CS 474
Homework 4

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Problem 1

Soln: A group is defined as a set S, along with a binary operation $\cdot : S \times S \to S$ and axioms:

1. $\forall a\,, b\,, c\,.\,(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. $\forall a\,.\,((a \cdot e) = a \wedge (e \cdot a) = a)$
3. $\forall a\,\exists b\,.\,((b \cdot a) = e \wedge (a \cdot b) = e)$

where e is the special constant that is the identity element of the group.

Task 1:

$$G:\ \forall e'\,.\,((\forall a\,.\,((e \cdot a) = a\ \wedge (e' \cdot a) = a)) \implies (e = e'))$$

The first two axioms are in prenex normal form, and do not contain any existential quantifiers. Skolemizing the third axiom (by replacing $\exists b$ with a function $f(a)$), we get:

$$\forall a\,.\,(a \cdot f(a)) = e \wedge (a \cdot f(a)) = e$$

Simplifying the goal G, to prenex normal form:

$$\begin{aligned}
&\forall e'\,\forall a\,.\,((a \cdot e') = a\ \wedge (e' \cdot a) = a) \implies (e = e')\\
\equiv\ &\forall e'\,\neg(\forall a\,.\,((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \vee (e = e')\\
\equiv\ &\forall e'\,(\exists a\,.\,\neg((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \vee (e = e')
\end{aligned}$$

To show the validity of the goal, we need to show that the negation of the goal is unsatisfiable along with all the axioms.

$$\begin{aligned}
\neg G &\equiv \neg(\forall e'\,(\exists a\,.\,\neg((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \vee (e = e'))\\
&\equiv \exists e'\,\neg(\exists a\,.\,\neg((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \wedge \neg(e = e')\\
&\equiv \exists e'\,(\forall a\,.\,((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \wedge \neg(e = e')
\end{aligned}$$

Skolemizing (with a constant e'' since $\exists e'$ is the outermost existential quantifier), we get:

$$\forall a\,.\,((a \cdot e'') = a\ \wedge (e'' \cdot a) = a \wedge \neg(e = e''))$$

Combining all the axioms, we have:

$$\begin{aligned}
\forall a\,, b\,, c\,.\,(a \cdot b) \cdot c = a \cdot (b \cdot c)\\
\forall a\,.\,((a \cdot e) = a \wedge (e \cdot a) = a)\\
\forall a\,.\,((a \cdot f(a)) = e \wedge (f(a) \cdot a) = e)\\
\forall a\,.\,((a \cdot e'') = a\ \wedge (e'' \cdot a) = a \wedge \neg(e = e''))
\end{aligned}$$

Instantiating with depth-0 terms (e, e''), we get a large set (conjunction) of formulae :

$$
\begin{aligned}
(e \cdot e) \cdot e = e \cdot (e \cdot e) \\
(e \cdot e) \cdot e'' = e \cdot (e \cdot e'') \\
(e \cdot e'') \cdot e = e \cdot (e'' \cdot e) \\
(e \cdot e'') \cdot e'' = e \cdot (e'' \cdot e'') \\
(e'' \cdot e) \cdot e = e'' \cdot (e \cdot e) \\
(e'' \cdot e) \cdot e'' = e'' \cdot (e \cdot e'') \\
(e'' \cdot e'') \cdot e = e'' \cdot (e'' \cdot e) \\
(e'' \cdot e'') \cdot e'' = e'' \cdot (e'' \cdot e'') \\
(e \cdot e) = e \wedge (e \cdot e) = e \\
(e'' \cdot e) = e \wedge (e \cdot e'') = e \\
(e \cdot f(e)) = e \wedge (f(e) \cdot (e)) = e \\
(e'' \cdot f(e'')) = e \wedge (f(e'') \cdot (e'')) = e \\
(e \cdot e'') = e \wedge (e'' \cdot e) = e \wedge \neg(e = e'') \\
(e'' \cdot e'') = e'' \wedge (e'' \cdot e'') = e'' \wedge \neg(e = e'')
\end{aligned}
$$

In this list, we can find two conjuncts that are contradictory:

$$
\begin{aligned}
(e \cdot e'') = e'' \wedge (e'' \cdot e) = e'' \\
(e'' \cdot e) = e \wedge (e \cdot e'') = e \wedge \neg(e = e'')
\end{aligned}
$$

Hence, there exists no model that satisfies all the formulae. Therefore, the goal G is valid.

The corresponding Z3 program is at the following URL: https://github.com/adharshkamath/logic-hw-4/blob/main/p1_1.py

Task 2: Given the skolemized set of axioms:

$$
\begin{aligned}
\forall a\,, b\,, c\,.\, (a \cdot b) \cdot c = a \cdot (b \cdot c) \\
\forall a\,.\, ((a \cdot e) = a \wedge (e \cdot a) = a) \\
\forall a\,.\, ((a \cdot f(a)) = e \wedge (a \cdot f(a)) = e)
\end{aligned}
$$

we can see that f is the inverse function. We can simplify the goal to

$$G:\ \forall a, b.(((a \cdot b = e) \wedge (b \cdot a = e)) \implies (b = f(a)))$$

Negating this goal gives us:

$$\neg G:\ \exists a, b.(((a \cdot b = e) \wedge (b \cdot a = e)) \wedge \neg(b = f(a)))$$

Skolemizing (with constants a', b') gives us:

$$((a' \cdot b' = e) \wedge (b' \cdot a' = e)) \wedge \neg(b' = f(a'))$$

Instantiating the axioms with depth-0 terms (a', b', e), we get a long list (conjunction) of formulae:

$$(a' \cdot a') \cdot a' = a' \cdot (a' \cdot a')$$
$$(a' \cdot a') \cdot b' = a' \cdot (a' \cdot b')$$
$$(a' \cdot b') \cdot a' = a' \cdot (b' \cdot a')$$
$$(a' \cdot b') \cdot b' = a' \cdot (b' \cdot b')$$
$$(b' \cdot a') \cdot a' = b' \cdot (a' \cdot a')$$
$$(b' \cdot a') \cdot b' = b' \cdot (a' \cdot b')$$
$$(b' \cdot b') \cdot a' = b' \cdot (b' \cdot a')$$
$$(b' \cdot b') \cdot b' = b' \cdot (b' \cdot b')$$
$$(a' \cdot a') \cdot e = a' \cdot (a' \cdot e)$$
$$(a' \cdot e) \cdot a' = a' \cdot (e \cdot a')$$
$$(a' \cdot e) \cdot e = a' \cdot (e \cdot e)$$
$$(e \cdot a') \cdot a' = e \cdot (a' \cdot a')$$
$$(e \cdot a') \cdot e = e \cdot (a' \cdot e)$$
$$(e \cdot e) \cdot a' = e \cdot (e \cdot a')$$
$$(e \cdot e) \cdot e = e \cdot (e \cdot e)$$
$$(b' \cdot b') \cdot e = b' \cdot (b' \cdot e)$$
$$(b' \cdot e) \cdot b' = b' \cdot (e \cdot b')$$
$$(b' \cdot e) \cdot e = b' \cdot (e \cdot e)$$
$$(e \cdot b') \cdot b' = e \cdot (b' \cdot b')$$
$$(e \cdot b') \cdot e = e \cdot (b' \cdot e)$$
$$(e \cdot e) \cdot b' = e \cdot (e \cdot b')$$
$$(a' \cdot b') \cdot e = a' \cdot (b' \cdot e)$$
$$(a' \cdot e) \cdot b' = a' \cdot (e \cdot b')$$
$$(e \cdot a') \cdot b' = e \cdot (a' \cdot b')$$
$$(e \cdot b') \cdot a' = e \cdot (b' \cdot a')$$
$$(b' \cdot a') \cdot e = b' \cdot (a' \cdot e)$$
$$(b' \cdot e) \cdot a' = b' \cdot (e \cdot a')$$
$$(a' \cdot e) = a' \wedge (e \cdot a') = a'$$
$$(b' \cdot e) = b' \wedge (e \cdot b') = b'$$
$$(e \cdot e) = e \wedge (e \cdot e) = e$$
$$(a' \cdot f(a')) = e \wedge (f(a') \cdot a') = e$$
$$(b' \cdot f(b')) = e \wedge (f(b') \cdot b') = e$$
$$(e \cdot f(e)) = e \wedge (f(e) \cdot e) = e$$

In this list, we can find two conjuncts that are contradictory:

$$(a' \cdot f(a')) = e \wedge (f(a') \cdot a') = e$$
$$(b' \cdot f(b')) = e \wedge (f(b') \cdot b') = e$$

The Z3 program for this task is at the following URL: https://github.com/adharshkamath/logic-hw-4/blob/main/p1_pt2.py

Problem 2

Soln:

(a)

The given formula φ

$$\varphi = y \leq x \wedge x \leq y \wedge f(y) = f(7) \wedge x \leq 5$$

contains terms from T_{UIF} and T_N (theory of UIF and theory of natural numbers).
Replacing $f(7)$ with $f(w)$, and adding $w = 7$ as an additional formula, we get:

$$\varphi' = y \leq x \wedge x \leq y \wedge x \leq 5 \wedge w = 7 \wedge f(y) = f(w)$$

The part of the formula in Σ_{UIF} :

$$F_{UIF} : f(y) = f(w)$$

and the part of the formula in Σ_N :

$$F_N : y \leq x \wedge x \leq y \wedge x \leq 5 \wedge w = 7$$

The conjunct $F_{UIF} \wedge F_N$ is $(T_{UIF} \cup T_N)$-equisatisfiable to φ.

(b)

The shared variables between F_{UIF} and F_N are:

$$V = \text{shared}(F_{UIF}, F_N) = \{y, w\}$$

The two possible equivalence relations and their arrangements are:

$$\begin{aligned} E_1 = \{\{y\}, \{w\}\}, &\quad \alpha_1(E_1, V) : y = w \\ E_2 = \{\{y, w\}\}, &\quad \alpha_2(E_2, V) : y \neq w \end{aligned}$$

Simplifying F_N, we can write:

$$\begin{aligned} F_N &= y \leq x \wedge x \leq y \wedge x \leq 5 \wedge w = 7 \\ &\equiv x = y \wedge x \leq 5 \wedge w = 7 \end{aligned}$$

We can see that F_N is satisfiable over natural numbers only under the arrangement α_2 (since $y \leq 5$ and $w = 7$). A satisfying model for F_N is: $\{x = 5, y = 5, w = 7\}$

We can see that F_{UIF} is satisfiable under this arrangement only if f maps all arguments to a constant (since we need to satisfy $f(y) = f(w) \wedge y \neq w$). A satisfying model for F_{UIF} is: $\{f(y) = c\}$ where c is any arbitrary constant.

From the above, we can derive a model for the original formula φ. A satisfying model for φ is: $\{x = 5, y = 5, f(t) = c\}$ where t is any term in the combined domains.

Problem 3

Soln:

(a)

The given formula f is defined over 2^N which is unbounded. Since f is monotone, there is no least fixed point for f.
