Properties of relations on a set determined by a matrix

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Abstract

This paper introduces a technique of converting a relation set into a matrix of $\mathbb{R}^n$ space using that matrix classifying the properties for the relational set. Using this technique, we can find the properties of a relation set by using some properties of the matrix. In programming, we can use this technique to determine our properties of a relation set by using the matrix. Code snippets are provided to show how this technique can be used in programming.

Contents

Chapter 1

Relations

1.0.1 What is a relation?

A (binary) relation R between sets A and B is a subset of $A \times B$. ($A \times B$ is a Cartesian product.)

Thus, a relation is a set of pairs.

The interpretation of this subset is that it contains all the pairs for which the relation is true. We write aRb if the relation is true for A and B (equivalently B, if $(A, B) \in R$).

A and B can be the same set, in which case the relation is said to be "on" rather than "between":

A binary relation R on a set A is a $\subseteq A \times A$. ($A \times A$ is a Cartesian product.)

Example of a relation using $A = \{0, 1, 2, 3\}$

$R = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

Relations may also be of other arities. An n-ary relation R between sets X_1, ... , and $X_n \subseteq n$-ary product $X_1 ... X_n$, in which case R is a set of n-tuples.

1.0.1.1 Some specific relations

The empty relation between sets X and Y, or on E, is the empty set $\emptyset$.

The empty relation is false for all pairs.

The full relation (or universal relation) between sets X and Y is the set X×Y.

The full relation on set E is the set E×E.

The full relation is true for all pairs.

The identity relation on set E is the set $(x, x)|x \in E$.

The identity relation is true for all pairs whose first and second element are identical.

1.0.2 Properties of relations

A relation R is..	if ...
reflexive	xRx
symmetric	xRy implies yRx
transitive	xRy and yRz implies xRz
irreflexive	xRy implies x≠y
antisymmetric	xRy and yRx implies x=y
triohotomy	xRy or x=y or yRx

1.0.2.1 Reflexive

Let Relation R on A
Where $R \subseteq A \times A$

The reflexive property of a relation is that $\forall a \in A$, aRa is true.

Also could be written as $R \subseteq A \times A$ and $\forall a \in A$, then $(a, a) \in R$

Example:

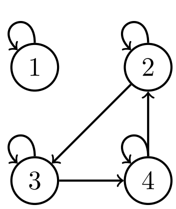
Let $A = \{1, 2, 3, 4\}$
Let $R = \{(1, 1), (2, 2), (2, 3), (3, 4), (3, 3), (4, 4), (4, 2)\}$

We could draw a matrix to show the relation R on A

To draw the matrix we plot the elements of A on the x-axis and y-axis.
Then we plot the pairs of R on the matrix as 1.
If the pair is not in R then we plot a 0.

	1	**2**	**3**	**4**
1	**1**	**0**	**0**	**0**
2	**0**	**1**	**1**	**0**
3	**0**	**0**	**1**	**1**
4	**0**	**1**	**0**	**1**

We can also draw a directional graph to show the relation R on A
We plot the elements of A as nodes.
We plot the pairs of R as directed edges.



Thus resulting in the relation R being reflexive.
This is because for all $a \in A$, aRa is true.
For example, $1R1$ is true, $2R2$ is true, $3R3$ is true, $4R4$ is true.
Therefore R is reflexive.

In the matrix, we can see that the diagonal is all 1's.
If you notice, the diagonal is the pairs (a, a) for all $a \in A$.
In programming we look at this problem as a 2D array.

Using Mathematics logic, we can write this as:
$R \subseteq A \times A$ and $\forall a \in A$, then $(a, a) \in R$

Where R is a 2D array.

Or in psuedocode:

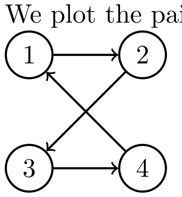
```
function isReflexive(R)
    bValid = True
    for i = 0 to R.length
        for j = 0 to R.length
            if R[i][j] = 0
                bValid = False
            end if
        end for
    end for
    return bValid
```

1.0.2.2 Irreflexive

Let Relation R on A
Where $R \subseteq A \times A$
The irreflexive property of a relation is that $\forall a \in A$, aRa is false.
Also could be written as $R \subseteq A \times A$ and $\forall a \in A$, then $(a, a) \notin R$
Example:
Let $A = \{1, 2, 3, 4\}$
Let $R = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$
We could draw a matrix to show the relation R on A
To draw the matrix we plot the elements of A on the x-axis and y-axis.
Then we plot the pairs of R on the matrix as 1.
If the pair is not in R then we plot a 0.

	1	**2**	**3**	**4**
1	**0**	**1**	**0**	**0**
2	**0**	**0**	**1**	**0**
3	**0**	**0**	**0**	**1**
4	**1**	**0**	**0**	**0**

We can also draw a directional graph to show the relation R on A
We plot the elements of A as nodes.
We plot the pairs of R as directed edges.



Thus resulting in the relation R being irreflexive.
This is because for all $a \in A$, aRa is false.
For example, $1R1$ is false, $2R2$ is false, $3R3$ is false, $4R4$ is false.
Therefore R is irreflexive.
In the matrix, we can see that the diagonal is all 0's.
If you notice, the diagonal is the pairs (a, a) for all $a \in A$.
In programming we look at this problem as a 2D array.

Using Mathematics logic, we can write this as:
$R \subseteq A \times A$ and $\forall a \in A$, then $(a, a) \notin R$
Where R is a 2D array.
Or in psuedocode:

```
function isIrreflexive(R)
    bValid = True
    for i = 0 to R.length
        for j = 0 to R.length
            if R[i][j] = 1
                bValid = False
            end if
        end for
    end for
    return bValid
```

If you want to be real and save time coding.

```
function isIrreflexive(R)
    return isReflexive(R)
end function
```

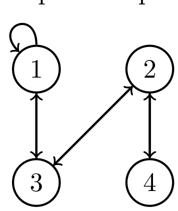
1.0.2.3 Symmetric

Let Relation R on A be symmetric if $\forall a, b \in A$ then $(a, b) \in R$ $(b, a) \in R$

Example: $R = (1, 1), (1, 3), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)$
We can draw a matrix to show the relation R on A
To draw the matrix we plot the elements of A on the x-axis and y-axis.
Then we plot the pairs of R on the matrix as 1.
If the pair is not in R then we plot a 0.

	1	**2**	**3**	**4**
1	**1**	**0**	**1**	**0**
2	**0**	**0**	**1**	**1**
3	**1**	**1**	**0**	**0**
4	**0**	**1**	**0**	**0**

We can also draw a directional graph to show the relation R on A
We plot the elements of A as nodes.
We plot the pairs of R as directed edges.



Notice that the relation R is symmetric.
This is because for all $a, b \in A$, aRb is true.
aRb is true if and only if bRa is true.
For example, $1R1$ is true, $1R3$ is true, $2R3$ is true, $2R4$ is true, $3R1$ is true, $3R2$ is true, $4R2$ is true.
Therefore R is symmetric.
In the matrix, we can see that the matrix is symmetric.

Using our knowledge of matrices, we can write this as:
$R \subseteq A \times A$ and $R = R^T$

Where R is a 2D array.

Remember that R^T is the transpose of R.
The transpose of a matrix is the matrix found by interchanging its rows into columns or columns into rows.
In programming we look at this problem as a 2D array.

```
function isSymmetric(R)
    RT = transpose(R)
    return R == RT
end function
```

1.0.2.4 Antisymmetric

Let Relation R on A be antisymmetric if $\forall a, b \in A$ then $(a, b) \in R$ $(b, a) \in R$
Or in psuedocode:

```
function isAntisymmetric(R)
    bSymmetric = isSymmetric(R)
    if bSymmetric == True
        return False
    else
        return True
    end if
end function
```

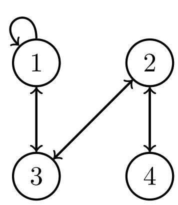
1.0.2.5 Transitive

A relation R on A is transitive if $\forall a, b, c \in A$ then $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Example: $R = (1,1), (1,3), (2,3), (2,4), (3,1), (3,2), (4,2)$
We can draw a matrix to show the relation R on A
To draw the matrix we plot the elements of A on the x-axis and y-axis.
Then we plot the pairs of R on the matrix as 1.
If the pair is not in R then we plot a 0.

	1	**2**	**3**	**4**
1	**1**	**0**	**1**	**0**
2	**0**	**0**	**1**	**1**
3	**1**	**1**	**0**	**0**
4	**0**	**1**	**0**	**0**

We can also draw a directional graph to show the relation R on A
We plot the elements of A as nodes.
We plot the pairs of R as directed edges.



For this problem we calculate the M^k where each element identifies the number of paths of length.
So for example M^2 is the number of paths of length 2.
$M^2 = M \cdot M$

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 1\cdot1+0\cdot0+1\cdot1+0\cdot0 & 1\cdot0+0\cdot0+1\cdot1+0\cdot1 & 1\cdot1+0\cdot1+1\cdot0+0\cdot0 & 1\cdot0+0\cdot1+1\cdot0+0\cdot0 \\ 0\cdot1+0\cdot0+1\cdot1+1\cdot1 & 0\cdot0+0\cdot0+1\cdot1+1\cdot1 & 0\cdot1+0\cdot1+1\cdot0+1\cdot0 & 0\cdot0+0\cdot1+1\cdot0+1\cdot0 \\ 1\cdot1+1\cdot0+0\cdot1+0\cdot0 & 1\cdot0+1\cdot0+0\cdot1+0\cdot1 & 1\cdot1+1\cdot1+0\cdot0+0\cdot0 & 1\cdot0+1\cdot1+0\cdot0+0\cdot0 \\ 0\cdot1+1\cdot0+0\cdot1+0\cdot0 & 0\cdot0+1\cdot0+0\cdot1+0\cdot1 & 0\cdot1+1\cdot1+0\cdot0+0\cdot0 & 0\cdot0+1\cdot1+0\cdot0+0\cdot0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

Looking at the matrix there must be at least one path of length 2 and length 1
Thus this is a transitive relation.

Looking at our psuedocode code.

```
def transitive(R):
    Rs = R**2
    bOnePass = False
    for i in range(len(Rs)):
        for j in range(len(Rs)):
            if Rs[i][j] == 1:
                bOnePass = True
    if bOnePass:
        bTwoPass = False
        for i in range(len(R)):
            for j in range(len(R)):
                if R[i][j] == 2:
                    bTwoPass = True
                    break
    if bOnePass and bTwoPass:
        return True
    else:
        return False
```

1.0.2.6 Equivalence relations

An **equivalence** relation is a relation that is **reflexive**, **symmetric**, and **transitive**.
An equivalence relation partitions its domain E into disjoint equivalence classes. Each equivalence class contains a set of elements of E that are equivalent to each other, and all elements of E equivalent to any element of the equivalence class are members of the equivalence class. The equivalence classes are disjoint: there is no x$\in E$ such that x is in more than one equivalence class. The equivalence classes exhaust E: there is no x$\in E$ such that x is in no equivalence class. Any element of an equivalence class may be its representative; the representative stands for all the members of its equivalence class.

1.0.2.7 Order relations

An order (or partial order) is a relation that is antisymmetric and transitive.

A **strict** order is one that is **irreflexive** and **transitive**; such an order is also trivially **antisymmetric** because there is no x and y such that xRy and yRx.

A **non-strict (weak)** order is one that is **reflexive**, **antisymmetric**, and **transitive**.

An order relation R on E is a **total order** if either xRy or yRx $\forall$x,y$\in E$.
An order relation R on E is a **partial order** if there is a x,y$\in E$ for which neither xRy nor yRx.
And order relation R on E is a **Triohotomy** if either xRy, yRx, or x=y $\forall$x,y$\in E$.
A **weak total order** is **reflexive**, **antisymmetric**, **transitive** and **trichotomy**.
A **strict total order** is **irreflexive**, **transitive**, and **trichotomy**.