CS 474

Homework 4

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Problem 1

Soln: A group is defined as a set S, along with a binary operation $\cdot : S \times S \to S$ and axioms:

1. $\forall a\,,b\,,c\,.\,(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. $\forall a\,.\,((a \cdot e) = a \wedge (e \cdot a) = a)$
3. $\forall a\,\exists b\,.\,((b \cdot a) = e \wedge (a \cdot b) = e)$

where e is the special constant that is the identity element of the group.

Task 1:

$$G:\ \forall e'\,.\,((\forall a\,.\,((e \cdot a) = a\ \wedge (e' \cdot a) = a))\implies (e = e'))$$

The first two axioms are in prenex normal form, and do not contain any existential quantifiers. Skolemizing the third axiom (by replacing $\exists b$ with a function $f(a)$), we get:

$$\forall a\,.\,(a \cdot f(a)) = e \wedge (a \cdot f(a)) = e$$

Simplifying the goal G, to prenex normal form:

$$\begin{aligned}
&\forall e'\,\forall a\,.\,((a \cdot e') = a\ \wedge (e' \cdot a) = a)\implies (e = e')\\
\equiv\ &\forall e'\,\neg(\forall a\,.\,((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \vee (e = e')\\
\equiv\ &\forall e'\,(\exists a\,.\,\neg((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \vee (e = e')
\end{aligned}$$

To show the validity of the goal, we need to show that the negation of the goal is unsatisfiable along with all the axioms.

$$\begin{aligned}
\neg G &\equiv \neg(\forall e'\,(\exists a\,.\,\neg((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \vee (e = e'))\\
&\equiv \exists e'\,\neg(\exists a\,.\,\neg((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \wedge \neg(e = e')\\
&\equiv \exists e'\,(\forall a\,.\,((a \cdot e') = a\ \wedge (e' \cdot a) = a)) \wedge \neg(e = e')
\end{aligned}$$

Skolemizing (with a constant e'' since $\exists e'$ is the outermost existential quantifier), we get:

$$\forall a\,.\,((a \cdot e'') = a\ \wedge (e'' \cdot a) = a \wedge \neg(e = e''))$$

Combining all the axioms, we have:

$$\begin{aligned}
\forall a\,,b\,,c\,.\,(a \cdot b) \cdot c = a \cdot (b \cdot c)\\
\forall a\,.\,((a \cdot e) = a \wedge (e \cdot a) = a)\\
\forall a\,.\,((a \cdot f(a)) = e \wedge (f(a) \cdot a) = e)\\
\forall a\,.\,((a \cdot e'') = a\ \wedge (e'' \cdot a) = a \wedge \neg(e = e''))
\end{aligned}$$

Instantiating with depth-0 terms (e, e''), we get a large set of formulae:

$$
\begin{aligned}
(e \cdot e) \cdot e = e \cdot (e \cdot e) \\
(e \cdot e) \cdot e'' = e \cdot (e \cdot e'') \\
(e \cdot e'') \cdot e = e \cdot (e'' \cdot e) \\
(e \cdot e'') \cdot e'' = e \cdot (e'' \cdot e'') \\
(e'' \cdot e) \cdot e = e'' \cdot (e \cdot e) \\
(e'' \cdot e) \cdot e'' = e'' \cdot (e \cdot e'') \\
(e'' \cdot e'') \cdot e = e'' \cdot (e'' \cdot e) \\
(e'' \cdot e'') \cdot e'' = e'' \cdot (e'' \cdot e'') \\
(e \cdot e) = e \wedge (e \cdot e) = e \\
(e'' \cdot e) = e \wedge (e \cdot e'') = e \\
(e \cdot f(e)) = e \wedge (f(e) \cdot (e)) = e \\
(e'' \cdot f(e'')) = e \wedge (f(e'') \cdot (e'')) = e \\
(e \cdot e'') = e \wedge (e'' \cdot e) = e \wedge \neg(e = e'') \\
(e'' \cdot e'') = e'' \wedge (e'' \cdot e'') = e'' \wedge \neg(e = e'')
\end{aligned}
$$

In this list, we can find two conjuncts that are contradictory:

$$
\begin{aligned}
(e \cdot e'') = e'' \wedge (e'' \cdot e) = e'' \\
(e'' \cdot e) = e \wedge (e \cdot e'') = e \wedge \neg(e = e'')
\end{aligned}
$$

Hence, there exists no model that satisfies all the axioms and the negation of the goal. Therefore, the goal is valid.

Task 2: Given the skolemized set of axioms:

$$
\begin{aligned}
\forall a\,, b\,, c\,.\,(a \cdot b) \cdot c = a \cdot (b \cdot c) \\
\forall a\,.\,((a \cdot e) = a \wedge (e \cdot a) = a) \\
\forall a\,.\,((a \cdot f(a)) = e \wedge (a \cdot f(a)) = e)
\end{aligned}
$$

we can see that f is the inverse function. We can simplify the goal to

$$G:\ \forall a, b.(((a \cdot b = e) \wedge (b \cdot a = e)) \implies (b = f(a)))$$

Negating this goal gives us:

$$\neg G:\ \exists a, b.(((a \cdot b = e) \wedge (b \cdot a = e)) \wedge \neg(b = f(a)))$$

Skolemizing (with constants a', b') gives us:

$$((a' \cdot b' = e) \wedge (b' \cdot a' = e)) \wedge \neg(b' = f(a'))$$

Instantiating the axioms with depth-0 terms (a', b'), we get a long list of formulae. In this list, we can find a conjunct that contradicts the above formula:

$$((a' \cdot f(a')) = e \wedge (f(a') \cdot a') = e)$$

Problem 2

Soln:

The given formula φ is

$$\varphi = y \leq x \land x \leq y \land f(y) = f(7) \land x \leq 5$$

Problem 3

Soln:

(a)

Given $f : 2^N \to 2^N$ that is defined as

$$f(S) = \{2\} \cup \{y | y = 2x, x \in S\}$$
