NEPAL COLLEGE OF INFORMATION TECHNOLOGY

Assessment Spring 2025

Level: Bachelor | Semester – Spring | Year: 2025

Programme: BE_COM(M & D)_IT(DAY) | Full Marks : 100

Course: Applied Mathematics | Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Attempt all the questions.

1\. a. State and prove cauchy-Reimann euation as necessary conditions for function to be analytic. 5

b. Define Harmonic function.show that u=cosx coshy is harmonicand find corresponding analytic function. 5

c. State and prove cauchy –integral formuia 5

2\. a. Solve: $\oint_C \frac{1}{z^2-1}\, dz$,the counter clockwise around the circle |z-1|=1.(chapter-1) 5

b. Define Laurent series.Find the Laurent expansion for f(z)=$\frac{1}{z-z^3}$ $in\ the\ region\ given\ by\ 1<|z+1|<2$ 5

c. Define Cauchy –Residue theorem. By using Cauchy residue theorem evalute the integration $\oint_C \frac{z-23}{z^2-4z-5} dz, where\ Cis\ the\ circle\ |z|=6$ 5

3\. a. Define bilinear transformation as well as fixed point.Find the bilinear transformation which makes the pointz=1.i.-i onto w=i,0,-i.$Also\ find\ the\ image\ of\ unit\ circle\ z=1$ 5

b. State and prove first shifting theorem of Z transform.Using it o find the z_transform of Z($e^{-at}coswt$). 5

c. State and prove Initial and final value problem of Z-transform.Find the z-transform of Z($a^n sinbt$). 5

4\. a. Prove that Z(y_{n+K})=$z^k(\bar{y}-y_0-\frac{y_1}{z}-\dots-\frac{y_{k-1}}{z^{k-1}})$ 5

b. Solve the difference equation: 5

$y_{n+2}+6y_{n+1}+9y_n=2^n \quad with\ y_0=y_1=0$

5 a. Using Fourier integral representation ,show that 5

$$\int_0^\infty \frac{cosxw+wsinxw}{1+w^2}\, dw = 0\ \ if\ x<0$$

$$=\frac{\pi}{2}\ if\ x=0$$

$$=\pi e^{-x}\ if\ x>0$$

b. Find the Fourier sine transform of 5

f(x)=e^{-x} $for\ x>0. Then\ prove\ that\ \int_0^\infty \frac{sinmx}{1+x^2}dx=\frac{\pi}{2}e^{-m}\ for\ m>0$.

6 a. Solve the following equation: 5

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0 \qquad by\ the\ seperation\ of\ variables.$$

b. A tightly stretche string with fixed end points x=0 and x=L is initially in a position given by u(x,0)=$sinx^3\left(\frac{\pi x}{L}\right)$.If it is realesed from rest from this position. Find the deflection u(x,t). 5

OR

Derive one dimensional Heat equation.

c. Determine the solution of one dimensional heat equation $\frac{\partial u}{\partial t}=c^2\frac{\partial 2_u}{\partial x^2}$ 5

Subject to the boundary condition u(0,t)=u(L,t)=0 and initial condition is u(x,0)=L ,being the length of the bar

OR

Derive one dimensional Wave equation with required assumptions.

7 a. When two dimensional Heat equation $\frac{\partial u}{\partial t}=c^2(\frac{\partial 2_u}{\partial x^2}+\frac{\partial 2_u}{\partial y^2})$ become Laplace equation.By the concepr of solution of laplace equation with rectangular boundaries,solve the problem: 1+7

A rectangular plate with insulated surface is 8 cm wide so long compared to its width that it may be considered infinite in length without introducing an appreciable.if the temperature along the short edge y=o is given by

u(x,0)=100 sin$\frac{\pi x}{8}$, $0 < x < 8$

while two long edges x=0 and x=8 as well as the other short edge are kept at o^0c.

Show that steady state temperature at any point of the plate is given by

u(x,y)=100$e^{-\pi y/8}\,sin\frac{\pi x}{8}$

OR

Derive two dimensional heat equation with required assumptions.

b. Derive polar form of Laplace equation. 7

OR

Aplate was insulated surface has the shape of quadrant of a circle of radius 10 cm.The bounding raddi $\theta = 0\ and\theta = \frac{\pi}{2}$ are kept at 0^0c and temperature along the circular quadrant is kept at 100$(\pi\theta - 2\theta^{2)0}c$ for $0 \le \theta \le \frac{\pi}{2}$

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AM assessment Qsn

Q1a)

Cauchy Reimann equation states that, "A function $f(z)$ is said to be analytic in domain D if and only if the partial derivatives of u and v satisfy the two conditions $u_x = v_y$, $u_y = -v_x$ at every point in D".

Let $f(z) = u(x,y) + i v(x,y)$ be a complex valued function.

or $f(z) = u + iv$ — (i)

Now, $f(z+\Delta z) = (u+\Delta u) + i(v+\Delta v)$ — (ii)

subtracting (i) from (ii),

$$f(z+\Delta z) - f(z) = \Delta u + i\Delta v$$

$$\text{or, } \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0}\left(\frac{\Delta u + i\Delta v}{\Delta z}\right)$$

$$\text{or, } f'(z) = \lim_{\Delta z \to 0}\left(\frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z}\right)$$

We know, $z = x + iy$

or, $\Delta z = \Delta x + i\Delta y$

(i) In real axis, $\Delta y = 0$ so, $\Delta z = \Delta x$

$$f'(z) = \lim_{\Delta z \to 0}\left(\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(ii) In Imaginary axis,

$$f'(z) = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{i\Delta y} + \frac{i\Delta v}{i\Delta y} \right)$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Now, we can write,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right)$$

Hence,

$$u_x = v_y \quad \text{and} \quad v_x = -u_y$$

This proves CR equation.

Q1b)

A complex valued function $f(x,y) = u(x,y) + iv(x,y)$ is said to be harmonic function if the second partial derivative of function exists i.e. $(u_{xx}, u_{yy}, v_{xx}, v_{yy})$ and satisfies Laplace equation (i.e. $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$).

Given, $u = \cos x \cosh y$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y \quad , \quad \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y \quad , \quad \frac{\partial^2 u}{\partial y^2} = \cos x . \cosh y$$

Here,

$$u_{xx} + u_{yy} = 0$$

or, $-\cos x \cosh y + \cos x . \cosh y = 0$

Hence, it is harmonic function.

By CR equation, we know,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Hence, $v_y = -\sin x \cosh y$

Integrating with respect to y, we get,

$$v = -\sin x . \sinh y + h(x)$$

where $h(x)$ is function of x only

Differentiating w.r.t. x,

$$v_x = -\cos x . \sinh y + h'(x)$$

Or, $-u_y = -\cos x . \sinh y + h'(x)$

Or, $-\cos x \sinh y = -\cos x . \sinh y + h'(x)$

$\therefore h'(x) = 0$

Integrating, we get,

$$h'(x) = c$$

Hence, $v = -\sin x . \sinh y + c$ is harmonic conjugate of u.

Hence, required analytic function is

$$f(z) = u + iv$$

$$= (\cos x \cosh y) + i(-\sin x . \sinh y + c)$$ Ans

Q1c)

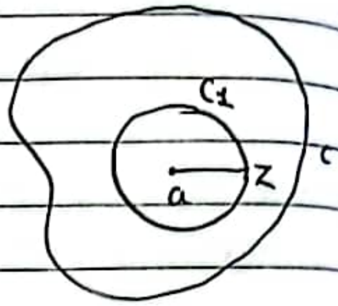
Cauchy Integral Formula:

Let $f(z)$ is analytic within and on a closed curve c. Let a be any singular point inside c where $f(z)$ is not analytic, then cauchy integral formula states that,

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)} dz$$

Proof:

We have, $f(z)$ which is analytic around a closed curve c except at point 'a'. construct a circle C_1 with center at a and small radius $(r \to 0)$.



Assume a point z on C_1. Now,

$$|z-a| = r$$
$$\text{or, } z-a = re^{i\theta}$$
$$\text{or, } z = a+re^{i\theta} \qquad \text{or, } dz = rie^{i\theta}d\theta$$

From cauch integral theorem for multiple connected regions,

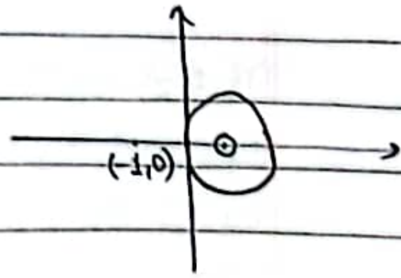
$$\oint_c \frac{f(z)}{(z-a)}dz = \oint_{C_1}\frac{f(z)}{(z-a)}dz$$
$$= \oint_{C_1}\frac{f(a+re^{i\theta})}{(a+re^{i\theta}-a)}\, rie^{i\theta}d\theta$$
$$= \oint_{C_1} f(a+re^{i\theta}) \times \frac{1}{\cancel{re^{i\theta}}} \times \cancel{r}i\cancel{e^{i\theta}}\, d\theta$$
$$= i\oint_{C_1} f(a+re^{i\theta})\, d\theta$$

Taking $r \to 0$,

$$= i\oint_{C_1} f(a)\, d\theta$$
$$= i \times f(a) \int_0^{2\pi} d\theta$$
$$\text{or, } \oint_c \frac{f(z)}{(z-a)}dz = i \times f(a)\, 2\pi$$
$$\therefore f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)}dz$$

proved.

) $\oint_c \frac{1}{z^2-1} dz$



Here, $f(z) = \frac{1}{z^2-1}$

To find singular point, $z^2 - 1 = 0$

or, $z^2 = 1$

or, $z = 1, -1$

$(1,0)$

Given circle, $|z-1| = 1$

$-1 \le z - 1 \le 1$

$0 \le z \le 2$

Center $= \frac{2+0}{2} = 1 = (1,0)$

Radius $= 1$

Here, $z = 1$ lies inside circle. so it is stationary point.

From Cauchy Residue theorem, Residue at $z=1$ is,

$$R_1 = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to 1} \cancel{(z-1)} \times \frac{1}{(z+1)\cancel{(z-1)}}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Hence, $\oint_c f(z)\, dz = 2\pi i$ (sum of Residue)

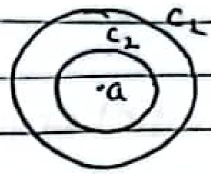
$$= 2\pi i \times \frac{1}{2}$$

$$\oint_c f(z)\, dz = \pi i$$

Ans

Q2b)

If $f(z)$ be any analytic function in circle C_1 and C_2 and in region R bounded by C_1 and C_2, then at any point z in region R, $f(z)$ can be written as series ~~in the~~ of positive and negative power of $(z-a)$ in form



$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

This is called laurent series.

Given, $f(z) = \dfrac{1}{z-z^3}$

$$= \frac{1}{z(1-z^2)}$$

$$f(z) = \frac{1}{z(1+z)(1-z)}$$

Given region is $1 < |z+1| < 2$

Let, $z+1 = u$

or, $z = u-1$

So, $f(u) = \dfrac{1}{(u-1)(1+u-1)(1-u+1)}$

$$f(u) = \frac{1}{u(u-1)(2-u)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{2-u}$$

Now, region is $1 < |u| < 2$

Or, $1 = A(u-1)(2-u) + B(u)(2-u) + C(u)(u-1)$

Put $u=1$,

$1 = B\times1\times(2-1)$

$\therefore B=1$

Put $u=2$,

$1 = C\times2\times1$

$\therefore C=\frac{1}{2}$

Put $u=0$,

$1 = A\times(-1)\times(2)$

$\therefore A=-\frac{1}{2}$

So, $f(u) = \frac{-1}{2u} + \frac{1}{u-1} + \frac{1}{2(2-u)}$

For the given region, $1<|u|<2$

$$f(u) = \frac{-1}{2u} + \frac{1}{u(1-\frac{1}{u})} + \frac{1}{2u(\frac{2}{u}-1)}$$

$$= \frac{-1}{2u} + \frac{1}{u}\times\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{2u}\left(1-\frac{2}{u}\right)^{-1}$$

$$= \frac{-1}{2u} + \frac{2}{2u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n - \frac{1}{2u}\sum_{n=0}^{\infty}\left(\frac{2}{u}\right)^n$$

$$f(u) = \frac{1}{2u}\left(-1+2\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n - \sum_{n=0}^{\infty}\left(\frac{2}{u}\right)^n\right)$$

$$\therefore f(z) = \frac{1}{2z+2}\left[-1+2\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \sum_{n=0}^{\infty}\left(\frac{2}{z+1}\right)^n\right].$$

is the required laurent series. Ans

Q2c)

Cauchy Residue theorem states that, If $f(z)$ is analytic function inside on a curve at any point except at number of poles (singular points) then $f(z)$ can be written as,

$$\oint f(z)\,dz = 2\pi i \times (\text{sum of Residue})$$

Here, $f(z) = \frac{z-23}{z^2-4z-5}$

For singular point, $z^2-4z-5=0$

or, $z^2-5z+z-5=0$

or, $z(z-5)+1(z-5)=0$

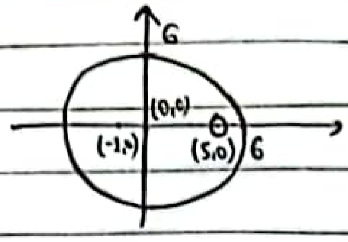
or, $(z-5)(z+1)=0$

$z=5$ of order 1, $z=-1$ of order 1

Given circle, $|z| = 6$

Center $= 0$

radius $= 6$



Thus, both points lie inside circle.

For $z = 5$,

$$R_1 = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to 5} \cancel{(z-5)} \times \frac{(z-23)}{\cancel{(z-5)}(z+1)}$$

$$= \frac{5-23}{5+1} \qquad \therefore R_1 = -3$$

For $z = -1$,

$$R_2 = \lim_{z \to -1} (z+1) f(z)$$

$$= \lim_{z \to -1} \cancel{(z+1)} \left(\frac{(z-23)}{\cancel{(z+1)}(z-5)} \right)$$

$$= \frac{-1-23}{-1-5} \qquad \therefore R_2 = 4$$

Hence, From cauchy Residue theorem,

$$\oint f(z)\, dz = 2\pi i \text{ (sum of residue)}$$

$$= 2\pi i \times (-3+4)$$

$$\therefore \oint f(z)\, dz = 2\pi i$$

Ans

2a)

A transformation of the form $f(z) = w = \dfrac{az+b}{cz+d}$ where $ad-bc \neq 0$

is said to be bilinear transformation.

A fixed point in a bilinear transformation is a point in the complex plane that is mapped to itself i.e. $w = z$.

If $w = \dfrac{az+b}{cz+d}$, in fixed point, $w = z$. So, $z = \dfrac{az+b}{cz+d}$

Given, $z_1 = 1 \qquad w_1 = i$

$z_2 = i \qquad w_2 = 0$

$z_3 = -i \qquad w_3 = -i$

By using cross ratio,

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_3-w_2)}$$

$$\text{or, } \frac{(z-1)(i+i)}{(z+i)(i-1)} = \frac{(w-i)(0+i)}{(w+i)(-i-0)}$$

$$\text{or, } \frac{(z-1)(2i)}{(z+i)(i-1)} = \frac{(w-i)}{(-w-i)}$$

$$\text{or, } (2iz-2i)(-w-i) = (w-i)(z+i)(i-1)$$

$$\text{or, } -2iwz+2i^2 = (wz+wi-iz-i^2)(i-1)$$

$$\text{or, } -2iwz-2 = iwz-wz+i^2w-iw-i^2z+iz-i^3+i^2$$

$$\text{or, } -2iwz-2 = iwz-wz-w-iw+z+iz+i-1$$

$$\text{or, } 3iwz+1-wz-w-iw+z+iz+i = 0$$

$$\cancel{\text{or, } z(3iw-w+1+i) = w+iw-i-1}$$

$$\cancel{\therefore z = \frac{w+iw-i-1}{3iw-w+1+i} = \frac{(1+i)w-(i+1)}{(3i-1)w+(i+1)}}$$

or, $w(3iz-z-1-i) = -1-z-iz-i$

~~is the required transformation.~~

$$\therefore w = \frac{1+z+iz+i}{z+1+i-3iz} = \frac{(1+i)z+(1+i)}{(1-3i)z+(1+i)}$$

The image of unit circle $z=1$ is,

$$z = 1$$

or, $\dfrac{w+iw-i-1}{3iw-w+1+i} = 1$

or, $w+iw-i-1 = 3iw-w+1+i$

or, $2iw-2w+2i+2=0$

or, $2iu+2i^2v-2u-2iv+2i+2=0$

or, $2iu-2v-2u-2iv+2i+2=0$

or, $u(2i-2)-v(2i+2)+(2i+2)=0$

or, $u(i-1)-v(i+1)+(i+1)=0$

or, $v(i+1)-u(i-1) = (i+1)$ is the image of unit circle

Q3b)

First shifting theorem of z transform states that,

If $z[f(t)] = f(z)$

then $z[e^{-at}f(t)] = f(ze^{aT})$

Proof:

From the definition of z-transform,

$$z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z)$$

Now,

$$z[e^{-at}f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{+aT}z)^{-n} f(nT)$$

$$= F(ze^{aT})$$ proved.

To find, $z(e^{-at} \cos \omega t)$

3c)

Initial value theorem:

If $Z[f(t)] = F(z)$, then $f(0) = \lim_{z\to\infty} F(z) = \lim_{t\to 0} f(t)$

Proof: By definition of z-transform,

$$F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$F(z) = f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \dots$$

Taking limit $z\to\infty$ on both sides,

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \dots\right]$$

$$= f(0)$$

$$= \lim_{t\to 0} f(t)$$

Final value theorem:

If $Z[f(t)] = F(z)$, then $f(\infty) = \lim_{z\to 1} (z-1)F(z)$

Proof: By definition of z-transform,

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \quad \text{—— (1)}$$

$$z[F(z) - f(0)] = Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T).\, z^{-n} \quad \text{—— (2)}$$

subtracting (1) from (2),

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n}$$

$$\text{or, } (z-1)F(z) = zf(0) + \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]\, z^{-n}$$

Taking limit $z \to 1$ on both sides,

$$\lim_{z\to 1} (z-1)F(z) = \lim_{z\to 1}\left\{ z f(0) + \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] \right\}$$

$$\therefore \lim_{z\to 1} (z-1)F(z) = f(0) + \lim_{k\to\infty} \sum_{n=0}^{k} f(nT+T) - f(nT)$$

$$= f(0) + \lim_{k\to\infty} [f(T) - f(0) + f(2T) - f(T) + f(3T) - f(2T) + \dots + f(kT) - f(k-1)T]$$

$$\therefore \lim_{k\to 1} (z-1)F(z) = f(0) + \lim_{k\to\infty} [-f(0) + f(k+1)T]$$

$$= \lim_{k\to\infty} f((k+1)T) = f(\infty)$$ Hence proved.

Here, $Z(a^n e^{ibt}) = [Z(e^{ibt})]_{z\to \frac{z}{a}}$

$$= \left(\frac{z}{z - e^{ibT}}\right)_{z\to\frac{z}{a}}$$

$$= \left(\frac{\frac{z}{a}}{\frac{z}{a} - e^{ibT}}\right)$$

$$= \frac{z}{z - ae^{ibT}}$$

or, $Z(a^n \cos bt + i \sin bt) = \dfrac{z}{z - a(\cos bT + i\sin bT)}$

$$= \frac{z}{(z - a\cos bT) + i(a\sin bT)} \times \frac{(z - a\cos bT) + i(a\sin bT)}{(z-a\cos bT) + i(a\sin bT)}$$

$$= \frac{z[(z - a\cos bT) + i(a\sin bT)]}{(z - a\cos bT)^2 + (a\sin bT)^2}$$

$$= \frac{z(z - a\cos bT)}{z^2 - 2az\cos bT + a^2} + \frac{iaz\sin bT}{z^2 - 2az\cos bT + a^2}$$

Here, we obtain,

$$Z(a^n \sin bt) = \frac{az\sin bT}{z^2 - 2az\cos bT + a^2}$$ Ans

Q4a) To Prove, $Z(y_{n+k}) = z^k\left(\bar{y} - y_0 - \frac{y_1}{z} - \dots - \frac{y_{k-1}}{z^{k-1}}\right)$

Proof:

By definition, we have,

$$Z(y_{n+k}) = \sum_{n=0}^{\infty} y_{n+k} z^{-n}$$

$$= \sum_{n=0}^{\infty} y_{n+k} z^{-(n+k)} z^k$$

$$= z^k \sum_{n=0}^{\infty} y_{n+k} z^{-(n+k)}$$

Putting $m = n+k$, when $n=0$, $m=k$

when $n=\infty$, $m=\infty$

$$Z(y_{n+k}) = z^k \sum_{m=k}^{\infty} y_m z^{-m}$$

$$= z^k\left[\sum_{m=0}^{\infty} y_m z^{-m} - \sum_{m=0}^{k-1} y_m z^{-m}\right]$$

$$= z^k\left[\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} - \dots - \frac{y_{k-1}}{z^{k-1}}\right]$$

where $\bar{y} = Z(y_n)$ Proved.

Q4b)

$$Z(y_{n+2}) + 6Z(y_{n+1}) + 9Z(y_n) = Z(2^n)$$

or, $z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 6(\bar{y} - y_0) + 9\bar{y} = \frac{z}{z-2}$

or, $\bar{y}(z^2 + 6z + 9) = \frac{z}{z-2}$

$$\therefore \bar{y} = \frac{z}{(z-2)(z+3)^2} \quad \text{——(i)}$$

$$= \frac{Az}{(z-2)} + \frac{Bz}{(z+3)} + \frac{Cz}{(z+3)^2}$$

On solving, $A = \frac{1}{25}$, $B = -\frac{1}{25}$, $C = -\frac{1}{5}$

From eq^n(i),

$$\bar{y} = \frac{1}{25}\left(\frac{z}{z-2}\right) - \frac{1}{25}\left(\frac{z}{z+3}\right) - \frac{1}{5}\left(\frac{z}{(z+3)^2}\right)$$

Hence,

$$y_n = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^n$$ Ans

Q5a)

Comparing with fourier integral, this problem is fourier integral of f(x) where,

$$f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

Here, $f(x) = \int_0^\infty (A(\omega)\cos\omega x + B(\omega)\sin\omega x)\, d\omega$

Now,

$$A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t \, dt$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0\cos\omega t\, dt + \int_0^0 \frac{\pi}{2}\cos\omega t\, dt + \int_0^\infty \pi e^{-t}\cos\omega t\, dt\right.$$

$$= \frac{e^{-t}}{(-1)^2+\omega^2}\left[-\cos\omega t + \omega\sin\omega t\right]_0^\infty$$

$$= \frac{1}{1+\omega^2}\left[\frac{-\cos\omega t}{e^t} + \frac{\omega\sin\omega t}{e^t}\right]_0^\infty$$

$$= \frac{1}{1+\omega^2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

$$= \frac{1}{\pi}\times\pi\int_{0}^{\infty} e^{-t}\sin\omega t\,dt$$

$$= \left[\frac{e^{-t}}{(-1)^2+\omega^2}\left[-\sin\omega t - \omega\cos\omega t\right]\right]_0^{\infty}$$

$$= \frac{\omega}{1+\omega^2}$$

Substituting,

$$f(x) = \int_0^{\infty} \frac{1}{1+\omega^2}\cos\omega x + \frac{\omega}{1+\omega^2}\sin\omega x\,d\omega$$

$$\therefore \int_0^{\infty} \frac{\cos x\omega + \omega\sin x\omega}{1+\omega^2}\,d\omega = 0 \quad \text{if } x<0$$

$$= \frac{\pi}{2} \quad \text{if } x=0$$

$$= \pi e^{-x} \quad \text{if } x>0$$

Q5b)

Given $f(x) = e^{-x}$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t\,dt$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-t}\sin\omega t\,dt$$

$$= \sqrt{\frac{2}{\pi}}\times\left[\frac{e^{-t}}{(-1)^2+\omega^2}\left((-1)\sin\omega t - \omega\cos\omega t\right)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\times\frac{1}{1+\omega^2}$$

We know, $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(\omega)\sin\omega x\,d\omega$

or, $e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(\omega)\sin\omega x\,d\omega$

~~Put x = m~~

or, $e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\omega^2} \sin\omega x \, d\omega$

Put $x = m$, $\omega = x$,

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+x^2} \sin xm \, dx$$

$$\therefore \int_0^\infty \frac{1}{1+x^2} \sin mx \, dx = \frac{\pi}{2} e^{-m}$$ Ans

Q6a). $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Let $z(x,y) = X(x)\,Y(y)$ be the solution.

Now, $\frac{\partial^2 z}{\partial x^2} = z_{xx} = X''Y$

$$\frac{\partial z}{\partial x} = z_x = X'Y$$

$$\frac{\partial z}{\partial y} = z_y = XY'$$

Now,

$$X''Y - 2X'Y + XY' = 0$$

~~or, X''Y~~

or, $(X'' - 2X')Y + XY' = 0$

or, $(X'' - 2X')Y = -XY'$

Or, $\frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = k$ (constant) —— (i)

Here, From (i)

$$\frac{X'' - 2X'}{X} = k$$

or, $X'' - 2X' - Xk = 0$

The auxiliary eq^n is, $M^2 - 2M - k = 0$

$ax^2+bx+c=0$

$$\text{or, } m^2-2m-K=0$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{+2 \pm \sqrt{4-4\times1\times(-K)}}{2\times1}$$

$$= \frac{2 \pm \sqrt{4+4K}}{2}$$

$$= 1 \pm \sqrt{1+K}$$

Hence, $m = 1+\sqrt{1+K}$ and $m = 1-\sqrt{1+K}$

So, the solution is,

$$X(x) = c_1 e^{(1+\sqrt{1+K})x} + c_2 e^{(1-\sqrt{1+K})x}$$

From (i),

$$\frac{-Y'}{Y} = K$$

$$\text{or, } -Y'-YK=0$$

$$\text{or, } Y'+YK=0$$

The auxillary eqn is $m+K=0$

$$\therefore m = -K$$

$$\text{So, } Y(y) = c_3 e^{-Ky}$$

Hence,

$$z(x,y) = X(x).Y(y)$$

$$= \left(c_1 e^{(1+\sqrt{1+K})x} + c_2 e^{(1-\sqrt{1+K})x}\right).c_3 e^{-Ky}$$

Ans

6(b) $x = 0$ and $x = L$

$u(x,0) = \sin x^3\left(\frac{\pi x}{L}\right) = f(x)$

$u(x,t) = ?$

$g(x) =$ Initial velocity $= 0$

We know, solution of 1D wave eq^n is,

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos \lambda_n t + b_n \sin \lambda_n t) \sin \frac{n\pi x}{L}$$

where, $a_n = \frac{2}{L}\int_0^L f(x) . \sin \frac{n\pi x}{L} \, dx$

$$b_n = \frac{2}{L . \lambda_n}\int_0^L g(x) . \sin \frac{n\pi x}{L} \, dx$$

As $g(x) = 0$, so, $b_n = 0$

Now, $a_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$

$$= \frac{2}{L}\int_0^L \sin x^3\left(\frac{\pi x}{L}\right) . \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_0^L 2\sin\left(\frac{\pi x^4}{L}\right) . \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_0^L \cos\left(\frac{\pi x^4}{L} - \frac{n\pi x}{L}\right) - \cos\left(\frac{\pi x^4}{L} + \frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_0^L \left[\cos \frac{\pi}{L}(x^4 - nx) - \cos \frac{\pi}{L}(x^4 + nx)\right] dx$$

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$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— } (*)$$

Boundary condition: $u(0,t) = u(L,t) = 0$

Initial condition: $u(x,0) = L$

Let $u(x,t) = X(x)\,T(t)$ be the soln of eqn $(*)$,

Now, $u_t = \frac{\partial u}{\partial t} = XT'$

$u_{xx} = \frac{\partial^2 u}{\partial x^2} = X''T$

Now, $XT' = c^2 X''T$

or, $\frac{X''}{X} = \frac{T'}{Tc^2} = k$ (constant)

Now, $X'' - kX = 0$

Auxillary eqn is $m^2 - k = 0$

$\cancel{\therefore m = \pm k}$

$\cancel{\text{So, } X(x) = c_1 e^{kx} + c_2 e^{-kx}}$

Then, $T' - Tc^2k = 0$

~~Auxillary eqn is~~ $\cancel{m - c^2k = 0}$

$\therefore m = c^2k$

(i) If constant k is negative i.e. $k = -p^2$, Then, eqn we get is now,

$X'' + p^2X = 0$

A.E. is $m^2 + p^2 = 0$

or, $m^2 = 0 - p^2$

$\therefore m = 0 \pm ip$

$T' + Tc^2p^2 = 0$

A.E. is $m + c^2p^2 = 0$

or, $m = -c^2p^2$

Hence, $T(t) = c_3 e^{-c^2p^2t}$

$X(x) = (c_1 \cos px + c_2 \sin px)\, e^{0x}$

$= c_1 \cos px + c_2 \sin px$

~~If t = 0 th~~ $\therefore u(x,t) = X(x).T(t)$

$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t}$ ——— (1)

If $t \to \infty$, then $u(x,t) = e^{-\infty} \to 0$. So, appropriate solⁿ of heat eqⁿ is with constant negative

Using boundary condition $u(0,t) = 0$ in (i),

$$u(0,t) = (C_1 \cos 0 + C_2 \sin 0) \cdot C_3 e^{-c^2p^2t}$$

$$0 = C_1 C_3 e^{-c^2p^2t}$$

As $C_3 \neq 0$, so, $C_1 = 0$.

Hence, in (i),

$$u(x,t) = C_2 \sin px \cdot C_3 e^{-c^2p^2t} \quad \text{—— (ii)}$$

Using Next BC $u(L,t)=0$ in (ii),

$$u(L,t) = C_2 \sin pL \cdot C_3 e^{-c^2p^2t}$$

$$\text{Or, } 0 = C_2 \sin pL \cdot C_3 e^{-c^2p^2t}$$

→ C_2, C_3 can't be zero and for $e^{-c^2p^2t}$ to be zero $t \to \infty$,

$$\text{So, } \sin pL = 0$$

$$\sin pL = \sin n\pi$$

$$\therefore p = \frac{n\pi}{L}$$

Now,

$$u(x,t) = C_2 \sin\left(\frac{\pi n}{L}\right)x \cdot C_3 e^{-c^2 t\left(\frac{n\pi}{L}\right)^2}$$

$$u(x,t) = b_n \sin\left(\frac{n\pi}{L}\right)x \cdot e^{-t\left(\frac{n\pi c}{L}\right)^2} \qquad \text{where } b_n = C_2 \cdot C_3$$

The general solution is,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \cdot e^{-t\left(\frac{n\pi c}{L}\right)^2}$$

Using initial condition,

Q7a)

A two dimensional heat equation becomes Laplace equation at steady state condition.

At steady state condition, u is independent with time t i.e. $\frac{\partial u}{\partial t} = 0$.

Thus, $u_{xx} + u_{yy} = 0$, This is Laplace equation.

Given, $L = 8\text{cm}$

$$u(x,0) = 100 \sin \frac{\pi x}{8} - f(x)$$

The solⁿ of 2D heat eqⁿ under given assumptions is given by,

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{8} \cdot e^{-\frac{n\pi y}{8}}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{8}\int_0^8 f(x)\sin\left(\frac{n\pi x}{8}\right)dx$$

$$= \frac{2}{8}\int_0^8 100\sin\frac{\pi x}{8}.\sin\frac{n\pi x}{8}\ dx.$$

$$= \frac{100}{8}\int_0^8 2\sin\frac{\pi x}{8}.\sin\frac{n\pi x}{8}\ dx$$

$$= \frac{100}{8}\int_0^8 \cos\left(\frac{\pi x}{8}-\frac{n\pi x}{8}\right) - \cos\left(\frac{\pi x}{8}+\frac{n\pi x}{8}\right)dx$$

$$= \frac{100}{8}\left[\frac{\sin x\left(\frac{\pi}{8}-\frac{n\pi}{8}\right)}{\frac{\pi}{8}-\frac{n\pi}{8}} - \frac{\sin x\left(\frac{\pi}{8}+\frac{n\pi}{8}\right)}{\frac{\pi}{8}+\frac{n\pi}{8}}\right]_0^8$$

$$= 100\left[\frac{\sin(\pi-n\pi)}{(\pi-n\pi)} - \frac{\sin(\pi+n\pi)}{(\pi+n\pi)}\right]$$

7b)

Here, the Laplace eqⁿ in polar form,

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

with boundary conditions

$$u(r,\theta) = 0^\circ c \qquad \text{in } 0 \le r \le 10$$

$$u(r,\pi) = 0^\circ c \qquad \text{in } 0 \le r \le 10$$