MSCA 32010
Linear Algebra And Matrix Analysis

Session 2

Shaddy Abado, Ph.D.

QUICK REVIEW OF SESSION 1

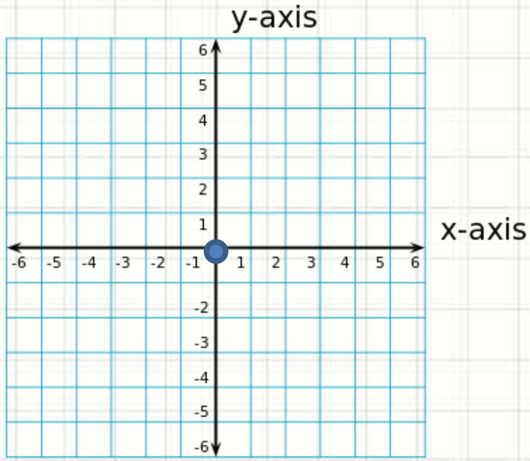
Vector Addition/Subtraction

$$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}1+4\\2+4\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

$$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}1-2\\2-2\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

Vector Addition/Subtraction

$$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \vec{0}$$



Null Vector
Or
Zero vector

Scalar Multiplication

$$3 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 \\ 2+2+2 \end{bmatrix} = \begin{bmatrix} 3*1 \\ 3*2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$3 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3*1 \\ 3*2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Core Idea - <u>Linear combination</u>

$$-0.5 * \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2 * \begin{bmatrix} 1 \\ 5 \end{bmatrix} =$$

$$= \begin{bmatrix} -0.5 * 2 \\ -0.5 * 4 \end{bmatrix} + \begin{bmatrix} 2 * 1 \\ 2 * 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

Core Idea – Independency and Dependency

$$a\vec{v} + b\vec{u}$$

- Vectors v and u are **Independent** if no combination except $0\vec{v} + 0\vec{u}$ gives $\vec{0}$
- Vectors v and u are **Dependent** if there is a combination $a\vec{v} + b\vec{u}$ that gives $\vec{0}$

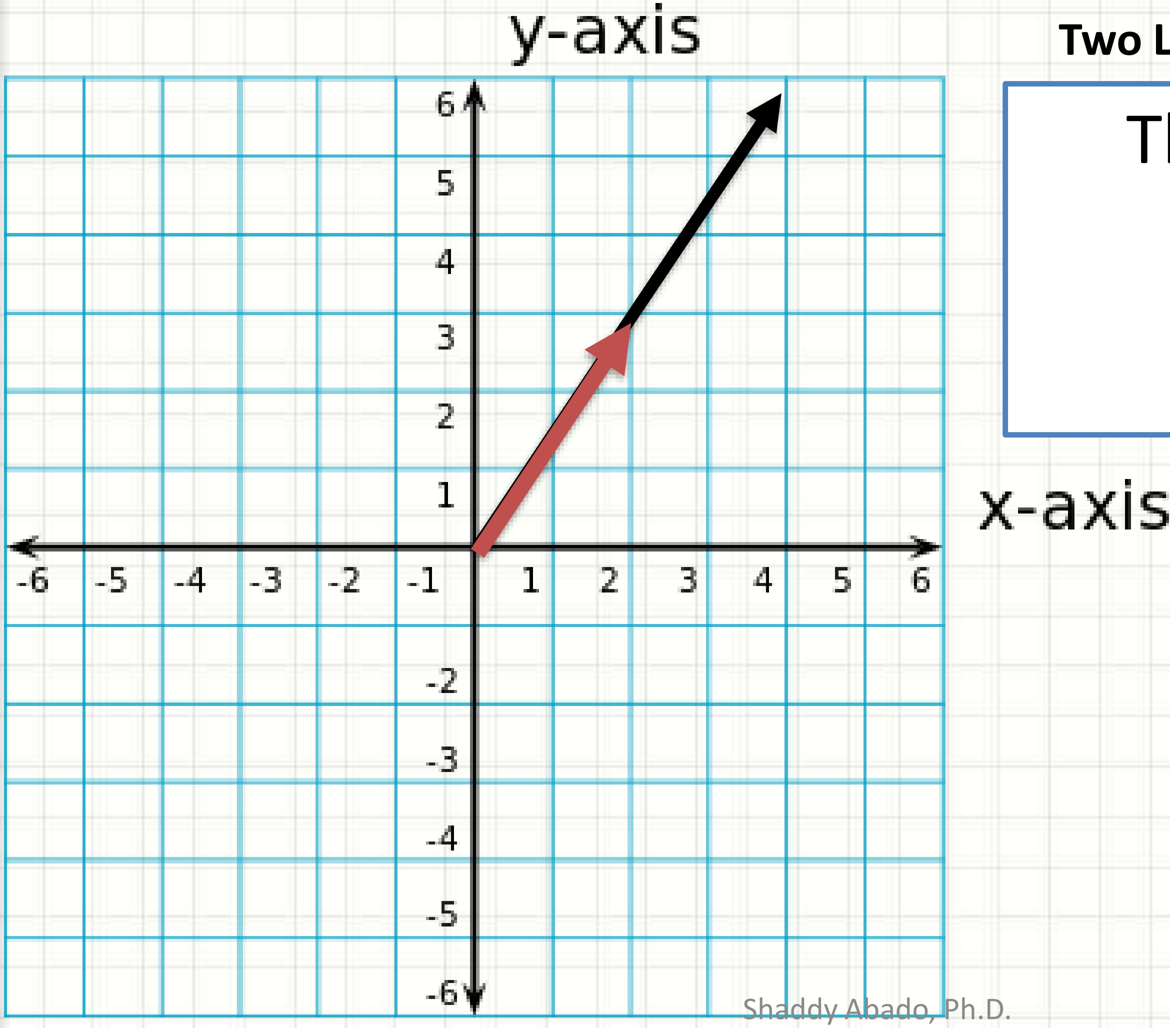
$$\begin{bmatrix}1\\0\end{bmatrix} \ and \begin{bmatrix}0\\1\end{bmatrix}$$

Independent

$$\begin{bmatrix}1\\1\end{bmatrix} \ and \begin{bmatrix}2\\2\end{bmatrix}$$

Dependent

Linear combination (2D case)



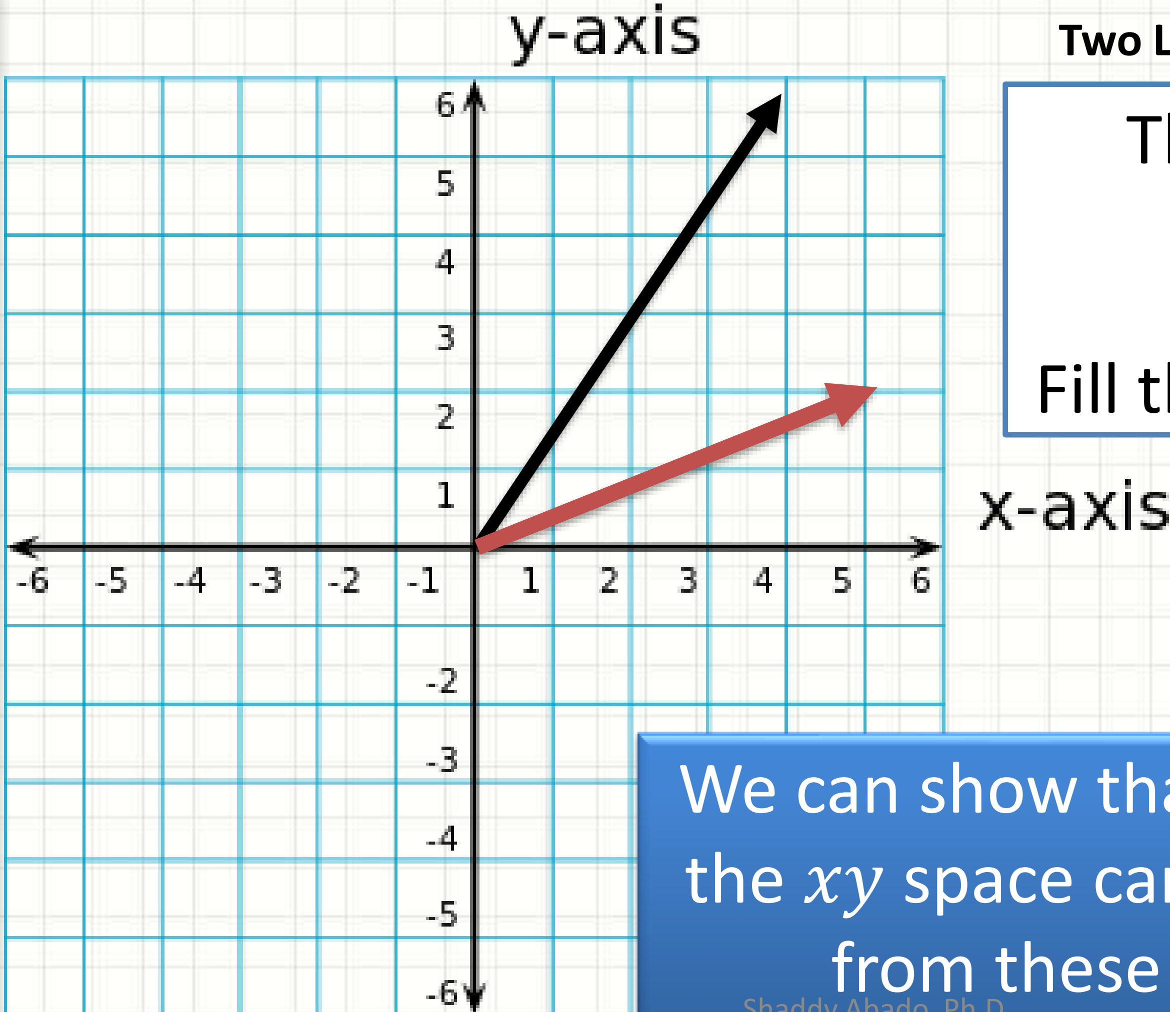
Two Linearly Dependent Vectors

The combination

$$a\begin{bmatrix}4\\6\end{bmatrix} + b\begin{bmatrix}\color{red}{2}\\\color{red}{3}\end{bmatrix}$$

Line

Linear combination (2D case)



Two Linearly Independent Vectors

The combination

$$a\begin{bmatrix}4\\6\end{bmatrix}+b\begin{bmatrix}5\\2\end{bmatrix}$$

Fill the whole 2D plane

We can show that every vector in the xy space can be regenerated from these two vectors

L_2 $Norm$

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Definition

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$$

Example

$$\left\|\begin{matrix} 2 \\ 3 \end{matrix}\right\| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$\left\|\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

Norm Property

The *norm* $||v||$ of a vector $v \in S$ is a real number that satisfies the following properties:

- $||v|| \geq 0$,
- $||v|| = 0$ if and only if $v = 0$,
- $||av|| = |a|\,||v||$, a ∈ R1, and
- $||v + w|| \leq ||v|| + ||w||$, (triangle or Minkowski inequality).

Unit Vector

Definition:

A unit vector is a vector whose length equals one

$$\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\| \quad \left\|\begin{bmatrix}0\\1\end{bmatrix}\right\| \quad \left\|\begin{bmatrix}1/\sqrt{3}\\\sqrt{2}/\sqrt{3}\end{bmatrix}\right\|, \dots$$

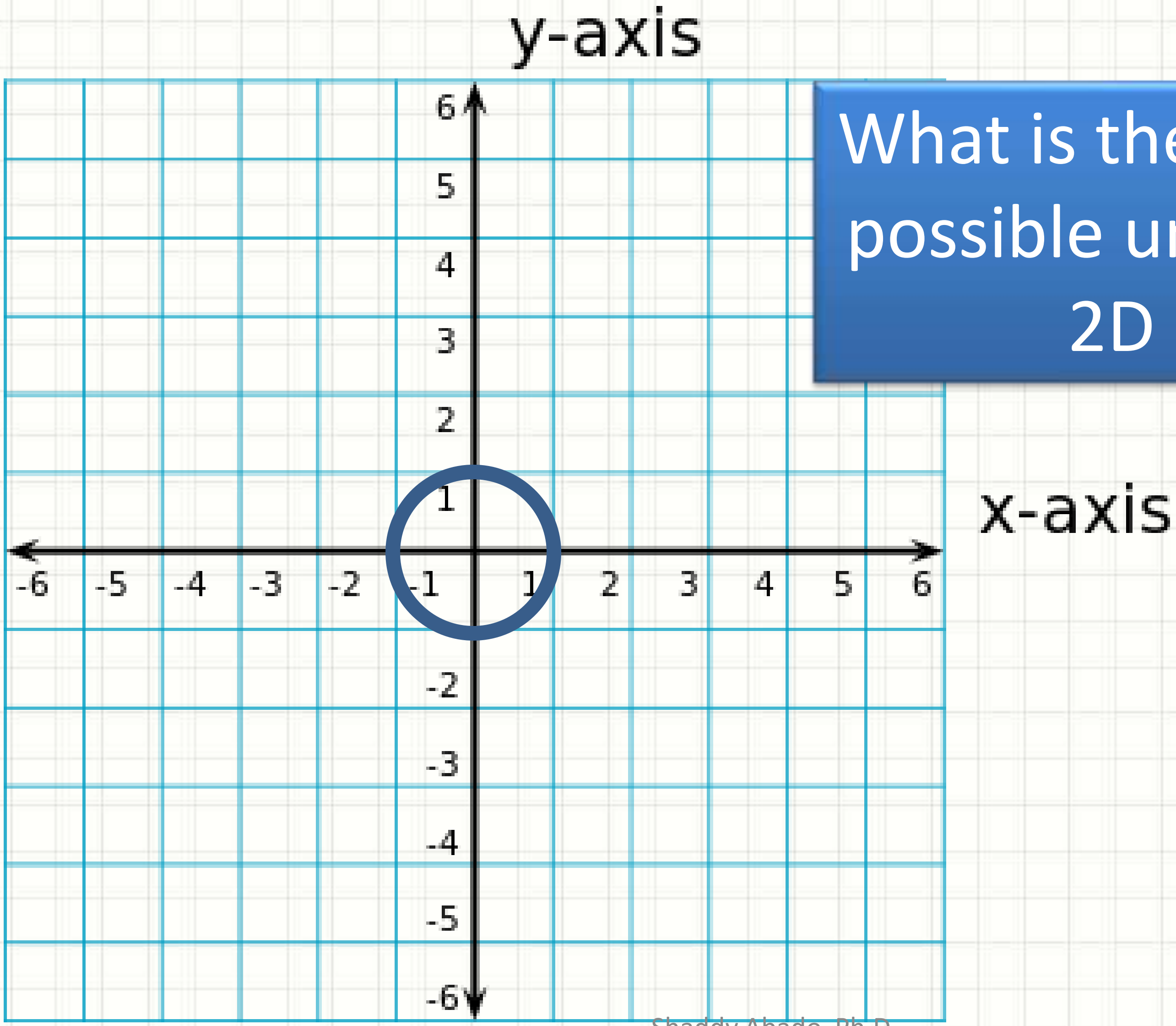
$$= 1$$

$$\|u\| = \sqrt{0^2 + 1^2} = \sqrt{0+1} = \sqrt{1} = 1$$

$$\|u\| = \sqrt{(1/\sqrt{3})^2 + (\sqrt{2}/\sqrt{3})^2} = \sqrt{1/3 + 2/3} = \sqrt{1} = 1$$

*

Unit Vectors



What is the shape of all possible unit vectors in 2D ? 3D?

Scalar product (dot product, inner product)

It is called 'scalar product' because the output is a scaler

$$y = \vec{v} \cdot \vec{w} = \sum_{i=1}^{N} v_i w_i$$

$$= v_1 w_1 + v_2 w_2 + \cdots + v_N w_N$$

Order makes no difference
$\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{w}$

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

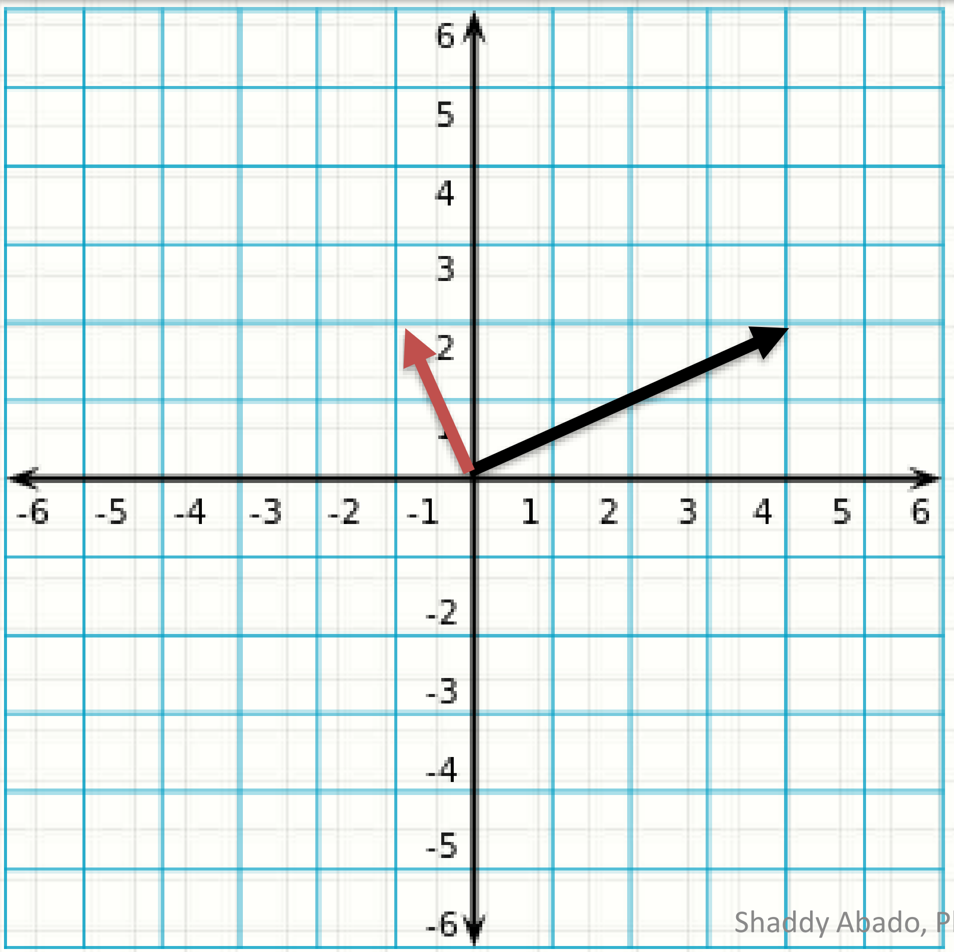
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$= 1 * 7 + 2 * 3 = 13$$

Perpendicular/Orthogonal Vectors

v **and** ***u*** **are said to be orthogonal to each other if**

$$v \cdot u = 0$$



$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$= 4 * -1 + 2 * 2$$
$$= 0$$

x-axis

Right angle (90°) between vectors

Dot product and Norm

The length (i.e., $||v||$) of a vector *v* is the square root of the dor product $\vec{v} \cdot \vec{v}$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + \cdots + v_N^2}$$

Example

$$\left\| \begin{matrix} 2 \\ 3 \end{matrix} \right\| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

Cosine Formula

$$\frac{\boldsymbol{v} \cdot \boldsymbol{u}}{\|\boldsymbol{v}\|\|\boldsymbol{u}\|} = \cos \boldsymbol{\theta}$$

$$\underbrace{\boldsymbol{v} \cdot \boldsymbol{u}}_{\textbf{Algebraic}} = \underbrace{\|\boldsymbol{v}\|\|\boldsymbol{u}\| \cos \boldsymbol{\theta}}_{\textbf{Geometric}}$$

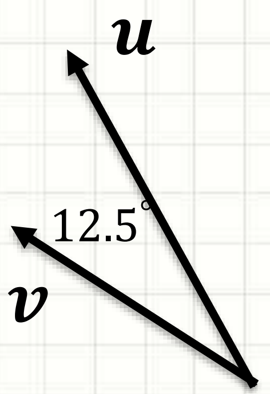
The significance of this property is that the left-hand side is purely algebraic and the right-hand side is purely geometric.

Cosine Formula - Example

$$\frac{\boldsymbol{v} \cdot \boldsymbol{u}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \operatorname{Cos}\boldsymbol{\theta}$$

$$\boldsymbol{v} \cdot \boldsymbol{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -1 * -1 + 2 * 4 = 9$$

$$\|\boldsymbol{v}\| = \sqrt{1+4} = \sqrt{5} \qquad \|\boldsymbol{u}\| = \sqrt{1+16} = \sqrt{17}$$



$$\frac{9}{\sqrt{17}\sqrt{5}} = \operatorname{Cos}\boldsymbol{\theta}$$

$$\operatorname{Cos}\boldsymbol{\theta} = 0.97$$

$$\boldsymbol{\theta} \sim 0.2\ Rad\ ; 12.5^{\circ}$$

Check

Session 2 – matrix algebra

Some of the slides are based on "LECTURE NOTES ON MATHEMATICAL METHODS" By Mihir Sen and Joseph M. Powers

Session 2 - Outline

- Brief overview of previous session

- Properties of Matrix
- Matrix Multiplication
  - Matrix – Scalar Multiplication
  - Matrix – Vector Multiplication
  - Matrix – Matrix Multiplication
- Definitions and properties:
  - Transpose, Identity, Diagonal, Symmetry, Anti-symmetry and Asymmetry, Triangular, Permutation, Inverse, and Orthogonal

Session 2 - Motivation

- Our ability to analyze dataset and solve equations depends on performing algebraic operations with matrices.
- Matrices are an efficient way to store information and a powerful tool for calculations involving linear transformations.
- Basic understanding of how to manipulate matrices is needed.

Keep in mind …

Matrices are the result of organizing information related to linear functions.

We are not studying matrices but rather linear functions; those linear functions can be represented as matrices under certain notational conventions.

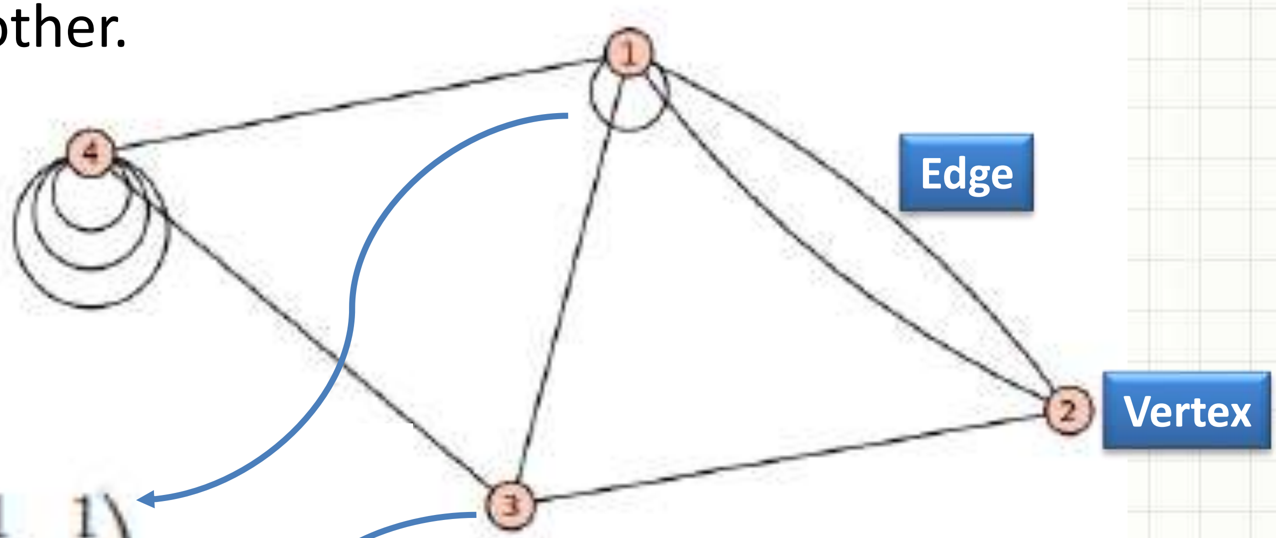
Example I - .GIF

- In computer graphics, .gif (Graphics Interchange Format) extension image files are actually just matrices: at the start of the file the size of the matrix is given, after which each number is a matrix entry indicating the color of a particular pixel in the image.

- This matrix then has its rows shuffled a bit: by listing, say, every eighth row, a web browser downloading the image can start displaying an incomplete version of the picture before the download is complete

Example II – Graph Theory

- In graph theory , a graph is a collection of vertices and some edges connecting vertices. Graphs occur in many applications, ranging from telephone networks to airline routes.
- A matrix can be used to indicate how many edges attach one vertex to another.



$$M = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

Matrix - Definition

3. MATHEMATICS

a rectangular array of quantities or expressions in rows and columns that is treated as a single entity and manipulated according to particular rules.

- an organizational structure in which two or more lines of command, responsibility, or communication may run through the same individual.

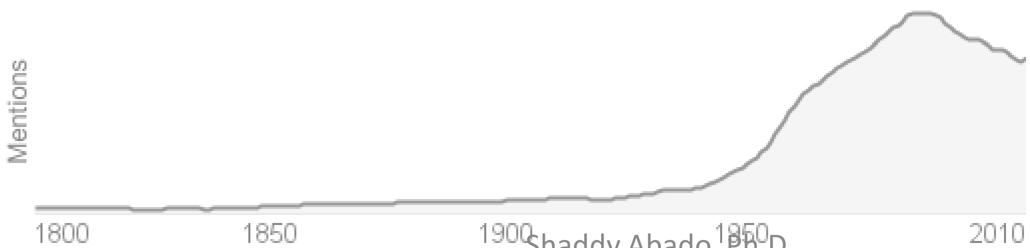
Origin

LATIN	LATIN	
mater matr- mother	→ matrix breeding female	→ matrix womb *late Middle English*

late Middle English (in the sense 'womb'): from Latin, 'breeding female,' later 'womb,' from *mater*, *matr-* 'mother.'

Translate matrix to Choose language

Use over time for: matrix



Matrix – Basic Definitions

Scalars → Vectors → Matrices

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$\boldsymbol{A} = [v \quad w \quad u] = \begin{bmatrix} v_1 & w_1 & u_1 \\ \vdots & \vdots & \vdots \\ v_n & w_n & u_n \end{bmatrix}$$

The columns of $\boldsymbol{A}$ are vectors in $\mathbb{R}^n$

Matrix – Basic Definitions

We will denote a matrix of size $N \times M$ as

$$\overbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M-1} & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M-1} & a_{2M} \\ & & \vdots & & \\ a_{N-1\,1} & a_{N-1\,2} & \cdots & a_{N-1M-1} & a_{N-1M} \\ a_{N1} & a_{N2} & \cdots & a_{NM-1} & a_{NM} \end{bmatrix}}^{M \text{ Columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \\ a \\ a \end{bmatrix}}\right\} \begin{matrix} N \\ \text{Rows} \end{matrix}$$

a_{ij} i – row
j - column

Matrix Element (Scalars)

Matrix – Example

We will denote a matrix of size 3×3 as

$$A_{3X3} = \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}^{\text{3 Columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \end{bmatrix}}\right\} \text{3 rows}$$

a_{ij} *i* – row, *j* - column

$$A_{3X3} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 5 \\ 7 & 9 & 0 \end{bmatrix}$$

Matrix – Basic Definitions

A vectors is a special types of matrix

- Will refer to an $nx1$ matrix as an n-dimensional column vector.
- Will refer to an $1xm$ matrix as an m-dimensional row vector.

$$v_{nx1} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad v_{1xm} = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}$$

Unless otherwise stated vectors are assumed to be column vectors.

Matrix – Basic Definitions

Main diagonal

$$A_{3X3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Square Matrix

$n = m$

$$A_{2X3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Rectangular Matrix

$n \neq m$
$n > m$
Or
$n < m$

Matrix - Examples

$$\boldsymbol{A}_{?X?} = \begin{bmatrix} 1 & 3 & 6 \\ 5 & -1 & 10 \\ 7 & 6 & 99 \end{bmatrix}$$

$\boldsymbol{a_{32}} = ?$

$\boldsymbol{Diagonal\ Elements}?$

$$\boldsymbol{B}_{?X?} = \begin{bmatrix} 2 & 4 & 2 & 3 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

$$\boldsymbol{C}_{?X?} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

Matrix Addition / Subtraction

Addition of matrices can be defined as

$$C_{nXm} = A_{nXm} + B_{nXm}$$

where the elements of C are obtained by adding the corresponding elements of $\boldsymbol{A}$ and $\boldsymbol{B}$.

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$\begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} & b_{13} + a_{13} \\ b_{21} + a_{21} & b_{22} + a_{22} & b_{23} + a_{23} \\ b_{31} + a_{31} & b_{32} + a_{32} & b_{33} + a_{33} \end{bmatrix}$$

- What about Subtraction?

Matrix Addition (Example)

$$\begin{bmatrix} 1 & 3 & -2 \\ -\frac{1}{3} & 6 & -1 \\ 2 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = ?$$

$$\begin{bmatrix} 1+3 & 3+3 & -2+3 \\ -\frac{1}{3}+3 & 6+3 & -1+3 \\ 2+3 & 2+3 & 5+3 \end{bmatrix} =$$

$$\begin{bmatrix} 4 & 6 & 1 \\ 2\frac{2}{3} & 9 & 2 \\ 5 & 5 & 8 \end{bmatrix}$$

Matrix subtraction (Example)

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -\frac{1}{3} & 6 & -1 \\ 2 & 2 & 5 \end{bmatrix} = ?$$

$$\begin{bmatrix} 3-1 & 3-3 & 3-(-2) \\ 3-(-\frac{1}{3}) & 3-6 & 3-(-1) \\ 3-2 & 3-2 & 3-5 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 & 5 \\ 3\frac{1}{3} & -3 & 4 \\ 1 & 1 & -2 \end{bmatrix}$$

Matrix Addition / Subtraction

What about?

$$\begin{bmatrix} 1 & 3 & -2 \\ -\frac{1}{3} & 6 & -1 \\ 2 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 5 \end{bmatrix} = ?$$

Null Matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ -\frac{1}{3} & 6 & -1 \\ 2 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -\frac{1}{3} & 6 & -1 \\ 2 & 2 & 5 \end{bmatrix} =?$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

Null Matrix

Matrix – Data Example

Let's assume we have different samples from a dataset with two attributes:

1) Number of cars per household

and

2) Number of cellphones per households

We also have a neighborhood with 3 houses

A household vector:

$$v_1 = \begin{bmatrix} \#Cars \\ \#Cells \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

A neighborhood matrix:

$$A_{2X3} = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Matrix Addition – Data Example

“Difference in a neighborhood’s content over the course of two years”

$$A_{2X3} - B_{2X3} =$$

$$\begin{array}{c} \begin{bmatrix} 3\ cars & 2\ cars & 3\ cars \\ 2\ Cells & 2\ Cells & 2\ Cells \end{bmatrix} \\ \begin{array}{ccc} \text{Household 1} & \text{Household 2} & \text{Household 3} \end{array} \end{array} - \begin{array}{c} \begin{bmatrix} 0\ cars & 1\ cars & 2\ cars \\ 2\ Cells & 1\ Cells & 1\ Cells \end{bmatrix} \\ \begin{array}{ccc} \text{Household 1} & \text{Household 2} & \text{Household 3} \end{array} \end{array}$$

Year # 2

Year # 1

$$\begin{array}{c} \begin{bmatrix} 3\ cars & 1\ cars & 1\ cars \\ 0 Cells & 0\ Cells & 1\ Cells \end{bmatrix} \\ \begin{array}{ccc} \text{Household 1} & \text{Household 2} & \text{Household 3} \end{array} \end{array}$$

Matrix Multiplication

Given matrix $\boldsymbol{A}$ we can multiply it by a:

- **Scalar**
- **Vector**
- **Matrix**

Matrix – Scalar Multiplication

Multiplication of a matrix A by a scalar b can be defined as

$$b\boldsymbol{A}_{nXm} = \boldsymbol{C}_{nXm}$$

where the elements of $\boldsymbol{C}$ are the corresponding elements of $\boldsymbol{A}$ multiplied by b.

Matrix

Scalar

$$b * \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$\begin{bmatrix} b * a_{11} & b * a_{12} & b * a_{13} \\ b * a_{21} & b * a_{22} & b * a_{23} \\ b * a_{31} & b * a_{32} & b * a_{33} \end{bmatrix}$$

Matrix – Scalar Multiplication (Example)

$$3 * \begin{bmatrix} 1 & 3 & -2 \\ -\frac{1}{3} & 6 & -1 \\ 2 & 2 & 5 \end{bmatrix} = ?$$

$$\begin{bmatrix} 3 * 1 & 3 * 3 & 3 * -2 \\ 3 * -\frac{1}{3} & 3 * 6 & 3 * -1 \\ 3 * 2 & 3 * 2 & 3 * 5 \end{bmatrix} = \begin{bmatrix} 3 & 9 & -6 \\ -1 & 18 & -3 \\ 6 & 6 & 15 \end{bmatrix}$$

Matrix – Scalar Multiplication (Data Example)

"Increasing the contain of each household in a neighborhood by three folds"

$$3 * \begin{bmatrix} 3\ cars & 2\ cars & 3\ cars \\ 2\ Cells & 2\ Cells & 1\ Cells \end{bmatrix}$$

Household 1 **Household 2** **Household 3**

$$\begin{bmatrix} 9\ cars & 6\ cars & 9\ cars \\ 6\ Cells & 6\ Cells & 3\ Cells \end{bmatrix}$$

Household 1 **Household 2** **Household 3**

Properties of Matrices –
Addition and Scalar multiplication

$$A + B = B + A$$

Commutative

$$\mathrm{A} + (B + C) = (A + B) + C$$

Associative

$$(A + \mathbf{0}) = A$$

$$a(A + B) = aA + aB$$

Distributive

$$(a + b)A = aA + bA$$

$$a(bA) = (ab)A$$

Associative
+
Distributive

Now you try…

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 2 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} =?$$

$$\begin{bmatrix} 1+1 & 1+0 & 0+3 \\ 2+0 & 1+2 & -1+0 \\ 2+3 & 3+1 & 1+4 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & -1 \\ 5 & 4 & 5 \end{bmatrix}$$

Now you try…

$$2 * \begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{2} & 6 & 4 \end{bmatrix} =?$$

$$\begin{bmatrix} 2 * 1 & 2 * 0 & 2 * -1 \\ 2 * \frac{1}{2} & 2 * 6 & 2 * 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 12 & 8 \end{bmatrix}$$

Now you try…

Matrix addition/subtraction and Scalar Multiplication

Calculate

$$2 * \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix} + 3 * \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -2 \\ 2 & 6 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 3 & 0 \\ 0 & 6 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 12 & 13 \end{bmatrix}$$

BREAK

Mathematical Notations - Summation

Capital Sigma $\sum$

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

***i* - Index of summation**

Examples

$$\sum_{i=3}^{6} i^2 = 3^2 + 4^2 + 5^2 + 6^2 = 86.$$

Recall and Example: Dot Product

$$y = \vec{v} \cdot \vec{w} = \sum_{i=1}^{N} v_i w_i$$

$$= v_1 w_1 + v_2 w_2 + \cdots + v_N\, w_N$$

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\sum_{i=1}^{2} v_i w_i = v_1 w_1 + v_2 w_2$$

$$= 1 * 7 + 2 * 3 = 13$$

Now you try …

$$y_1 = \sum_{j=1}^{n} a_{1j} v_j = ?$$

$$y_1 = \sum_{j=1}^{n} a_{1j} v_j = a_{11} v_1 + a_{12} v_2 + \cdots + a_{1n} v_n$$

Matrix – Vector Multiplication

Matrix – vector Multiplication

Multiplication of Matrix $\boldsymbol{A}$ and vector $\vec{\boldsymbol{v}}$ can be defined only if they are of the proper sizes.

$$y_{mx1} = \boldsymbol{A}_{mxn} \cdot v_{nx1}$$

Notice that # of columns of A = # of rows of v

Matrix – Vector Multiplication

$$y_{mx1} = A_{mxn} \cdot v_{nx1}$$

Notice that # of columns of A = # of rows of v

The general formula for a matrix-vector product is

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

The vector y is a linear combination of the columns of A.

Matrix – Vector Multiplication (Example)

$$\boldsymbol{A}_{2X2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad v_{2X1} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \longrightarrow \text{“Row Weights”}$$

First, multiply Row 1 of the matrix by Column 1 of the vector.

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 1 * 5 + 2 * 6 = 17$$

Next, multiply Row 2 of the matrix by Column 1 of the vector.

$$\begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 3 * 5 + 4 * 6 = 39$$

Finally, write the matrix-vector product.

$$\boldsymbol{A} \cdot v = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

Matrix – Vector Multiplication (Example)

$$A_{2X3} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 4 \end{bmatrix} \qquad v_{3X1} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

“Weights”

What is the expected dimension of the output vector?

$$A_{2X3} \cdot v_{3X1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 4 \end{bmatrix}_{2X3} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}_{3X1}$$

$$= \begin{bmatrix} 1 * 2 + 2 * (-2) + (-1) * 1 \\ 2 * 2 + 0 * (-2) + 4 * 1 \end{bmatrix}_{2X1}$$

$$= \begin{bmatrix} -3 \\ 8 \end{bmatrix}_{2X1}$$

Matrix – Vector Multiplication

Let $\boldsymbol{A}$ be a matrix and v a column vector.

If $A \cdot v = \vec{0}$ then vector v is orthogonal to the rows of $\boldsymbol{A}$.

$$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Matrix – Vector Multiplication Text Mining Example

Word – Document Matrix

	Doc 1	Doc 2	Doc 3
Word 1	3	0	0
Word 2	2	0	0
Word 3	0	2	0
Word 4	1	0	1

- The documents are represented by a vector in $\mathbb{R}^3$
- In reality, we may have thousands of documents and words

Matrix – Vector Multiplication Text Mining Example

$$w = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Represents a document dealing primarily with document 1 and secondarily with document 3.

$$\begin{array}{cc} & \text{Document} \\ \text{Words} & \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 3 \end{bmatrix} \; \text{New Document}$$

This is a linear combination of the columns of the Word – Document Matrix.

Matrix – Matrix Multiplication

Matrix – Matrix Multiplication

Multiplication of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ can be defined only if they are of the proper sizes.

$$Y_{mxn} = \boldsymbol{A}_{mxk} \cdot \boldsymbol{B}_{kxn}$$

Matrix – Matrix Multiplication

Given matrix $\boldsymbol{A}_{mxk}$, and matrix $\boldsymbol{B}_{kxn}$ with columns $[b_1, b_2, \dots b_n]$, then the product $\boldsymbol{AB}$ is the mxn matrix whose columns are $\boldsymbol{A}b_1, \boldsymbol{A}b_2, \dots \boldsymbol{A}b_n$

$$\boldsymbol{AB} = \boldsymbol{A}[b_1, b_2, \dots b_n] = [\boldsymbol{A}b_1, \boldsymbol{A}b_2, \dots \boldsymbol{A}b_n]$$

Each column in $\boldsymbol{AB}$ is a linear combination of the columns of $\boldsymbol{A}$ using weights from the corresponding columns of $\boldsymbol{B}$

Matrix – Matrix Multiplication

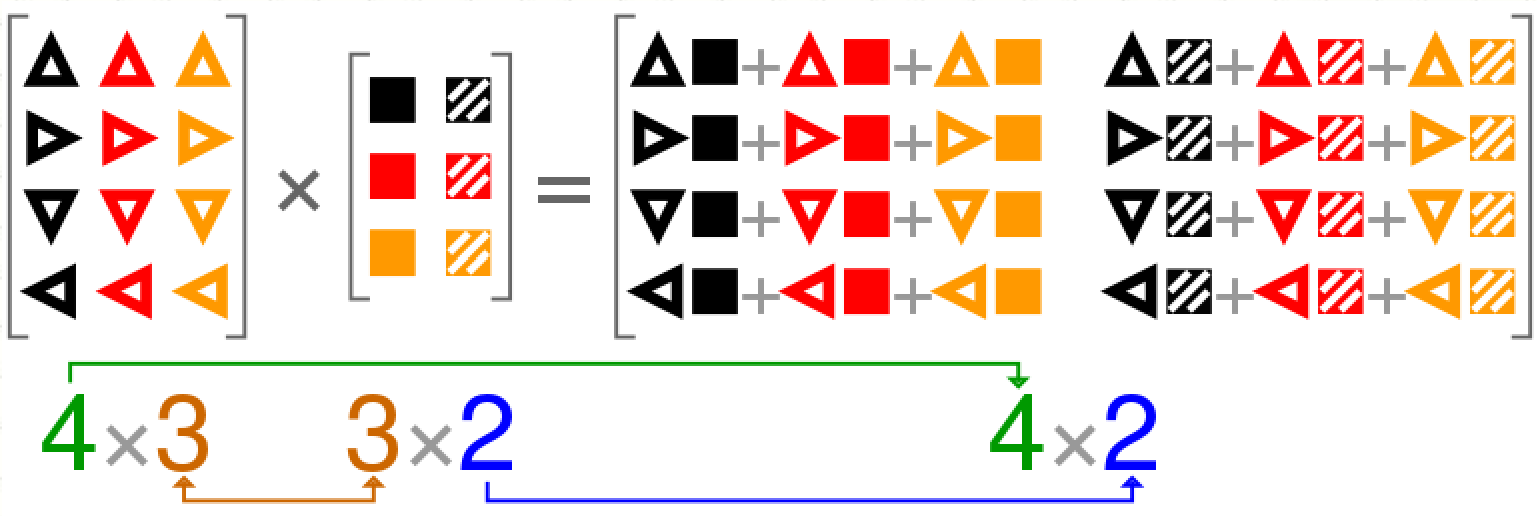
The product element in row i and column j (i.e., a_{ij}) is the sum of the products of corresponding elements from row i of $\boldsymbol{A}$ and column j of $\boldsymbol{B}$

Example for multiplying matrix $\boldsymbol{A}$ which contains 2 rows and matrix $\boldsymbol{B}$ which contains 3 columns

$$\begin{bmatrix} Row1 \\ Row2 \end{bmatrix} \begin{bmatrix} Col1 & Col2 & Col3 \end{bmatrix}$$
$$= \begin{bmatrix} Row1\ X\ Col1 & Row1\ X\ Col2 & Row1\ X\ Col3 \\ Row2\ X\ Col1 & Row2\ X\ Col2 & Row2\ X\ Col3 \end{bmatrix}$$

Matrix – Matrix Multiplication

The product element in row i and column j (i.e., a_{ij}) is the sum of the products of corresponding elements from row i of $\boldsymbol{A}$ and column j of $\boldsymbol{B}$



Source: https://en.wikipedia.org/wiki/File:Matrix_multiplication_principle.svg

Matrix – Matrix Multiplication (Example)

$$\boldsymbol{AB} = \boldsymbol{A}[b_1, b_2, \dots b_n] = [\boldsymbol{A}b_1, \boldsymbol{A}b_2, \dots \boldsymbol{A}b_n]$$

$$\boldsymbol{A}_{2X2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \qquad \boldsymbol{B}_{2X2} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$$

What is the expected dimension ?

$$\boldsymbol{A}_{2X2} \cdot \boldsymbol{B}_{2X2}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 * (-1) + 2 * 5 \ , & 1 * 0 + 2 * 1 \\ 4 * (-1) + 3 * 5 \ , & 4 * 0 + 3 * 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 \\ 11 & 3 \end{bmatrix}$$

Matrix – Matrix Multiplication (Example)

The product element in row i and column j (i.e., a_{ij}) is the sum of the products of corresponding elements from row i of $\boldsymbol{A}$ and column j of $\boldsymbol{B}$

$$A_{2x2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \qquad B_{2x2} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$A_{2x2} \cdot B_{2x2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$$

Row 2 → $i = 2$
Column 1 → $j = 1$

$$= \begin{bmatrix} 9 & 2 \\ 11 & 3 \end{bmatrix}$$

$$4 * -1 + 3 * 5 = 11$$

Matrix – Matrix Multiplication (Example)

$$A_{2X3} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \qquad B_{3X2} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

What is the expected Dimension ?

$$\begin{bmatrix} Row1 \\ Row2 \end{bmatrix} \begin{bmatrix} Col1 & Col2 \end{bmatrix} = \begin{bmatrix} Row1\ X\ Col1 & Row1\ X\ Col2 \\ Row2\ X\ Col1 & Row2\ X\ Col2 \end{bmatrix}$$

$$A_{2X3} \cdot B_{3X2}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 * 2 + 2 * 1 + 3 * 0 & 1 * 4 + 2 * (-1) + 3 * 0 \\ 4 * 2 + 5 * 1 + 6 * 0 & 4 * 4 + 5 * (-1) + 6 * 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 13 & 11 \end{bmatrix}$$

Matrix – Matrix Multiplication (Example)

Given

$$A_{2X2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$B_{2X2} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$A_{2X2} \cdot B_{2X2} = \begin{bmatrix} 9 & 2 \\ 11 & 35 \end{bmatrix}$$

Check

$$B_{2X2} \cdot A_{2X2} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 9 & 13 \end{bmatrix}$$

Check

$$\boldsymbol{B_{MXN} \cdot A_{NXM} \neq A_{NXM} \cdot B_{MXN}}$$

Not Commutative

Matrix – Matrix Multiplication (Example)

A vector operating on a vector can yield a scalar or a matrix, depending on the order of operation.

$$\begin{bmatrix} Row1 \\ Row2 \end{bmatrix} [Col1 \quad Col2 \quad Col3] = \begin{bmatrix} Row1\ X\ Col1 & Row1\ X\ Col2 & Row1\ X\ Col3 \\ Row2\ X\ Col1 & Row2\ X\ Col2 & Row2\ X\ Col3 \end{bmatrix}$$

$$A_{3X1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}; B_{1X3} = [2 \quad 0 \quad 3]$$

Matrix

$$A_{3X1} \cdot B_{1X3} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} [2 \quad 0 \quad 3] = \begin{bmatrix} 4 & 0 & 6 \\ 2 & 0 & 3 \\ -2 & 0 & -3 \end{bmatrix}$$

Scalar

$$B_{1X3} \cdot A_{3X1} = [2 \quad 0 \quad 3] \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2 * 2 + 1 * 0 + 3 * (-1) = 1$$

Your Turn …

Calculate $A \cdot v$

$$A_{3X2} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \qquad v_{2X1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A_{2X3} \cdot v_{3X1} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}_{3X2} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2X1}$$

$$= \begin{bmatrix} -1 * -1 + 1 * 1 \\ 1 * -1 - 1 * 1 \\ 2 * -1 + 2 * 1 \end{bmatrix}_{3X1}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}_{3X1}$$

Your Turn …

$$A_{3X2} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad B_{2X3} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

Calculate $A \cdot B$

$$A_{3X2} \cdot B_{2X3} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}_{3X2} \cdot \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}_{2X3}$$

$$= \begin{bmatrix} 1*1+3*-1 & 1*2+3*3 & 1*0+3*1 \\ 2*1+1*-1 & 2*2+1*3 & 2*0+1*1 \\ 1*1+2*-1 & 1*2+2*3 & 1*0+2*1 \end{bmatrix}_{3x3}$$

$$= \begin{bmatrix} -2 & 11 & 3 \\ 1 & 7 & 1 \\ -1 & 8 & 2 \end{bmatrix}$$

Your Turn …

$$A_{2X3} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix} \qquad B_{3X4} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

Calculate $A \cdot B$

$$A_{2X3} \cdot B_{3X4} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}_{2X3} \cdot \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix}_{3X4}$$

$$= \begin{bmatrix} 5 & 5 & -1 & 1 \\ 6 & 2 & -1 & -3 \end{bmatrix}_{2X4}$$

Matrices – properties, definitions and special matrices

Properties of Matrices –
Matrix Multiplication

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$ **Associative**

$$A \cdot (B + C) = A \cdot B + A \cdot C$$ **Left Distributive**

$$(A + B) \cdot C = A \cdot C + B \cdot C$$ **Right Distributive**

$$A \cdot B \neq B \cdot A$$ **In general (not Commutative)**

$$a(A \cdot B) = (aA) \cdot B = A \cdot (aB)$$

Properties of Matrices –

Matrix Multiplication

If $\boldsymbol{AB} = \boldsymbol{AC}$ then it is **not true** that $\boldsymbol{B} = \boldsymbol{C}$

If $\boldsymbol{AB} = \boldsymbol{0}$, we **cannot** conclude that either

$$\boldsymbol{A} = \boldsymbol{0} \ or \ \boldsymbol{B} = \boldsymbol{0}$$

$$A_{2X2} \cdot B_{2X2} = \begin{bmatrix} -1 & 4 \\ 3 & -12 \end{bmatrix} \cdot \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Check

$$A_{2X2} \cdot C_{2X2} = \begin{bmatrix} -1 & 4 \\ 3 & -12 \end{bmatrix} \cdot \begin{bmatrix} 12 & 16 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Transpose

The transpose A^T of a matrix A is an operation in which the terms above and below the diagonal are interchanged.

Example I

$$A_{2X3} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \quad A^T_{3X2} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Example II

$$A_{2X2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \quad A^T_{2X2} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

For any matrix A_{NXM}: $A^T \cdot A$ and $A \cdot A^T$ are square matrices of size M and N, respectively.

This will be very helpful in sessions 6 and 8.

Transpose - Examples

$$A = \begin{bmatrix} 3 & 9 & -6 \\ -1 & 18 & -3 \\ 6 & 6 & 15 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & -1 & 6 \\ 9 & 18 & 6 \\ -6 & -3 & 15 \end{bmatrix}$$

Transpose - Examples

$$A = \begin{bmatrix} 3 & 9 & -6 \\ -1 & 18 & -3 \\ 6 & 6 & 15 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & -1 & 6 \\ 9 & 18 & 6 \\ -6 & -3 & 15 \end{bmatrix}$$

Transpose - Properties

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

Sum

$$(A \cdot B)^T = B^T \cdot A^T$$

Product

The order matters

Inner and Outer products

A vector operating on a vector can yield a scalar or a matrix, depending on the order of operation.

$$\begin{bmatrix} Row1 \\ Row2 \end{bmatrix} \begin{bmatrix} Col1 & Col2 & Col3 \end{bmatrix} = \begin{bmatrix} Row1\ X\ Col1 & Row1\ X\ Col2 & Row1\ X\ Col3 \\ Row2\ X\ Col1 & Row2\ X\ Col2 & Row2\ X\ Col3 \end{bmatrix}$$

$$v_{3X1} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Inner Product

$$v^T{}_{1X3} \cdot v_{3X1} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 * 2 + 0 * 0 + 1 * 1 = 5$$

Scalar

Outer Product

$$v_{3X1} \cdot v^T{}_{1X3} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Matrix

Diagonal Matrix

A diagonal matrix has nonzero terms only along its main diagonal.

The matrix A_{NXN} is diagonal if $a_{ij} = 0 \; if \; i \neq j$

- **The sum and product of diagonal matrices are also diagonal.**

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Diagonal Matrix - Examples

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Recall:
The sum and product of diagonal matrices are also diagonal.

$$AB = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Diagonal Matrix - Data Example

" A neighborhood with Three households, each household owns a unique type of items"

$$\begin{array}{ccc} & & \\ \left[\begin{array}{ccc} 9\ cars & 0 & 0 \\ 0 & 2\ cells & 0 \\ 0 & 0 & 4\ Bikes \end{array}\right] \\ \text{Household 1} \quad \text{Household 2} \quad \text{Household 3} \end{array}$$

Identity Matrix

The *identity* matrix I is a square diagonal matrix with 1 on the main diagonal.

Example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A \cdot I = I \cdot A = A}$$

Identity Matrix - Data Example

" A neighborhood with Three households, each household owns a single unique item"

$$\begin{bmatrix} 1\,cars & 0 & 0 \\ 0 & 1\,cells & 0 \\ 0 & 0 & 1\,Bikes \end{bmatrix}$$

Household 1 **Household 2** **Household 3**

Trace

The **Trace** of a square matrix A_{nXn} is the sum of its diagonal elements

$$tr\ A = \sum_{i=1}^{N} a_{ii}$$

Example

$$\operatorname{tr}\begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix} = 2 + 5 + 8 = 15\,.$$

Now you try …

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 5 & 9 \\ 1 & 0 & 2 \end{bmatrix}$$

$$Trace(A) = ?$$

$$Trace(A) = 1 + 5 + 2 = 8$$

Now you try …

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 0 & -1 & 7 \\ 4 & -1 & 0 & -2 \end{bmatrix}$$

$$A^T = ?$$

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \\ 5 & 7 & -2 \end{bmatrix}$$

BREAK

Symmetry, Anti-symmetry and Asymmetry Matrices

- A symmetric matrix is one for which $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$.
- An anti-symmetric matrix is one for which $\mathbf{A}^{\mathrm{T}} = -\boldsymbol{A}$
- An asymmetric matrix is neither symmetric nor anti-symmetric.

Any matrix A can be written as

$$\boldsymbol{A} = \underbrace{\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{A} + \boldsymbol{A}^{T}\right)}_{\text{Symmetric matrix}} + \underbrace{\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{A} - \boldsymbol{A}^{T}\right)}_{\text{Anti-Symmetric matrix}}$$

Symmetric and Anti-symmetry Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}$$

$$or$$

$$a_{ij} = a_{ji}$$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$$

$$or$$

$$a_{ij} = -a_{ji}$$

Transpose of a Symmetric Matrix (Example)

$$a_{ij} = a_{ji}$$

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 5 & -1 \end{bmatrix}$$

$$A^T = ?$$

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 5 & -1 \end{bmatrix}$$

$$A^T = A$$

Symmetric Matrix - Examples

Are these matrices symmetric?

Recall:

$$a_{ij} = a_{ji}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 5 & -1 \end{bmatrix} ?$$ Yes

$$\begin{bmatrix} 10 & 1 & 3 \\ 2 & -22 & 3 \\ 2 & 2 & 4 \end{bmatrix} ?$$ No

$$\begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} ?$$ Yes

Symmetry Matrices - Applications

Symmetric matrices have many applications. For example, if we consider the shortest distance between pairs of cities, we might get a table like the following:

	City A	City B	City C
City A	0	150	120
City B	150	0	300
City C	120	300	0

$$\begin{bmatrix} 0 & 150 & 120 \\ 150 & 0 & 300 \\ 120 & 300 & 0 \end{bmatrix}$$

Triangular Matrices

A **lower triangular** matrix is one in which all entries above the main diagonal are zero. Lower triangular matrices are often denoted by L.

$$L = \begin{bmatrix} \ell_{1,1} & & & & 0 \\ \ell_{2,1} & \ell_{2,2} & & & \\ \ell_{3,1} & \ell_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

The transpose of a **lower triangular** matrix is an **upper triangular** matrix (Check)

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 0 \end{bmatrix}$$

Triangular Matrices

An **upper triangular** matrix is one in which all entries <u>below</u> the main diagonal are zero. Upper triangular matrices are often denoted by U.

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \ldots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

The transpose of an **upper triangular** matrix is **lower triangular** matrix (Check)

Example

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

Triangular Matrices - Examples

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Upper triangular

$$\begin{bmatrix} -1 & 0 & 0 \\ 2 & -2 & 0 \\ 9 & 9 & -1 \end{bmatrix}$$

Lower triangular

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

Lower and upper triangular

$$\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Neither

(Not a Square Matrix)

Elementary Matrices

Differs from the identity matrix by one single elementary row operation

- Row switching (Permutation Matrix)
- Row addition (Elimination Matrix)
- Row multiplying by constant

This will be very useful in Sessions 3 and 4

Permutation Matrix

A **permutation matrix** $\boldsymbol{P}$ is a square matrix composed of zeroes and a single one in each column. None of the ones occur in the same row.

- P_{ij} is the identity matrix with rows i and j reversed
- It affects a row exchange when it operates on a general matrix A.
- There are $n!$ permutation matrices of size n
- $PP^T = I$ (Check)
- $\|P\|_2 = 1$; $\|P\|_\infty = 1$
- $Det\ (P) = 1$

Permutation matrices of order two

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Permutation matrices of order three

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Permutation Matrix (Example)

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_{3X4} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

$$P \cdot A = ?$$

Permutation Matrix (Example)

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_{3X4} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

Permutation matrix which affects the exchange of the first and third rows

$$P \cdot A =$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & -2 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & -1 & 1 \end{bmatrix}$$

Check

Elimination Matrix

The **elimination matrix E_{ij}** multiplies the j^{th} row by l_{ij} and subtracts it from the i^{th} row.

- To define an elimination matrix start with the identity matrix I and change one of its zeros to multiplier $-l$.
- $E^{-1}E = I$ (Check)

Examples

$$\boldsymbol{E_{21}} = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{a_{21}}$

Multiplies the $1st$ row by l and subtracts it from the $2nd$ row

$\boldsymbol{a_{13}}$

$$\boldsymbol{E_{13}} = \begin{bmatrix} 1 & 0 & -l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplies the $3rd$ row by l and subtracts it from the $1st$ row

Elimination Matrix (Example)

$$\boldsymbol{E_{21}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{3X4} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

$$\boldsymbol{E_{21}} \cdot \boldsymbol{A} = ?$$

Elimination Matrix (Example)

$$\boldsymbol{E_{21}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{3X4} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

The elimination matrix $\boldsymbol{E_{ij}}$ multiplies the j^{th} row by l_{ij} and subtract it from the i^{th} row

The elimination matrix $\boldsymbol{E_{21}}$ multiplies the $\underline{\boldsymbol{1^{st}}}$ row by $\underline{\boldsymbol{2}}$ and subtract from the $\underline{\boldsymbol{2^{nd}}}$ row

$$\boldsymbol{E_{21}} \cdot A =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -5 & 2 & -2 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

$-l_{21}$

Unchanged

Elimination Matrix (Example)

$$\boldsymbol{E_{21}} \cdot A =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -5 & 2 & -2 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

The elimination matrix E_{21} multiplies the 1^{st} row by 2 and subtract from the 2^{nd} row

$$[-2 \quad 1 \quad 0]\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = -2 * \underbrace{1}_{1^{st}\text{ row}} + 1 * \overbrace{2}^{2^{nd}\text{ row}} + 0 * 1 = 0$$

$$[-2 \quad 1 \quad 0]\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = -2 * \underbrace{3}_{1^{st}\text{ row}} + 1 * \overbrace{1}^{2^{nd}\text{ row}} + 0 * 2 = -5$$

Multiplying Row by constant

- Matrix D multiplies all elements on the i^{th} row by non-zero scalar m
- $\det(D) = m$
- $||D|| = m$

Examples

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m \end{bmatrix}$$

Multiplies the $2nd$ row by m

Multiplies the $3rd$ row by m

Multiplying Row by constant (Example)

$$\mathrm{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{3X4} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 3 & 5 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

$$\boldsymbol{D} \cdot \boldsymbol{A} = ?$$

Multiplying Row by constant (Example)

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{3X4} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 3 & 5 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & 3 & 5 \\ 1 & 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 4 & 2 & 6 & 10 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

Unchanged

Orthogonal and Orthonormal Matrices

A matrix is orthogonal if all its columns are orthogonal to each other

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \left[\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} \right]$$

Check

If the columns are also normalized (Size =1) then the matrix is called orthonormal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

Check

Inverse Matrix

- What is the inverse of 3 ?

1/3 ➔ 3*1/3 =1

- What should the inverse of matrix A do?

$$I = A \cdot (Inverse\ Matrix)$$

Definition:

Matrix A has an inverse A^{-1} if

$$I = A \cdot A^{-1} = A^{-1} \cdot A$$

Inverse Matrix - Properties

$$(A^{-1})^{-1} = A$$

$$(A^{-1})^{T} = (A^{T})^{-1}$$

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

The order matters

More in Session 4

Inverse Matrix (Example)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\underset{\boldsymbol{A}}{\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}} \cdot \underset{\boldsymbol{A^{-1}}}{\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Invertible and Singular Matrices

- If A^{-1} exists ➔ A is **Invertible**
- If A^{-1} doesn't exist ➔ A is **Singular**

- **Independent** column vectors ➔ A is **Invertible**
- **Dependent** column vectors ➔ A is **Singular**

Python Example

QUIZ

EXTRA SLIDES

Mathematical Notations - Product

Capital Pi $\prod$

$$\prod_{i=m}^{n} x_i = x_m \cdot x_{m+1} \cdot x_{m+2} \cdot \cdots \cdot x_{n-1} \cdot x_n$$

i **- Index of Product**

Examples

$$\prod_{i=1}^{6} i = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

Matrix – Vector Multiplication

What is the expected dimension of the output vector?

$$y_m = \boldsymbol{A}_{mXn} \cdot v_n$$

$$y_i = \sum_{j=1}^{n} a_{ij} v_j \,; \quad i = 1, \dots m$$

Column index

Row index

Example

$$y_1 = \sum_{j=1}^{n} a_{1j} v_j = a_{11} v_1 + a_{12} v_2 + \cdots + a_{1n} v_n$$

$$= 1^{st}\, row \cdot v$$

The vector y is a linear combination of the columns of A.

Matrix – Matrix Multiplication

Multiplication of matrices A and B can be defined only if they are of the proper sizes.

$$Y_{mxn} = A_{mxk} \cdot B_{kxn}$$

$$y_{ij} = \sum_{s=1}^{k} a_{is} b_{sj}$$

$$i = 1, \ldots m; j = 1, \ldots n$$

The product element in row i and column j (i.e., y_{ij}) is the sum of the products of corresponding elements from row i of A and column j of B

Matrix – Matrix Multiplication

$$y_{ij} = \sum_{s=1}^{k} a_{is} b_{sj}$$

$$i = 1, \ldots m; \;\; j = 1, \ldots n$$

Example (Assume $k = 3$)

$$y_{11} = \sum_{s=1}^{k=3} a_{1s} b_{s1} = a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31}$$

$$y_{12} = \sum_{s=1}^{k=3} a_{1s} b_{s2} = a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32}$$

The product element in row i and column j (i.e., y_{ij}) is the sum of the products of corresponding elements from row i of $\boldsymbol{A}$ and column j of $\boldsymbol{B}$

Symmetry, Anti-symmetry (Example)

Any matrix A can be written as $\boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{A}+\boldsymbol{A}^T) + \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{A}-\boldsymbol{A}^T)$

Symmetric matrix **Anti-Symmetric matrix**

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 2 \\ 4 & 1 & -3 \end{bmatrix}$$

$$\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{A}+\boldsymbol{A}^T) = \begin{bmatrix} 1 & 3/2 & 5/2 \\ 3/2 & 2 & 3/2 \\ 5/2 & 3/2 & -3 \end{bmatrix}$$

✓ **Symmetric matrix**

$$\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{A}-\boldsymbol{A}^T) = \begin{bmatrix} 0 & 3/2 & 3/2 \\ -3/2 & 0 & -1/2 \\ -3/2 & 1/2 & 0 \end{bmatrix}$$

✓ **Anti-Symmetric matrix**

$$\boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{A}+\boldsymbol{A}^T) + \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{A}-\boldsymbol{A}^T)$$

✓ **Check**

Matrix Norm

Similarly to vectors, The *norm* ||**A**|| of a matrix **A** is a real number that satisfies the following properties:

- $||\boldsymbol{A}|| \geq 0$,
- $||\boldsymbol{A}|| = 0$ if and only if $\boldsymbol{A} = 0$,
- $||a\boldsymbol{A}|| = |a| \, ||v||$, $a \in R1$, and
- $||\boldsymbol{A} + \boldsymbol{B}|| \leq ||\boldsymbol{A}|| + ||\boldsymbol{B}||$, (triangle or Minkowski inequality).

Matrix Norm (Examples)

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|;$$

Which is the maximum absolute <u>column</u> sum of the matrix

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|;$$

which is the maximum absolute <u>row</u> sum of the matrix

$$A = \begin{bmatrix} -3 & 5 & 7 \\ 2 & 6 & 4 \\ 0 & 2 & 8 \end{bmatrix}$$

$$\|A\|_1 = \max(|-3| + 2 + 0, 5 + 6 + 2, 7 + 4 + 8) = \max(5, 13, 19) = 19$$

$$\|A\|_\infty = \max(|-3| + 5 + 7, 2 + 6 + 4, 0 + 2 + 8) = \max(15, 12, 10) = 15.$$

Vector and Matrix Norm - Properties

$$\|\boldsymbol{A}v\| \leq \|\boldsymbol{A}\|\|v\|$$

$$\|\boldsymbol{AB}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$$