DynaBase: Mathematical Background

Benjamin Hutz – Saint Louis University

1. Mathematical Background

Let S be a set with a self-map $f : S \to S$. For any integer $n \geq 1$ we can define the n-th iterate f^n recursively as

$$f^n = f \circ f^{n-1} \quad \text{where } f^0 \text{ is the identity map.}$$

In arithmetic dynamics, the set S is an algebraic object, such as the set of rational points of an algebraic variety and the map f is an algebraic map often defined by a tuple of polynomials.

Example 1. For the polynomial $f(z) = z^2 - 1$, there are three points with finite forward orbit (preperiodic) $\{0, -1, 2\}$. We can represent these preperiodic points as a directed graph.

$$\begin{tikzcd} 1 \arrow[r] & 0 \arrow[r, bend left] & -1 \arrow[l, bend left] \end{tikzcd}$$

The database will focus on self-maps of projective space defined by a tuple of homogeneous polynomials. There are two main reasons for this. The first is practicality. The known algorithms and their implementations in Sage are almost entirely for self-maps of projective space. The second is that many of the main conjectures in arithmetic dynamics can be reduced to the case of self-maps of projective space. These conjectures are already quite hard and beautiful in this case.

A self-map of projective space $f : \mathbb{P}^N \to \mathbb{P}^N$ can be represented by an $(N+1)$-tuple of homogeneous polynomials of the same degree d with no common factors. We denote the set of such maps as Rat_d^N. The subset where the tuple of polynomials also has no common zeros are the endomorphisms of $\mathbb{P}^N$ denoted Hom_d^N. For $N = 1$, rational maps and morphisms are the same and we have $\mathrm{Hom}_d^1 = \mathrm{Rat}_d^1$; but for $N \geq 2$, there are additional elements in Rat_d^N where the defining polynomials have common zeros. The elements of Hom_d^N are, in general, more well-behaved and many of the algorithms only apply to elements of Hom_d^N. We primarily focus on elements of Hom_d^N for the database.

The projective linear group PGL is the group of automorphisms of projective space, which can be represented by matrices up to scalar multiple. Acting on $\mathbb{P}^N$ by $\alpha \in \mathrm{PGL}_{N+1}$ forms a commutative diagram

$$\begin{tikzcd} \mathbb{P}^N \arrow[r, "f"] \arrow[d, "\alpha"'] & \mathbb{P}^N \arrow[d, "\alpha"] \\ \mathbb{P}^N \arrow[r, "f^\alpha"] & \mathbb{P}^N \end{tikzcd}$$

where

$$f^\alpha = \alpha^{-1} \circ f \circ \alpha.$$

Note that this conjugation action on Hom_d^N preserves dynamical properties since for any positive integer n

$$(f^\alpha)^n = (f^n)^\alpha.$$

The quotient by this action forms the *moduli space* $\mathcal{M}_d^N$ of degree d endomorphisms of projective space [Levy, 2011; Petsche et al., 2009; Silverman, 1998]. Thus, it is possible to have two different maps $f, g : \mathbb{P}^N \to \mathbb{P}^N$ that have different representations as tuples of polynomials which are dynamically the same. **These maps have one entry in the database.** The representatives f and g are called *models* in the database. Much effort is spent computing good models for the database.

2. DEFINITIONS

Field of Definition or Base Field: The field of definition of $f : \mathbb{P}^N \to \mathbb{P}^N$ is the field that contains all the coefficients of the representation as a pair of homogeneous polynomials. For number fields, we utilize the LMFDB number field database and labelling system.

Resultant: A function $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree d given by the tuple of polynomials $(f_0, \dots, f_N)$, we can compute its *resultant* as the MacCaulay resultant of the representing polynomials. The prime factors of the resultant are the primes of bad reduction.

Representation of Functions: A function $f : \mathbb{P}^N \to \mathbb{P}^N$ is given by a tuple $(f_0, \dots, f_N)$ of homogeneous polynomials of the same degree with no common factors. However, since conjugation by $\alpha \in \mathrm{PGL}_{N+1}$ fixes (most) dynamical properties, we need to choose what representation of the function to store in the database. Since conjugation can affect the field of definition of the representation, properties such as rational preperiodic points that depend intrinsically on the field of definition may be affected. For $f : \mathbb{P}^1 \to \mathbb{P}^1$, the following representations are both interesting and currently computable.

Monic-Centered Representation: The map f is *polynomial* if it has a totally ramified fixed point. Conjugating so that the totally ramified fixed point is the point at infinity, the leading term is monic, and second order term has coefficient zero, is called *monic centered form*. This is the standard form for a polynomial map, however, moving the totally ramified fixed point via conjugation may require an extension of the field of definition.

Reduced Representation: Hutz-Stoll developed an effective algorithm that determines the representation of a map which has both smallest resultant and smallest coefficients. This representation is unique to the conjugacy class, [Hutz and Stoll, 2019].

Newton representation: The map f is in *Newton form* if there is a polynomial F so that the dehomogenization of f can be written as

$$\tilde{f}(z) = z - \frac{F(z)}{F'(z)}$$

where $F(z)$ is a single variable polynomial.

Chebyshev representation: We say f is Chebyshev if f is conjugate to a Chebyshev polynomial.

If a map can be be put in more than one of these forms, then the elements $\alpha \in \mathrm{PGL}_2$ that conjugate between each of these forms will be included in the data.

Multiplier Invariants: To each fixed point Q of f, we can assign a value called the *multiplier* as

$$\lambda_Q := \tilde{f}'(Q)$$

where $\tilde{f}$ is a dehomogenization of f and $'$ represents the derivative. The coefficients of the polynomial

$$\tau(t) = \prod_{Q \in \mathrm{Fix}(f)} (t - \lambda_Q)$$

are (up to sign change) the elementary symmetric functions evaluated at the multipliers of the fixed points. Label them

$$\boldsymbol{\sigma}^{(1)} = (\sigma_1^{(1)}, \dots, \sigma_{d+1}^{(1)}).$$

These symmetric functions on the multipliers are invariant under conjugation [Hutz, 2020]. Similarly for $n > 1$, we can utilize the fixed points of the iterate f^n to define $\boldsymbol{\sigma}^{(n)}$ which are also invariant under conjugation. Note that even though λ_Q is potentially defined over an extension field, every multiplier invariant $\sigma_i^{(n)}$ is defined over the field of definition of f. They can be computed via a resultant calculation so can be computed for families of dynamical systems as functions of the parameters. Furthermore, every conjugacy class in the moduli space is determined up to finitely many choices by the set $\{\boldsymbol{\sigma}^{(i)} : 1 \le i \le n\}$ for $n = 3$ large enough [McMullen, 1987; **?**].

Label: Every conjugacy class in the database is assigned a unique label. This label is of the form

$$dim.sig1.sig2.sig3.ord$$

These quantities are

- ‘dim‘ an integer representing the dimension of the domain/codomain
- sig1, sig2, sig3 - The shake_256 hash of $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$, and $\boldsymbol{\sigma}^{(3)}$ of length 8
- ‘ord‘ - since the multiplier invariants only determine the conjugacy class up to finitely many choices, this is an integer used to distinguish the distinct conjugacy classes.

Postcritically Finite: A map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is *postcritically finite* if all of its critical points are preperiodic. The directed graph representing the orbits of the critical points is called the *critical point portrait*.

A similar definition can be made for $\mathbb{P}^N$ in regards to the preperiodicity of the irreducible components of the critical locus.

Rational Preperiodic Points: The rational perperiodic points are the points with finite forward orbit defined over the given field. Such a point has period (m, n) where m is the strictly preperiodic portion, called the *tail* and n is the length of the periodic cycle.

For any given map $f : \mathbb{P}^N \to \mathbb{P}^N$, the set of rational preperiodic points is finite by Northcott's Theorem and can be described as a directed graph.

Automorphism Group: For a given $f : \mathbb{P}^N \to \mathbb{P}^N$, the $\alpha \in \mathrm{PGL}_{N+1}$ that fix f under conjugation are called *automorphisms of* f. These automorphisms form a group

$$\mathrm{Aut}(f) = \{\alpha \in \mathrm{PGL}_{N_1} : f^{\alpha} = f\}.$$

Since the points of a given period are invariant under an automorphism, $\mathrm{Aut}(f)$ is finite when f is a morphism. Faber, Manes, and Viray developed an algorithm to compute the automorphism group for endomorphisms $\mathbb{P}^1$ [Faber et al., 2014] efficieintly. Less efficient methods exists for $\mathbb{P}^N$.

References

Faber, X., Manes, M., and Viray, B. (2014). Computing conjugating sets and automorphism groups of rational functions. *Journal of Algebra*, 423:1161–1190.

Hutz, B. (2020). Multipliers and invariants of endomorphisms of projective space in dimension greater than 1. *Journal de Théorie des Nombres de Bordeaux*, 32(2):439–469.

Hutz, B. and Stoll, M. (2019). Smallest representatives of sl(2,z)-orbits of binary forms and endomorphisms of p1. *Acta Arithmetica*, 189:283–308.

Levy, A. (2011). The space of morphisms on projective space. *Acta Arithmetica*, 146:12–31.

McMullen, C. (1987). Families of rational maps and iterative root-finding algorithms. *Ann. of Math.*, 125(3):467–493.

Petsche, C., Szpiro, L., and Tepper, M. (2009). Isotriviality is equivalent to potential good reduction for endomorphisms of $\mathbb{P}^N$ over function fields. *Journal of Algebra*, 322(9):3345–3365.

Silverman, J. H. (1998). The space of rational maps on P1. *Duke Math. J.*, 94:41–118.