Mid Point Ellipse Algorithm

Definition: An ellipse is defined as set of points such that sum of the distances from the two fixed points is the same for all points. If the distance to two fixed points from any point P(x,y) on ellipse are d_1 , d_2 then general equation of an ellipse is $d_1 + d_2 = \text{constant}$

Or expressing distance d_1 , d_2 in terms of focal coordinates $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$

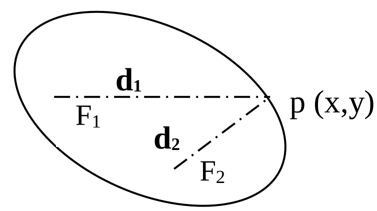
We have,

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = \text{constant}$$

Mid point ellipse method is applied throughout first quadrant in two parts(according to the slope of ellipse)

The equation of an ellipse is given by

$$x^2 / r^2_x + y^2 / r^2_y = 1$$



or $F_{ellipse}(x,y) = r^2_y x^2 + r^2_x y^2 - r^2_x r^2_y$

now $F_{ellipse}(x,y)$ < 0 if(x,y) is inside the ellipse boundary

$= 0$ if(x,y) is on the ellipse boundary

> 0 if(x,y) is outside the ellipse boundary

This ellipse function $F_{ellipse}(x,y)$ serves as the decision parameter

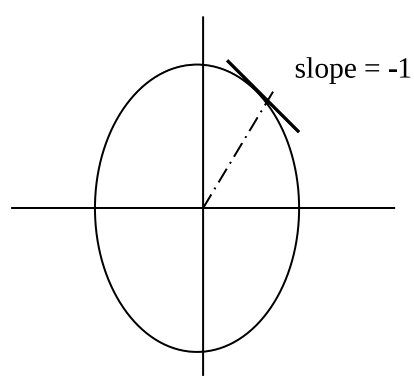
We select next pixel along the ellipse path according to the sign of ellipse function evaluated at the midpoint between two candidate pixels.

Start at $(0, r_y)$ take unit steps in 'x' direction until we reach boundary region 1 and 2 then switch to unit steps in 'y' direction for remainder of curve in the first quadrant.

At each step test the value of the slope of the curve. The ellipse slope is given by

$$2 r^2_y x + 2 r^2_x y \ dy/dx = 0$$

$$dy/dx = - 2 r^2_y x / 2 r^2_x y$$



At the boundary between region 1 and 2 $dy/dx = -1$

so, $2 r^2_y x = 2 r^2_x y$

and we move out of the region 1 when

$$2 r^2_y x >= 2 r^2_x y$$

Region 1.

Assuming position (x_k , y_k) has been selected at previous step we determine next position (x_{k+1} , y_{k+1}) as either (x_{k+1} , y_k) or $(x_{k+1} , y_k - 1)$ along elliptic path by evaluating the decision parameter(elliptic function)

$$P1_k = F_{ellipse}(x_k + 1 , y_k - ½)$$
$$= r^2_y (x_k + 1)^2 + r^2_x (y_k - ½)^2 - r^2_x r^2_y \text{ -- } (i)$$

At the next sampling position $(x_{k+1} +1 = x_k + 2)$, the decision parameter for region 1 is evaluated as

$$P1_{k+1} = F_{ellipse}(x_{k+1} + 1 , y_{k+1} - ½)$$
$$= r^2_y [(x_k + 1) +1]^2 + r^2_x (y_{k+1} - ½)^2 - r^2_x r^2_y \text{ ---------------------------- } (ii)$$

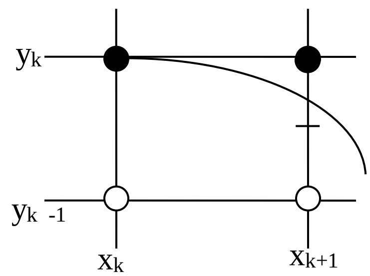
Now subtracting eq (i) and (ii),

$$P1_{k+1} = P1_k + 2r^2_y (x_k + 1) + r^2_y + r^2_x [(y_{k+1} - ½)^2 -(y_k - ½)^2] \text{ ---------- } (iii)$$

where y_{k+1} is either y_k or $y_k - 1$ depending on the sign of $P1_k$.

<u>Case 1:</u>

if $P1_k < 0$ then the mid point is inside the ellipse, so pixel on scanline 'y_k' is closer to the ellipse boundary and $y_{k+1} = y_k$



so the increment will be $2r^2_y x_{k+1} + r^2_y$

i.e. from equation (iii)

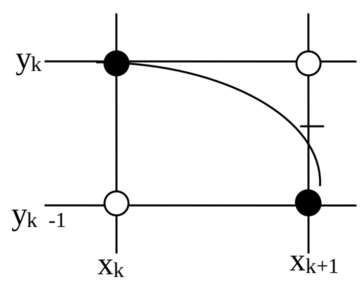
Or $P1_{k+1} = P1_k + 2r^2_y x_{k+1} + r^2_y$ ---------- (a)

Where $x_{k+1} = x_k + 1$

or $2r^2_y x_{k+1} = 2r^2_y x_k + 2 r^2_y$

<u>Case 2:</u>

if $P1_k >= 0$ then the mid point is outside or on the boundary of the ellipse, so we select the pixel on scan line '$y_k - 1$' then $y_{k+1} = y_k - 1$



so the increment will be $2r^2_y x_{k+1} - 2r^2_x y_{k+1}$

i.e. from equation (iii)

Or $P1_{k+1} = P1_k + 2r^2_y x_{k+1} - 2r^2_x y_{k+1} + r^2_y$ ---------- (b)

Where $2r^2_x y_{k+1} = 2r^2_x y_k – 2 r^2_x$

or $2r^2_y x_{k+1} = 2r^2_y x_k + 2 r^2_y$

Initial decision parameter for Region 1 $= P1_0$

The starting position is $(0,r_y)$

Next pixel to plot is either $(1,r_y)$ or $(1,r_y - 1)$

So, midpoint coordinate position is $(1, r_y - ½)$

$F_{ellipse}(1, r_y - ½) = r^2_y + r^2_x (r_y - ½)^2 - r^2_y r^2_y$

Thus,

$$P1_0 = r^2_y + ¼\ r^2_x - r^2_x r_y$$

Region 2.

Sample at unit steps in 'y' direction, the midpoint is taken between horizontal pixels at each step now.

Assuming, (x_k, y_k) has been plotted, next pixel to plot is (x_{k+1}, y_{k+1}) where

x_{k+1} is either x_k or x_{k+1}

and y_{k+1} is $y_k - 1$

i.e. we choose either $(x_k, y_k - 1)$ or $(x_{k+1}, y_k - 1)$

So, midpoint coordinate position is $(x_k + ½, y_k - 1)$

$F_{ellipse}(x_k + ½, y_k - 1)$

Or, $P2_k = r^2_y (x_k + ½)^2 + r^2_x (y_k - 1)^2 - r^2_x r^2_y$ -- (iv)

now, at next sampling position , the next pixel to plot will either be

$(x_{k+1}, y_{k+1} - 1)$ or $(x_{k+1} + 1, y_{k+1} - 1)$

thus,

$F_{ellipse}(x_{k+1} + ½, y_{k+1} - 1)$

Or, $P2_{k+1} = r^2_y (x_{k+1} + ½)^2 + r^2_x (y_{k+1} - 1)^2 - r^2_x r^2_y$

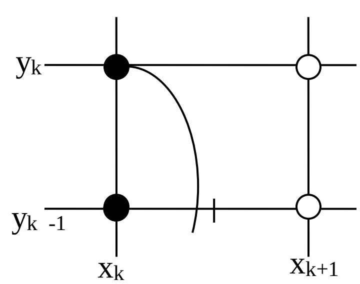
$= r^2_y (x_{k+1} + ½)^2 + r^2_x [(y_k - 1) - 1]^2 - r^2_x r^2_y$ -- (v)

Now subtracting eq (iv) and (v),

$P2_{k+1} = P2_k - 2r^2_x (y_k - 1) + r^2_x + r^2_y [(x_{k+1} + ½)^2 - (x_k + ½)^2]$ ---------- (vi)

where x_{k+1} is either x_k or $x_k + 1$ depending on the sign of $P2_k$.

Case 1:



if $P2_k > 0$ then the mid point is outside the boundary of the ellipse, so we select the pixel at 'x_k'

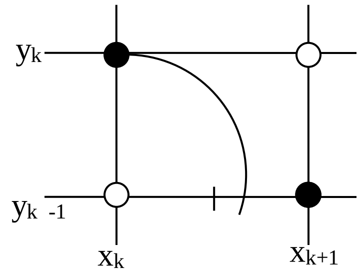
Or $P2_{k+1} = P2_k - 2r^2_x (y_k - 1) + r^2_x$

$= P2_k - 2r^2_x y_{k+1} + r^2_x$ ---------- (c) Where $y_{k+1} = y_k - 1$

or $2r^2_x y_{k+1} = 2r^2_x y_k - 2 r^2_x$

Case 2:

if $P2_k <= 0$ then the mid point is inside or on the boundary of the ellipse , so we select pixel at '$x_k + 1$'



i.e. from equation (vi)

Or $P2_{k+1} = P2_k - 2r^2_x (y_k – 1) + r^2_x + r^2_y [(x_{k+1} + ½)^2 - (x_k + ½)^2]$

$= P2_k - 2r^2_x (y_k – 1) + r^2_x + r^2_y [(x_k +1) + ½)^2 - (x_k + ½)^2]$

$= P2_k - 2r^2_x (y_k – 1) + r^2_x + r^2_y [(x_k +3/2)^2 - (x_k + ½)^2]$

$= P2_k - 2r^2_x (y_k – 1) + r^2_x + r^2_y [x^2_k + 3x_k + 9/4 - x^2_k - x – 1/4]$

$= P2_k - 2r^2_x (y_k – 1) + r^2_x + r^2_y [2x_k + 2]$

$= P2_k - 2r^2_x y_{k+1} + r^2_x + 2r^2_y x_{k+1}$ ---------- (d)

Where $2r^2_x y_{k+1} = 2r^2_x y_k – 2 r^2_x$

or $2r^2_y x_{k+1} = 2r^2_y x_k + 2 r^2_y$

For region 2, the initial position (x_0 , y_0) is taken as the last position selected in region 1 and thus the initial decision parameter in region 2 is

$P2_0 = F_{ellipse}(x_0 + ½ ,y_0 - 1)$

$= r^2_y (x_0 + ½)^2 + r^2_x (y_0 - 1)^2 - r^2_x r^2_y$