CS 474

Assignment 2

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Problem 1

Soln:

Let us use the symbol ψ to refer to the given formula.

$$\psi = (p \wedge (p \Rightarrow q)) \Rightarrow q$$

In order to show that ψ is valid, we can show that

$$\neg\psi = \neg\left((p \wedge (p \Rightarrow q) \Rightarrow q)\right)$$

is unsatisfiable. Rewriting the above formula:

$$\begin{aligned}
\neg\psi &= \neg\left((p \wedge (p \Rightarrow q) \Rightarrow q)\right) \\
&\equiv \neg\left((p \wedge (\neg p \vee q)) \Rightarrow q\right) \\
&\equiv \neg\left(\neg(p \wedge (\neg p \vee q)) \vee q\right) \\
&\equiv (p \wedge (\neg p \vee q)) \wedge \neg q
\end{aligned}$$

The last step is due to De Morgan's Law. We can now convert this to CNF, and construct a resolution refutation to show that it is unsatisfiable. To convert to CNF, we use the Tseitin transformation. We only need three new propositional variables, x_ψ, x_1, x_2, where x_ψ corresponds to ψ, x_1 corresponds to $(\neg p \vee q)$ and x_2 corresponds to $(p \wedge x_1)$. This gives us the following set of clauses:

$$\begin{Bmatrix}
\{\neg x_\psi\}, \\
\{\neg\neg x_\psi, x_2\}, \{\neg\neg x_\psi, \neg q\}, \{\neg x_\psi, \neg x_2, \neg\neg q\} \\
\{\neg x_2, p\}, \{\neg x_2, x_1\}, \{x_2, \neg p, \neg x_1\}, \\
\{x_1, \neg\neg p\}, \{x_1, \neg q\}, \{\neg x_1, \neg p, q\},
\end{Bmatrix}$$

Simplifying the set by replacing $\neg\neg p$ with p for all propositional variables, we get:

$$\begin{Bmatrix}
\{\neg x_\psi\}, \\
\{x_\psi, x_2\}, \{x_\psi, \neg q\}, \{\neg x_\psi, \neg x_2, q\} \\
\{\neg x_2, p\}, \{\neg x_2, x_1\}, \{x_2, \neg p, \neg x_1\}, \\
\{x_1, p\}, \{x_1, \neg q\}, \{\neg x_1, \neg p, q\},
\end{Bmatrix}$$

We can now create a resolution refutation to show that this set is unsatisfiable:

$$
\begin{aligned}
&1.\{\neg x_\psi\}\\
&2.\{x_\psi, x_2\}\\
&3.\{x_2\} \quad \text{Resolvent of 1 and 2}\\
&4.\{\neg x_2, p\}\\
&5.\{p\} \quad \text{Resolvent of 3 and 4}\\
&6.\{\neg x_1, \neg p, q\}\\
&7.\{\neg x_1, q\} \quad \text{Resolvent of 5 and 6}\\
&8.\{x_\psi, \neg q\}\\
&9.\{x_\psi, \neg x_1\} \quad \text{Resolvent of 7 and 8}\\
&10.\{\neg x_1\} \quad \text{Resolvent of 1 and 9}\\
&11.\{\neg x_2, x_1\}\\
&12.\{\neg x_2\} \quad \text{Resolvent of 3 and 11}\\
&13.\{\} \quad \text{Resolvent of 10 and 12}
\end{aligned}
$$

By creating this resolution refutation, we have shown that there is no valuation that can satisfy $\neg\psi$. Since $\neg\psi$ is unsatisfiable, we can conclude that ψ is valid.

The above proof can be validated using the resolution tool provided as a part of the assignment. The initial set of clauses, and the resulting resolution proof object can be found at https://github.com/adharshkamath/cs474-hw/tree/main/resolution_proofs/p1

Problem 2

Soln:

We are given the formula:

$$\psi = (q \vee \neg r) \wedge (\neg p \vee r) \wedge (\neg q \vee r \vee p) \wedge (p \vee q \vee \neg q) \wedge (\neg r \vee q)$$

The set of clauses representing this formula is:

$$\Big\{\{q, \neg r\}, \{\neg p, r\}, \{\neg q, r, p\}, \{p, q, \neg q\}, \{\neg r, q\} \quad \Big\}$$

We shall consider resolutions that (i) lead to new clauses and are (ii) not trivial. For example, resolving $\{q, \neg r\}$ and $\{\neg q, r \vee p\}$ gives us $\{r, \neg r, p\}$, which is trivially true and hence not included in the resolution steps below. we will use the given formula as is, without modifying the given formula (removing any trivial or redundant conjuncts).

We can find resolvents in the following way:

$$
\begin{aligned}
&1.\{q, \neg r\}\\
&2.\{\neg p, r\}\\
&3.\{q, \neg p\} \quad \text{Resolvent of 1 and 2}\\
&4.\{p, q, \neg q\}\\
&5.\{p, q, \neg r\} \quad \text{Resolvent of 1 and 4}\\
&6.\{\neg q, r, p\}\\
&7.\{r, \neg q\} \quad \text{Resolvent of 2 and 6}
\end{aligned}
$$

The final set of clauses is:

$$\left\{\begin{matrix} \{q, \neg r\}, \{\neg p, r\}, \{\neg q, r, p\}, \{p, q, \neg q\}, \{\neg r, q\} \\ \{q, \neg p\}, \{p, q, \neg r\}, \{r, \neg q\} \end{matrix}\right\}$$

Note that, if we had removed redundant and trivial clauses from the initial set of clauses, the final set would be different in the following ways:

- The clause $\{p, q, \neg q\}$ would be removed since it is trivial
- The clause $\{\neg r, q\}$ would be removed since it is redundant
- The resolvent $\{p, q, \neg r\}$ would not be present in the final set since $\{p, q, \neg q\}$ was removed from the initial set (step 4 in the resolution)

The final set of clauses in that case would be:

$$\left\{\begin{matrix} \{q, \neg r\}, \{\neg p, r\}, \{\neg q, r, p\}, \\ \{q, \neg p\}, \{r, \neg q\} \end{matrix}\right\}$$

We can use this set of clauses in place of the previously derived set, since we have removed only trivial and redundant clauses.

Problem 3

Soln:
