Mathematics of Pricing Options

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Ideas of Mathematics

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1 Introduction

Mathematics and probability theory play important roles in the financial markets; from basic economic analysis to portfolio management, from risk management to market predictions. One of the most renowned achievements in finance is the Black-Scholes formula. The Black-Scholes model was developed by Fischer Black and Myron Scholes. It is a standard model used for valuing financial derivatives. In this paper, we will focus our attention on call options and how Black-Scholes model is used to value them.

What are call options? A call option on a stock is a financial contract between the buyer and the seller. In a call option, the buyer and the seller agree to exchange the underlying asset at an agreed upon price (exercise price) and at an agreed upon date (maturity/exercise date) sometime in the future. In general, each individual call option contract controls a hundred unit of the underlying asset. Each contract is also associated with a purchase price, known as the premium price. To get a thorough understanding of a call option consider the following example.

Example To keep the example simple, suppose each contract only controls one unit of the underlying asset and the purchase price of the contract is \$0. In our example, the underlying asset will be stocks of ABC Inc. We define the following:

- S_0 is the stock price at the time contract was initiated, time $t = 0$.
- E is the exercise price, where E is greater than S_0.
- T is the exercise date sometime in the future.
- S_T is the stock price on the exercise date.
- C_T is the value of the contract on the exercise date.

Suppose Person A buys a call option for ABC Inc. from Person B. The details of the contract are as follows:

Call Option		
Underlying Asset	**Exercise Price**	**Exercise date**
ABC Inc. Stock	E	T

Now there are two possible outcomes that arise from this exchange, displayed in Figure 1 below. Our stock price can follow two paths: path (1) or path (2).

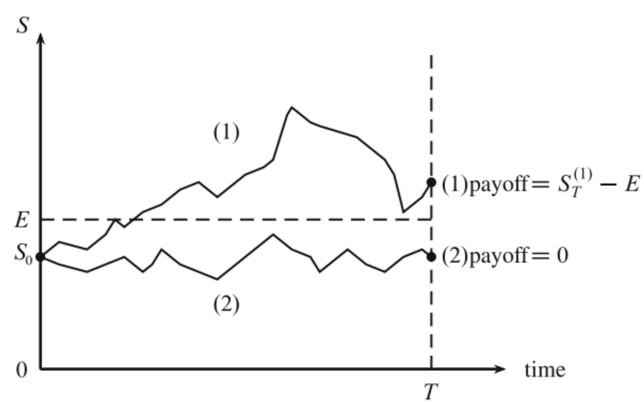

Figure 1: Call Option Outcome

Suppose it follows path (1), then S_T will be greater than E on the exercise date T. In this scenario, the buyer has a positive payoff. The buyer will purchase the stock from the seller at \$$E$, then sell it back in the market at \$$S_T$. Here, we say the value of the option is $C_T = S_T^{(1)} - E$. Instead, suppose it follows path

(2), then S_T will be less than E on the exercise date T. In this scenario, the buyer's payoff is zero. Since the stock price in the market is less than the exercise price, it does not make rational sense for the buyer to exercise the call option. Here, we say the value of the option is $C_T = 0$. In general, the options payoff function can be written $C_T = \max(S_T - E, 0)$.

As you may have realized, the stock price is not limited to these two paths. There are many paths that can be taken, arguably infinitely many different paths. Here we must stop and think, "what does the distribution of S_T look like? What is the expected value of C_T?" These are the two main challenges investors face. Stock prices move stochastically, determining the distribution is not a trivial task. Figure 2 below gives a visualization of the possible distribution of S_T. Here, we have simulated a number of stock price paths.

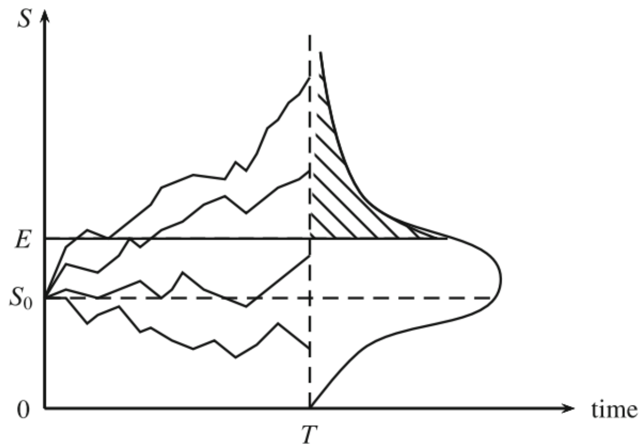

Figure 2: Distribution of S_T

We are interested in the probability that the stock price at T finishes above E, this is the shaded area under the bell curve. Observe that the distribution of S_T is roughly normal. Intuitively, this makes sense. Our stock price S_T is conditional on two things: S_0 and T. The shorter the time period to maturity date, the less likely it is for the stock price to deviate far away from S_0. In other words, there is a high probability that the stock price is near S_0 at T and a low probability that the stock price is far away from S_0 at T. This creates the bell shaped curve.

Now that we have built an intuition behind the distribution of S_T, let us dive into the mathematical concepts and tools needed to determine the precise distribution of S_T. We will then use the distribution of S_T to calculate the expected value of the payoffs C_T. Figures 1 and 2, and explanation of call option are referenced from [1].

2 Stochastic Process of Stock Prices

Finance is in the world of uncertainty. Take stock prices for instance, they do not follow a deterministic path. That is, we cannot determine tomorrow's stock price with a hundred percent certainty. There is a random error term involved which makes the process stochastic. We say stock prices behave stochastically.

Definition (Stochastic Process)**.** *A stochastic process is a collection of random variables* $\{X_t, t \in I\}$*. The set* I *is the index set of the process. The random variables are defined on a common state space* S [2].

This idea of a stochastic process is illustrated in Figure 3 below. In this case, the stock prices are our random variables and the collection of all 256 realizations is our stochastic process. This stochastic process can be written as $\{100, 100.12, 100.43, \ldots, 122.20, 123.51\}$. The index set is $\{1, 2, 3, \ldots, 256\}$ and the state space is the real numbers.

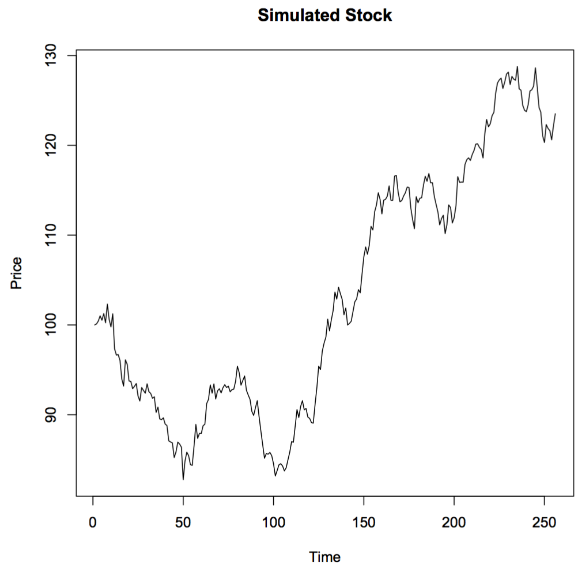

Figure 3: Stochastic sequence of stock price

As stated earlier, stock prices behave stochastically. We cannot determine the future stock price with hundred percent certainty. However, we can create a model which best mimics the past realizations of our stock price and use it to predict the future stock price with a certain margin of error.

Once we have attained a model, a fair question to ask would be, "what is the probability of observing a certain price sequence in the future?" More formally, we want to obtain a joint probability density function

$$p((x_k, t_k), (x_{k-1}, t_{k-1}), \dots, (x_1, t_1)); \quad t_1 < t_2 < \dots < t_k$$

where x_i is the stock price at time t_i in the future. Since our model is based on historic data, the probability of observing (x_1, t_1) is conditional on the past stock prices. Let y_i be the stock price in the past at time τ_i. Then observing the sequence $\{(x_1, t_1), (x_2, t_2), \dots, (x_k, t_k)\}$ is conditional on the previously observed sequence $\{(y_1, \tau_1), (y_2, \tau_2), \dots, (y_n, \tau_n)\}$. See Figure 4.

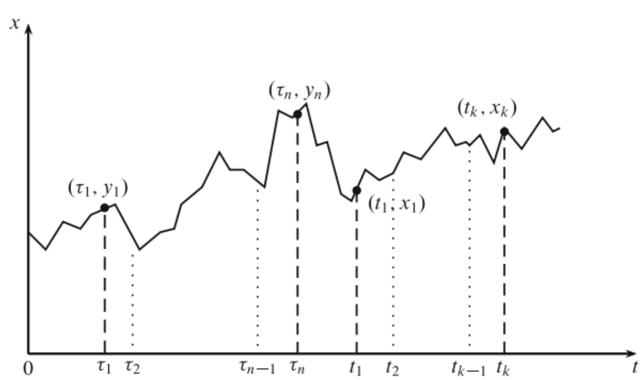

Figure 4: Conditional probability of a sample path

Definition (Conditional Probability)**.** *Given two events A and B. The conditional probability of B given A is defined as the quotient of the joint probability of events A and B, and probability of A: $P(B|A) = \frac{P(A \cap B)}{P(A)}$.* [3]

Using our definition of conditional probability, we can define our conditional probability function as follows:

$$
\begin{aligned}
&p((x_k, t_k), (x_{k-1}, t_{k-1}), \ldots, (x_1, t_1) \mid (y_n, \tau_n), (y_{n-1}, \tau_{n-1}), \ldots, (y_1, \tau_1)) \\
&\qquad = \frac{p((x_k, t_k), (x_{k-1}, t_{k-1}), \ldots, (x_1, t_1), (y_n, \tau_n), (y_{n-1}, \tau_{n-1}), \ldots, (y_1, \tau_1))}{p((y_1, \tau_1), (y_2, \tau_2), \ldots, (y_n, \tau_n))}.
\end{aligned}
$$

The left-hand side of our equation is the probability of observing the price sequence $\{(x_1, t_1), (x_2, t_2), \ldots, (x_k, t_k)\}$ given that the price sequence $\{(y_1, \tau_1), (y_2, \tau_2), \ldots, (y_n, \tau_n)\}$ has just been observed over the previous time period.

A stochastic process which relies on all of the previous realizations can become extremely complex. For simplicity, we are going to make the assumption that our future price sequence is only dependent on the most recent observed stock price. That is, our stock prices are Markov chains, a subtype of a stochastic process.

Definition (Markov Chain)**.** *A Markov chain is a sequence of random variables $X_0, X_1, \ldots$ with the property that*

$$P(X_{n+1}, X_{n+2}, \ldots, X_{n+k} \mid X_0, X_1, \ldots, X_n) = P(X_{n+1}, X_{n+2}, \ldots, X_{n+k} \mid X_n). \text{ [4]}$$

This simplifies our conditional probability function. We obtain the following result:

$$
\begin{aligned}
&p((x_k, t_k), (x_{k-1}, t_{k-1}), \ldots, (x_1, t_1) \mid (y_n, \tau_n), (y_{n-1}, \tau_{n-1}), \ldots, (y_1, \tau_1)) \\
&\qquad = p((x_k, t_k), (x_{k-1}, t_{k-1}), \ldots, (x_1, t_1) \mid (y_n, \tau_n)) \\
&\qquad = \frac{p((x_k, t_k), (x_{k-1}, t_{k-1}), \ldots, (x_1, t_1), (y_n, \tau_n))}{p((y_n, \tau_n))}.
\end{aligned}
$$

Notice, it does not matter which path our process realizes in the past, as long as the last observed data point is (y_n, τ_n), then every process produces the same conditional probability function from the point (y_n, τ_n) given that the future price sequence is the same. See Figure 5.

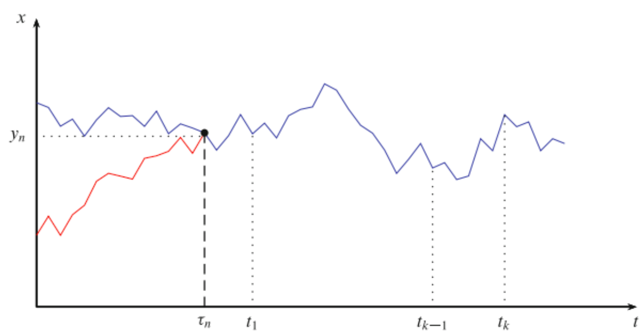

Figure 5: A Markov process

This section and figures are referenced from [5].

3 Conditional Probability and Distribution of Stock Prices

To begin discussing the distribution of the stock prices, we need to impose the following condition:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-z|>\varepsilon} p((x, t + \Delta t) \mid (z, t))dx = 0, \forall \varepsilon > 0. \tag{1}$$

This condition, known as Lindeberg condition, allows our sample paths to be continuous function of time. Why? This condition is saying that the probability the stock price, x, jumps "significantly" at time $t+\Delta t$ as $\Delta t \to 0$ is zero. In other words, the stock price, x, is going to stays within the band $|x-z| < \varepsilon$ at time $t+\Delta t$ as $\Delta t \to 0$. This establishes continuity. See Figure 6.

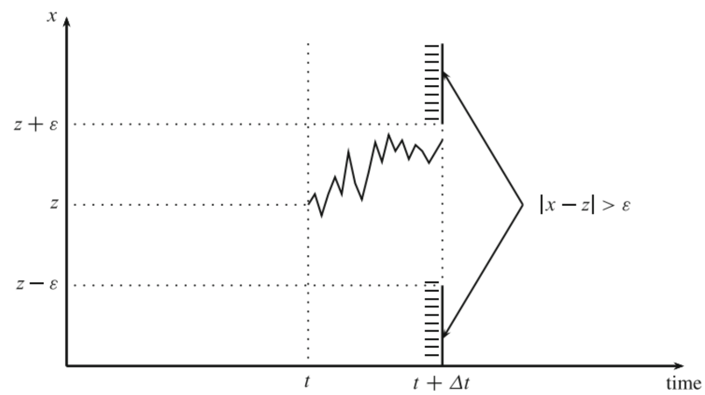

Figure 6: Lindeberg Condition

Earlier we had developed the intuition that the stock prices are normally distributed, therefore establishing continuity condition was important. Through empirical studies we have discovered the conditional probability density is given by:

$$p((x, t+\Delta t) \mid (z,t)) = \frac{1}{\sqrt{2\pi\Delta t}\sigma} \exp\Big[\frac{-(x-z)^2}{2\sigma^2\Delta t}\Big], \tag{2}$$

where $x \sim N(z, \sigma^2\Delta t)$. The conditional probability density given above is known as Brownian Motion. Again, through empirical studies it suggests that many asset prices are modelled with Brownian motion. Notice Brownian motion satisfies Lindeberg's condition. As $\Delta t \to 0$, the distribution becomes peaked, implying the stock price is going to be within the band $|x-z| < \varepsilon, \forall\varepsilon > 0$. See Figure 7

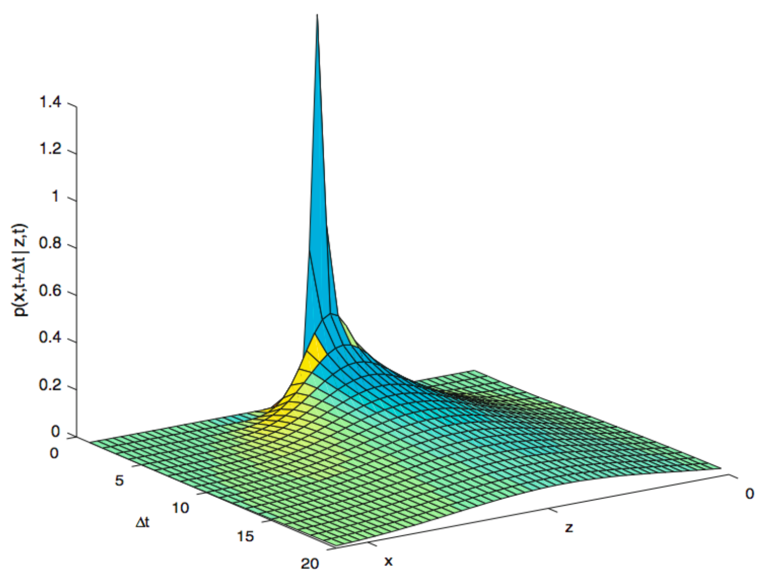

Figure 7: Lindeberg Condition

This section and figures are referrenced from [6].

4 Black-Scholes Equation and Black-Scholes Formula

4.1 Black-Scholes Equation

The Black-Scholes equation describes the price of an option over time. The derivation of this equation is complex and exceeds the scope of this paper, so we simply provide the equation.

$$\frac{\partial \mathrm{C}}{\partial \mathrm{t}} + \frac{1}{2}\sigma^2 \mathrm{S}^2 \frac{\partial^2 \mathrm{C}}{\partial \mathrm{C}^2} + \mathrm{rS}\frac{\partial \mathrm{C}}{\partial \mathrm{S}} = \mathrm{rC} \tag{3}$$

Notice that that equation (3) is a partial differential equation. The solution of this equation gives us the Black-Scholes formula.

4.2 Black-Scholes formula for European option price

The Black-Scholes formula allows us the calculate the price of European call and put options.

$$\mathrm{C}(\mathrm{S},\mathrm{t}) = \mathrm{N}(\mathrm{d}_1)\mathrm{S} - \mathrm{N}(\mathrm{d}_2)\mathrm{K}\mathrm{e}^{-rt} \tag{4}$$

$$\mathrm{d}_1 = \frac{1}{\sigma\sqrt{\mathrm{t}}}\left[\ln\left(\frac{S}{K}\right) + t\left(r + \frac{\sigma^2}{2}\right)\right] \tag{5}$$

$$\mathrm{d}_2 = \frac{1}{\sigma\sqrt{\mathrm{t}}}\left[\ln\left(\frac{S}{K}\right) + t\left(r - \frac{\sigma^2}{2}\right)\right] \tag{6}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}z^2} dz \tag{7}$$

The Black-Scholes equation and formula are referenced from [7].

Example You want to buy an Toronto–Dominion Bank European call option with a strike price of \$210. The stock is currently trading at a price of \$208.99 . You calculate the volatility of the stock to be 17%. The rate at which you can borrow and lend money is 5% Assume the rate is risk-free. The time to maturity of the option is 77 days. What price should you pay (per share) for the option contract?

$$\mathrm{d}_1 = \frac{1}{0.17\sqrt{0.21095}}\left[\ln\left(\frac{208.99}{210}\right) + 0.21095\left(0.05 + \frac{0.17^2}{2}\right)\right] = 0.1123799$$
$$\mathrm{d}_2 = \frac{1}{0.17\sqrt{0.21095}}\left[\ln\left(\frac{208.99}{210}\right) + 0.21095\left(0.05 - \frac{0.17^2}{2}\right)\right] = 0.0343001$$

Now that we've calculated d_1 and d_2 we will calculate $\mathrm{N}(\mathrm{d}_1)$ and $\mathrm{N}(\mathrm{d}_2)$. It should be noted that it's not possible to evaluate equation (7) by normal means so you must use a table or use computer software to evaluate the following integrals:

$$N(d_1) = N(0.1123799) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.1123799} \mathrm{e}^{-\frac{1}{2}z^2} dz = 0.3516077$$
$$N(d_2) = N(0.0343001) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.0343001} \mathrm{e}^{-\frac{1}{2}z^2} dz = 0.3135993$$

Now we plug these in equation (4) to calculate the price of the call option (per share).

$$\mathrm{C}(208.99, 0.21095) = (0.3516077)(208.99) - (0.3135993)(210)\mathrm{e}^{-(0.05)(0.21095)} \approx 2.3454$$

You should pay around \$2.35 (per share) for the options contract. As a side note, options contracts are sold in lots of 100 shares, so this particular contract would sell for roughly \$235.

5 Limitation of Black-Scholes Formula

In the real world, the precise distribution of asset prices is slightly different from the assumed normal distribution. It is difficult to accurately estimate the parameters of the distribution.

From a scientific point of view, we statistically evaluate the outcomes of the past to estimate the parameters. However, this implicitly assumes that the past is a reasonable predictor of the future. In the Black-Scholes formula, there are many variables that are unaccounted for such as human activities, real world factors like news, policies and even rumours have a strong effect in the financial markets.

Even the use of Black-Scholes formula may change the price of stock in the market due to the feedback effect. If everyone is using Black-Scholes formula, we will modify the stock price in the market. This is another variable which is unaccounted for, thus reducing the accuracy of the model. These limitations are referenced from [8].

References

[1] Chiarella, Carl, Xue-Zhong He and Sklibosios Christina Nikitopoulos. *Derivative Security Pricing*, 2015, pp. 3-5.

[2] Dobrow, Robert. *Introduction to Stochastic Processes with R*, 2016, p. 6.

[3] Wikipedia contributors. "Conditional probability." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 18 Mar. 2019. Web. 1 Apr. 2019.

[4] Dobrow, Robert. *Introduction to Stochastic Processes with R*, 2016, p. 41.

[5] Chiarella, Carl, Xue-Zhong He and Sklibosios Christina Nikitopoulos. *Derivative Security Pricing*, 2015, pp. 9-11.

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