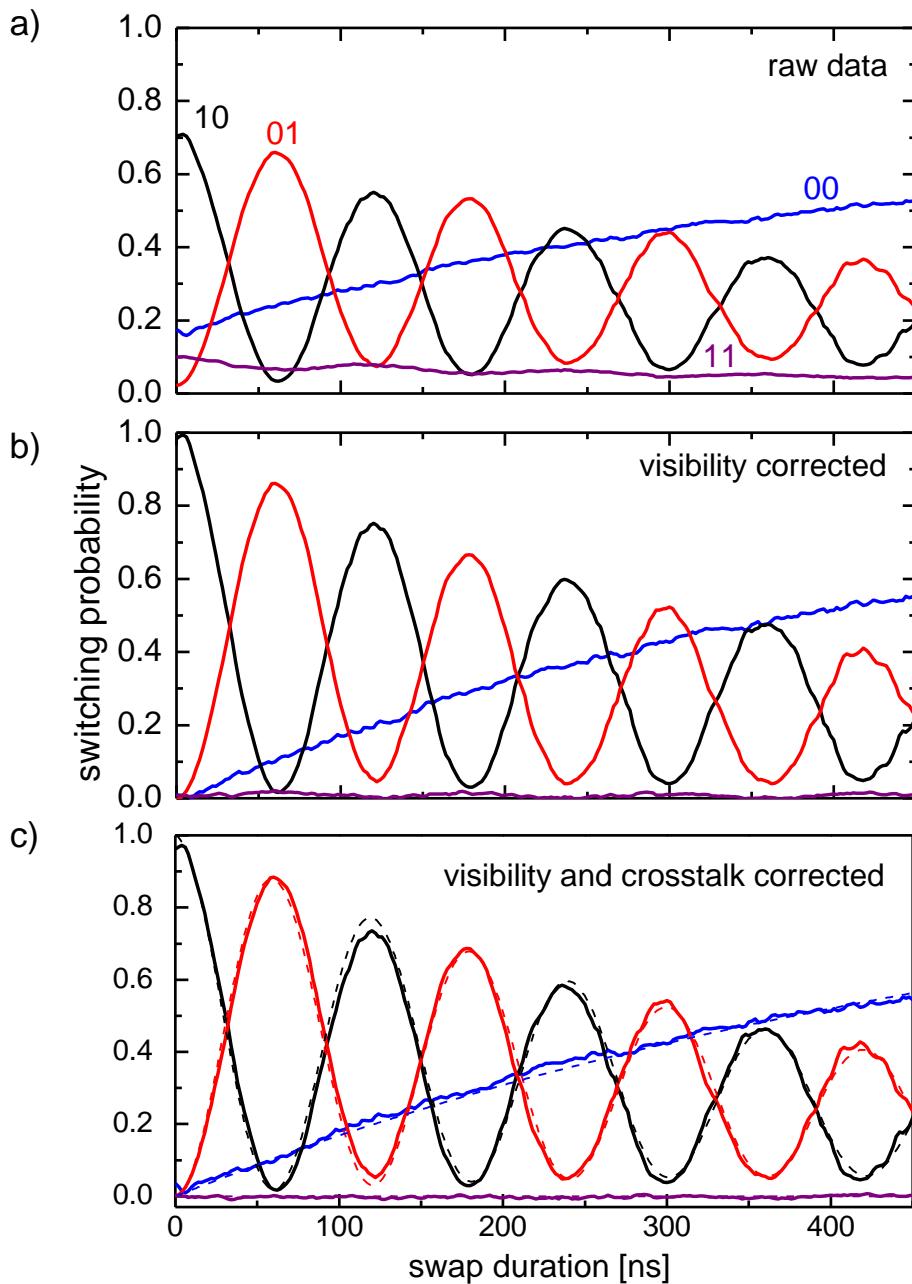


Figure 5.8: Raw, visibility corrected and fully corrected state occupation probabilities for the two-qubit register during a coherent energy swap. a) Shows the raw state probabilities corresponding to the states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$, b) shows the same data corrected for the limited visibility of each qubit readout and c) shows the fully corrected readout data where we account for both visibility and inter-qubit readout crosstalk.

which describes a system of two spin 1/2 particles that is in a superposition between the $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ states. Now, when separating the two particles by a large distance and measuring the spin value of both of them simultaneously –such that by the assumption of locality, no information exchange can take place between the two systems–, the wavefunction of the system will collapse in either of the two states $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$. Hence, measuring one of the particles seems to somehow instantaneously change the state of the other particle, thereby violating the principle of locality. In their paper, Einstein, Podolsky and Rosen called this effect *spooky action at a distance*.

However, the violation of locality can be resolved by assuming that the state of the quantum system does actually possess a well-defined value even before measuring it. This would mean that our description of the quantum-mechanical state as given by eq. (5.2) just misses some additional, *hidden variables* that would contain the information on the spin of each particle and which would just not accessible by us. A theory that would complete the quantum mechanical description of the particle and remove the ambiguity from its quantum state is often referred to as a *hidden variable theory*.

In 1964, J. Bell provided a mathematical description of locality and realism and devised an experimental test of the hypothesis formulated by Einstein *et. al.* [3]. Building on his work, J. Clauser, M. Horne, A. Shimony and R. Holt proposed an actual experiment to test quantum mechanics against *local hidden-variable theories* as formulated in the Bell and EPR papers [9]. The first realization of this proposed experiment was then achieved by A. Aspect *et. al.* [2] and confirmed the validity of quantum mechanics, thereby showing that either the assumption of realism or that of locality must be false. In the following decades, several more experiments have been performed to test the Bell hypothesis, trying to close several so-called *loopholes* that can put in doubt the results obtained in a Bell experiment. Such a loophole is a problem of the experimental setup or design that affects the validity of the experimental findings. Typical loopholes affecting Bell-type experiments are the *detection efficiency* or *fair sampling*, the *communication* or *locality* and the *rotational invariance* loophole. We will discuss the loopholes most relevant to our experiment in the following paragraph.

To test the Bell inequality with our two-qubit setup, we use the experimental procedure proposed by Clauser *et. al.* [9]. According to this procedure, we first prepare an entangled two-qubit state of the form

$$|\phi\rangle = \frac{1}{\sqrt{2}} (|01\rangle + e^{-i\varphi} |10\rangle) \quad (5.3)$$

On this quantum state we measure then the expectation value of the operator

$$\text{CHSH} = \text{QS} + \text{RS} + \text{RT} - \text{QT} \quad (5.4)$$

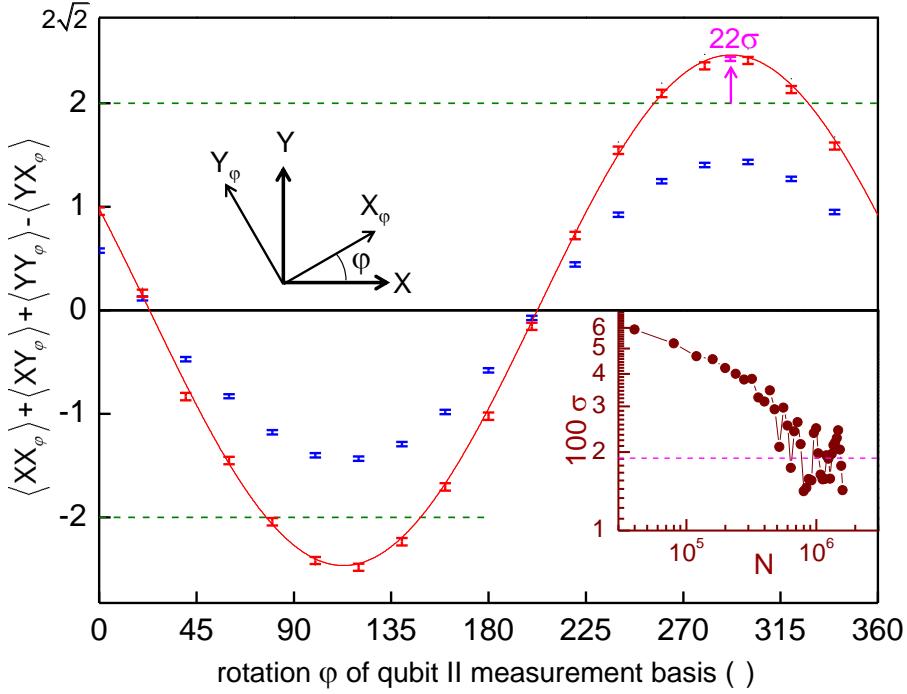


Figure 5.9: Measured value of the $\langle \text{CHSH} \rangle$ operator measured for an ensemble of identically prepared Bell states $1/\sqrt{2}(|01\rangle + e^{-i\varphi}|10\rangle)$, plotted as a function of the rotation ϕ of the measurement basis. Blue markers correspond to raw measurement data, red markers to data corrected for readout errors. The solid line represents the best fit to the theory. The inset shows the standard deviation σ of the maximum value of the $\langle \text{CHSH} \rangle$ operator as a function of the sample size. For large samples, σ is limited by experimental drift of the measurement basis.

where the individual operators Q, R, S, T are given as

$$\begin{aligned} Q &= \sigma_z^1 & S &= \sigma_z^2 \cdot \cos \phi + \sigma_x^2 \cdot \sin \phi \\ R &= \sigma_x^1 & T &= -\sigma_z^2 \cdot \sin \phi + \sigma_x^2 \cdot \cos \phi \end{aligned} \quad (5.5)$$

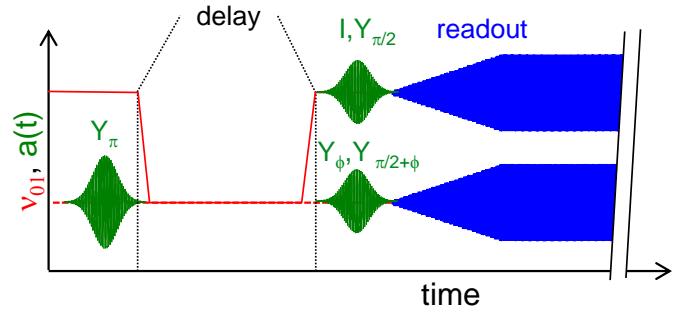
Here, the angle ϕ is a parameter that gives the rotation of the measurement basis in respect to the z axis and should be chosen in accordance to the phase φ of the entangled qubit state to which the operator is applied. To obtain the value of $\langle \text{CHSH} \rangle$, we measure $\langle QS \rangle$, $\langle RS \rangle$, $\langle RT \rangle$ and $\langle QT \rangle$ on an ensemble of identically prepared quantum states. For a classical state, the operator $\langle \text{CHSH} \rangle$ is bound by

$$|\langle \text{CHSH} \rangle| \leq 2. \quad (5.6)$$

However, for an entangled quantum state, the maximum value of the operator is given as $2\sqrt{2}$.

The pulse sequence used for generating an entangled two-qubit state and measuring the CHSH operator on it is shown in fig. 5.10. As before, we generate an entangled

Figure 5.10: Experimental pulse sequence used in the CHSH experiment. It consists in exciting one of the qubits to the state $|1\rangle$, bringing it in resonance with the other qubit for a well-defined duration, performing single-qubit rotations to align the qubit state with the desired measurement basis and finally measuring the $\langle \sigma_z^I \cdot \sigma_z^{II} \rangle$ operator of the two-qubit register.



two-qubit state by exciting one of the qubits to the state $|1\rangle$ and bringing it in resonance with the other qubit for a well-defined amount of time. Afterwards, we apply single-qubit rotations to align the qubit state with the axis along which we want to perform the measurement. Finally we measure the $\langle \sigma_z^I \cdot \sigma_z^{II} \rangle$ of the rotated qubit state. We then repeat this procedure on an ensemble of identically prepared input states for all of the four individual operators of CHSH operator. Figure 5.9 shows the results $\langle \text{CHSH} \rangle$ of such a measurement. As can be seen, the value of the operator $\langle \text{CHSH} \rangle$ varies sinusoidally as a function of the rotation ϕ of the measurement basis. The maximum and minimum value of the raw value of $\langle \text{CHSH} \rangle$ reaches a value of ≈ 1.4 , thereby failing to violate the boundary for non-classical states as given by eq. (5.6). However, when accounting for the readout errors in our experiment, the corrected CHSH data reaches a maximum value ≈ 2.52 , thereby violating the classical boundary of the CHSH equation. We can hence violate the Bell inequality with our qubit processor, however we are not able to close the detector efficiency loophole. In addition, due to the measurement time required to determine the state of each qubit and the close proximity of the two qubits we are not able to close the communication or locality loophole either. The violation of the Bell inequality with the generated two-qubit states also serves as an entanglement witness for the generated states and confirms that we are able to generate highly entangled quantum states with our processor.

Errors

Besides obvious readout errors, the main source of errors in our experiment is drift of the measurement equipment. Most importantly, the phase of the arbitrary waveform generator (AWG) can drift in respect to the phases of the microwave sources, thereby changing the effective phase of the measurement basis of the CHSH operator. Fig. 5.11 illustrates the effect of this drift on the phase of the generated Bell state. Shown is the phase of the state $|\phi\rangle$ as given by eq. (5.3) extracted from a full CHSH data set as shown in fig. 5.9. When repeating this measurement over a long time period,

oscillations of the phase φ with an amplitude of $\approx 40^\circ$ can be observed. This amplitude can be explained by a time shift of the AWG waveform driving the microwave sources by an order of 200 ps, which has indeed been observed experimentally.

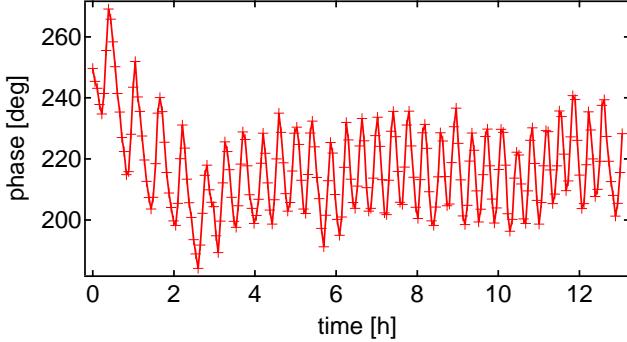


Figure 5.11: Phase φ of an experimentally generated Bell state as a function of time. The phase exhibits oscillatory drift of the order of $\approx 40^\circ$ being caused by a time offset drift of the AWG.

5.3.3 Quantum State Tomography

The experiments described in the following paragraphs often require us to extract the density matrix of an experimental two-qubit state. The method that we use for this purpose is called *quantum state tomography* (QST) (see e.g. [41] for an overview). In this section, we therefore explain in detail the procedure that we use to perform QST. Furthermore, we also discuss several sources of errors occurring in standard QST and how we can estimate their effect on our results.

The density matrix of an n -qubit system can be written as

$$\rho = \sum_{v_1, v_2 \dots v_n} \frac{c_{v_1, v_2 \dots v_n} \sigma_{v_1} \otimes \sigma_{v_2} \dots \otimes \sigma_{v_n}}{2^n} \quad (5.7)$$

$$c_{v_1, v_2 \dots v_n} = \text{tr} (\sigma_{v_1} \otimes \sigma_{v_2} \dots \otimes \sigma_{v_n} \rho) \quad (5.8)$$

where $v_i \in \{X, Y, Z, I\}$ and n gives the number of qubits in the system and where the $c_{v_1, v_2 \dots v_n}$ are real-valued coefficients that describe the given density matrix. To reconstruct the density matrix of an experimental quantum system in a well-prepared state it is hence sufficient to measure the expectation values of these $n^2 - 1$ coefficients on an ensemble of identically prepared states. For our two qubit system, this means measuring all possible combinations of the operators $\{I, \sigma_x^I, \sigma_y^I, \sigma_z^I\} \otimes \{I, \sigma_x^{II}, \sigma_y^{II}, \sigma_z^{II}\}$. In our experiments, rather than measuring σ_x and σ_z operators directly, we rotate the quantum state of each qubit around such that the state vector along the desired measurement axis coincides with the z-axis of the Bloch sphere and measure σ_z afterwards.

When using this direct measurement method, statistical and systematic measurement errors can yield a set of coefficients v_i that corresponds to a *non-physical* density matrix, which violates either the positivity $\langle \psi | \rho | \psi \rangle > 0$ for all valid states $|\psi\rangle$ or the

unity-trace condition $\text{Tr}(\rho) = 1$. To alleviate this problem, several techniques can be employed. In the following section we discuss an estimation technique which is able to overcome the problem of non-physical density matrices, the so-called *maximum likelihood estimation*.

Maximum Likelihood Estimation of Quantum States

As the name implies, *maximum likelihood estimation* is a method that numerically or analytically maximizes a likelihood function that is usually based on a number of measured outcomes and a set of parameters that need to be estimated. The set of parameters corresponding to the maximum of this probability function can then be interpreted as the one with the highest probability of generating the observed outcomes. When estimating the parameters of a density matrix with this method, the probability function to be maximized is the joint probability of obtaining the measured values $\{c_{X,X,\dots,X}, c_{Y,X,\dots,X}, \dots, c_{I,I,\dots,I}\}$ for a given estimate of a density matrix $\hat{\rho}$. By numerically or analytically maximizing this joint probability over the set of possible density matrices we obtain the density matrix which is most likely to have produced the set of measurement outcomes that we have observed.

The joint measurement operators $\Sigma_j = \sigma_{v_1} \otimes \sigma_{v_2} \dots \otimes \sigma_{v_n}$ have the eigenvalues ± 1 and can thus be written as

$$\sigma_{v_1} \otimes \sigma_{v_2} \dots \otimes \sigma_{v_n} = |+_j\rangle\langle+_j| - |-_j\rangle\langle-_j| \quad (5.9)$$

where $|+_j\rangle$ and $|-_j\rangle$ are the eigenstates corresponding to the eigenvalues ± 1 of Σ_j . When performing l measurements of Σ_j , the expectation value $\langle \Sigma_j \rangle$ can be estimated by

$$\widehat{\langle \Sigma_j \rangle}_\rho = \frac{1}{l} \sum_{i=1}^l M_i(\Sigma_j, \rho) \quad (5.10)$$

where $M_i(M, \rho)$ denotes the outcome of the i -th measurement of the operator M on the state described by the density matrix ρ . Since each outcome $M_i(\Sigma_j, \rho)$ is Bernoulli distributed, the sum of them is binomially distributed with the expectation value $E(\widehat{\langle \Sigma_j \rangle}_\rho) = \langle \Sigma_j \rangle_\rho$ and the variance $\sigma^2(\widehat{\langle \Sigma_j \rangle}_\rho) = 1/l \cdot (1 - \langle \Sigma_j \rangle_\rho^2)$. For large sample sizes l , the binomial distribution can be well approximated by a normal distribution with the same expectation value and variance. The joint probability of obtaining a set of measurement values $\{s_1, \dots, s_{n^2-1}\}$ for the set of operators $\{\widehat{\langle \Sigma_1 \rangle}_\rho, \dots, \widehat{\langle \Sigma_{n^2-1} \rangle}_\rho\}$ is thus given as

$$P\left(\widehat{\langle \Sigma_1 \rangle}_\rho = s_1; \dots; \widehat{\langle \Sigma_{n^2-1} \rangle}_\rho = s_{n^2-1}\right) = \prod_{i=1}^{n^2-1} \exp\left(-\frac{l}{2} \frac{(s_i - \langle \Sigma_i \rangle_\rho)^2}{1 - \langle \Sigma_i \rangle_\rho^2}\right) \quad (5.11)$$

By maximizing this probability (or the logarithm of it) we obtain an estimate of the pa-

rameters of the density matrix ρ of the quantum state. This technique also allows us to include further optimization parameters when calculating the joint probability. This is useful for modeling e.g. systematic errors of the measurement or preparation process, which can be described by modifying the operators contained in the probability sum. A common source of errors in our tomography measurements are shifts in the amplitude and phase of the microwave pulses used to drive the qubit. We are obliged to rotate the qubit state when performing state tomography since our measurement apparatus permits us only to measure the σ_z operator of each qubit. We can therefore replace the operators σ_x and σ_y in eq. (5.9) with an effective measurement of σ_z preceded by a rotation R_{ν_i} given as

$$R_X = \exp(-i\sigma_y\pi/4) \quad (5.12)$$

$$R_Y = \exp(+i\sigma_x\pi/4) \quad (5.13)$$

Within this model, phase and amplitude errors can be modeled by including additional phase factors in the exponential of these expression, such that we obtain new operators

$$R_X = \exp(-i[+\sigma_y \cos \alpha + \sigma_x \sin \alpha][\pi/4 + \gamma]) \quad (5.14)$$

$$R_Y = \exp(+i[-\sigma_y \sin \beta + \sigma_x \cos \beta][\pi/4 + \delta]) \quad (5.15)$$

Here, α and β represent phase errors whereas γ and δ represent amplitude errors in the drive pulses. !12!

To Do 12: expand this section and add a figure with the measured probability distributions and a Pauli set

5.3.4 Creation of Bell states

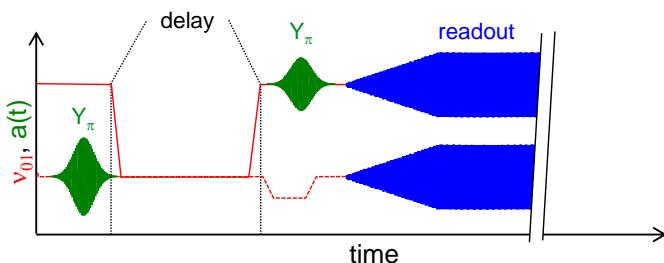


Figure 5.12: Pulse sequence used to experimentally generate entangled Bell states. The sequence consists in exciting one of the qubits to the state $|1\rangle$, creating an entangled state using the swapping interaction and correcting the acquired phase of the state. For the $|\Phi_{\pm}\rangle$ states, we use an additional $X(\pi)$ pulse at the end of the sequence.

We can use the ML method presented above to perform state tomography of entangled Bell states that we create by using the swapping interaction between the qubits. In

general, the four Bell states are

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \quad (5.16)$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \quad (5.17)$$

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad (5.18)$$

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \quad (5.19)$$

The experimental protocol for generating a Bell state is shown in fig. 5.12. In essence, we use the swapping interaction between the qubits to create an entangled state, compensate the acquired phase of that state by a short z-pulse. To create the Φ_+ and Φ_- states we use a $X(\pi)$ pulse at the end of the sequence in addition. Finally we perform quantum state tomography on the created state to reconstruct its density matrix. In fig. 5.13 we show the measured density matrices of the four created Bell states. The trace fidelity between the experimentally created and ideal state is shown on top of the density matrix, ranging between 83% and 87%.

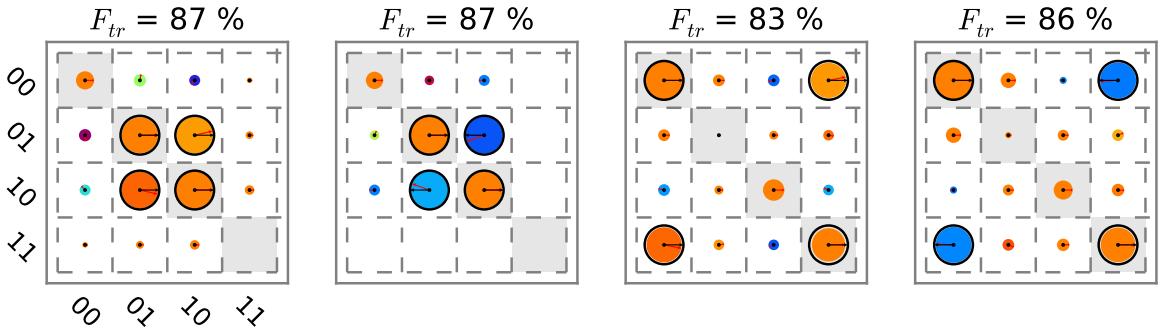


Figure 5.13: Reconstructed density matrices of experimentally created Bell states. Shown are the states $|\Psi_{\pm}\rangle = 1/\sqrt{2}(|01\rangle \pm |10\rangle)$ and $|\Phi_{\pm}\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$. The trace fidelities of the density matrix $\langle\psi|\rho|\phi\rangle$ range between 83 % and 87 %.

5.3.5 Measuring the Evolution of an Entangled Qubit States

Using quantum state tomography, we can repeat the swapping experiment described in section 5.3.1, reconstruction the full experimental density matrix of our system during the swap. Fig. 5.14 shows the result of such an experiment. We plot the measured values of all Pauli operators as a function of the swapping time between the qubits. The red and green curves correspond to the single-qubit Pauli operators $\sigma_{X,Y,Z}^I \otimes I$ and $I \otimes \sigma_{X,Y,Z}^{II}$, the blue curves to the two-qubit Pauli operators $\sigma_{X,Y,Z}^I \otimes \sigma_{X,Y,Z}^{II}$. The dashed line corresponds to a master equation simulation of the swapping experiment where

the qubit coupling, relaxation and dephasing parameters are chosen in accordance with experimental values. As can be seen, the simulation agrees well with the experimental data. Deviation between data and simulation can be seen in the $\sigma_{X,Y}^I \otimes \sigma_{X,Y}^{II}$ operators is caused by an additional phase acquired by the qubits during the swap. This phase is due to the fact that we have to shift the frequency of the qubits to bring them in resonance, hence we acquire a phase in the frame rotating at the uncoupled qubit frequencies.

From the measured Pauli set we can obtain the density matrix of the two-qubit system at any point during the swap. This density matrices are (will be soon :) shown in the margins of this thesis book. When following the evolution of the quantum state as a function of time it can be seen that the coherence between the two qubits decreases steadily during the swapping sequence, due to relaxation and dephasing.

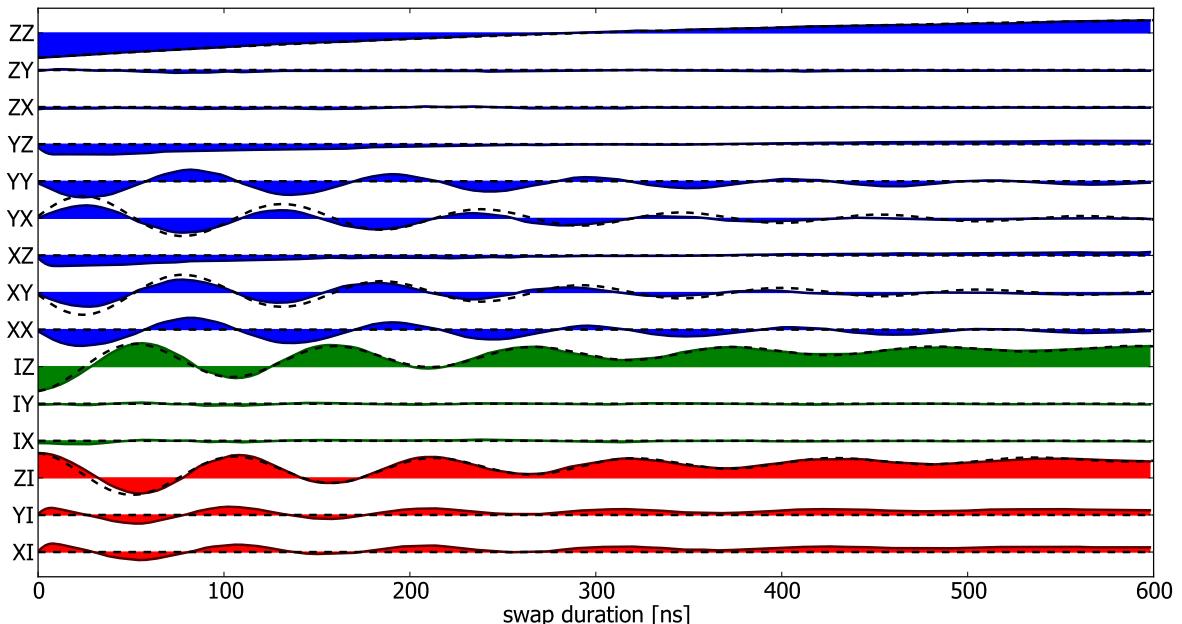


Figure 5.14: Measured Pauli operators $\sigma_i \otimes \sigma_j$ with $i, j \in \{X, Y, Z, I\}$ as a function of the interaction time. Shown are the 6 single-qubit operators as well as the 9 two-qubit correlation operators. The dashed line represents a master-equation simulation of the experiment.

5.4 Realizing a Two-Qubit Gate

We have demonstrated that it is possible to create highly entangled qubit states with our processor by creating Bell states, performing quantum state tomography and violating the Bell inequality. However, to realize a universal quantum processor, it is necessary to implement a so-called *universal two-qubit gate* on our processor. In the following sections we discuss how we can realize and characterize such a gate on our processor.

5.4.1 Principle

5.4.2 Experimental Implementation

5.4.3 Quantum Process Tomography of the Gate

Introduction & Principle

Implementation

A quantum process can be described as a map $\mathcal{E} : \rho_{\mathcal{H}} \rightarrow \rho_{\mathcal{H}}$ that maps a density matrix ρ defined in a Hilbert space Q_1 to another density matrix $\mathcal{E}(\rho)$ defined in a target Hilbert space Q_2 and fulfilling three axiomatic properties [41, 27]:

Axiom 5.0.1. $\text{tr} [\mathcal{E}(\rho)]$ is the probability that the process represented by \mathcal{E} occurs, when ρ is the initial state.

Axiom 5.0.2. \mathcal{E} is a *convex-linear map* on the set of density matrices, that is, for probabilities $\{p_i\}$,

$$\mathcal{E} \left(\sum_i p_i \rho_i \right) = \sum_i p_i \mathcal{E}(\rho_i) \quad (5.20)$$

Axiom 5.0.3. \mathcal{E} is a *completely-positive* map. That is, if \mathcal{E} maps density operators of system Q_1 to density operators of system Q_2 , then $\mathcal{E}(A)$ must be positive for any positive operator A . Furthermore, if we introduce an extra system R of arbitrary dimensionality, it must be true that $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive for any positive operator A on the combined system RQ_1 , where \mathcal{I} denotes the identity map on system R .

As shown in [41], any quantum process fulfilling these criteria can be written in the form

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \quad (5.21)$$

for some set of operators $\{E_i\}$ which map the input Hilbert space to the output Hilbert space, and $\sum_i E_i^\dagger E_i \leq I$.

Now, if we express the operators E_i in a different operator basis \tilde{E}_j such that $E_i = \sum_j a_{ij} \tilde{E}_j$ and insert into eq. (5.21), we obtain

$$\mathcal{E}(\rho) = \sum_i \sum_j a_{ij} \tilde{E}_j \rho \sum_k a_{ik}^* \tilde{E}_k^\dagger \quad (5.22)$$

$$= \sum_{j,k} \tilde{E}_j \rho \tilde{E}_k^\dagger \sum_i a_{ij} a_{ik}^* \quad (5.23)$$

$$= \sum_{j,k} \tilde{E}_j \rho \tilde{E}_k^\dagger \chi_{jk} \quad (5.24)$$

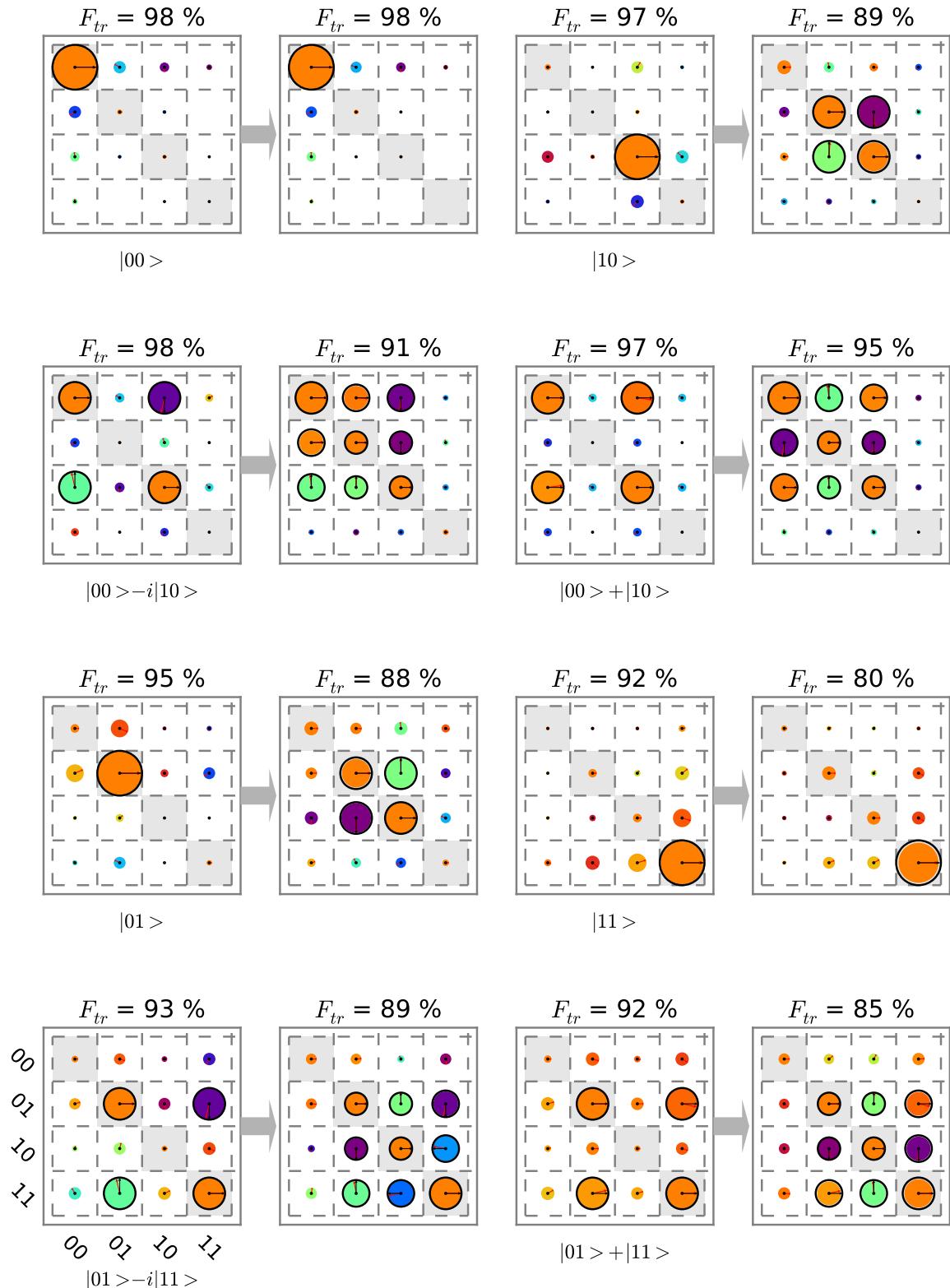


Figure 5.15: The input-output density matrix of the quantum process tomography of the $\sqrt{i\text{SWAP}}$ gate. Shown are the measured density matrices of 16 different input states and the corresponding output matrices with their state fidelities. The ideal matrices are overlaid in red.

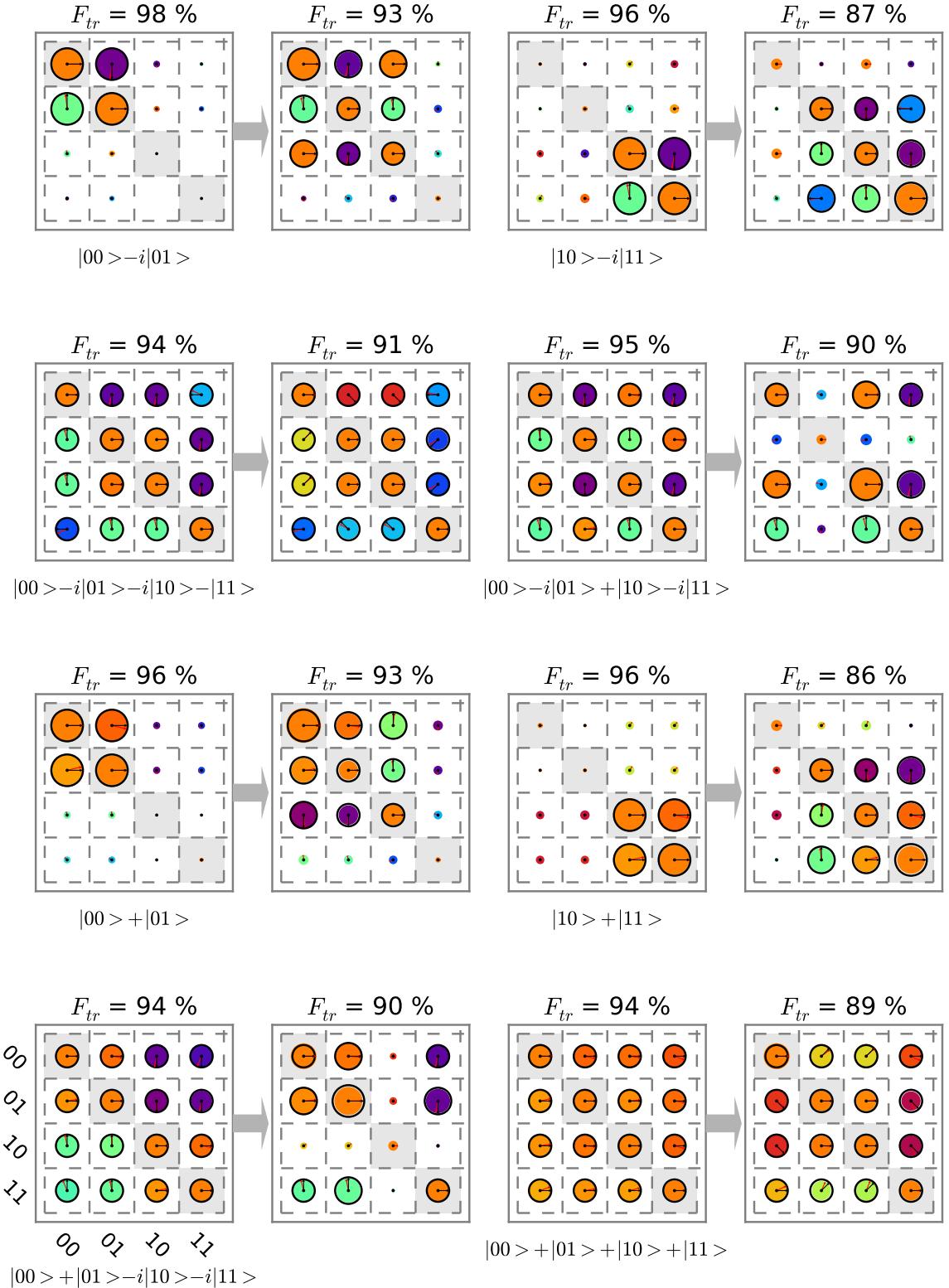


Figure 5.16: The input-output density matrix of the quantum process tomography of the $\sqrt{i\text{SWAP}}$ gate. Shown are the measured density matrices of 16 different input states and the corresponding output matrices with their state fidelities. The ideal matrices are overlaid in red.

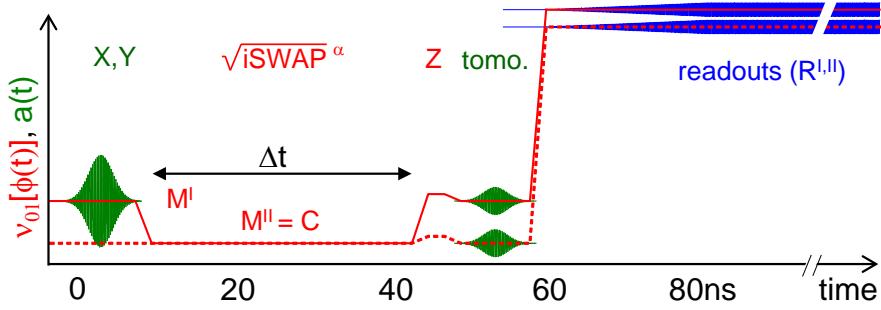


Figure 5.17

where we defined $\chi_{jk} = \sum_i a_{ij} a_{ik}^*$. This is the so-called χ -matrix representation of the quantum process. Here, all the information on the process is contained in the χ matrix, which controls the action of the process-independent operators \tilde{E}_i on the initial density matrix ρ .

Now, the goal of *quantum process tomography* is to obtain the coefficients of the χ -matrix – or any other complete representation of the process – from a set of experimentally measured density matrices ρ and $\mathcal{E}(\rho)$.

To achieve this, several techniques have been developed. The technique used in this work is the so-called *standard quantum process tomography (SQPT)*. This technique proceeds as follows:

1. Choose a set of operators E_i that forms a full basis of $\mathcal{M} : Q_1 \rightarrow Q_2$. For n-qubit process tomography we usually choose $E_{i_1, i_2 \dots i_n} = \sigma_{i_1} \otimes \sigma_{i_2} \dots \otimes \sigma_{i_n}$, where σ_i are the single-qubit Pauli operators and $i \in \{I, X, Y, Z\}$.
2. Choose a set of pure quantum states $|\phi_i\rangle$ such that $|\phi_i\rangle \langle \phi_i|$ span the whole space of input density matrices ρ . Usually, for a n-qubit system we choose $\phi = \{|0\rangle, |1\rangle, (|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle + i|1\rangle)/\sqrt{2}\}^{\otimes n}$, where \otimes^n denotes the n-dimensional Kronecker product of all possible permutations.
3. For each of the $|\phi_i\rangle$, determine $\mathcal{E}(|\phi_i\rangle \langle \phi_i|)$ by quantum state tomography. Usually we also determine $|\phi_i\rangle \langle \phi_i|$ experimentally since the preparation of this state already entails small preparation errors that should be taken into account when performing quantum process tomography.

After having obtained the ρ_i and $\mathcal{E}(\rho_i)$ one obtains the χ -matrix by writing $\mathcal{E}(\rho_i) = \sum_j \lambda_{ij} \tilde{\rho}_j$, with some arbitrary basis $\tilde{\rho}_j$ and letting $\tilde{E}_m \tilde{\rho}_j \tilde{E}_n^\dagger = \sum_k \beta_{jk}^{mn} \tilde{\rho}_k$. We can then insert into eq. (5.24) and obtain

$$\sum_k \lambda_{ik} \tilde{\rho}_k = \sum_{m,n} \chi_{mn} \sum_k \beta_{ik}^{mn} \tilde{\rho}_k \quad (5.25)$$

This directly yields $\lambda_{ik} = \sum_{m,n} \beta_{ik}^{mn} \chi_{mn}$, which, by linear inversion, gives χ .

The Kraus Representation of the Quantum Process

Besides the χ -matrix representation, there is another useful way of expressing a quantum map, the so called *Kraus representation*, which is given as

$$\mathcal{E}(\rho) = \sum_i M_i \rho M_i^\dagger \quad (5.26)$$

It can be shown [27] that this sum contains at most N elements, where N is the dimension of the Hilbert space of the density matrix ρ . We can go from the χ representation to the Kraus representation by changing the basis \tilde{E}_i such that

$$\tilde{E}_i = \sum_l a_{il} \check{E}_l \quad (5.27)$$

which, for eq. (5.24), yields

$$\mathcal{E}(\rho) = \sum_{j,k} \sum_l a_{jl} \check{E}_l \rho \sum_m a_{km}^* \check{E}_m^\dagger \chi_{jk} \quad (5.28)$$

$$= \sum_{l,m} \check{E}_l \rho \check{E}_m^\dagger \sum_{j,k} a_{jl} a_{km}^* \chi_{jk} \quad (5.29)$$

The last sum on the right side of eq. (5.29) corresponds to a change of coordinates of the matrix χ . Now, we can pick the a such that χ is diagonal in the new basis \check{E} and obtain

$$\mathcal{E}(\rho) = \sum_l \lambda_l \check{E}_l \rho \check{E}_l^\dagger \quad (5.30)$$

$$= \sum_l M_l \rho M_l^\dagger \quad (5.31)$$

with λ_l being the l -th eigenvalue of the χ matrix with the eigen-operator \check{E}_l and $M_l = \sqrt{\lambda_l} \check{E}_l$.

5.4.4 Gate Fidelity

5.4.5 Gate Error Analysis

Tomographic errors are removed from the process map of our \sqrt{iSWAP} gate using the following method. The measured Pauli sets corresponding to the sixteen input states are

first fitted by a model including errors both in the preparation of the state (index *prep*) and in the tomographic pulses (index *tomo*). The errors included are angular errors $\varepsilon_{I,II}^{\text{prep}}$ on the nominal π rotations around $X_{I,II}$, $\eta_{I,II}^{\text{prep,tomo}}$ and $\delta_{I,II}^{\text{prep,tomo}}$ on the nominal $\pi/2$ rotations around $X_{I,II}$ and $Y_{I,II}$, a possible departure $\xi_{I,II}$ from orthogonality of (\vec{X}_I, \vec{Y}_I) and $(\vec{X}_{II}, \vec{Y}_{II})$, and a possible rotation $\mu_{I,II}$ of the tomographic XY frame with respect to the preparation one. The rotation operators used for preparing the states and doing their tomography are thus given by

$$\begin{aligned} X_{I,II}^{\text{prep}}(\pi) &= e^{-i(\pi+\varepsilon_{I,II}^{\text{prep}})\sigma_x^{I,II}/2}, \\ X_{I,II}^{\text{prep}}(-\pi/2) &= e^{+i(\pi/2+\eta_{I,II}^{\text{prep}})\sigma_x^{I,II}/2}, \\ Y_{I,II}^{\text{prep}}(\pi/2) &= e^{-i(\pi/2+\delta_{I,II}^{\text{prep}})[\cos(\xi_{I,II})\sigma_y^{I,II}-\sin(\xi_{I,II})\sigma_x^{I,II}]/2}, \\ X_{I,II}^{\text{tomo}}(\pi/2) &= e^{-i(\pi/2+\eta_{I,II}^{\text{tomo}})[\sin(\mu_{I,II})\sigma_x^{I,II}+\cos(\mu_{I,II})\sigma_y^{I,II}]/2}, \\ Y_{I,II}^{\text{tomo}}(-\pi/2) &= e^{+i(\pi/2+\delta_{I,II}^{\text{tomo}})[\cos(\mu_{I,II}+\xi_{I,II})\sigma_y^{I,II}-\sin(\mu_{I,II}+\xi_{I,II})\sigma_x^{I,II}]/2}. \end{aligned}$$

The sixteen input states are then $\{\rho_{\text{in}}^e = U|0\rangle\langle 0|U^\dagger\}$ with $\{U\} = \{I_I, X_I^{\text{prep}}(\pi), Y_I^{\text{prep}}(\pi/2), X_I^{\text{prep}}(-\pi/2)\} \otimes \{I_{II}, X_{II}^{\text{prep}}(\pi), Y_{II}^{\text{prep}}(\pi/2), X_{II}^{\text{prep}}(-\pi/2)\}$, and each input state yields a Pauli set $\{\langle P_k^e \rangle = \text{Tr}(\rho_{\text{in}}^e P_k^e)\}$ with $\{P_k^e\} = \{I_I, X_I^e, Y_I^e, Z_I\} \otimes \{I_{II}, X_{II}^e, Y_{II}^e, Z_{II}\}$, $X^e = Y^{\text{tomo}}(-\pi/2)^\dagger \sigma_z Y^{\text{tomo}}(-\pi/2)$, and $Y^e = X^{\text{tomo}}(\pi/2)^\dagger \sigma_z X^{\text{tomo}}(\pi/2)$. Figure S5.1 shows the best fit of the modelled $\{\langle P_k^e \rangle\}$ set to the measured input Pauli sets, yielding $\varepsilon_I^{\text{prep}} = -1^\circ$, $\varepsilon_{II}^{\text{prep}} = -3^\circ$, $\eta_I^{\text{prep}} = 3^\circ$, $\eta_{II}^{\text{prep}} = 4^\circ$, $\delta_I^{\text{prep}} = -6^\circ$, $\delta_{II}^{\text{prep}} = -3^\circ$, $\eta_I^{\text{tomo}} = -6^\circ$, $\eta_{II}^{\text{tomo}} = -4^\circ$, $\lambda_I^{\text{tomo}} = 12^\circ$, $\lambda_{II}^{\text{tomo}} = 5^\circ$, $\xi_I = 1^\circ$, $\xi_{II} = -2^\circ$, and $\mu_I = \mu_{II} = -11^\circ$.

Knowing the tomographic errors and thus $\{\langle P_k^e \rangle\}$, we then invert the linear relation $\{\langle P_k^e \rangle = \text{Tr}(\rho P_k^e)\}$ to find the 16×16 matrix B that links the vector $\overrightarrow{\langle P_k^e \rangle}$ to the columnized density matrix $\overrightarrow{\rho}$, i.e. $\overrightarrow{\rho} = B \cdot \overrightarrow{\langle P_k^e \rangle}$. The matrix B is finally applied to the measured sixteen input and sixteen output Pauli sets to find the sixteen $(\rho_{\text{in},k}, \rho_{\text{out},k})$ couples to be used for calculating the gate map.

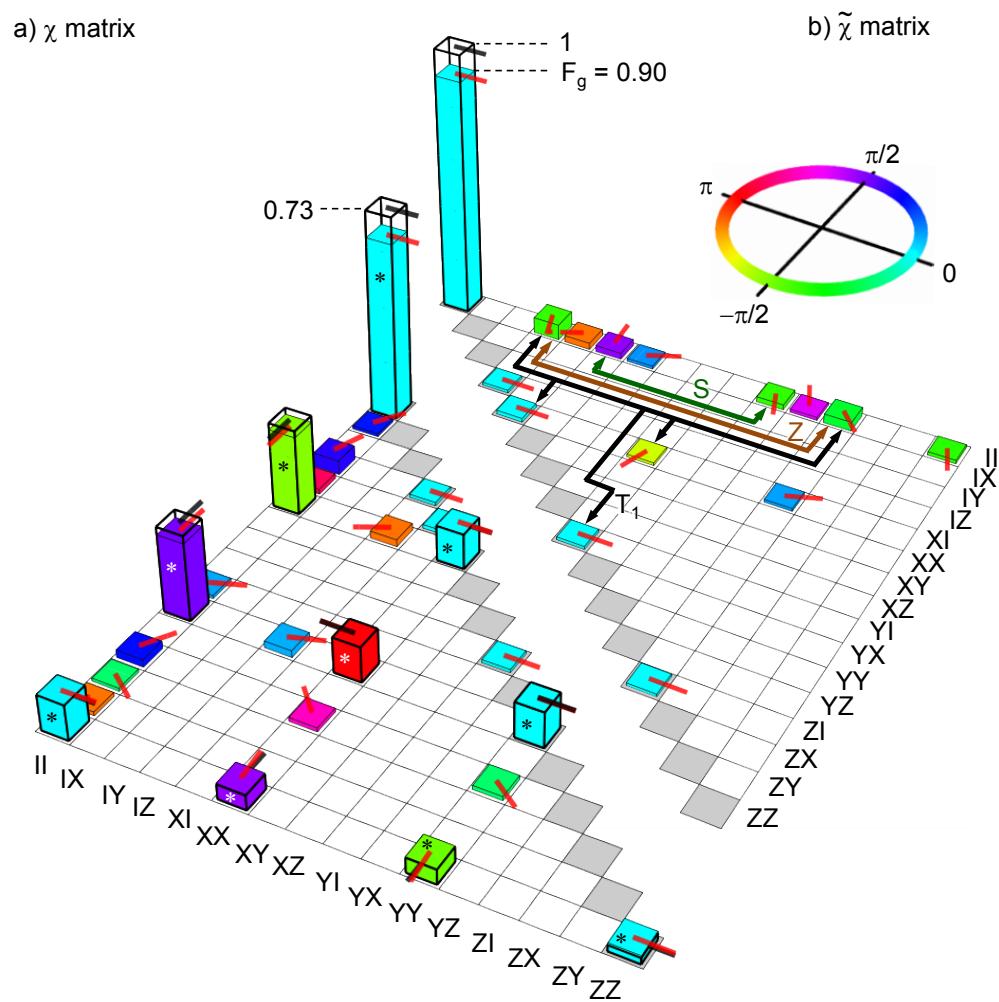


Figure 5.18

Chapter 6

Running the Grover Search Algorithm

This chapter describes an experimental implementation of the so-called *Grover search algorithm* with our two-qubit quantum processor. The first section provides a short introduction of the algorithm and motivate the interest in realizing it. The following sections then discuss the details of the experimental realization of the algorithm. We will discuss the results that we obtained and compare the algorithm fidelity and runtime to that of an equivalent, classical algorithm. Finally, we will analyze all relevant unitary and non-unitary error sources relevant to our experiment and provide a quantitative error model of our implementation of the Grover algorithm.

6.1 Introduction & Motivation

Search algorithms are of great importance in many domains of mathematics and computer science. One such search problem that often arises and which will be discussed in the following sections can be formulated in simple terms as follows:

Theorem 6.1. Assume that we have a search space \mathcal{S} that consists of a finite number N of states $s \in \mathcal{S}$. The solution to our search problem corresponds to a subset of M states of the search space $\mathcal{T} \subset \mathcal{S}$. We can then define a search function $\mathcal{C}(s) : \mathcal{S} \rightarrow \{0, 1\}$ that discriminates between states that solve the search problem and states that don't, such that $\mathcal{C}(s) = 1$ for $s \in \mathcal{T}$ and $\mathcal{C}(s) = 0$ otherwise. In accordance with the general convention in the research literature on the Grover search algorithm we will often refer to this search function as the *Oracle function* or (in a quantum-mechanical context) as the *Oracle operator* in the following sections.

Using this definition of the search problem, the goal of a search algorithm is to find all states $t \in \mathcal{S}$ for which $\mathcal{C}(t) = 1$. In the following sections, for the sake of simplicity we assume that the solution set \mathcal{T} contains only one single state t . This special case

can be generalized to cases where more than one solution to the search problem exists (see e.g. [41] for a detailed review of the general-case Grover search algorithm)

The first step in order to solve a search problem of the kind described above using classical or quantum computation is to map the problem to a form suitable for solution by a digital (quantum) computer. For this, we first number and encode the N input states $i \in \mathcal{S}$ in binary form as $i = (b_l^i, \dots, b_0^i)_B$, where l is the length of the binary register able to hold all N input states. With this definition, it is then trivial to find a mathematical representation of \mathcal{C} that operates on a binary input register.

Using these assumptions and definitions, it can then be shown that the most efficient classical search algorithm for solving the search problem above will use $\mathcal{O}(N)$ calls of the function \mathcal{C} to find all solutions t of the search problem. Assuming that the time to evaluate the function \mathcal{C} is far superior to the time needed to perform any other operation during the search algorithm, the time needed for calling n -times the function \mathcal{C} corresponds approximately to the runtime of the whole search algorithm.

Amazingly, in 1997, Lev Grover found a quantum algorithm that could solve this search problem with only $\mathcal{O}(\sqrt{N})$ calls to the function \mathcal{C} [25]. His algorithm achieves this by repeatedly calling a quantum-mechanical implementation of the function \mathcal{C} starting from a highly superposed qubit register $\sum_i^N |i\rangle$ and applying a special operator to the output state afterwards. The individual steps of his algorithm are straightforward and are given as follows:

1. Initialize a qubit register to the state $|\psi\rangle = |0\rangle$ (corresponding to a binary input state $|0000\dots 0_B\rangle$)
2. Apply the generalized Hadamard operation to the qubit register, producing a fully superposed quantum state

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle$$

3. Repeat the following sequence $\mathcal{O}(\sqrt{N})$ times:
 - a) Apply the *Oracle operator* $|i\rangle \rightarrow (-1)^{\mathcal{C}(i)} |i\rangle$ to the state $|\psi\rangle$
 - b) Apply the so-called *diffusion operator* $|i\rangle \rightarrow -|i\rangle + \frac{2}{N} \sum_{j=0}^{N-1} |j\rangle$ to $|\psi\rangle$
4. Measure the state of the quantum register in the computational basis $|i\rangle$.

For the description above we have enumerated the states of the qubit register from $|0\rangle$ to $|N-1\rangle$. Basically, the Grover algorithm makes use of quantum parallelism to solve the search problem $\mathcal{O}(\sqrt{N})$ times faster than the most efficient classical algorithm. To understand better the strategy it uses to solve the search problem, the different steps of the algorithm can be rephrased in the following more intuitive way:

- First, the algorithm creates a fully superposed quantum state which contains all possible solutions to the search problem at once. The amplitudes and phases of each individual state are all equal in the beginning.
- Then, it applies the Oracle operator to this superposed state. The effect of the Oracle is to turn the phase of the states t by an angle π for which $C(t) = 1$. As will be shown later, such an Oracle operator can be implemented in a straightforward way for any classical search function.
- In the next step, it applies a diffusion operator to the quantum state which transfers a fraction of the amplitude from states with zero phase to the states with π phase, increasing thus the amplitude of the latter. In this process, the phases of all states gets also turned back to zero, allowing the algorithm to repeat the sequence above.
- Repeating these two operations increases each time the amplitude of the states that correspond to a solution of the search problem until the amplitudes of all the other states vanish. After that point, the process reverses and the amplitude is transferred back to the original states. It is therefore crucial to stop the repetition sequence given above after the right number of iterations.

By implementing the search function as a quantum operator, the Grover algorithm is able to evaluate it in one single call for all possible input states. This so-called *quantum parallelism* provides the basis for the speed-up of the search in comparison to a classical algorithm. However, being able to encode the result of the search function in the phase of a multi-qubit state does not directly translate to a speed advantage since it is usually very hard to extract this phase information from the quantum state. Indeed, to extract the values of all phases from an N -qubit state, it would be necessary to perform $\mathcal{O}(2^N)$ measurements on an ensemble of identically prepared quantum states. However, extracting the state amplitudes from such a state takes only $\mathcal{O}(N)$ measurements, which in addition can usually be carried out in parallel. It is for this that the Grover algorithm uses an operator that transforms the information encoded in the phases of the qubits to an information encoded in their amplitude. However, since the conversion between phase to amplitude information through the application of an unitary operator is limited by certain physical constraints, the algorithm needs to repeat the encode-and-transfer sequence described above $\mathcal{O}(\sqrt{N})$ times.

To analyze further the constraints and principles of the algorithm, we will discuss a more detailed derivation of it starting from the Schrödinger equation and we will also explain what limits the efficiency of the phase-to-amplitude conversion in the algorithm.

6.1.1 Deriving the Grover Algorithm from Schrödinger's Equation

An interesting derivation of the Grover algorithm starting from Schrödinger's equation has been detailed by Grover himself in a seminal paper [26] and shall be briefly rediscovered here since it sheds light on the basic principles on which the algorithm is based. The derivation begins by considering a quantum system governed by Schrödinger's equation, which can be written as (setting $\hbar = 1$ for clarity)

$$-i\frac{\partial}{\partial t}\psi(x, t) = \frac{\partial^2}{\partial x^2}\psi(x, t) - V(x)\psi(x, t) \quad (6.1)$$

Here $\psi(x, t)$ describes the wave-function and V is a time-independent potential. Let us assume that the potential $V(x)$ is shaped as in fig. 6.1, i.e. possessing a global minimum of energy. When one initializes the system to a state $\psi_0(x, t_0)$ and lets it evolve for a given time, $\psi(x, t)$ will be attracted by the minimum of potential energy and “fall into it” much like a classical particle in such a potential would¹. We might thus ask if we can encode the solution to a search problem as a point of minimum energy x_0 of a potential $V(x)$, take an initial state $\psi_0(x, t_0)$ and let it evolve into a state that has a high probability around x_0 , thereby solving the search problem. To answer this question, it is first necessary to discretize the wavefunction $\psi(x, t)$ such that it can represent the search problem stated in the last section, which is defined over a finite number of states. In the most simple case, we can use a regular grid of points x_i with a spacing Δx for this, as shown in fig. 6.1b. Discretizing the time evolution of eq. 6.1 in steps Δt as well and defining $\epsilon = \Delta t/\Delta x^2$, we obtain a new equation of the form

$$-\frac{\psi_i^{t+\Delta t} - \psi_i^t}{\Delta t} = \frac{\psi_{i+1}^t + \psi_{i-1}^t - 2\psi_i^t}{\Delta x^2} - V(x_i)\psi_i^t \quad (6.2)$$

where we have written $\psi(x_i, t) = \psi_i^t$. For a circular grid with N points we can write this equation in matrix form as

$$\vec{\psi}^{t+\Delta t} = S^{\Delta t} \cdot \vec{\psi}^t \quad (6.3)$$

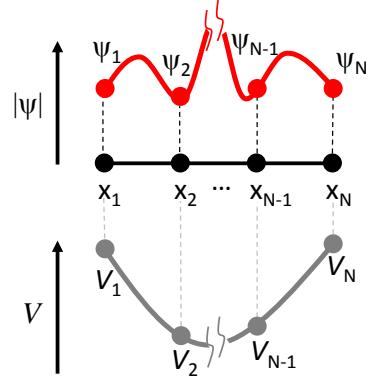


Figure 6.1: A wavefunction $\psi(x)$ and potential $V(x)$ defined on a grid of points x_1, \dots, x_N .

¹of course, since there is no dissipation, the state will not come to rest at the minimum point of energy but rather oscillate around it conserving its total potential and kinetic energy

with S being a state transition matrix of the form

$$S^{\Delta t} = \begin{pmatrix} 1 - 2i\epsilon - iV(x_1)\Delta t & i\epsilon & 0 & \dots & i\epsilon \\ i\epsilon & 1 - 2i\epsilon - iV(x_2)\Delta t & i\epsilon & \dots & 0 \\ 0 & i\epsilon & \ddots & & \vdots \\ \vdots & & \ddots & & \vdots \\ i\epsilon & 0 & \dots & i\epsilon & 1 - 2i\epsilon - iV(x_N)\Delta t \end{pmatrix} \quad (6.4)$$

For infinitesimal times Δt we can separate the effect of the potential $V(x)$ on the wavefunction from the spatial dispersion by writing $S^{\Delta t} \approx D \cdot R$ with

$$D = \begin{pmatrix} 1 - 2i\epsilon & i\epsilon & 0 & 0 & \dots & i\epsilon \\ i\epsilon & 1 - 2i\epsilon & i\epsilon & 0 & \dots & 0 \\ \dots & \ddots & & & & \vdots \\ i\epsilon & 0 & 0 & \dots & i\epsilon & 1 - 2i\epsilon \end{pmatrix} \quad (6.5)$$

and

$$R = \begin{pmatrix} e^{-iV(x_1)\Delta t} & 0 & \dots & 0 \\ 0 & e^{-iV(x_2)\Delta t} & \dots & 0 \\ 0 & \dots & 0 & e^{-iV(x_N)\Delta t} \end{pmatrix} \quad (6.6)$$

This approximation is correct to $\mathcal{O}(\epsilon)$ up to an unimportant renormalization factor. We can now repeatedly apply the matrix product $D \cdot R$ to the wavefunction to obtain its state after a given finite time t by writing

$$\vec{\psi}^{t_0+t} = \left(\prod_{i=1}^{t/\Delta t} D \cdot R \right) \cdot \vec{\psi}^t \quad (6.7)$$

This technique of splitting up the full evolution operator into a product of two or more non-commuting operators that are applied repeatedly to the wavefunction to obtain its state after a finite time is sometimes referred to as *Trotterification* – in reference to the so-called *Lie-Trotter formula* on which it is based – and is widely used in digital quantum simulation [32, 31].

As can be seen in eq. (6.7), the evolution of the wavefunction at infinitesimal times is governed by two processes: The interaction with the potential V and a diffusion process that mixes different spatial parts of the wavefunction with each other. The operator D resembles a Markov diffusion process since each row and column of the matrix sums up to unity, whereas R changes the phase of each element of the wavefunction in accordance with the local potential seen by it. If we apply R to a fully superposed initial state of the

form $\psi_i = 1$ (omitting the normalization factor for clarity) and assume that $V_i = 0$ for $i \neq j$ and $V_j \Delta t = \pi/2$ (the potential thus encoding a search function with $\mathcal{C}(j) = 1$ and $\mathcal{C}(i) = 0$ for $i \neq j$), the element ψ_j will get turned according to $\psi_j \rightarrow i\psi_j$, whereas all other elements ψ_i will remain unchanged. Applying the operator D to the resulting state will transform ψ_j according to $\psi_j \rightarrow \psi_j(i + 2\epsilon(1 + i))$ with a corresponding amplitude $\sqrt{1 + 4\epsilon + \mathcal{O}(\epsilon^2)}$ and the adjacent states $\psi_{j\pm 1}$ according to $\psi_{j\pm 1} \rightarrow \psi_{j\pm 1}(1 - \epsilon(1 + i))$ with an amplitude $\sqrt{1 - 2\epsilon + \mathcal{O}(\epsilon^2)}$. Hence there is a transfer of amplitude between the state whose phase has been turned and its neighboring states. If we reset the phases of all the ψ_i to zero afterwards, we can iterate the application of $D \cdot R$ until all of the amplitude has been transferred to the element ψ_j which corresponds to a solution to the search problem. This is, in essence, exactly what the Grover algorithm does, the only difference being that it replaces the matrix D with an unitary matrix that maximizes the amplitude transfer to the states solving the search problem, thereby speeding up the algorithm. As stated before, the efficiency with which the algorithm can transfer amplitude between different states is limited by physical constraints, in the next section we will therefore discuss exactly what limits this efficiency and which unitary matrix one should choose to maximize it.

Efficiency of Quantum Searching

It is interesting to ask which is the maximum amount of amplitude that can be transferred in a single step of the Grover search algorithm and which matrix D should be chosen to maximize this transfer. To answer this question and derive the ideal diffusion matrix, we will assume first that the matrix R which encodes the value of the search function \mathcal{C} in the quantum state of the qubit register can be written in the most general case as

$$R = \sum_{j=0}^{N-1} \exp[i\alpha\mathcal{C}(j)] |j\rangle\langle j| \quad (6.8)$$

Here, α is a factor which we can choose arbitrarily. So, without loss of generality, we can choose $\alpha = \pi$, yielding an Oracle operator of the form

$$R = I - 2 \sum_{j=0}^{N-1} \mathcal{C}(j) |j\rangle\langle j| \quad (6.9)$$

This operator will flip the sign of all states for which $\mathcal{C}(j) = 1$. Now, the next step consists in finding a diffusion or state transfer matrix which will maximize the amplitude transfer to states marked by the Oracle operator above and which will also reset the phases of the quantum register to zero afterwards, such that we might apply the Oracle operator to the resulting state again. In the most general case, such a state transfer matrix will

have the form

$$D_c = \begin{pmatrix} b & a & a & \dots & a \\ a & b & a & \dots & a \\ \vdots & \ddots & & & \vdots \\ a & \dots & a & b \end{pmatrix} \quad (6.10)$$

Here, we assume that all non-diagonal elements of the matrix are equal, which is well justified since we have no knowledge of the structure of the search space of the problem and therefore want to treat all basis states equally during the phase-to-amplitude conversion. Furthermore, since both the initial quantum state and the Oracle operator as given by eq. (6.9) contain only real numbers and we demand that the quantum state after applying D_c may contain only positive real numbers as well it is easy to show that a, b must be real numbers. Finally, the unitarity of quantum operators demands that $D_c^\dagger D_c = I$, which for the matrix above is equivalent to the two conditions

$$1 = b^2 + (N - 1)a^2 \quad (6.11)$$

$$0 = 2ab + (N - 2)a^2 \quad (6.12)$$

Solving these two equations for a, b yields the trivial solution $b = \pm 1, a = 0$ and the more interesting one $b = \pm(1 - 2/N), a = \mp 2/N$. As can be checked easily, the solution $b = 1 - 2/N, a = 2/N$ results in a maximum amplitude transfer from states $|i\rangle$ for which $\mathcal{C}(i) = 0$ to states $|j\rangle$ for which $\mathcal{C}(j) = 1$. Thus the ideal diffusion matrix to be used in the Grover algorithm is given as

$$D = \begin{pmatrix} -1 + 2/N & 2/N & 2/N & \dots & 2/N \\ 2/N & -1 + 2/N & 2/N & \dots & 2/N \\ \vdots & & \ddots & & \vdots \\ 2/N & 2/N & 2/N & \dots & -1 + 2/N \end{pmatrix} \quad (6.13)$$

This matrix, together with an Oracle operator R as given by eq. (6.9) will yield the maximum amplitude transfer from states not solving the search problem to states that solve it. Repeating the application of $D \cdot R$ on an initially fully superposed quantum states for $\mathcal{O}(\sqrt{N})$ times will transform the input state to a state containing only the solutions of the search problem, thus solving the problem quadratically faster than possible with any classical algorithm.

6.1.2 Implementation & Comparability to Classical Algorithms

In the following sections we discuss a two-qubit implementation of the Grover search algorithm that we implemented in this work. To be able to compare the Grover algorithm

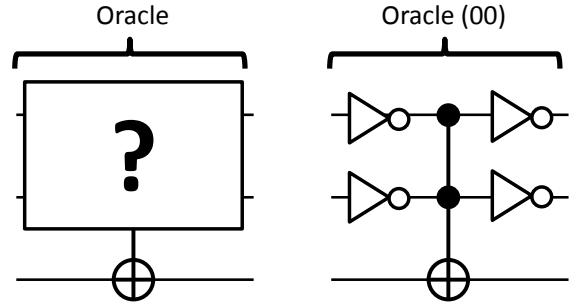


Figure 6.2: a) Two-qubit ancilla-based implementation of the Oracle function \mathcal{C} . The state of the third bit get flipped if the search function $\mathcal{C}(i) = 1$ for the given input state i . b) An example of an ancilla-based search function that returns a true value for the input state 00.

as outlined here to a classical version solving the same search problem, we will now discuss another variant of the algorithm that uses an ancilla qubit to encode the result of the search function. This implementation will make it possible to devise a classical algorithm that can then be compared to the quantum algorithm.

6.1.3 Ancilla-based Implementation of the Algorithm

The implementation of the Grover search algorithm as outlined above encodes the value of the search function \mathcal{C} directly in the phase of the input state supplied to this function. This makes it hard to compare the algorithm to a classical search algorithm which operates on a binary input states and, in general, cannot encode the result of the search function directly in the input state. It is therefore useful to formulate a version of the Grover algorithm where the Oracle function does not directly encode the marked state in the input qubit register but rather uses an ancilla qubit to store the result of calling \mathcal{C} . Such a representation of the algorithm, although of little practical relevance, is very useful since it allows us to directly compare the quantum algorithm to its classical counterpart implemented using reversible logic gates, thus making it possible to benchmark the quantum algorithm and provide an estimation of the quantum speed-up that can be achieved.

Exemplary implementations of ancilla-based search functions \mathcal{C} implemented using reversible (quantum) gates are shown in fig. 6.2 for the two-qubit case. There, a two-qubit Toffoli gate in combination with several single-qubit NOT gates (that can be easily implemented as single-qubit X_π rotations) is used to flip the state of an ancilla-qubit conditionally on the input state of the gate. Using a similar approach, any arbitrary classical search function \mathcal{C} that can be implemented with a set of universal reversible logic gates (e.g. the Toffoli gate and the NOT gate) can be directly mapped to a corresponding quantum operator that works on quantum-mechanical input states and implements the classical search function.

Now, to use the Grover algorithm with such an ancilla-based quantum Oracle, it is necessary to re-encode the result of the Oracle in the qubit input state. Fig. 6.3 shows

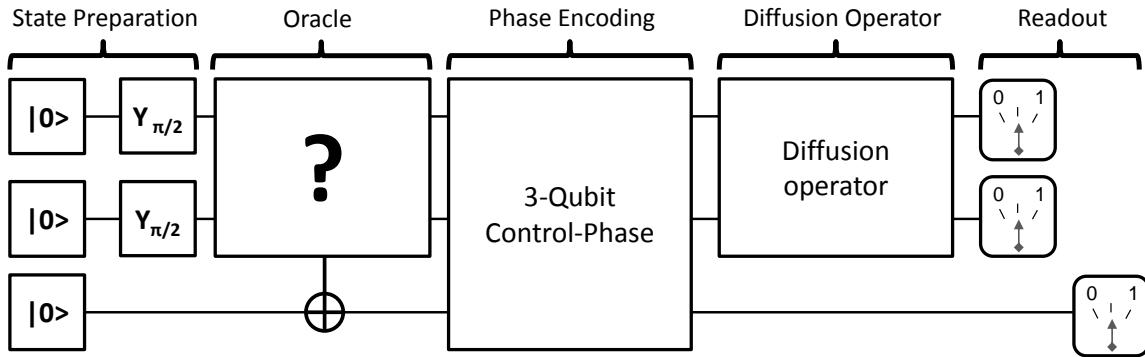


Figure 6.3: A full version of an ancilla-based implementation of the two-qubit Grover search algorithm. The algorithm works on a two-qubit input state and flips the state of a control qubit for one of the four possible input states in accordance to an unknown Oracle function. It then applies a 3-qubit control-phase operation of that maps $|xy1\rangle \rightarrow -|xy1\rangle$, $|xy0\rangle \rightarrow |xy0\rangle$ to encode the state of the control qubit directly in the two input qubits and then uses a diffusion operator to determine the state which has been marked by the Oracle function.

a version of the two-qubit Grover algorithm that achieves exactly this by using a three-qubit control-not (CNOT) gate C of the form

$$C = I^{n \otimes n} - 2 \sum_{ij} |ij1\rangle \langle ij1| \quad (6.14)$$

to phase-encode the state of the ancilla qubit in the state of the input qubit register. After the re-encoding of the result, the ancilla qubit is not needed during the remainder of the algorithm and must not be read out before the algorithm terminates.

6.2 Comparision to a Classical Algorithm

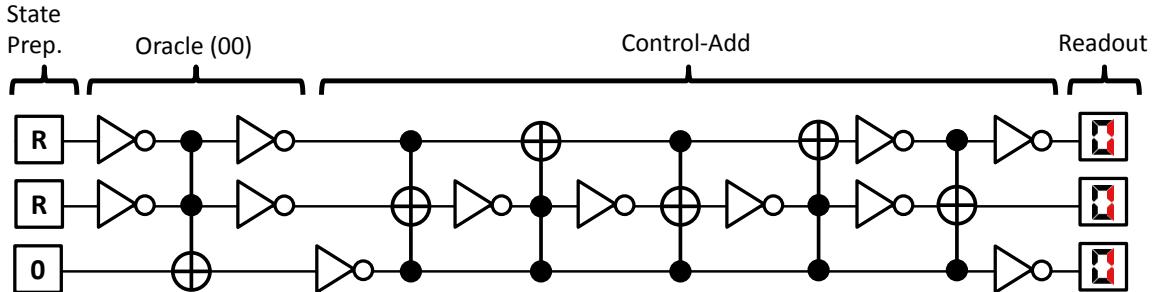


Figure 6.4: Classical reversible implementation of a search algorithm on a two-bit input register. An exemplary Oracle function can be implemented using two single-bit NOT operations and a Toffoli gate. R designates the generation of a random binary value at the beginning of the algorithm. If the Oracle does not yield the correct answer, the test state is incremented. The average success probability of the algorithm is 50 %.

In order to quantify the speed-up achieved by a quantum algorithm it is necessary to map the problem it solves to an equivalent problem that can be solved by a classical algorithm. For the Grover algorithm, this is the search problem that we discussed in the first section of this chapter. Using the reversible, ancilla-based implementation of the search function that was introduced in the last section we can directly formulate a classical algorithm that solves the same problem as the Grover algorithm and compare the runtime of the two. Since the two-qubit Grover algorithm evaluates the search function \mathcal{C} only once it is interesting to ask what would be the success probability of an equivalent classical algorithm that calls \mathcal{C} once and returns an estimate of the state solving the search problem afterwards. Such an algorithm is shown in fig. 6.4 and it achieves a success probability of 50 % by evaluating once the function \mathcal{C} for a randomly generated two-bit input value r and returning r if it found $\mathcal{C}(r) = 1$ or $r + 1 \pmod 4$ otherwise. The 50 % success rate of this algorithm provides a benchmark against which we will measure the speed-up of our implementation of the Grover algorithm.

Alternatively, we can also compare the success probability of the quantum algorithm with either a simple classical “query” algorithm or a “query-and-guess” algorithm, plotting the success probability of either of them as a function of the number of calls to the Oracle function \mathcal{C} . This makes it possible to compare the performance of the quantum algorithm to the classical ones for an arbitrary number of calls n to \mathcal{C} . Such a comparison will be made later in this chapter when reviewing the experimental data obtained for our implementation of the Grover algorithm.

6.3 Experimental Implementation

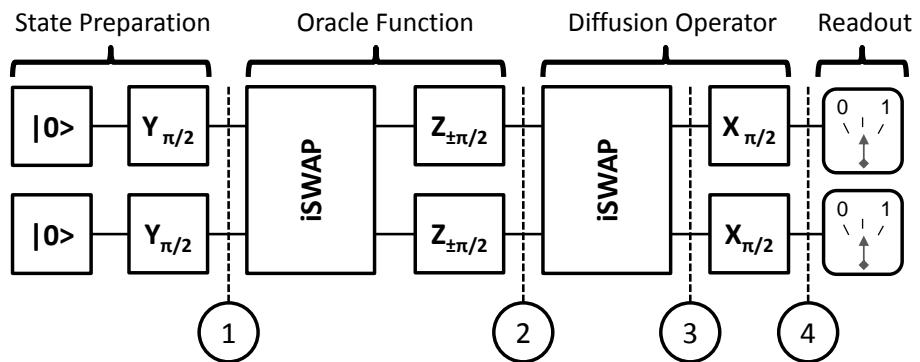


Figure 6.5: Schematic of our implementation of the Grover search algorithm. The algorithm consists in generating a fully superposed input state, applying the Oracle function to it and analyzing the resulting state by applying the Diffusion transform to it and reading out the value of the qubit register afterwards.

We implemented a compiled version of the two-qubit Grover algorithm using our two-

qubit quantum processor. The gate sequence of the algorithm is shown in fig. 6.5 and consists in two *iSWAP* gates and six single-qubit gates applied to an initial state $|00\rangle$. Here, the first *iSWAP* gate together with the two single-qubit $Z_{\pm\pi}$ rotations implements the Oracle function $f(x)$ as given by eq. (6.9), where the signs of the rotation operations determines the state which is marked by the Oracle. This state can be either $|00\rangle$ (corresponding to a $Z_{-\pi/2}^1 \cdot Z_{-\pi/2}^2$ rotation), $|01\rangle$ ($Z_{-\pi/2}^1 \cdot Z_{\pi/2}^2$), $|10\rangle$ ($Z_{\pi/2}^1 \cdot Z_{-\pi/2}^2$) or $|11\rangle$ ($Z_{\pi/2}^1 \cdot Z_{\pi/2}^2$). After the encoding, the second *iSWAP* operation together with the following $X_{\pi/2}^1 \cdot X_{\pi/2}^2$ single-qubit operations implement the diffusion operator as given by eq. (6.13). The final step of the algorithm consists in reading out the two-qubit register.

6.3.1 Pulse Sequence

To implement the gate sequence described above we need to realize a sequence of microwave and flux pulses which realize the individual quantum gates of the sequence. To eliminate possible gate errors, we perform a series of calibration measurements before to tune-up the individual single- and two-qubit gates needed for the algorithm. In addition, we run individual parts of the algorithm successively and perform quantum state tomography to characterize the state of the quantum register after each step of the algorithm and correct the gate operations applied to the qubit in order to maximize the fidelity of the measured states in respect to the ideal ones. Fig. 6.6 shows an experimental pulse sequence for the Grover algorithm with an Oracle operator marking the state $|00\rangle$. Shown are the frequencies of the two qubits during the runtime of the algorithm and the microwave drive and readout pulses applied to them.

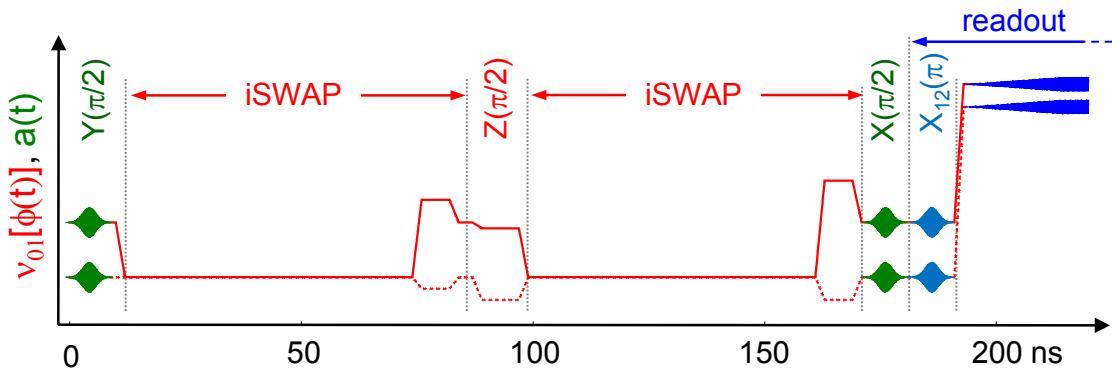


Figure 6.6: The pulse sequence used in realizing Grover's quantum search algorithm. First, a $Y_{\pi/2}$ pulse is applied to each qubit to produce the fully superposed state $1/2(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Then, an *iSWAP* gate is applied, followed by a $Z_{\pm\pi/2}$ gate on each qubit, which corresponds to the application of the oracle function. The resulting state is then analyzed using another *iSWAP* gate and two $X_{\pi/2}$ gates to extract the state which has been marked by the oracle function. Optionally, a Y_{π}^{12} pulse is used on each qubit to increase the readout fidelity.

6.4 Results

Here we discuss the results obtained when running the Grover search algorithm with our two-qubit processor. In the first section we will analyze the quantum state of the qubit register during the algorithm by performing quantum state tomography. In the second section we will present and discuss the single-run results obtained in our experiment.

6.4.1 State Tomography of the Quantum Register

Fig. 6.7 shows the experimentally measured density matrices of the two-qubit register when running the Grover search algorithm for the four possible Oracle functions. For each of those four cases, density matrices were measured at the steps 1-4 of the Grover algorithm as indicated in fig. 6.5. The black outlined circles in the density matrices represent the ideal theoretical states and the colored solid circles represent the measured states. The trace fidelities of all states with respect to the ideal states are noted above each density matrix. As can be seen, the fidelity diminishes as a function of the runtime of the algorithm due to dephasing and relaxation of the qubit register, as we will discuss later.

6.4.2 Single Run Results

The experimental state tomographies discussed in the last section show that we are able to implement the Grover search algorithm with adequate fidelity using our two-qubit processor. However, the analysis of the two-qubit register by quantum state tomography at the end of the algorithm does not prove that we can achieve real quantum speed-up with our processor. For this, it is necessary to directly read out the state of the qubit register at the end of the algorithm *without* performing any kind of error correction afterwards. By looking at this “raw” outcome data and generating outcome statistics over many single runs of the processor we can then quantify the success rate and the fidelity of the algorithm we implemented. The results of such measurements that we performed for the four possible Oracle functions are shown in fig. 6.8. Besides the single-run probabilities for all four Oracle functions, the diagram shows for comparison the expected outcome probabilities calculated based on the quantum state tomographies discussed above and the readout matrix of the two-qubit processor. As can be seen, the agreement between the measured and calculated probabilities is rather good. The dashed line in the diagrams corresponds to the success probability of a classical single-step “query-and-guess” algorithm as described above, which is bound to 50 % and provides the benchmark against which we measure the quantum speed-up in this system. As can be seen, our implementation of the Grover search algorithm outperforms such a

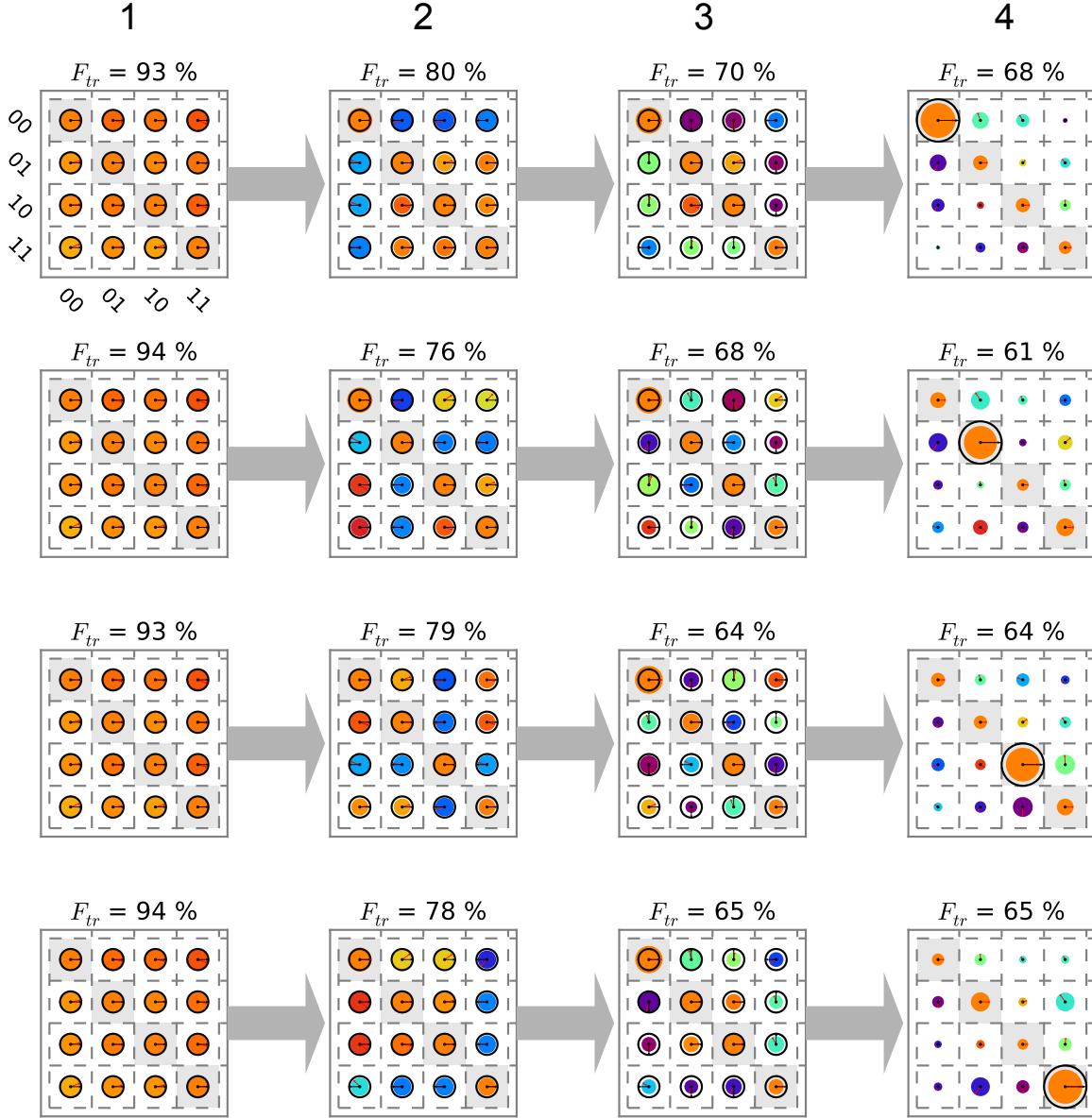


Figure 6.7: Quantum state tomographies at steps 1-4 of the Grover search algorithm, as indicated in fig. 6.6. The density matrices show the experimentally measured states in color and the theoretical states in black. For each state, the trace fidelity $F_{tr}(\rho_A, \rho_B) = \text{Tr}\{\rho_A \cdot \rho_B\}$ is shown above the density matrix.

classical search algorithm for all four Oracle functions, if only by a small margin.

6.5 Algorithm Fidelity

We can define the average fidelity of the algorithm in a single run, which corresponds to the conditional probability of finding the correct state $|i\rangle$ given a certain Oracle function C_i . Table 6.1 shows these single-run probabilities along with the so-called *user fidelities*,

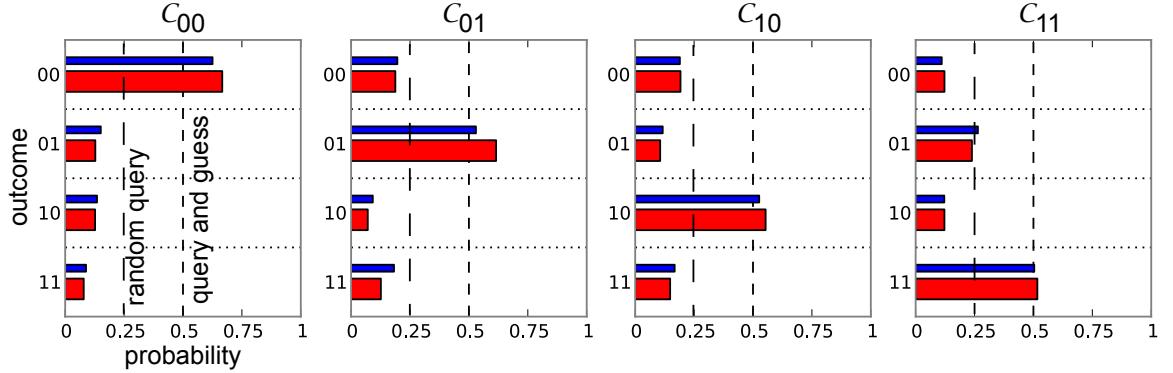


Figure 6.8: The single-run success probabilities of our implementation of the Grover search algorithm. Shown are the averaged probabilities for the four possible Oracle functions. The red bars correspond to measured values, the blue ones to expected probabilities calculated using the reconstructed density matrices after the final step of the algorithm and the measured two-qubit readout matrix. The dashed line indicates the average success probability of a classical query-and-guess algorithm for comparison.

which are given as

$$f_{ab} = p(|ab\rangle |ab) = \frac{p(ab| |ab\rangle)}{\sum_{uv} p(uv| |uv\rangle)} \quad (6.15)$$

, where $p(ab| |ab\rangle)$ is the conditional probability of obtaining the search result ab given the Oracle operator $|ab\rangle$. The user fidelity is complementary to the average fidelity and correspond to the conditional probability of having found the correct state given a certain measured state, averaged over all possible Oracle functions. For all four Oracles, both the single-run and user fidelities are $> 50\%$, hence demonstrating quantum speed-up in comparison with a classical query-and-guess algorithm, as will be discussed below.

$ab/ uv\rangle$	$ 00\rangle$	$ 01\rangle$	$ 10\rangle$	$ 11\rangle$	Σ	f_{ab}
00	0.666	0.192	0.188	0.122	1.168	57.0 %
01	0.127	0.554	0.071	0.122	0.874	63.4 %
10	0.128	0.106	0.615	0.239	1.088	56.5 %
11	0.079	0.148	0.126	0.517	0.870	59.4 %

Table 6.1: Conditional probabilities $p_{ab/|uv\rangle}$ and user fidelities f_{ab} for all possible outcomes ab for our implementation of Grover's algorithm.

6.6 Comparison to a Classical Search Algorithm

As discussed above, we can compare the success probability of our implementation of the Grover search algorithm to different classical algorithm in order to quantify the quantum speed-up that we achieve. When repeatedly running the quantum search algorithm

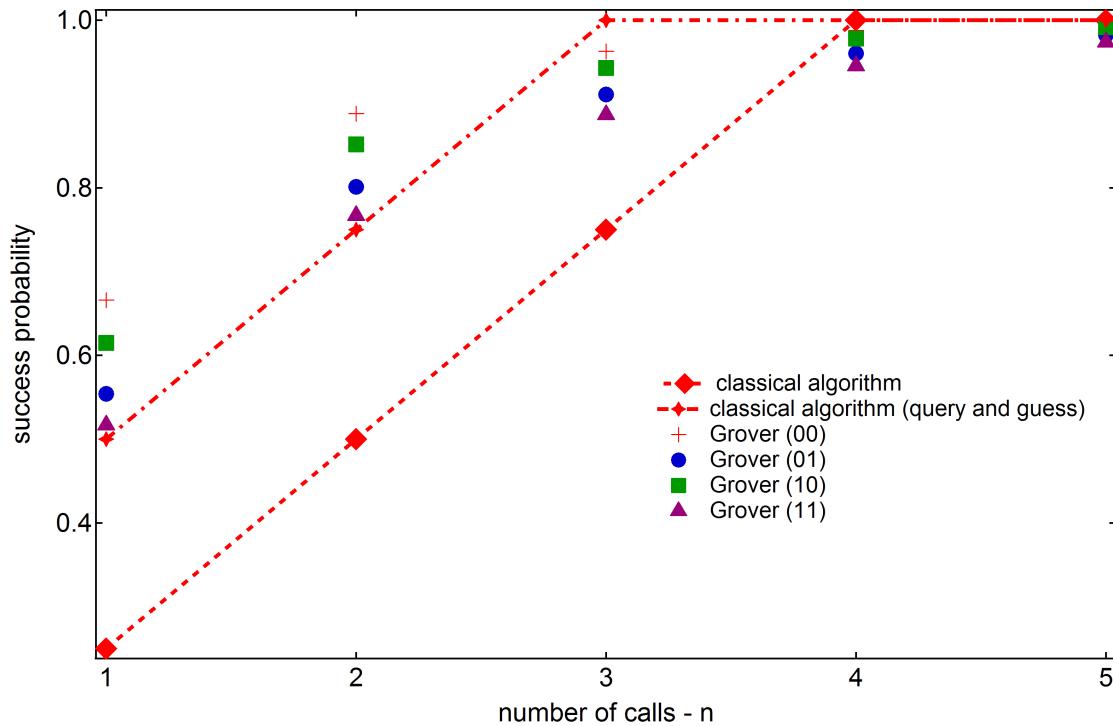


Figure 6.9: Success probability of our implementation of the Grover search algorithm, plotted against the number of single runs of the algorithm n . For comparison we show the success probabilities of a classical query and a classical query-and-guess search algorithm as a function of n .

we can define the success probability as the probability of obtaining the right answer to the search problems within n single runs, which is given as

$$p_s(n) = \sum_{i=1}^n (1 - p_s)^{i-1} p_s \quad (6.16)$$

Here, p_s is the single-run success probability of the algorithm. We can plot $p_s(n)$ as a function of the runs of the algorithm and compare it with different classical algorithms. Fig. 6.9 shows such a comparison for our implementation of the Grover search algorithm. For comparison with the quantum algorithm we plot the success probability of a classical query search algorithm and a classical query-and-guess algorithm as a function of Oracle calls n . The difference between the query and the query-and-guess algorithm is that the former doesn't return a result if it does not find the solution to the search problem after a number of n calls, whereas the latter takes an "educated guess" at the right state even if it did not obtain it directly by calling the Oracle function n times. Hence the statistical success probability of the query-and-guess-algorithm is bigger than that of the simple query algorithm.

As can be seen, our implementation of the Grover search algorithm beats the classical query algorithm for $n < 4$ single runs and the classical query-and-guess algorithm

for $n < 3$ single runs. However, unlike the classical algorithms it never converges to 100 % success probability due to always-present unitary and non-unitary errors in our system. We will analyze all such different errors relevant to our experiment in the next section.

6.7 Error Analysis

There are three kind of errors arising in our implementation of the Grover search algorithm that we will analyze in the following section. These errors are:

1. Deterministic, unitary gate errors
2. Stochastic errors introduced due to qubit decoherence during the runtime of the algorithm
3. Readout errors due to qubit relaxation during readout, insufficient readout sensitivity or retrapping of the readout resonator state during measurement.

We analyze the contributions of all these three error sources for the implementation of the algorithm below.

6.7.1 Gate Errors & Decoherence

Gate errors are unitary errors that arise due to misshaped or mistuned gate pulses and qubit frequencies. Usually the effect of these errors is combined with stochastic, non-unitary errors arising due to qubit decoherence during the runtime of the algorithm and therefore has to be analyzed together with the latter. Hence, in order to quantify these errors it is necessary to generate an error model of our algorithm that takes into account both unitary and as non-unitary errors. The parameters of such a model can be obtained by numerically fitting it to our experimental results.

Modeling Decoherence

I13! We could again model decoherence processes in our algorithm by formulating an effective master equation of the two-qubit system that includes relaxation and dephasing processes as we did when analyzing the universal quantum gate that we implemented.

For our implementation of the Grover algorithm, however, we chose to rather use a set of discrete decoherence operators that model amplitude (i.e. T_1) and phase damping (i.e. T_ϕ) processes and which we can directly integrate in a more simple, operator-based model of the algorithm. We can then model the decoherence in our algorithm by applying these operators to the calculated quantum states after each individual step of

To Do 13: replace this analytical calculating with a master equation simulation modeling the qubit as a three-level system and capturing all relevant errors

the algorithm. Like this we can generate an error model incorporating the most relevant experimental decoherence processes without the need to numerically integrate an effective master equation, thereby greatly speeding up the process of fitting our experimental data to the formulated error model. In the following paragraphs we introduce the reader to the operators we use to model relaxation and dephasing processes in our error model.

To model qubit relaxation, we can use a pair of single-qubit operators describing amplitude-damaging of the qubit state, which given as [41]

$$E_1^{T_1} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma_{T_1}} \end{pmatrix} \quad E_2^{T_1} = \begin{pmatrix} 0 & \sqrt{\gamma_{T_1}} \\ 0 & 0 \end{pmatrix} \quad (6.17)$$

On the other hand, phase-damaging operators describing qubit dephasing can be written analogously as

$$E_1^{T_{\phi 1}} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma_{\phi}} \end{pmatrix} \quad E_2^{T_{\phi 1}} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma_{\phi}} \end{pmatrix} \quad (6.18)$$

Both operators are applied to a quantum state ρ according to

$$\rho \rightarrow E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger \quad (6.19)$$

and yield a trace-preserving, non-unitary evolution of the quantum state of ρ . The decoherence fraction γ that is used in the operators can be calculated from the corresponding relaxation and dephasing rates as $\gamma_{T_{1,2}}(t) = 1 - \exp(-t\Gamma_{1,2}^{T_1})$ and $\gamma_{\phi_{1,2}} = 1 - \exp(-t\Gamma_{1,2}^{\phi}/2)$, where t is the time during which the state is exposed to the given decoherence process.

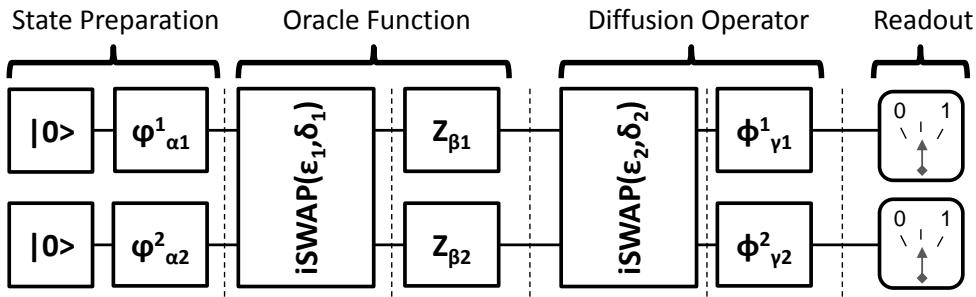


Figure 6.10: The error model we use to analyze the different gate and decoherence errors present when running the Grover search algorithm. The dotted lines indicate the points at which the quantum state has been measured by state tomography.

Using these two operators combined with a set of unitary operators describing the quantum operations performed during the algorithm we formulate a full error model that

we use to model our experimental data, as shown in fig. 6.10. This model takes into account the following error sources:

- **Energy relaxation and phase decoherence:** Energy relaxation and dephasing of the qubit is modeled using the processes given in eqs. (6.17) and (6.18), applying these operators with an adapted γ after each unitary operation performed during the algorithm.
- **Single-qubit gate errors:** We model rotation angle and rotation phase errors of our single-qubit X_α and Y_α gates by replacing them with operators of the form $X_\alpha \rightarrow \phi_{\alpha'} = \cos \phi X_{\alpha'} + \sin \phi Y_{\alpha'}$ and $Y_\alpha \rightarrow \varphi_{\alpha'} = \sin \varphi X_{\alpha'} + \cos \varphi Y_{\alpha'}$. For Z -type single-qubit operators we model only rotation angle errors by replacing $Z_\alpha \rightarrow Z_{\alpha'}$.
- **Two-qubit gate errors:** We model both detuning and gate-length errors of our $i\text{SWAP}$ 2-qubit gates.

For the two-qubit gates, we model the errors present in the $i\text{SWAP}$ operation by using the representation of the gate

$$i\text{SWAP}(t, \Delta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t g_e - i \frac{\Delta}{g_e} \sin t g_e & i \frac{g}{g_e} \sin t g_e & 0 \\ 0 & i \frac{g}{g_e} \sin t g_e & \cos t g_e + i \frac{\Delta}{g_e} \sin t g_e & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.20)$$

where $g_e = \sqrt{g^2 + \Delta^2}$ is the effective swap frequency at a qubit frequency detuning $f_{01}^1 - f_{01}^2 = 2\Delta$. Often it is practical to replace t and Δ with $\beta = t g_e$ and $\delta = \Delta/g$. Using this notation of the $i\text{SWAP}$ gate and the definition of the single-qubit gates as discussed before, the full algorithm with only gate errors can be written as (for right-multiplication)

$$\text{Grover} = \phi_{\gamma_1}^1 \otimes \phi_{\gamma_2}^2 \cdot i\text{SWAP}(\epsilon_2, \delta_2) \cdot Z_{\beta_1} \otimes Z_{\beta_2} \cdot i\text{SWAP}(\epsilon_1, \delta_1) \cdot \varphi_{\alpha_1}^1 \otimes \varphi_{\alpha_2}^2 \quad (6.21)$$

In addition, we add dephasing and relaxation error after each step of the algorithm to model the decoherence during the runtime of the algorithm. Numerical optimization is then used to produce a fit of all the gate errors, which is shown in tab. 6.2. Here, the qubit relaxation and dephasing times were measured independently and are not part of the fit.

The resulting fitted error models obtained for our experimental data are shown in tab. 6.2. As can be seen, the phase and gate-time errors for the first gates are comparatively small and grow bigger during the following steps of the algorithm. Curiously, the phase errors are bigger for the states $|10\rangle$ and $|11\rangle$, as are the gate-length errors for the two $i\text{SWAP}$ gates used in the algorithm. This fact might be explained by a drift of the

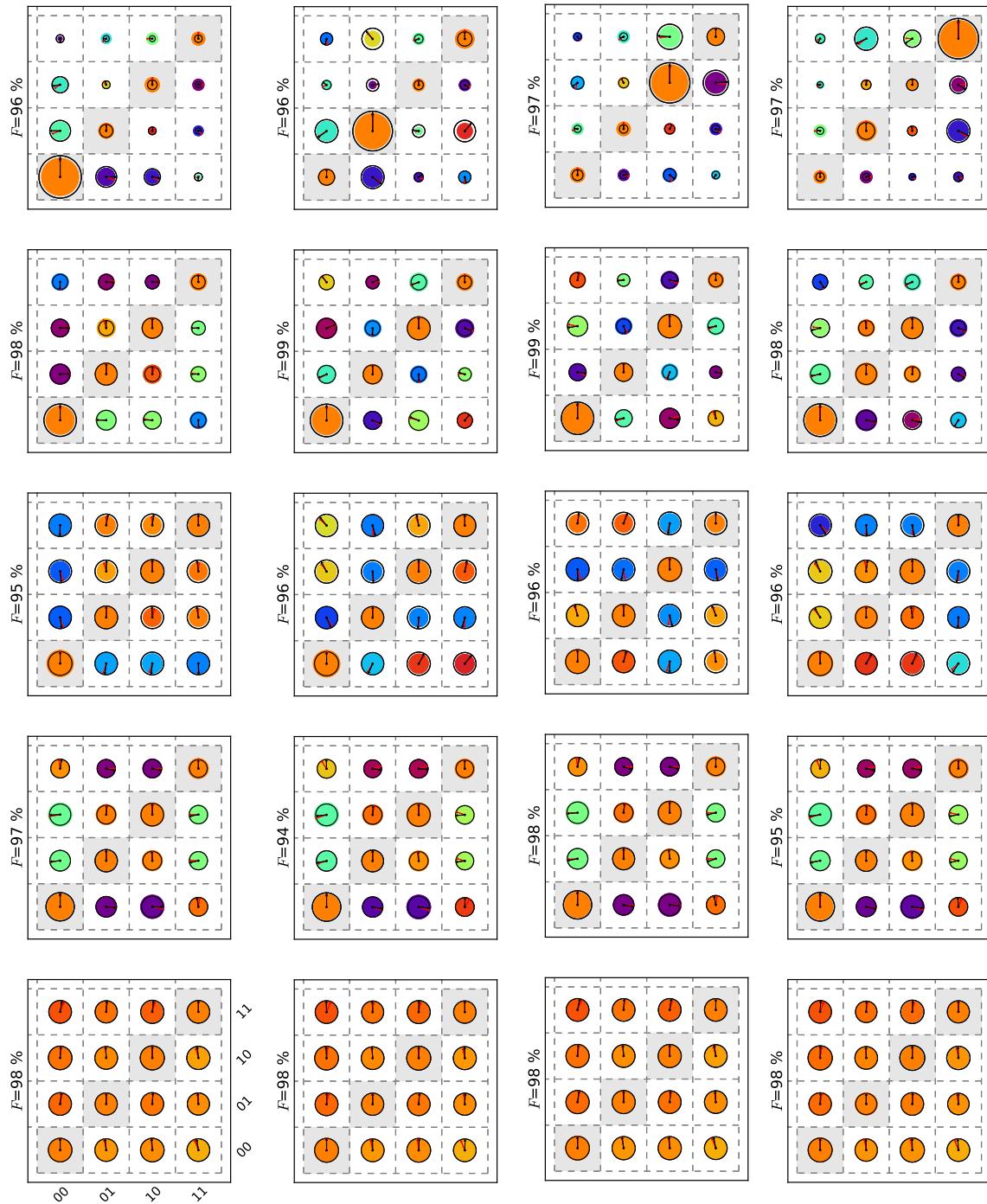


Figure 6.11: Comparison of the fitted error model as given by eq. (6.21) to our experimental data. Experimental data is shown in color, the fitted density matrices as black outlines. As before, we show the state fidelity according to eq. (??) between experimental and fitted states.

state	δ_1	δ_2	α_1	α_2	φ_1	φ_2	ϵ_1	β_1	β_2	ϵ_2	γ_1	γ_2	ϕ_1	ϕ_2
$ 00\rangle$	0.06	-0.06	-2.5	2.7	6.1	3.1	-7.3	-3.3	-4.1	7.5	29	9.3	0.66	-1.7
$ 01\rangle$	0.04	-0.3	-0.1	0.1	7.9	3.6	-11	-5.9	2.2	-6.9	28	-19	9	2
$ 10\rangle$	0.09	-0.2	-3.1	1.7	1	-2.5	-6.5	-15	-22	-7.5	-15	32	3.6	5.2
$ 11\rangle$	0.16	0.13	-6	3.9	2.2	0.9	-9.5	-20	-15	17	-12	-32	-7	-8.9

Table 6.2: Fitted error parameters for the measured density matrices, modeled according to the error model given in eq. (6.21). All angles are given in deg.

operating point of our microwave setup during the time it took to take the data for the four possible Oracle operators, during which the parameters of individual qubit gates were not recalibrated.

Fig. 6.11 shows again the measured density matrices for our realization of the Grover search algorithm, this time overlaid with the numerically optimized error model according to eq. (6.21). As can be seen, our error model is able to capture most of the observed experimental errors and can reproduce to very good accuracy the observed density matrices. The state fidelities according to eq. ?? between the measured density matrices and those of the fitted error model are shown above each density matrix.

Fidelity of the Oracle and diffusion operators

It is interesting to analyze the individual experimental fidelities of the Oracle and diffusion operators that make up the Grover algorithm achieved in our experiment. For this, we compare the action of the ideal operators D' and R' with that of the experimentally implemented versions D'_e and R'_e , taking the measured quantum states before applying each of the operators as input. We take then as the fidelity of each operator the average state fidelity of the measured output states as compared to the calculated ones, i.e.

$$F(D'_e) = F(D' \rho_{in} D'^\dagger, D'_e \rho_{in} D'_e^\dagger) \quad (6.22)$$

$$F(R'_e) = F(R' \rho_{in} R'^\dagger, R'_e \rho_{in} R'_e^\dagger) \quad (6.23)$$

where we also make use of the state fidelity according to eq. ?. By this method, we obtain the following experimental fidelities for the two gate operations:

Operator / State	$ 00\rangle$	$ 01\rangle$	$ 10\rangle$	$ 11\rangle$	Average
D'	92.3	93.4	94.3	91.7	92.9
R'	94.5	93.6	88.5	87.7	91.1

Table 6.3: Measured fidelities of the quantum Oracle and diffusion operators used in the Grover search algorithm according to eqs. (6.22) and (6.23). All fidelities are given in percent.

As can be seen, on average we are able to implement both the diffusion operator and the quantum Oracle with a fidelity $> 90\%$.

6.7.2 Readout Errors

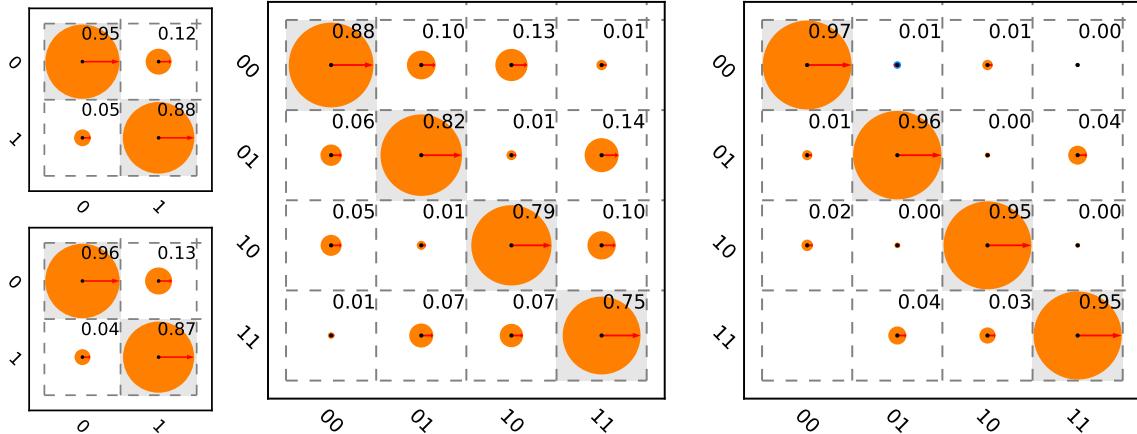


Figure 6.12: a.)The measured single-qubit readout matrices, showing the readout outcome probabilities as a function of the prepared state for both qubits. b.)The measured two-qubit readout matrix, showing again the detector outcome probabilities versus the prepared qubit states. c.) The crosstalk matrix, corresponding to the product of the inverse two-qubit readout matrix and the Kronecker product of the single-qubit readout matrices. Note that the $|1\rangle \rightarrow |2\rangle$ shelving method is used for reading out the qubit state, which increases readout fidelity but also inter-qubit readout crosstalk up to 4 % in the worst case.

Another source of errors affecting the single-run fidelities of the algorithm arises due to the imperfection of our qubit readout. Here, mostly qubit relaxation during the readout process reduces the visibility of individual qubit states and introduces errors when reading out the qubit register in the final step of the algorithm. We can easily quantify those readout errors by using the readout matrix that was introduced in the last chapter. When running the Grover algorithm we use the $|1\rangle \rightarrow |2\rangle$ shelving method described in the last chapter to increase the readout contrast and thereby the algorithm fidelity. This technique reduces single-qubit readout errors but increases inter-qubit readout crosstalk. To quantify all single-qubit and inter-qubit readout errors, we first model the readout matrix R of the two-qubit system as a product $R = R_v \cdot R_{ct}$, where R_v is the so-called *visibility matrix* and R_{ct} a matrix describing the readout crosstalk. The visibility matrix can be written as the Kronecker product $R_v = R_v^1 \otimes R_v^2$ of the two single-qubit readout matrices, which have the form

$$R_v^{1,2} = \begin{pmatrix} p_{00}^{1,2} & 1 - p_{11}^{1,2} \\ 1 - p_{00}^{1,2} & p_{11}^{1,2} \end{pmatrix} \quad (6.24)$$

Here, $p_{00}^{1,2}$ ($p_{11}^{1,2}$) corresponds to the probability to measure the value 0 (1) at the readout after having prepared the qubit in state $|0\rangle$ ($|1\rangle$). Usually, the full two-qubit readout matrix R and the single-qubit readout matrices $R_v^{1,2}$ are measured experimentally which allows us then to calculate the crosstalk matrix as $R_{ct} = R_v^{-1} \cdot R$. The three matrices measured in our experiment are shown in fig. 6.12. As can be seen, the single-qubit

readout fidelities range between 87 - 96 % and the combined two-qubit readout fidelities between 75 - 85 %. Depending on the qubit state we also observe between 3-5 % inter-qubit readout crosstalk in our system.

Fig. 6.8 shows the single-run probabilities when running the Grover algorithm for the four different Oracle functions. In blue, the expected readout outcome probabilities, as calculated using the state tomography of the final states given in fig. 6.7d and the measured readout matrix of our system are shown along the measured readout outcome probabilities. The readout error model shows good quantitative agreement with the measured data, with deviations most probably due to parameter drifts occurred between the measurement of the quantum state tomography and the single-run experiment.

6.8 Conclusions

To summarize, we have shown that we can implement the Grover search algorithm with our quantum processor and achieve a single-run fidelity that is sufficient to demonstrate simple probabilistic quantum speed-up as compared to a classical, reversible search algorithm. The error model formulated in this chapter is able to account for most of the observed imperfections and can explain the data we observed. Unfortunately, the coherence times of our qubits does not permit the realization of more complex algorithm with this system, but nevertheless it provides a proof-of-principle of our approach to build a superconducting quantum computer with individual-qubit single shot readout.

In the following chapter, we will discuss the extension of this approach to a system of four qubits and explain different strategies for scaling up such system to an even larger number of qubits.

Chapter 7

Designing a Scalable Architecture for Quantum Bits

7.1 Definition & Requirements

7.2 Qubit Design

7.2.1 Qubit Parameters

7.2.2 Qubit-Qubit Coupling

7.3 Readout Design

7.3.1 Readout Parameters

7.3.2 Qubit-Readout Coupling

7.4 Single-Qubit Manipulation

7.4.1 Error Analysis

7.5 Multi-Qubit Manipulation

7.5.1 Error Analysis

7.6 Implementing a Universal Set of Quantum Gates

7.7 Realizing A Four-Qubit Architecture

7.8 Scaling Up

Chapter 8

Conclusions & Perspective

8.1 Future Directions in Superconducting QC

8.1.1 3D Circuit Quantum Electrodynamics

8.1.2 Hybrid Quantum Systems

8.1.3 Quantum Error Correction & Feedback

Appendix A

Modeling of Multi-Qubit Systems

A.1 Analytical Approach

A.1.1 Multi-Qubit Hamiltonian

A.1.2 Energies and Eigenstates

A.2 Master Equation Approach

$$\hat{H} = \begin{pmatrix} e_0^I + e_0^{II} & & & & & \\ & e_0^I + e_1^{II} & & & & \\ & & e_0^I + e_2^{II} & & & \\ & & & e_1^I + e_0^{II} & & \\ & & & & e_1^I + e_1^{II} & \\ & & & & & e_1^I + e_2^{II} \\ & & & & & & e_2^I + e_0^{II} \\ & & & & & & & e_2^I + e_1^{II} \\ & & & & & & & & e_2^I + e_2^{II} \end{pmatrix} \quad (\text{A.1})$$

Without loss of generality we can assume $e_0^I = 0$, $e_0^{II} = 0$. We define $\omega_{01}^I = e_1^I$, $\omega_{01}^{II} = e_1^{II}$, $\Delta_{01} = \omega_{01}^{II} - \omega_{01}^I$ and $\alpha^I = e_2^I - 2e_1^I$ and $\alpha^{II} = e_2^{II} - 2e_1^{II}$. The interaction

Hamiltonian is given as

$$\hat{H}_i = \begin{pmatrix} 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}g & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}g & 0 & 0 & 0 & \sqrt{2}g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2g & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.2})$$

Going to the interaction picture with $\hat{H}_0 = \hat{H}$ the interaction Hamiltonian \hat{H}_i becomes

$$\hat{H}_i = \begin{pmatrix} 0 & \dots & & & & & & & \\ 0 & 0 & \dots & & & & & & \\ 0 & 0 & 0 & \dots & & & & & \\ 0 & ge^{-i\Delta t} & 0 & 0 & \dots & & & & \\ 0 & 0 & \sqrt{2}ge^{-i(\Delta-\alpha^{II})t} & 0 & 0 & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & & \\ 0 & 0 & 0 & 0 & \sqrt{2}ge^{-i(\Delta+\alpha^I)t} & 0 & 0 & \dots & \\ 0 & 0 & 0 & 0 & 0 & 2ge^{-i(\Delta+\alpha^I-\alpha^{II})t} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.3})$$

Relaxation and Dephasing

We can model relaxation and dephasing of the three-level system using a set of 6 Lindblad operators. The relaxation operators are based on the three matrices

$$\sigma_{01}^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \sigma_{12}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \sigma_{02}^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{A.4})$$

and the dephasing operators on

$$\sigma_{01}^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \sigma_{12}^z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \sigma_{02}^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (\text{A.5})$$

Using these matrices, we define a set of three relaxation operators $L_{01}^r = \sqrt{\Gamma_{10}^r}\sigma_{01}^-$, $L_{12}^r = \sqrt{\Gamma_{12}^r}\sigma_{12}^-$ and $L_{02}^r = \sqrt{\Gamma_{02}^r}\sigma_{02}^-$ as well as a set of dephasing operators $L_{01}^\phi = \sqrt{\Gamma_{01}^\phi/2}\sigma_{01}^z$, $L_{12}^\phi = \sqrt{\Gamma_{12}^\phi/2}\sigma_{12}^z$, $L_{02}^\phi = \sqrt{\Gamma_{02}^\phi/2}\sigma_{02}^z$.

A.2.1 Direct Integration

We can use the Hamiltonian and the Lindblad operators discussed above to directly integrate the master equation of the two-qubit system, which is given as

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_j [2L_j \rho L_j^\dagger - \{L_j^\dagger L_j, \rho\}] \quad (\text{A.6})$$

Here $-i/\hbar [H, \rho]$ corresponds to the unitary evolution of the state of the system and the part right to it to the non-unitary part, i.e. relaxation and dephasing. For numerical analysis it is usually convenient to rewrite this equation in the form

$$\frac{d\vec{\rho}}{dt} = \mathcal{L}(t)\vec{\rho}(t) \quad (\text{A.7})$$

where $\vec{\rho}$ is a column vector containing all elements of ρ and \mathcal{L} is the so-called “superoperator” which acts on the vectorized density matrix. For a density matrix with dimension $N \times N$, the superoperator has dimensions $N^2 \times N^2$, which makes the numerical solution of eq. (A.7) computationally expensive for large N . An alternative method which avoids this inefficiency is the so-called “quantum monte carlo” method which we will discuss in the next section.

From these matrices, the corresponding two-qubit operator can be obtained as $L_i^I = L_i \otimes \mathcal{I}$ and $L_i^{II} = L_i \otimes \mathcal{I}$.

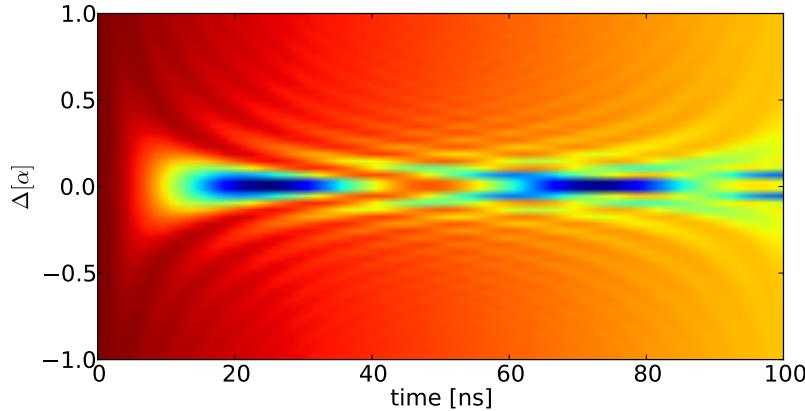


Figure A.1: ...

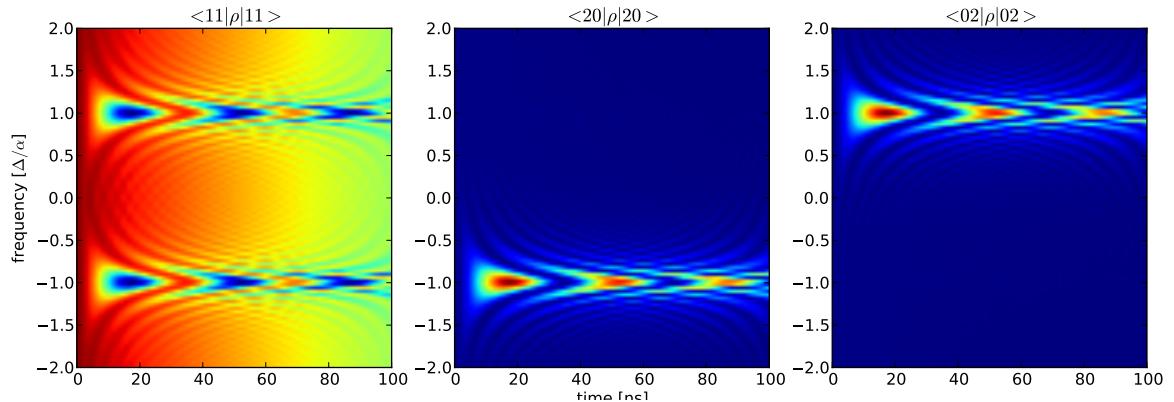


Figure A.2: ...

A.2.2 Monte Carlo Simulation

A.2.3 Speeding Up Simulations

Appendix B

Data Acquisition & Management

B.1 Data Acquisition Infrastructure

B.2 Data Management Requirements

B.3 PyView

B.3.1 Overview

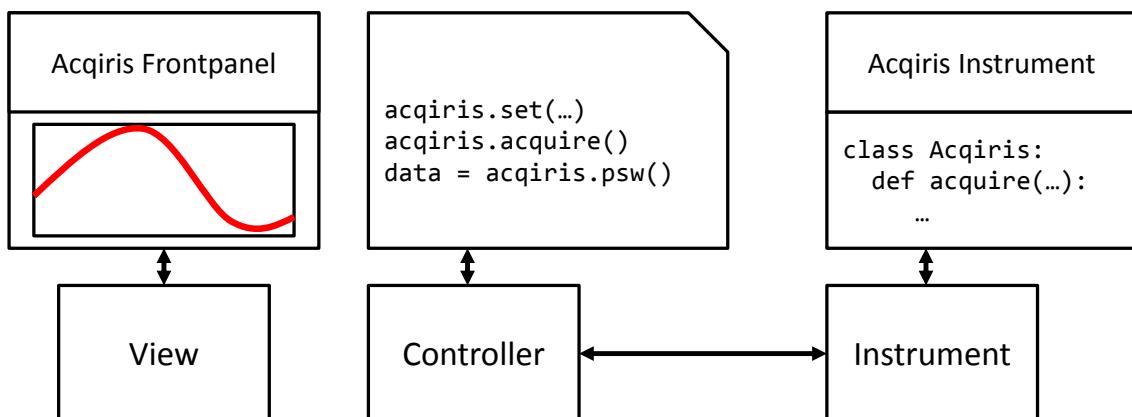


Figure B.1

B.3.2 Instrument Management**B.3.3 Data Acquisition****B.3.4 Data Management****B.3.5 Data Analysis**

Appendix C

Design & Fabrication

C.1 Mask Design

C.2 Optical Lithography

C.3 Electron Beam Lithography

Bibliography

- [1] M. Ansmann, H. Wang, R. C. Bialczak, M. Hofheinz, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, M. Weides, J. Wenner, A. N. Cleland, and J. M. Martinis. Violation of bell’s inequality in josephson phase qubits. *Nature*, 461(7263):504–506, 2009.
- [2] A. Aspect, P. Grangier, and G. Roger. Experimental realization of Einstein-Podolsky-Rosen-Bohm gedankenexperiment: A new violation of bell’s inequalities. *Physical Review Letters*, 49(2):91–94, July 1982.
- [3] J. Bell. On the einstein podolsky rosen paradox. *Physics*, pages 195–200, 1964.
- [4] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf. Quantum-information processing with circuit quantum electrodynamics. *Physical Review A*, 75(3):032329, Mar. 2007.
- [5] A. Blais, R. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf. Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation. *Physical Review A*, 69(6):062320, June 2004.
- [6] V. Bouchiat, D. Vion, P. Joyez, D. Esteve, and M. H. Devoret. Quantum coherence with a single cooper pair. *Physica Scripta*, T76(1):165, 1998.
- [7] I. Chiorescu, Y. Nakamura, C. J. P. M. Harmans, and J. E. Mooij. Coherent quantum dynamics of a superconducting flux qubit. *Science*, 299(5614):1869 –1871, Mar. 2003.
- [8] J. Clarke, A. N. Cleland, M. H. Devoret, D. Esteve, and J. M. Martinis. Quantum mechanics of a macroscopic variable: The phase difference of a josephson junction. *Science*, 239(4843):992–997, Feb. 1988.
- [9] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed experiment to test local Hidden-Variable theories. *Physical Review Letters*, 23(15):880–884, Oct. 1969.

- [10] E. Collin, G. Ithier, A. Aassime, P. Joyez, D. Vion, and D. Esteve. NMR-like control of a quantum bit superconducting circuit. *Physical Review Letters*, 93(15):157005, Oct. 2004.
- [11] A. Cottet. *Implementation of a quantum bit in a superconducting circuit*. PhD thesis, Université Paris VI, Paris, 2002.
- [12] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 400(1818):97–117, July 1985.
- [13] M. Devoret. Quantum fluctuations in electrical circuits. In *Les Houches Session on Quantum Fluctuations*, volume LXIII, pages 351–386. 1995.
- [14] M. H. Devoret, J. M. Martinis, and J. Clarke. Measurements of macroscopic quantum tunneling out of the Zero-Voltage state of a Current-Biased Josephson junction. *Physical Review Letters*, 55(18):1908–1911, Oct. 1985.
- [15] L. DiCarlo, J. M. Chow, J. M. Gambetta, L. S. Bishop, B. R. Johnson, D. I. Schuster, J. Majer, A. Blais, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf. Demonstration of two-qubit algorithms with a superconducting quantum processor. *Nature*, 460(7252):240–244, July 2009.
- [16] L. DiCarlo, M. D. Reed, L. Sun, B. R. Johnson, J. M. Chow, J. M. Gambetta, L. Frunzio, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf. Preparation and measurement of three-qubit entanglement in a superconducting circuit. *Nature*, 467(7315):574–578, 2010.
- [17] D. P. DiVincenzo. The physical implementation of quantum computation. *Fortschritte der Physik*, 48(9-11):771–783, Sept. 2000.
- [18] D. P. DiVincenzo. Fault-tolerant architectures for superconducting qubits. *Physica Scripta*, T137:014020, Dec. 2009.
- [19] A. Einstein, B. Podolsky, and N. Rosen. Can Quantum-Mechanical description of physical reality be considered complete? *Physical Review*, 47(10):777–780, May 1935.
- [20] A. Fedorov, L. Steffen, M. Baur, and A. Wallraff. Implementation of a toffoli gate with superconducting circuits. *arXiv:1108.3966*, Aug. 2011.
- [21] R. Feynman. Simulating physics with computers. *International Journal of Theoretical Physics*, 21(6):467–488, 1982.

- [22] A. Galiautdinov, A. N. Korotkov, and J. M. Martinis. Resonator–zero-qubit architecture for superconducting qubits. *Physical Review A*, 85(4):042321, Apr. 2012.
- [23] M. Göppl, A. Fragner, M. Baur, R. Bianchetti, S. Filipp, J. M. Fink, P. J. Leek, G. Puebla, L. Steffen, and A. Wallraff. Coplanar waveguide resonators for circuit quantum electrodynamics. *Journal of Applied Physics*, 104(11):113904–113904–8, Dec. 2008.
- [24] L. K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, STOC ’96, page 212–219, New York, NY, USA, 1996. ACM.
- [25] L. K. Grover. Quantum mechanics helps in searching for a needle in a haystack. *Physical Review Letters*, 79(2):325–328, July 1997.
- [26] L. K. Grover. From schrödinger’s equation to the quantum search algorithm. *American Journal of Physics*, 69(7):769–777, 2001.
- [27] S. Haroche and J. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons*. Oxford University Press, 2006.
- [28] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland. Synthesizing arbitrary quantum states in a superconducting resonator. *Nature*, 459(7246):546–549, May 2009.
- [29] B. Josephson. Possible new effects in superconductive tunnelling. *Physics Letters*, 1(7):251–253, July 1962.
- [30] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. Charge-insensitive qubit design derived from the cooper pair box. *Physical Review A*, 76(4):042319, Oct. 2007.
- [31] B. P. Lanyon, C. Hempel, D. Nigg, M. Müller, R. Gerritsma, F. Zähringer, P. Schindler, J. T. Barreiro, M. Rambach, G. Kirchmair, M. Hennrich, P. Zoller, R. Blatt, and C. F. Roos. Universal digital quantum simulation with trapped ions. *Science*, 334(6052):57–61, July 2011.
- [32] S. Lloyd. Universal quantum simulators. *Science*, 273(5278):1073–1078, Aug. 1996.
- [33] F. Mallet, F. R. Ong, A. Palacios-Laloy, F. Nguyen, P. Bertet, D. Vion, and D. Esteve. Single-shot qubit readout in circuit quantum electrodynamics. *Nat Phys*, 5(11):791–795, Nov. 2009.

- [34] M. Mariantoni, H. Wang, T. Yamamoto, M. Neeley, R. C. Bialczak, Y. Chen, M. Lenander, E. Lucero, A. D. O'Connell, D. Sank, M. Weides, J. Wenner, Y. Yin, J. Zhao, A. N. Korotkov, A. N. Cleland, and J. M. Martinis. Implementing the quantum von neumann architecture with superconducting circuits. *Science*, 334(6052):61 –65, Oct. 2011.
- [35] J. M. Martinis, M. H. Devoret, and J. Clarke. Energy-Level quantization in the Zero-Voltage state of a Current-Biased josephson junction. *Physical Review Letters*, 55(15):1543–1546, Oct. 1985.
- [36] J. M. Martinis, M. H. Devoret, and J. Clarke. Experimental tests for the quantum behavior of a macroscopic degree of freedom: The phase difference across a josephson junction. *Physical Review B*, 35(10):4682–4698, Apr. 1987.
- [37] J. M. Martinis, S. Nam, J. Aumentado, and C. Urbina. Rabi oscillations in a large Josephson-Junction qubit. *Physical Review Letters*, 89(11):117901, 2002.
- [38] J. E. Mooij, T. P. Orlando, L. Levitov, L. Tian, C. H. van der Wal, and S. Lloyd. Josephson Persistent-Current qubit. *Science*, 285(5430):1036 –1039, 1999.
- [39] F. Motzoi, J. M. Gambetta, P. Rebentrost, and F. K. Wilhelm. Simple pulses for elimination of leakage in weakly nonlinear qubits. *Physical Review Letters*, 103(11):110501, 2009.
- [40] Y. Nakamura, Y. A. Pashkin, and J. S. Tsai. Coherent control of macroscopic quantum states in a single-Cooper-pair box. *Nature*, 398(6730):786–788, Apr. 1999.
- [41] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
- [42] H. Paik, D. I. Schuster, L. S. Bishop, G. Kirchmair, G. Catelani, A. P. Sears, B. R. Johnson, M. J. Reagor, L. Frunzio, L. I. Glazman, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf. Observation of high coherence in josephson junction qubits measured in a Three-Dimensional circuit QED architecture. *Physical Review Letters*, 107(24):240501, Dec. 2011.
- [43] A. Palacios-Laloy. *Superconducting Qubit in a Resonator: Test of the Leggett-Garg Inequality and Single-Shot Readout*. PhD thesis, Commissariat à l'Energie Atomique, Saclay, 2010.
- [44] D. M. Pozar. *Microwave Engineering*. Wiley, 4 edition, Nov. 2011.

- [45] M. D. Reed, L. DiCarlo, S. E. Nigg, L. Sun, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf. Realization of Three-Qubit quantum error correction with superconducting circuits. *arXiv:1109.4948*, Sept. 2011.
- [46] D. I. Schuster. *Circuit Quantum Electrodynamics*. PhD thesis, Yale University, New Haven, 2007.
- [47] I. Siddiqi, R. Vijay, M. Metcalfe, E. Boaknin, L. Frunzio, R. J. Schoelkopf, and M. H. Devoret. Dispersive measurements of superconducting qubit coherence with a fast latching readout. *Physical Review B*, 73(5):054510, Feb. 2006.
- [48] I. Siddiqi, R. Vijay, F. Pierre, C. M. Wilson, M. Metcalfe, C. Rigetti, L. Frunzio, and M. H. Devoret. RF-Driven josephson bifurcation amplifier for quantum measurement. *Physical Review Letters*, 93(20):207002, Nov. 2004.
- [49] S. J. Srinivasan, A. J. Hoffman, J. M. Gambetta, and A. A. Houck. Tunable coupling in circuit quantum electrodynamics using a superconducting charge qubit with a V-Shaped energy level diagram. *Physical Review Letters*, 106(8):083601, Feb. 2011.
- [50] R. Vijay, M. H. Devoret, and I. Siddiqi. Invited review article: The josephson bifurcation amplifier. *Review of Scientific Instruments*, 80(11):111101–111101–17, Nov. 2009.
- [51] R. Vijay, C. Macklin, D. H. Slichter, S. J. Weber, K. W. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi. Quantum feedback control of a superconducting qubit: Persistent rabi oscillations. *arXiv:1205.5591*, May 2012.
- [52] D. Vion, A. Aassime, A. Cottet, P. Joyez, H. Pothier, C. Urbina, D. Esteve, and M. H. Devoret. Manipulating the quantum state of an electrical circuit. *Science*, 296(5569):886 –889, May 2002.
- [53] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf. Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics. *Nature*, 431(7005):162–167, 2004.
- [54] T. Yamamoto, M. Neeley, E. Lucero, R. C. Bialczak, J. Kelly, M. Lenander, M. Mariantoni, A. D. O’Connell, D. Sank, H. Wang, M. Weides, J. Wenner, Y. Yin, A. N. Cleland, and J. M. Martinis. Quantum process tomography of two-qubit controlled-Z and controlled-NOT gates using superconducting phase qubits. *Physical Review B*, 82(18):184515, Nov. 2010.
- [55] B. Yurke and J. S. Denker. Quantum network theory. *Physical Review A*, 29(3):1419–1437, Mar. 1984.