

CS 474

Homework 4

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Problem 1

Soln: A group is defined as a set S , along with a binary operation $\cdot : S \times S \rightarrow S$ and axioms:

1. $\forall a, b, c. (a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. $\forall a. ((a \cdot e) = a \wedge (e \cdot a) = a)$
3. $\forall a \exists b. ((b \cdot a) = e \wedge (a \cdot b) = e)$

where e is the special constant that is the identity element of the group.

Task 1:

$$G : \forall e'. ((\forall a. ((e \cdot a) = a \wedge (e' \cdot a) = a)) \implies (e = e'))$$

The first two axioms are in prenex normal form, and do not contain any existential quantifiers. Skolemizing the third axiom (by replacing $\exists b$ with a function $f(a)$), we get:

$$\forall a. (a \cdot f(a)) = e \wedge (a \cdot f(a)) = e$$

Simplifying the goal G , to prenex normal form:

$$\begin{aligned} & \forall e' \forall a. ((a \cdot e') = a \wedge (e' \cdot a) = a) \implies (e = e') \\ & \equiv \forall e' \neg(\forall a. ((a \cdot e') = a \wedge (e' \cdot a) = a)) \vee (e = e') \\ & \equiv \forall e' (\exists a. \neg((a \cdot e') = a \wedge (e' \cdot a) = a)) \vee (e = e') \end{aligned}$$

To show the validity of the goal, we need to show that the negation of the goal is unsatisfiable along with all the axioms.

$$\begin{aligned} \neg G & \equiv \neg(\forall e' (\exists a. \neg((a \cdot e') = a \wedge (e' \cdot a) = a)) \vee (e = e')) \\ & \equiv \exists e' \neg(\exists a. \neg((a \cdot e') = a \wedge (e' \cdot a) = a)) \wedge \neg(e = e') \\ & \equiv \exists e' (\forall a. ((a \cdot e') = a \wedge (e' \cdot a) = a)) \wedge \neg(e = e') \end{aligned}$$

Skolemizing (with a constant e'' since $\exists e'$ is the outermost existential quantifier), we get:

$$\forall a. ((a \cdot e'') = a \wedge (e'' \cdot a) = a \wedge \neg(e = e''))$$

Combining all the axioms, we have:

$$\begin{aligned} & \forall a, b, c. (a \cdot b) \cdot c = a \cdot (b \cdot c) \\ & \forall a. ((a \cdot e) = a \wedge (e \cdot a) = a) \\ & \forall a. ((a \cdot f(a)) = e \wedge (f(a) \cdot a) = e) \\ & \forall a. ((a \cdot e'') = a \wedge (e'' \cdot a) = a \wedge \neg(e = e'')) \end{aligned}$$

Instantiating with depth-0 terms (e, e'') , we get a large set of formulae:

$$\begin{aligned}
(e \cdot e) \cdot e &= e \cdot (e \cdot e) \\
(e \cdot e) \cdot e'' &= e \cdot (e \cdot e'') \\
(e \cdot e'') \cdot e &= e \cdot (e'' \cdot e) \\
(e \cdot e'') \cdot e'' &= e \cdot (e'' \cdot e'') \\
(e'' \cdot e) \cdot e &= e'' \cdot (e \cdot e) \\
(e'' \cdot e) \cdot e'' &= e'' \cdot (e \cdot e'') \\
(e'' \cdot e'') \cdot e &= e'' \cdot (e'' \cdot e) \\
(e'' \cdot e'') \cdot e'' &= e'' \cdot (e'' \cdot e'') \\
(e \cdot e) &= e \wedge (e \cdot e) = e \\
(e'' \cdot e) &= e \wedge (e \cdot e'') = e \\
(e \cdot f(e)) &= e \wedge (f(e) \cdot (e)) = e \\
(e'' \cdot f(e'')) &= e \wedge (f(e'') \cdot (e'')) = e \\
(e \cdot e'') &= e \wedge (e'' \cdot e) = e \wedge \neg(e = e'') \\
(e'' \cdot e'') &= e'' \wedge (e'' \cdot e'') = e'' \wedge \neg(e = e'')
\end{aligned}$$

In this list, we can find two conjuncts that are contradictory:

$$\begin{aligned}
(e \cdot e'') &= e'' \wedge (e'' \cdot e) = e'' \\
(e'' \cdot e) &= e \wedge (e \cdot e'') = e \wedge \neg(e = e'')
\end{aligned}$$

Hence, there exists no model that satisfies all the formulae. Therefore, the goal G is valid.

The corresponding Z3 program is at the following URL: (<https://tinyurl.com/abcd>)

Task 2: Given the skolemized set of axioms:

$$\begin{aligned}
\forall a, b, c. (a \cdot b) \cdot c &= a \cdot (b \cdot c) \\
\forall a. ((a \cdot e) = a \wedge (e \cdot a) &= a) \\
\forall a. ((a \cdot f(a)) = e \wedge (a \cdot f(a)) &= e)
\end{aligned}$$

we can see that f is the inverse function. We can simplify the goal to

$$G : \forall a, b. ((a \cdot b = e) \wedge (b \cdot a = e)) \implies (b = f(a))$$

Negating this goal gives us:

$$\neg G : \exists a, b. ((a \cdot b = e) \wedge (b \cdot a = e)) \wedge \neg(b = f(a))$$

Skolemizing (with constants a', b') gives us:

$$((a' \cdot b' = e) \wedge (b' \cdot a' = e)) \wedge \neg(b' = f(a'))$$

Instantiating the axioms with depth-0 terms (a', b') , we get a long list of formulae. In this list, we can find a conjunct that contradicts the above formula:

$$((a' \cdot f(a')) = e \wedge (f(a') \cdot a') = e)$$

Problem 2

Soln:

The given formula φ is

$$\varphi = y \leq x \wedge x \leq y \wedge f(y) = f(7) \wedge x \leq 5$$

Problem 3

Soln:

(a)

Given $f : 2^N \rightarrow 2^N$ that is defined as

$$f(S) = \{2\} \cup \{y \mid y = 2x, x \in S\}$$
