CS 474

Homework 3

Name: Adharsh Kamath NetID: ak128

Problem 1

Soln: We are given three models with signatures $(\mathrm{U}, 0, 1, +, *)$ where U is $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$.

(a)

For each formula given below, we shall see if the formula is true or false over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$.

1. $\exists y.(y * y = 1 + 1)$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
False	False	True

This formula is true over $\mathbb{R}$ since there is a real number ($\sqrt{2}$) which satisfies the formula. However, the same is not true for $\mathbb{N}$ and $\mathbb{Q}$.

2. $\forall x.\exists y.(x + y = 0)$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
False	True	True

This formula is true over $\mathbb{Q}$ and $\mathbb{R}$ since for any $x \in \mathbb{Q}$ (or $\mathbb{R}$), there exists a $y \in \mathbb{Q}$ (or $\mathbb{R}$) such that $x + y = 0$. (We can take $y = -x$ when $x \neq 0$ and $y = 0$ otherwise). However, this is not true for $\mathbb{N}$ since there is no $y \in \mathbb{N}$ such that $x + y = 0$ for any $x \in \mathbb{N}$.

3. $\forall x.\forall y.(\neg(y = 0) \Rightarrow (\exists z.x * y = x + z))$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
True	True	True

This formula is true over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ since for any $x, y \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$), we can find a $z \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$) such that $x * y = x + z$. (We can take $z = x * (y - 1)$, and still ensure $z \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$). This is possible since the premise of the implication is that $y \neq 0$).

4. $\exists x.\exists y.(x + 1 = 0 \wedge y * y = x)$

$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
False	False	False

This formula is false over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ since there is no $x, y \in \mathbb{N}$ (or $\mathbb{Q}$ or $\mathbb{R}$) such that $x + 1 = 0$ and $y * y = x$. This formula can evaluate to true only if $x = -1$ and $y = \pm\sqrt{-1}$, which is possible only in $\mathbb{C}$ (complex numbers) and not in $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{R}$.

(b)

Task 1:

We need to write a formula $gt_{\mathbb{N}}(x, y)$ that is true precisely when $x > y$ for any $x, y \in \mathbb{N}$.

We can write the formula as $gt_{\mathbb{N}}(x, y) = \exists z.(z \neq 0 \land x = y + z)$.

Task 2:
We need to write a formula $gt_{\mathbb{R}}(x, y)$ that is true precisely when $x > y$ for any $x, y \in \mathbb{R}$.
We can write the formula as $gt_{\mathbb{R}}(x, y) = \exists z.(z \neq 0 \land x = y + (z * z))$.

(c)
We need to write a formula that is true over $\mathbb{N}$, but false over $\mathbb{R}$.
Using the definitions of $gt_{\mathbb{N}}(x, y)$ and $gt_{\mathbb{R}}(x, y)$ from above, we can write the following formula:

$$\forall x.(x > 0 \lor x = 0)$$

where $x > 0$ is $gt_{\mathbb{N}}(x, 0)$ for $\mathbb{N}$ and $x > 0$ is $gt_{\mathbb{R}}(x, 0)$ for $\mathbb{R}$. This formula is true over $\mathbb{N}$ since for any $x \in \mathbb{N}$, x is greater than or equal to 0, which is not true for $\mathbb{R}$.

Problem 2
Soln: (a)
Task 1:
We are given the following formula over the logic $(\mathbb{R}, 0, 1, <)$:

$$\begin{aligned} &\forall z.(l_1 < z \land z < u_1 \land l_2 < z \land z < u_2 \\ \Rightarrow &(\exists w.\, l_1 < w \land w < u_1 \land l_2 < w \land w < u_2 \land w \neq z)) \end{aligned}$$

Consider the inner formula that is quantified by $\exists w$:

$$\exists w.(l_1 < w \land w < u_1 \land l_2 < w \land w < u_2 \land w \neq z)$$

Rewriting $w \neq z$ as $\neg(w = z)$, which is equivalent to $w < z \lor w > z$, we get:

$$\exists w.(l_1 < w \land w < u_1 \land l_2 < w \land w < u_2 \land (w < z \lor w > z))$$

Applying the distributive property and simplifying, we get the DNF form:

$$\begin{aligned} &\exists w.(l_1 < w \land w < u_1 \land l_2 < w \land w < u_2 \land w < z) \\ \lor &\exists w.(l_1 < w \land w < u_1 \land l_2 < w \land w < u_2 \land w > z) \end{aligned}$$

Consider the first disjunct:

$$\exists w.(l_1 < w \land w < u_1 \land l_2 < w \land w < u_2 \land w < z)$$

In the context of the Q.E procedure for DLOWE, the lower bounds in this case are l_1, l_2 and the upper bounds are u_1, u_2, z. Eliminating the existential quantifier, we get:

$$(l_1 < u_1 \land l_2 < u_1 \land l_1 < u_2 \land l_2 < u_2 \land l_1 < z \land l_2 < z)$$

Consider the second disjunct:

$$\exists w.(l_1 < w \land w < u_1 \land l_2 < w \land w < u_2 \land w > z)$$

In the context of the Q.E procedure for DLOWE, the lower bounds in this case are l_1, l_2, z and the upper bounds are u_1, u_2. Eliminating the existential quantifier, we get:

$$(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2)$$

Combining the two disjuncts, we get:

$$\begin{aligned}(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)\\ \vee(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2)\end{aligned}$$

Thus we have eliminated the existential quantifier $\exists w$ from the original formula.

Therefore, the original formula can be simplified to:

$$\begin{aligned}\forall z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2\\ \Rightarrow ((l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)\\ \vee(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2)))\end{aligned}$$

Rewriting the implication as a disjunction, we get:

$$\begin{aligned}\forall z.\neg(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2)\\ \vee((l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)\\ \vee(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2))\end{aligned}$$

Rewriting the universal quantifier as an existential quantifier, and pushing the negation inside, we get:

$$\begin{aligned}\neg(\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2))\\ \wedge\neg((l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)\\ \vee(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2))\end{aligned}$$

Consider the inner existential formula:

$$\begin{aligned}\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2)\\ \wedge\neg((l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)\\ \vee(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2))\end{aligned}$$

Applying De Morgan's law, we get:

$$\begin{aligned}\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2)\\ \wedge(\neg(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)\\ \wedge\neg(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2))\end{aligned}$$

Consider the first inner negated formula:

$$\neg(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge l_1 < z \wedge l_2 < z)$$

Using De Morgan's law, we get:

$$\begin{aligned}(l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2)\\ \vee(l_1 > z \vee l_1 = z) \vee (l_2 > z \vee l_2 = z)\end{aligned}$$

Similarly for the other inner negated formula:

$$\neg(l_1 < u_1 \wedge l_2 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_2 \wedge z < u_1 \wedge z < u_2)$$

Applying De Morgan's law, we get:

$$\begin{aligned}(l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2)\\ \vee(z > u_1 \vee z = u_1) \vee (z > u_2 \vee z = u_2)\end{aligned}$$

The conjunction of the above two formulae is:

$$\begin{aligned}((l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2)\\ \vee(l_1 > z \vee l_1 = z) \vee (l_2 > z \vee l_2 = z))\\ \wedge\\ ((l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2)\\ \vee(z > u_1 \vee z = u_1) \vee (z > u_2 \vee z = u_2))\end{aligned}$$

Applying the distributive property and simplifying, we get:

$$\begin{aligned}(l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2)\\ \vee((l_1 > z \vee l_1 = z) \wedge (z > u_1 \vee z = u_1)) \vee ((l_2 > z \vee l_2 = z) \wedge (z > u_1 \vee z = u_1))\\ \vee((l_1 > z \vee l_1 = z) \wedge (z > u_2 \vee z = u_2)) \vee ((l_2 > z \vee l_2 = z) \wedge (z > u_2 \vee z = u_2))\end{aligned}$$

(The term after applying distributive property and before simplification is omitted for brevity). Putting the above back into the original formula, we get:

$$\begin{aligned}\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2)\\ \wedge(l_1 > u_1 \vee l_1 = u_1 \vee l_2 > u_1 \vee l_2 = u_1 \vee l_1 > u_2 \vee l_1 = u_2 \vee l_2 > u_2 \vee l_2 = u_2\\ \vee((l_1 > z \vee l_1 = z) \wedge (z > u_1 \vee z = u_1)) \vee ((l_2 > z \vee l_2 = z) \wedge (z > u_1 \vee z = u_1))\\ \vee((l_1 > z \vee l_1 = z) \wedge (z > u_2 \vee z = u_2)) \vee ((l_2 > z \vee l_2 = z) \wedge (z > u_2 \vee z = u_2)))\end{aligned}$$

Applying distributive property and simplifying, we get:

$$\begin{aligned}\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > u_1)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = u_1)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > u_1)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = u_1)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > u_2)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = u_2)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > u_2)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = u_2)\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge (l_1 > z \vee l_1 = z) \wedge (z > u_1 \vee z = u_1))\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge (l_2 > z \vee l_2 = z) \wedge (z > u_1 \vee z = u_1))\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge (l_1 > z \vee l_1 = z) \wedge (z > u_2 \vee z = u_2))\\ \vee\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge (l_2 > z \vee l_2 = z) \wedge (z > u_2 \vee z = u_2))\end{aligned}$$

Simplifying the last four disjuncts, we get:

$$
\begin{aligned}
&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z > u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z > u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z = u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z = u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z = u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z = u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z = u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z > u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 = z \wedge z = u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z = u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z > u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 = z \wedge z = u_2)
\end{aligned}
$$

The following terms can be pulled out from the first eight disjuncts since they do not refer to z:

$$(l_1 > u_1) \vee (l_1 = u_1) \vee (l_2 > u_1) \vee (l_2 = u_1) \vee (l_1 > u_2) \vee (l_1 = u_2) \vee (l_2 > u_2) \vee (l_2 = u_2)$$

Upon removing the above terms, the first eight disjuncts all become the same:

$$\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2)$$

The existential can be eliminated using the Q.E procedure for DLOWE, and we get:

$$l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2$$

Combining this formula with the terms that were pulled out, we get the simplified formula:

$$
\begin{aligned}
&((l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2))\\
&\qquad \wedge(l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2)
\end{aligned}
$$

The next four disjuncts are:

$$\begin{aligned}
&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z > u_1)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_1 > z \wedge z > u_2)\\
\vee&\exists z.(l_1 < z \wedge z < u_1 \wedge l_2 < z \wedge z < u_2 \wedge l_2 > z \wedge z > u_2)
\end{aligned}$$

and they can be eliminated since they contain terms that are not satisfiable
(e.g. $l_1 < z \wedge l_1 > z$ which cannot be satisfied).
The next twelve disjuncts contain terms of the form $l_1 = z$ and also $l_1 < z$ or $l_1 > z$ which are not satisfiable together.

Bringing back the outermost negation, we get the simplified formula:

$$\begin{aligned}
\neg((l_1 > u_1 \vee l_1 = u_1) \vee (l_2 > u_1 \vee l_2 = u_1) \vee (l_1 > u_2 \vee l_1 = u_2) \vee (l_2 > u_2 \vee l_2 = u_2))\\
\wedge(l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2)
\end{aligned}$$

Simplifying the above formula (and using the fact that $\neg(l_1 > u_1 \vee l_1 = u_1) \equiv l_1 < u_1$), we get:

$$\begin{aligned}
&(l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2)\\
\vee&\neg(l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2)
\end{aligned}$$

which is a equivalent to the original formula.
We can also see that the formula is a tautology of the form $A \vee \neg A$, where A is $l_1 < u_1 \wedge l_1 < u_2 \wedge l_2 < u_1 \wedge l_2 < u_2$.
Hence, the formula simplifies to $\top$.

Task 2:
Encoding the above formula in Z3 and using the Q.E tactic:

```
    from z3 import *

l1 = Real("l1")
l2 = Real("l2")
u1 = Real("u1")
u2 = Real("u2")
w = Real("w")
z = Real("z")

goal = Goal()
goal.add(
    ForAll(
        [z],
        Implies(
            And(l1 < z, z < u1, l2 < z, z < u2),
            Exists([w], And(l1 < w, w < u1, l2 < w, w < u2, w != z)),
        ),
    )
)

tactic = Tactic("qe")
```

```
result = tactic(goal)
print(result)
```

The output of the above code is:

```
[[]]
```

(b)

Task 1:

Let us name the vertices of the given graph as A, B, C, D where A is the topmost vertex and D is the bottommost vertex. C is the vertex on the left and B is the vertex on the right.

The following vertices are connected by an edge: A and B, A and C, B and D, C and D.

Let us represent each vertex as an interval, using (l_A, u_A) to represent vertex A, and so on.

If G is an interval graph, for two intervals (l_A, u_A) and (l_B, u_B) to be connected by an edge there must exist a $z \in \mathbb{R}$ such that $l_A < z < u_A$ and $l_B < z < u_B$.

Written as a formula, this is:

$$\exists z.(l_A < z \wedge z < u_A) \wedge (l_B < z \wedge z < u_B)$$

For the given graph, the conjunction of the following formulae must be true (for it to be an interval graph):

$$\begin{aligned} &\exists z.(l_A < z \wedge z < u_A) \wedge (l_B < z \wedge z < u_B) \\ \wedge &\exists z_1.(l_A < z_1 \wedge z_1 < u_A) \wedge (l_C < z_1 \wedge z_1 < u_C) \\ \wedge &\exists z_2.(l_B < z_2 \wedge z_2 < u_B) \wedge (l_D < z_2 \wedge z_2 < u_D) \\ \wedge &\exists z_3.(l_C < z_3 \wedge z_3 < u_C) \wedge (l_D < z_3 \wedge z_3 < u_D) \end{aligned}$$

Each existential formula represents an edge in the graph.

We also need to ensure that if two vertices are not connected by an edge, then their intervals do not intersect.

We can state the above as:

$$\begin{aligned} &\neg(\exists z.(l_A < z \wedge z < u_A) \wedge (l_D < z \wedge z < u_D)) \\ \wedge &\neg(\exists z.(l_B < z \wedge z < u_B) \wedge (l_C < z \wedge z < u_C)) \end{aligned}$$

Combining the above formulae, we get the required formula:

$$\begin{aligned} \alpha_G = &\exists z.(l_A < z \wedge z < u_A) \wedge (l_B < z \wedge z < u_B) \\ \wedge &\exists z_1.(l_A < z_1 \wedge z_1 < u_A) \wedge (l_C < z_1 \wedge z_1 < u_C) \\ \wedge &\exists z_2.(l_B < z_2 \wedge z_2 < u_B) \wedge (l_D < z_2 \wedge z_2 < u_D) \\ \wedge &\exists z_3.(l_C < z_3 \wedge z_3 < u_C) \wedge (l_D < z_3 \wedge z_3 < u_D) \\ \wedge &\neg(\exists z_4.(l_A < z_4 \wedge z_4 < u_A) \wedge (l_D < z_4 \wedge z_4 < u_D)) \\ \wedge &\neg(\exists z_5.(l_B < z_5 \wedge z_5 < u_B) \wedge (l_C < z_5 \wedge z_5 < u_C)) \end{aligned}$$

Task 2:

Encoding the above formula, α_G, in Z3:

```
from z3 import *

la = Real("la")
ua = Real("ua")
lb = Real("lb")
ub = Real("ub")
lc = Real("lc")
uc = Real("uc")
ld = Real("ld")
ud = Real("ud")
z = Real("z")
z1 = Real("z1")
z2 = Real("z2")
z3 = Real("z3")
z4 = Real("z4")
z5 = Real("z5")

s = Solver()
s.add(
  And(
    Exists([z], And(la < z, z < ua, lb < z, z < ub)),
    Exists([z1], And(la < z1, z1 < ua, lc < z1, z1 < uc)),
    Exists([z2], And(ld < z2, z2 < ud, lb < z2, z2 < ub)),
    Exists([z3], And(ld < z3, z3 < ud, lc < z3, z3 < uc)),
    Not(Exists([z4], And(la < z4, z4 < ua, ld < z4, z4 < ud))),
    Not(Exists([z5], And(lb < z5, z5 < ub, lc < z5, z5 < uc))),
  )
)
print(s.check())
```

The output of the above code is:

```
unsat
```

Thus, the given graph is an not an interval graph.

Problem 3

Soln:

We are given the following formula over rationals with addition and less than symbols:

$$\psi \equiv \forall x.\exists y.((2y > 3x) \wedge (4y < 8x + 10))$$

Consider the inner formula that is quantified by $\exists y$:

$$(2y > 3x) \wedge (4y < 8x + 10)$$

We can rewrite the above formula as:

$$(y > \frac{3}{2}x) \wedge (y < \frac{8}{4}x + \frac{10}{4})$$

Now we can define the sub-formulae in the Ferrante-Rackoff procedure:

$$\psi_1 = (\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 > \frac{3}{2}x \wedge (\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 < \frac{8}{4}x + \frac{10}{4}$$
$$\psi_2 = ((\frac{3}{2}x + \frac{3}{2}x)/2 > \frac{3}{2}x) \wedge ((\frac{3}{2}x + \frac{3}{2}x)/2) < \frac{8}{4}x + \frac{10}{4}$$
$$\psi_3 = ((\frac{8}{4}x + \frac{10}{4}) + (\frac{8}{4}x + \frac{10}{4}))/2 > \frac{3}{2}x \wedge ((\frac{8}{4}x + \frac{10}{4}) + (\frac{8}{4}x + \frac{10}{4}))/2 < \frac{8}{4}x + \frac{10}{4}$$

We also need to include the following:

$$\psi_{-\infty} = \bot \wedge \top$$
$$\psi_{+\infty} = \top \wedge \bot$$

The final formula can be written as:

$$\psi' = \psi_1 \vee \psi_2 \vee \psi_3 \vee \psi_{-\infty} \vee \psi_{+\infty}$$

But as we can see from above, $\psi_{-\infty}$ and $\psi_{+\infty}$ and evaluate to $\bot$. Hence, we can eliminate them from the formula.
Therefore, the formula can be written as:

$$\begin{aligned}\psi' = \psi_1 \vee \psi_2 \vee \psi_3 = ((\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 > \frac{3}{2}x \wedge (\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 < \frac{8}{4}x + \frac{10}{4}) \\ \vee((\frac{3}{2}x + \frac{3}{2}x)/2 > \frac{3}{2}x \wedge (\frac{3}{2}x + \frac{3}{2}x)/2 < \frac{8}{4}x + \frac{10}{4}) \\ \vee(((\frac{8}{4}x + \frac{10}{4}) + (\frac{8}{4}x + \frac{10}{4}))/2 > \frac{3}{2}x \wedge ((\frac{8}{4}x + \frac{10}{4}) + (\frac{8}{4}x + \frac{10}{4}))/2 < \frac{8}{4}x + \frac{10}{4})\end{aligned}$$

We can eliminate ψ_2 and ψ_3 since they cannot be satisfied (They contain a term of the form $p > p$ or $p < p$, where p is $\frac{3}{2}x$ or $\frac{8}{4}x + \frac{10}{4}$).

Therefor, the formula without the $\exists y$ quantifier can be written as:

$$\psi' = \psi_1 = ((\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 > \frac{3}{2}x \wedge (\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 < \frac{8}{4}x + \frac{10}{4})$$

Putting the above back into the original formula, we get:

$$\forall x.((\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 > \frac{3}{2}x \wedge (\frac{3}{2}x + \frac{8}{4}x + \frac{10}{4})/2 < \frac{8}{4}x + \frac{10}{4})$$

Rewriting the above to contain terms of the form $x < c$ and $x > c$, we get:

$$
\begin{aligned}
\forall x.((\frac{6}{4}x + \frac{8}{4}x + \frac{10}{4})/2 > \frac{3}{2}x) \wedge ((\frac{6}{4}x + \frac{8}{4}x + \frac{10}{4})/2 < \frac{8}{4}x + \frac{10}{4}) \\
= \forall x.((\frac{6}{8}x + \frac{8}{8}x + \frac{10}{8}) > \frac{3}{2}x) \wedge ((\frac{6}{8}x + \frac{8}{8}x + \frac{10}{8}) < \frac{8}{4}x + \frac{10}{4}) \\
= \forall x.((\frac{10}{8} > \frac{3}{2}x - \frac{6}{8}x - \frac{8}{8}x) \wedge (\frac{10}{8} < \frac{8}{4}x + \frac{10}{4} - \frac{6}{8}x - \frac{8}{8}x)) \\
= \forall x.((\frac{10}{8} > \frac{12}{8}x - \frac{6}{8}x - \frac{8}{8}x) \wedge (\frac{10}{8} < \frac{16}{8}x - \frac{6}{8}x - \frac{8}{8}x)) + \frac{10}{4} \\
= \forall x.((\frac{10}{8} > \frac{-2}{8}x) \wedge (\frac{10}{8} - \frac{10}{4} < \frac{2}{8}x)) \\
= \forall x.((\frac{10}{8} * \frac{8}{-2} > x) \wedge (\frac{-10}{8} * \frac{8}{2} < x)) \\
= \forall x.((\frac{10}{-2} > x) \wedge (\frac{-10}{2} < x)) \\
= \forall x.((20 > x) \wedge (20 < x))
\end{aligned}
$$

We have simplified the formula to:

$$\forall x.(20 > x) \wedge (20 < x)$$

Rewriting the formula as an existential formula, we get:

$$\neg \exists x.\neg((20 > x) \wedge (20 < x))$$

Consider the inner quantified formula:

$$\neg((20 > x) \wedge (20 < x))$$

Defining the sub-formulae in the Ferrante-Rackoff procedure:

$$
\begin{aligned}
\psi_1 = (20 > (20 + 20)/2) \wedge (20 < (20 + 20)/2) \\
\psi_{-\infty} = \perp \wedge \top \\
\psi_{+\infty} = \top \wedge \perp
\end{aligned}
$$

The final formula can be written as:

$$\psi' = \psi_1 \vee \psi_{-\infty} \vee \psi_{+\infty}$$

But as we can see from above, $\psi_{-\infty}$ and $\psi_{+\infty}$ and evaluate to $\perp$. Hence, we can eliminate them from the formula.
Therefore, the formula can be written as:

$$
\begin{aligned}
\psi' = \psi_1 = (20 > (20 + 20)/2) \wedge (20 < (20 + 20)/2) \\
= (20 > 20) \wedge (20 < 20) \\
= \perp \wedge \perp
\end{aligned}
$$