CS 474

Homework 3

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Problem 1

Soln: We are given three models with signatures $(\mathrm{U}, 0, 1, +, *)$ where U is $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$.

(a)

For each formula given below, we shall see if the formula is true or false over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$.

1. $\exists y.(y * y = 1 + 1)$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
False	False	True

This formula is true over $\mathbb{R}$ since there is a real number ($\sqrt{2}$) which satisfies the formula. However, the same is not true for $\mathbb{N}$ and $\mathbb{Q}$.

2. $\forall x.\exists y.(x + y = 0)$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
False	True	True

This formula is true over $\mathbb{Q}$ and $\mathbb{R}$ since for any $x \in \mathbb{Q}$ (or $\mathbb{R}$), there exists a $y \in \mathbb{Q}$ (or $\mathbb{R}$) such that $x + y = 0$. (We can take $y = -x$ when $x \neq 0$ and $y = 0$ otherwise). However, this is not true for $\mathbb{N}$ since there is no $y \in \mathbb{N}$ such that $x + y = 0$ for any $x \in \mathbb{N}$.

3. $\forall x.\forall y.(\neg(y = 0) \Rightarrow (\exists z.x * y = x + z))$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
True	True	True

This formula is true over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ since for any $x, y \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$), we can find a $z \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$) such that $x * y = x + z$. (We can take $z = x * (y - 1)$, and still ensure $z \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$). This is possible since the premise of the implication is that $y \neq 0$).

4. $\exists x.\exists y.(x + 1 = 0 \land y * y = x)$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
False	False	False

This formula is false over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ since there is no $x, y \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$) such that $x + 1 = 0$ and $y * y = x$. This formula can evaluate to true only if $x = -1$ and $y = \pm\sqrt{-1}$, which is possible only in $\mathbb{C}$ (complex numbers) and not in $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{R}$.

(b)

Task 1:

We need to write a formula $gt_{\mathbb{N}}(x, y)$ that is true precisely when $x > y$ for any $x, y \in \mathbb{N}$.

We can write the formula as $gt_{\mathbb{N}}(x, y) = \exists z.(z \neq 0 \wedge x = y + z)$.

Task 2:
We need to write a formula $gt_{\mathbb{R}}(x, y)$ that is true precisely when $x > y$ for any $x, y \in \mathbb{R}$.
We can write the formula as $gt_{\mathbb{R}}(x, y) = \exists z.(z \neq 0 \wedge x = y + (z * z))$.

(c)
We need to write a formula that is true over $\mathbb{N}$, but false over $\mathbb{R}$.
Using the definitions of $gt_{\mathbb{N}}(x, y)$ and $gt_{\mathbb{R}}(x, y)$ from above, we can write the following formula:

$$\forall x.(x > 0 \vee x = 0)$$

where $x > 0$ is $gt_{\mathbb{N}}(x, 0)$ for $\mathbb{N}$ and $x > 0$ is $gt_{\mathbb{R}}(x, 0)$ for $\mathbb{R}$. This formula is true over $\mathbb{N}$ since for any $x \in \mathbb{N}$, x is greater than or equal to 0, which is not true for $\mathbb{R}$.

Problem 2
Soln: (a)
Task 1:
We are given the following formula over the logic $(\mathbb{R}, 0, 1, <)$:

$$\begin{aligned} &\forall z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \\ \Rightarrow &(\exists w.\, l_1 < w \wedge w < u_1 \wedge l_2 < w \wedge w < u_2 \wedge w \neq z)) \end{aligned}$$

Consider the inner formula that is quantified by $\exists w$:

$$\exists w.(l_1 < w \wedge w < u_1 \wedge l_2 < w \wedge w < u_2 \wedge w \neq z)$$

Rewriting $w \neq z$ as $\neg(w = z)$, which is equivalent to $w < z \vee w > z$, we get:

$$\exists w.(l_1 < w \wedge w < u_1 \wedge l_2 < w \wedge w < u_2 \wedge (w < z \vee w > z))$$

Applying the distributive property and simplifying, we get the DNF form:

$$\begin{aligned} &\exists w.(l_1 < w \wedge w < u_1 \wedge l_2 < w \wedge w < u_2 \wedge w < z) \\ \vee &\exists w.(l_1 < w \wedge w < u_1 \wedge l_2 < w \wedge w < u_2 \wedge w > z) \end{aligned}$$

Consider the first disjunct:

$$\exists w.(l_1 < w \wedge w < u_1 \wedge l_2 < w \wedge w < u_2 \wedge w < z)$$

In the context of the Q.E procedure for DLOWE, the lower bounds in this case are l_1, l_2 and the upper bounds are u_1, u_2, z. Eliminating the existential quantifier, we get:

$$(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)$$

Consider the second disjunct:

$$\exists w.(l_1 < w \wedge w < u_1 \wedge l_2 < w \wedge w < u_2 \wedge w > z)$$

In the context of the Q.E procedure for DLOWE, the lower bounds in this case are l_1, l_2, z and the upper bounds are u_1, u_2. Eliminating the existential quantifier, we get:

$$(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2)$$

Combining the two disjuncts, we get:

$$\begin{aligned}
&(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z) \\
\lor&(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2)
\end{aligned}$$

Thus we have eliminated the existential quantifier $\exists w$ from the original formula.

Therefore, the original formula can be simplified to:

$$\begin{aligned}
&\forall z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \\
\Rightarrow &((l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z) \\
\lor&(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2)))
\end{aligned}$$

Rewriting the implication as a disjunction, we get:

$$\begin{aligned}
&\forall z.\neg(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2) \\
\lor&((l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z) \\
\lor&(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2))
\end{aligned}$$

Rewriting the universal quantifier as an existential quantifier, and pushing the negation inside, we get:

$$\begin{aligned}
&\neg(\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2)) \\
\land\neg&((l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z) \\
\lor&(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2))
\end{aligned}$$

Consider the inner existential formula:

$$\begin{aligned}
&\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2) \\
\land\neg&((l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z) \\
\lor&(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2))
\end{aligned}$$

Applying De Morgan's law, we get:

$$\begin{aligned}
&\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2) \\
\land&(\neg(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z) \\
\land\neg&(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2))
\end{aligned}$$

Consider the first inner negated formula:

$$\neg(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z)$$

Using De Morgan's law, we get:

$$\begin{aligned}(l_1 > u_1 \lor l_1 = u_1) \lor (l_2 > u_1 \lor l_2 = u_1) \lor (l_1 > u_2 \lor l_1 = u_2) \lor (l_2 > u_2 \lor l_2 = u_2)\\ \lor (l_1 > z \lor l_1 = z) \lor (l_2 > z \lor l_2 = z)\end{aligned}$$

Similarly for the other inner negated formula:

$$\neg(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land z < u_1 \land z < u_2)$$

Applying De Morgan's law, we get:

$$\begin{aligned}(l_1 > u_1 \lor l_1 = u_1) \lor (l_2 > u_1 \lor l_2 = u_1) \lor (l_1 > u_2 \lor l_1 = u_2) \lor (l_2 > u_2 \lor l_2 = u_2)\\ \lor (z > u_1 \lor z = u_1) \lor (z > u_2 \lor z = u_2)\end{aligned}$$

The conjunction of the above two formulae is:

$$\begin{aligned}((l_1 > u_1 \lor l_1 = u_1) \lor (l_2 > u_1 \lor l_2 = u_1) \lor (l_1 > u_2 \lor l_1 = u_2) \lor (l_2 > u_2 \lor l_2 = u_2)\\ \lor (l_1 > z \lor l_1 = z) \lor (l_2 > z \lor l_2 = z))\\ \land\\ ((l_1 > u_1 \lor l_1 = u_1) \lor (l_2 > u_1 \lor l_2 = u_1) \lor (l_1 > u_2 \lor l_1 = u_2) \lor (l_2 > u_2 \lor l_2 = u_2)\\ \lor (z > u_1 \lor z = u_1) \lor (z > u_2 \lor z = u_2))\end{aligned}$$

Applying the distributive property and simplifying, we get:

$$\begin{aligned}(l_1 > u_1 \lor l_1 = u_1) \lor (l_2 > u_1 \lor l_2 = u_1) \lor (l_1 > u_2 \lor l_1 = u_2) \lor (l_2 > u_2 \lor l_2 = u_2)\\ \lor ((l_1 > z \lor l_1 = z) \land (z > u_1 \lor z = u_1)) \lor ((l_2 > z \lor l_2 = z) \land (z > u_1 \lor z = u_1))\\ \lor ((l_1 > z \lor l_1 = z) \land (z > u_2 \lor z = u_2)) \lor ((l_2 > z \lor l_2 = z) \land (z > u_2 \lor z = u_2))\end{aligned}$$

(The term after applying distributive property and before simplification is omitted for brevity). Putting the above back into the original formula, we get:

$$\begin{aligned}\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2\\ \land (l_1 > u_1 \lor l_1 = u_1 \lor l_2 > u_1 \lor l_2 = u_1 \lor l_1 > u_2 \lor l_1 = u_2 \lor l_2 > u_2 \lor l_2 = u_2\\ \lor ((l_1 > z \lor l_1 = z) \land (z > u_1 \lor z = u_1)) \lor ((l_2 > z \lor l_2 = z) \land (z > u_1 \lor z = u_1))\\ \lor ((l_1 > z \lor l_1 = z) \land (z > u_2 \lor z = u_2)) \lor ((l_2 > z \lor l_2 = z) \land (z > u_2 \lor z = u_2)))\end{aligned}$$

Applying distributive property and simplifying, we get:

$$\begin{aligned}\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_1 > u_1)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_1 = u_1)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_2 > u_1)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_2 = u_1)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_1 > u_2)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_1 = u_2)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_2 > u_2)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_2 = u_2)\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land (l_1 > z \lor l_1 = z) \land (z > u_1 \lor z = u_1))\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land (l_2 > z \lor l_2 = z) \land (z > u_1 \lor z = u_1))\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land (l_1 > z \lor l_1 = z) \land (z > u_2 \lor z = u_2))\\ \lor \exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land (l_2 > z \lor l_2 = z) \land (z > u_2 \lor z = u_2))\end{aligned}$$

Simplifying the last four disjuncts, we get:

$$
\begin{aligned}
&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z > u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z > u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z > u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z > u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z = u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z > u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z = u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z = u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z > u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z = u_1) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z = u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z > u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z = u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z = u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z > u_2) \\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z = u_2)
\end{aligned}
$$

The following terms can be pulled out from the first eight disjuncts since they do not refer to z:

$$(l_1 > u_1) \vee (l_1 = u_1) \vee (l_2 > u_1) \vee (l_2 = u_1) \vee (l_1 > u_2) \vee (l_1 = u_2) \vee (l_2 > u_2) \vee (l_2 = u_2)$$

Upon removing the above terms, the first eight disjuncts all become the same:

$$\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2)$$

The existential can be eliminated using the Q.E procedure for DLOWE, and we get:

$$l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2$$

Combining this formula with the terms that were pulled out, we get the simplified formula:

$$
\begin{aligned}
&((l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2)) \\
&\qquad \wedge(l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2)
\end{aligned}
$$

The next four disjuncts are:

$$
\begin{aligned}
&\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_1 > z \land z > u_1)\\
\lor&\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_2 > z \land z > u_1)\\
\lor&\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_1 > z \land z > u_2)\\
\lor&\exists z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \land l_2 > z \land z > u_2)
\end{aligned}
$$

and they can be eliminated since they contain terms that are not satisfiable (e.g. $l_1 < z \land l_1 > z$ which cannot be satisfied).
The next twelve disjuncts contain terms of the form $l_1 = z$ and also $l_1 < z$ or $l_1 > z$ which are not satisfiable together.

Bringing back the outermost negation, we get the simplified formula:

$$
\begin{aligned}
\neg((l_1 > u_1 \lor l_1 = u_1) \lor (l_2 > u_1 \lor l_2 = u_1) \lor (l_1 > u_2 \lor l_1 = u_2) \lor (l_2 > u_2 \lor l_2 = u_2))\\
\land(l_1 < u_1 \land l_1 < u_2 \land l_2 < u_1 \land l_2 < u_2)
\end{aligned}
$$

Simplifying the above formula (and using the fact that $\neg(l_1 > u_1 \lor l_1 = u_1) \equiv l_1 < u_1$), we get:

$$
\begin{aligned}
&(l_1 < u_1 \land l_1 < u_2 \land l_2 < u_1 \land l_2 < u_2)\\
\lor\neg&(l_1 < u_1 \land l_1 < u_2 \land l_2 < u_1 \land l_2 < u_2)
\end{aligned}
$$

which is a equivalent to the original formula.
We can also see that the formula is a tautology of the form $A \lor \neg A$, where A is $l_1 < u_1 \land l_1 < u_2 \land l_2 < u_1 \land l_2 < u_2$.
Hence, the formula simplifies to $\top$.

Task 2:
Encoding the above formula in Z3 and using the Q.E tactic:

```
    from z3 import *

l1 = Real("l1")
l2 = Real("l2")
u1 = Real("u1")
u2 = Real("u2")
w = Real("w")
z = Real("z")

goal = Goal()
goal.add(
    ForAll(
        [z],
        Implies(
            And(l1 < z, z < u1, l2 < z, z < u2),
            Exists([w], And(l1 < w, w < u1, l2 < w, w < u2, w != z)),
        ),
    )
)

tactic = Tactic("qe")
```

```
result = tactic(goal)
print(result)
```

The output of the above code is:

```
[[]]
```

(b)
Task 1:
Let us name the vertices of the given graph as A, B, C, D where A is the topmost vertex and D is the bottommost vertex.
C is the vertex on the left and B is the vertex on the right.
The following vertices are connected by an edge: A and B, A and C, B and D, C and D.
Let us represent each vertex as an interval, using (l_A, u_A) to represent vertex A, and so on.

Problem 3
Soln:
