$$\therefore \quad I_1 = \left(\frac{T_1^2}{T_2^2 - T_1^2}\right) I_2 \qquad \ldots \text{(ix)}$$

And, moment of inertia of circular ring $I_2 = \frac{M(R_1^2 + R_2^2)}{2}$

R_1 and R_2 are the internal and external radius of the circular ring.

Note: Here to derive the relation for T no approximation is used like in the case of compound or simple pend [Where we used sin θ ~ θ for small amplitude] The time period of a torsional pendulum therefore remains un ef even if the amplitude be large].

Difference between Compound and Torsion Pendulum

Compound Pendulum	Torsion Pendulum
1. Definition: A compound pendulum is a rigid body of whatever shape capable of oscillating about a horizontal axis passing through it.	1. Definition: A torsion pendulum is a rigid b capable of executing torsional vibration abou axis passing through it's c.g.
2. It oscillates about any axis passing through it other than c.g.	2. It oscillates about an axis along the passing through c.g.
3. It executes angular vibration.	3. It executes torsional vibration.
4. c.g. moves in an arc.	4. c.g remains fixed.
5. Time period $T = 2\pi\sqrt{\frac{k^2/l + l}{g}}$	5. Time period, $T = 2\pi\sqrt{I/C}$ where, $C = \frac{\pi\eta r^4}{2l}$
6. Time period is dependent of 'g'.	6. Time period is independent of g.
7. The approximation, for small θ, sin θ ≈ θ is used to derive the time period. If θ is large the time period of compound pendulum is affected and the motion remain no longer simple harmonic. Therefore θ should be small.	7. No approximation is used to derive the time peri Therefore the time period of torsion pendul remains unaffected even if the amplitude be la provided the elastic limit of the suspension wir not exceeded.

Solved Example

Example 1: If the relaxation time of a damped harmonic oscillator is 50 sec. Find the time perioc for which the amplitude and energy of oscillator falls to $\frac{1}{e}$ times the initial value.

Solution:

Here relaxation time $\tau = 50$ sec

We know,

$$x = x_m\, e^{\frac{-bt}{2m}} = x_m \cdot e^{\frac{-t}{2\tau}} \qquad \left[\tau = \frac{m}{b}\right]$$

Here, $x = \frac{x_m}{e}$ and $t = T$

$$\therefore \quad \frac{x_m}{e} = x_m\, e^{\frac{-T}{2\tau}} \Rightarrow e^{-1} = e^{\frac{-T}{2\tau}}$$

$$\Rightarrow \quad \frac{T}{2\tau} = 1 \Rightarrow T = 2\tau = 2 \times 50 = 100 \text{ sec}$$

We know $E = E_0 e^{\frac{-bt}{m}} = E_0 e^{\frac{-t}{\tau}}$

When $E = \frac{E_0}{e}$, $t = T$

$$\therefore \quad \frac{E_0}{e} = E_0 e^{\frac{-t}{\tau}} \Rightarrow e^{-1} = e^{-\frac{t}{\tau}}$$

$$\Rightarrow \quad \frac{-t}{\tau} = 1 \Rightarrow T = \tau = 50 \text{ sec}$$

Example 2: The amplitude of lightly damped oscillator decreases by 3% during each cycle. What fraction of the energy of the oscillator is lost in each cycle.

Solution:

Given, Amplitude, $x_m = A - 3\%$ of $A = A - \frac{3A}{100} = \frac{97}{100} A = 0.97\ A$

$\therefore$ Energy lost in each full oscillation, $= \frac{1}{2}\ m\omega^2 A^2 - \frac{1}{2} m\omega^2 x_m{}^2$

$$= \frac{1}{2}\ m\omega^2 A^2 - \frac{1}{2} mw^2\ (0.97\ A)^2$$

$$= \frac{1}{2}\ m\ \omega^2 A^2\ [1 - 0.97^2] = E_0 \times 0.591 = 0.0571\ E_0.$$

Fraction of energy lost in each cycle $= \frac{\Delta E}{E_0} = 0.0591$ Ans.

→ force oscillation

Example 3: What is amplitude of the oscillation of a system acted by a sinusoidally varying force of amplitude 0.01 N and frequency 100 Hz acting on a system with the mass of 50 gm, restoring capacity 64 Nm^{-1} and damping constant 20gms^{-1}?

Solution

Amplitude of varying force $(F_0) = 0.01$ N

Frequency (f) = 100 HZ

$\therefore$ $\omega = 2\pi \times 100 = 628$ rad/sec

Mass (m) = 50 gm = 0.05 kg

Restoring constant (k) = 64 Nm^{-1}

Damping constant (b) = 20 gms^{-1} = 0.02 kg s^{-1}

$\therefore$ The amplitude of oscillation is given by

$$x_m = \frac{F_0/m}{\sqrt{(\omega o^2 - \omega^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

$$\Rightarrow \quad \text{Here, } \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{64}{0.05}} = 35.78$$

$$= \frac{0.01/0.05}{\sqrt{(628^2 - 35.78^2)^2 + \left(\frac{0.02 \times 628}{0.05}\right)^2}} = \frac{0.2}{393103.87} = 5.087 \times 10^{-7}\ m$$

Example 4: A Damped oscillator has mas 250 gm, spring constant 85N/m and damping cons g/s (i) How long does it take for the amplitude of the damped oscillator to drop to h its initial value (ii) How long does it take for the mechanical energy to drop to half initial value?

Solution:

Mass of block m = 250 gm = 0.25 kg, spring constant k = 85 N/m

Damping constant b = 70 g/s = 0.07 kg/s

(i) The amplitude is $A = x_m e^{\frac{-bt}{2m}}$

According to questions, $A = \frac{x_m}{2} = x_m e^{\frac{-bt}{2m}}$

or, $t = \frac{2 \times 0.25}{0.07} \ell n\, 2 = 5$ second

(ii) The energy is, $E = E_0 e^{\frac{-bt}{m}}$

Here, $E = E_0/2 = E_0 e^{-bt/m}$

or, $e^{bt/m} = 2$

$\frac{bt}{m} = \ell n\, 2 \quad \Rightarrow t = \frac{m}{b} \ell n\, 2 = \frac{0.025}{0.07} \times \ell n 2$

T = 2.5 second

Example 5: A vibrating system consists of mass of 400 gm and spring of spring constant 250 N The frictional force acting on it is directly propositional to velocity with the coefficie g/sec. What is the amplitude of oscillation if a periodic force of amplitude 1N and angu frequency of 20 Hz is applied to the system.

Solution:

Mass of body m = 400 gm = 0.4 kg, spring constant k = 250 N/m

∴ Natural frequency, $(\omega_0) = \sqrt{\frac{k}{m}} = 25$ rad/sec

Damping constant (b) = 4 gm/sec = 0.004 kg/sec.

Static force (F_0) = 1N

Frequency (f) = 20 Hz, $\omega = 2\pi f = 2\pi \times 20 = 125.6$

$$X_m = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

$$= \frac{1/0.4}{\sqrt{(25^2 - 125.6^2)^2 + \left(\frac{0.004 \times 125.6}{0.4}\right)^2}} = 1.65 \times 10^{-4} \text{ m}$$

Example 6: A 0.4 kg mass is moving on the end of a spring with force constant k = 300 N/m and is acted on by a damping force F = – bv (i) calculate the frequency of oscillation if b = 9 kg/s (ii) for what value of 'b' will motion will be critically damped.

Solution:

(i) Damping constant b= 9 kg/s

$$\omega = \sqrt{\frac{k}{m}-\left(\frac{b}{2m}\right)^2} = \sqrt{\frac{300}{0.4}-\left(\frac{9}{2\times 0.4}\right)^2}$$

$$\omega = \sqrt{750-126.6} = 24.97$$

$\therefore$ Frequency (f) $= \frac{\omega}{2\pi} = \frac{24-97}{2\times 3.24} = 3.98\text{Hz.}$

(2) For critical damping we have

$$\frac{b}{2m} = \omega_0$$

$$\Rightarrow \quad b = 2m\,\omega_0$$

$$= 2\times 0.4\times\sqrt{\frac{k}{m}}$$

$$= 0.8\times\sqrt{\frac{300}{0.4}} = 0.8\times\sqrt{750} = 21.9$$

Level 1

$\therefore$ damping constant 'b' = 21.9 kg/s

Example 7: If the relaxation time of a damped harmonic oscillator is 50 sec, find the time in which the amplitude falls to $\frac{1}{e^3}$ of its initial value and energy of the system falls to $\frac{1}{e^4}$ of its initial value.

Solution:

Here, relaxation time $\tau = 50$ sec

The amplitude is $x = x_m e^{\frac{-bt}{2m}}\ x_m e^{\frac{-t}{2\tau}}$ $\qquad \left[\because 2\delta = \frac{b}{m} = \frac{1}{\tau}\right]$

Here, $x = \frac{x_m}{e^3}$, $t = T$

$$\therefore \quad \frac{x_m}{e^3} = x_m e^{\frac{-bt}{2m}} \Rightarrow e^{-3} = e^{\frac{-bt}{2m}}$$

$$\frac{-bt}{2m} = 3 \Rightarrow T = \frac{5m}{b} = b\tau = 6\times 50 = 300 \text{ sec}$$

Again, Energy $E = E_0 = e^{\frac{-bt}{m}} = E_0 e^{\frac{-t}{\tau}}$

Here, $E = \frac{E_0}{e^4}$ $\qquad t = T$

$$\therefore \quad \frac{E_0}{e^4} = E_0 e^{\frac{-T}{\tau}} \Rightarrow e^{-4} = e^{\frac{-T}{\tau}}$$

$$\therefore \quad T = 4\tau = 4\times 50 = 200 \text{ sec.}$$

Example 10: A linear spring whose force constant is 0.2 N/m hangs vertically supporting a 1kg mass at rest. The mass is pulled down a distance 0.2m and then released. What will be its maximum velocity? Also find the frequency of vibration.

Solution:

Spring constant (k) = 0.2 N/m

Mass (m) = 1kg

Amplitude (A) = 0.2m

1. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.2}{1}} = 0.447$

Maximum velocity $V_{max} = A\omega = 0.2 \times 0.447 = 0.089$ m/sec.

2. Frequency (f) $= \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 0.071$Hz.

Example 11: A particle of mass 10 gm moves under a potential $V_x = 8\times10^5 x^2$ ergs/gm, where x is in cm. Deduce the time displacement relation when the total energy is 8×10^5 ergs.

Solution:

Here potential $V_x = 8\times10^5\, x^2$ ergs/gm

Potential energy acquired by particle,

$U \quad = m\,V_x = 10\times8\times10^5 x^2$

$= 8\times10^6\, x^2$ ergs

Since P.E. is maximum at extreme position, that is when x = A (maximum displacement or amplitude). Where total energy is only P.E.

$\therefore \quad 8\times10^6 A^2 = 8\times10^5$

$$A^2 = \frac{1}{10}$$

$$A = \frac{1}{\sqrt{10}}\text{ cm}$$

Now, force acting on particle, $F = -\frac{dU}{dx} = -\,16\times10^6 x$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -\,16\times10^6 x$$

$$m\frac{d^2x}{dt^2} + (16\times10^6)x = 0$$

$$\frac{d^2x}{dt^2} + \left(\frac{16\times10^6}{m}\right)x = 0$$

Therefore, $\omega^2 = \frac{16\times10^6}{m} = \frac{16\times10^6}{10} = 16\times10^5$

$\omega = \sqrt{16\times10^5} = 400\sqrt{10}$

The time displacement relation is given by

$$x = \frac{1}{\sqrt{10}}\sin(400\sqrt{10}\,t + \phi)$$

Example 12: A sinusoidal wave travels along a string. The time for a particular point to move from maximum displacement to zero is 0.17 S. What are the

(i) Period and frequency?

(ii) The wavelength is 1.40 m; what is the wave speed?

Solution:

Let T be the time period of oscillation.

The time interval between two extreme (maximum) position $= \frac{T}{2}$

Then, the time interval between mean position and one maximum position $= \frac{T}{4}$

(i) Time period (T) = 4 × (Time for maximum displacement to zero) = 4 × 0.17 S = 0.68 sec

and Frequency $(f) = \frac{1}{T} = \frac{1}{0.68} = 1.47$ Hz

(ii) $\lambda = 1.40$ m, wave speed $(v) = f\lambda = 1.47 \times 1.40 = 2.05$ m/S

Example 13: A small body of mass 0.1 kg is undergoing a SHM of amplitude 0.1m and period 2s. (1) What is the maximum force on the body? (2) If the oscillations are produced in a spring, what should be the force constant?

Solution:

Mass of the body, m = 0.1kg

Amplitude, A = 0.1m

Time period, T = 2sec

i. $F_{max} = ?$

ii. $k = ?$

We have,

$$T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow T^2 = \frac{4\pi^2 m}{k}$$

$$k = \frac{4\pi^2 m}{T^2} = 0.986 \text{ N/m}$$

$$F_{max} = kA = 0.0986 \text{ N.}$$

Example 14: If a particle moves in a potential energy field $U = U_0 - ax + bx^2$, where a and b are positive constants, a) obtain an expression for the force acting on it as a function of position b) At what point does the force vanish? c) Is this a point of stable equilibrium? calculate the force constant, time period and frequency.

Solution:

a) The force acting on the particle is given by

$$F = -\frac{dU}{dx} = -\frac{d}{dx}(U_0 - ax + bx^2) = a - 2bx$$

b) The force vanishes at the point where $\frac{dU}{dx} = 0$

i.e. $a - 2bx = 0$

$$x = \frac{a}{2b}$$

c) Here, $\frac{d^2U}{dx^2} = \frac{d}{dx}\left(\frac{dU}{dx}\right) = \frac{d}{dx}(-a + 2bx) = 2b$, which is positive. The point $x = \frac{a}{2b}$ represents the point of minimum potential energy. It is therefore, a point of stable equilibrium.

d) We have $F = a - 2bx$, from this relation if is clear that F is a linear restoring force with force constant $k = 2b$.

Therefore, time period, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{2b}}$

and frequency, $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{2b}{m}}$

Example 15: Show that the time period of oscillation of loaded spring is $T = 2\pi\sqrt{\frac{x}{g}}$.

Solution:

If k is the force constant, we have

$$mg = kx$$

$$\frac{m}{k} = \frac{x}{g}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{x}{g}}$$

Example 16: Show that if a uniform stick of length 'l' is mounted so as to rotate about a horizontal axis perpendicular to the stick and at a distance 'd' from the centre of gravity the period has minimum value when $d = 0.289l$.

Solution:

Here, Total length of stick = l

We have for a rod, $I = mk^2 = \frac{ml^2}{12} \Rightarrow k = \frac{l}{\sqrt{12}} = 0.289l$.

The time period of compound pendulum is minimum when length of pendulum (d) = radius of gyration (k)

Therefore, $d = k = 0.289l$.

Example 17: A uniform circular disc of radius 'R' oscillates in a vertical plane about horizontal axis. Find the distance of the axis of rotation from the centre for which the time period is minimum. Find the value of time period also.

Solution:

For a circular disc, $I = mk^2 = \frac{mR^2}{2}$

$$\Rightarrow \quad k = \frac{R}{\sqrt{2}}$$

Time period is minimum when, $l = k = \frac{R}{\sqrt{2}}$

Since, $T = 2\pi\sqrt{\frac{\frac{k^2}{l}+l}{g}}$

Therefore,

$$T_{min} = 2\pi\sqrt{\frac{k+k}{g}}$$

$$= 2\pi\sqrt{\frac{2k}{g}}$$

$$= 2\pi\sqrt{\frac{2}{g}\cdot\frac{R}{\sqrt{2}}}$$

$$= 2\pi\sqrt{\frac{1.41R}{g}}$$

Example 18: A metal disc of radius 0.5m oscillates in its own plane about an axis passing throug point on its edge. Find the length of equivalent simple pendulum.

Solution:

Radius of disc (R) = 0.5m

Therefore, length of pendulum $l = R = 0.5m$

For disc we have,

$$I = mk^2 = \frac{mR^2}{2} \Rightarrow k = \frac{R}{\sqrt{2}}$$

Therefore, Length of equivalent simple pendulum $L = \frac{k^2}{l} + l$

$$L = \frac{R^2}{2}\cdot\frac{1}{R} + R = \frac{R}{2} + R = \frac{3R}{2}$$

$L = 0.75m$

Example 19: A thin straight uniform rod of length $l = 1m$ and mass m = 160 gm hangs from a pivot one end. (a) What is it's time period for small amplitude oscillation? (b) What is th length of a simple pendulum that will have the same time period?

Solution:

Here, length of rod, $l' = 1m$

Therefore, length of pendulum $l = \frac{l'}{2} = 0.5m$

Mass, m = 160 gm = 0.16kg

Since, the moment of inerfra rod about an axis passing through one end is

$$I = \frac{1}{3}ml'^2 = \frac{1}{3} \times 0.16 \times 1 = 0.053\ kgm^2$$

a. $T = 2\pi\sqrt{\dfrac{I}{mgl}} = 2\pi\sqrt{\dfrac{0.053}{0.16\times9.8\times0.5}} = 1.63$ sec.

b. $T = 2\pi\sqrt{\dfrac{k^2/l + l}{g}} = 2\pi\sqrt{\dfrac{L}{g}}$

$\Rightarrow \quad L = \dfrac{gT^2}{4\pi^2}$

$= \dfrac{9.8\times1.63^2}{4\times\pi^2}$

$= 0.66\text{m}$

Example 20: A wire has torsional constant of 2 Nm/rad. A disc of radius 5cm and mass 100 gm is suspended at it's centre. What is the frequency?

Solution:

Here,

$C = 2\text{Nm/rad}, R = 5\text{cm} = 0.05\text{m}, m = 100 \text{ gm} = 0.1\text{kg}$

We have, $I = \dfrac{1}{2}mR^2 = \dfrac{1}{2}\times0.1\times0.05^2 = 1.25\times10^{-4}\ \text{kgm}^2$

and, $T = 2\pi\sqrt{\dfrac{I}{C}}$

$= 2\pi\sqrt{\dfrac{1.25\times10^{-4}}{2}}$

$= 0.04965$ Sec.

Therefore,

Frequency (f) $= \dfrac{1}{T} = 20.14$ Hz.

Example 21: A flat uniform circular disc has mass of 3kg and radius of 70cm. It is suspended in a horizontal plane by a wire attached to it's centre. If the disc is rotated through 2.5 radian about the wire, a torque of 0.6Nm is required to maintain that orientation. Calculate i) The rotational inertia of the disc about the wire ii) The torsion constant iii) The angular frequency of the torsion pendulum when it is set oscillating.

Solution:

Here, $m = 3\text{kg}, R = 70\text{cm} = 0.7\text{m}, \theta = 2.5$ radian and $\tau = 0.6$ Nm.

i. $I = \dfrac{1}{2}mR^2 = \dfrac{1}{2}\times3\times0.7^2 = 0.735\ \text{kgm}^2$

ii. $\tau = C\theta \Rightarrow C = \dfrac{\tau}{\theta} = \dfrac{0.6}{2.5} = 0.24$ Nm/rad

iii. $\omega = \sqrt{\dfrac{C}{I}} = \sqrt{\dfrac{0.24}{0.735}} = 0.571$ rad/sec.

Example 22: A solid sphere of radius 0.3m executes torsional oscillation of time period $2\pi\sqrt{12}$ sec, at the end of suspension wire where upper end is fixed in a rigid support. If the torsional constant of the wire be 6×10^{-3} Nm/rad, calculate the mass of the sphere.

Solution:

Here, $R = 0.3m$, $T = 2\pi\sqrt{12}$ sec, $C = 6\times10^{-3}$ Nm/rad, $m = ?$

We have, $T = 2\pi\sqrt{\frac{I}{C}} \Rightarrow I = \frac{CT^2}{4\pi^2} = \frac{6\times10^{-3}\times4\pi^2\times12}{4\pi^2}$

$I = 7.2\times10^{-2}\,kg\,m^2$

Since, for sphere, $I = \frac{2}{5}mR^2$

$m = \frac{5I}{2R^2} = 2kg.$

THEORETICAL ANSWER QUESTIONS

1. Derive the differential equation of damped oscillatory motion and show that mechanical energy decrease with increase in time.
2. Explain clearly the terms free, damped and forced vibration. Develop the differential equation of a particle executing damped vibration in a medium. Explain the physical meaning of each term and each constant in the equations.
3. What is forced oscillation? Derive differential equation for forced oscillation and show the amplitude at resonance is inversely proportional to the damping constant of medium.
4. Explain how the amplitude of oscillation vary with time.
5. Explain the sharpness of resonance.
6. Give the necessary theory of forced vibration and deduce the condition for amplitude and velocity resonance. Relate sharpness of resonance with quality factor.
7. How does the amplitude of forced oscillation vary with the frequency of external period force. Derive the relation for sinusoidally varying periodic external force.
8. Derive an expression for the time period of a physical pendulum and establish the interchangeability of centre of oscillation and suspension.
9. Distinguish between simple and physical pendulum. Deduce the time period of a compound pendulum and show that it is minimum when the length of the pendulum is equal to radius of gyration.
10. Derive an expression for the time period and radius of gyration of compound pendulum.
11. What are the limitation of a simple pendulum?
12. Why compound pendulum is preferred than simple pendulum?
13. Derive time period of torsion pendulum and find an expression for modulus of rigidity of the suspension wire.
14. In which condition time period of compound pendulum is maximum and minimum.

2

Chapter

WAVE MOTION

CHAPTER OUTLINE

After completion of this chapter, you will be able to know:

- Introduction of wave, wave velocity and particle velocity, types of waves and their applications, Speed of transverse wave in stretched string, energy, power and intensity of plane progressive wave, standing wave and resonance, sonometer,.

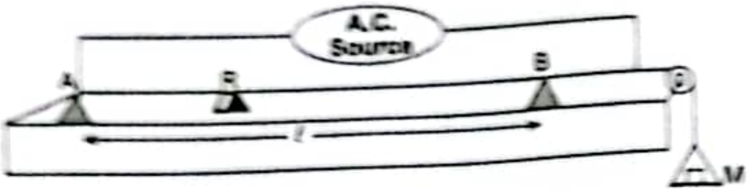


Figure 4 : Sonometer

When the wire resonates the frequency of a.c. mains is equal to the frequency of vibration of the string. According to laws of transverse vibration of string, the frequency of fundamental mode is,

$$f = \frac{1}{2l}\sqrt{\frac{T}{\mu}} \qquad \ldots \text{(i)}$$

Where, l is length of the vibrating pigment of the wire

$T =$ mg, is the tension in the wire, m is mass placed in the pan and μ is mass per unit length of wire.

$$T = 4\mu f^2 l^2 \qquad \ldots \text{(ii)}$$

If d be the diameter of the wire, then area of cross-section of wire $= \frac{\pi d^2}{4}$

Volume of wire = Area of cross-section × Length

$\therefore$ Mass of wire = Volume × Density

$$= \frac{\pi d^2}{4} \times l \times \rho$$

$\therefore$ Mass per unit length of wire $(\mu) = \frac{\pi d^2}{4} \times \rho$

Here, $\rho =$ density of material of wire.

Solved Example

Example 1: The equation of transverse wave travelling in a rope is given by $y = 10 \sin \pi (0.01\,x -$ centimeters. Find the amplitude, frequency, velocity and wave length of the wave.

Solution:

The given equation is $y = 10 \sin \pi (0.01x - 2t)$

Comparing this equation with the general displacement equation for a transverse wave,

$y = A \sin (kx - \omega t)$

a. $A = 10$ cm

b. $k = 0.01\,\cancel{\pi} \Rightarrow \frac{2\pi}{\lambda} = 0.01\,\cancel{\pi} \Rightarrow \lambda = \frac{2}{0.01} = 200 \text{ cm} = 2\text{m}$

c. $\omega = 2\cancel{\pi} \Rightarrow 2\pi f = 2\pi \Rightarrow f = 1 \text{ Hz}$

d. The wave speed is $v = f\lambda = 2\text{m/s}$

Example 2: **Calculate the wavelength, frequency, speed of the wave and maximum particle velocity in the wave represented by $y = 20 \sin\pi (2t - 0.05x)$, the values of x and y are in centimeters.**

do it

Solution:

We have,

$$y = 20 \sin\pi (2t - 0.05x) \qquad \ldots (i)$$

Comparing equation (1) with the equation of progressive wave $y = A \sin(\omega t - kx)$

$A = 20 \text{ cm} = 20 \times 10^{-2}$ m

$\omega = 2\pi \Rightarrow 2\pi f = 2\pi \Rightarrow f = 1$ Hz

$k = \pi\, 0.05\ (\text{cm}^{-1}) = \pi\, 0.05 \times 100\ \text{m}^{-1} = 5\pi\ \text{m}^{-1}$

i. Wave length, $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{5\pi} = 0.4$ m

ii. Speed of wave, $v = f\lambda = 1 \times 0.4 = 0.4$ m/sec — aω = Vmax

iii. Maximum particle velocity (V_{max}) $= A\omega = 20 \times 10^{-2} \times 2\pi = 1.25$ m/sec

Example 3: **What is amplitude, the wave length and the velocity of the wave represented by $y = 5\sin(6\pi t + 4x)$. Where distance and time are measured in SI units?**

Solution:

do

$$y = 5 \sin (6\pi t + 4x) \qquad \ldots (i)$$

Comparing this equation with ... (ii)

$$y = A \sin (\omega t + kx)$$

we get,

$A = 5\text{m}, \omega = 6\pi, k = 4$

we have,

$\omega = 2\pi f$

$6\pi = 2\pi f$

$\therefore \quad f = 3$ Hz

Also, $k = \dfrac{2\pi}{\lambda}$; $\qquad \lambda = \dfrac{2p}{4} = 1.57$ m

Velocity of wave (v) $= f\lambda$

$= 3 \times 1.57$

$= 4.712$ m/s.

Example 4: A boy claps his hands once every second and hears the echo from a distance building. He hears the echo of each clap mid-way between it and the next clap. If the velocity of sound is 340 m/s what is the distance of the building from the body?

do

Solution:

V = 340 m/s, Let d be the distance between building and the boy.

Total distance travel by the sound (D) = 2d

Time interval (t) $= \dfrac{1}{2}$ sec. = 0.5 sec.

We have,

Maximum particle velocity = $A\omega = 2 \times 4 = 8$

For third wave,

$A = 2, K = 3, \omega = 3$

Wave speed = $\omega/k = 3/3 = 1$

Maximum particle velocity = $A\omega = 2 \times 3 = 6$

The descending order is, 2nd, 3rd and 1st wave.

Example 7: Calculate the ratio of intensity of following two waves $y_1 = 6 \sin(0.4t - 25x)$ cm and $y_2 = 2.5 \sin(3.2t - 200x)$ cm.

Solution:

DC

For first wave, $y_1 = 6 \sin(0.4t - 25x)$

$$a = 6\text{cm}, \omega = 0.4, k = 25 \Rightarrow v = \frac{\omega}{k} = \frac{0.4}{25} = 0.016 \text{ m/s}$$

For second wave, $y_2 = 2.5 \sin(3.2t - 200x)$

$$a = 2.5 \text{ cm}, \omega = 3.2, k = 200 \Rightarrow v = \frac{3.2}{200} = 0.016 \text{ m/s}$$

$$\therefore \quad \frac{I_1}{I_2} = \frac{1/2\, v_1 \rho \omega_1^2 a_1^2}{1/2\, v_2 \rho \omega_2^2 a_2^2} = \frac{0.4^2 \times 6^2}{2.5^2 \times 3.2^2} = 0.09$$

Example 8: If the intensity of wave is 1.0×10^6 W/m^2 at 50km from a source, what is the intensity at 10 km from the source.

Solution:

Since Intensity, $I \propto \frac{1}{(\text{distance})^2}$ i.e. $I \propto \frac{1}{r^2}$

do

Therefore, $\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \Rightarrow \frac{1 \times 10^6}{I_2} = \frac{(10 \times 10^3)^2}{(50 \times 10^3)^2} = \frac{1}{25}$

$I_2 = 25 \times 10^6$ Watt/m^2 = 2.5×10^7 Watt/m^2

Example 9: Calculate frequency of vibration of air particles in plane progressive wave of amplitude 2.18×10^{-10} m and intensity 10^{-10} W/m^2, the velocity of sound in air is 340 m/S and density of air is 0.00129 gm/cc.

Solution:

do

$A = 2.18 \times 10^{-10}$ m, $I = 10^{-10}$ W/m^2, $v = 340$ m/sec,

$$\rho = 0.00129 \text{ gm/cc} = \frac{0.00129 \times 10^{-3}}{10^{-6}} = 1.29 \text{ kg/m}^3$$

We have,

$$I = \frac{1}{2} v\rho\, \omega^2 a^2$$

$$\omega^2 = \frac{2I}{v \rho a^2}$$

$$\omega = \sqrt{\frac{2I}{v \rho a^2}}$$

$$2\pi f = \sqrt{\frac{2I}{v \rho a^2}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{2I}{v\rho a^2}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{2 \times 10^{-10}}{340 \times 1.29 \times (2.18 \times 10^{-10})^2}}$$

$f = 493$ Hz.

Example 10: A progressive and stationary, simple harmonic wave having frequency 250 Hz and each having same velocity 30 m/S
(i) Determine the phase difference between two vibrating point in a progressive wave at a distance of 10 cm.
(ii) Wave equation of progressive wave if amplitude is 0.03 m.
(iii) Distance between nodes in stationary wave.

Imp

Solution:

Here, $f = 250$ Hz, $v = 30$ m/S

We have, $v = f\lambda \Rightarrow \lambda = \frac{v}{f} = \frac{30}{250}$

$\lambda = 0.12$ m

i. Since, path difference, $\lambda \Rightarrow$ Phase difference, 2π

Path difference, $1 \Rightarrow$ Phase difference, $2\pi/\lambda$

Path difference 10 cm (= 0.1m) $\Rightarrow$ Phase difference, $\frac{2\pi}{\lambda} \times 0.1$

Therefore, phase difference $= \frac{2\pi}{\lambda} \times 0.1 = \frac{2\pi}{0.12} \times 0.1 = 5.24°$

ii. Here, amplitude, $A = 0.03$ m

The wave equation is given by,

$y = A \sin(\omega t - kx)$

$$y = 0.03 \sin\left(2\pi f t - \frac{2\pi}{\lambda}x\right)$$

$$y = 0.03 \sin\left(2\pi \times 250\, t - \frac{2\pi}{0.12}x\right)$$

$y = 0.03 \sin(500\,\pi t - 52.36x)$

$y = 0.03 \sin(1570.8\, t - 52.36\, x)$

iii. The distance between two adjacent nodes in a stationary wave is $\frac{\lambda}{2} = \frac{0.12}{2} = 0.06$ m.

Example 11: A wave of frequency 500 cycles /sec has a phase velocity of 350 ms^{-1} (i) How far ap are two points 60° out phase (ii) what is the phase difference between t displacements at certain points at times 10^{-3} sec apart.

Imp

Solution:

Here, $f = 500$ cycles/sec, $v = 350\ ms^{-1}$

Since, $v = f\lambda \Rightarrow \lambda = \frac{v}{f}$

$$\lambda = \frac{350}{500} = 0.7\text{ m}$$

i) We have, phase difference, $2\pi \Rightarrow$ path difference, λ.

Phase difference, $1 \Rightarrow$ path difference, $\frac{\lambda}{2\pi}$

Phase difference, $60^\circ \Rightarrow$ path difference, $\frac{\lambda}{2\pi} \times 60$

Therefore, path difference $= \frac{0.7}{2 \times 180} \times 60 = 0.116$ m

ii) Here the time period, $T = \frac{1}{f} = \frac{1}{500} = 2 \times 10^{-3}$ sec

For time period 'T' seconds the phase difference $= 2\pi$

For time period 1 sec the phase difference $= \frac{2\pi}{T}$

For time 10^{-3} sec the phase difference $= \frac{2\pi}{T} \times 10^{-3} = \frac{2\pi}{2 \times 10^{-3}} \times 10^{-3} = \pi$ rad

Example 12: A piano wire with mass 5 gm and length 90 cm is stretched with tension of 25 N. A wave with frequency 100 Hz and amplitude 1.6 mm travels along the wire (a) calculate the average power carried by the wave (b) What happens to the power if the wave amplitude is halved.

Solution:

Mass (m) = 5 gm = 5×10^{-3} kg, length (l) = 90 cm = 0.9 m

Then, $\mu = \frac{m}{l} = \frac{5 \times 10^{-3}}{0.9} = 5.55 \times 10^{-3}$ kg/m

Tension (T) = 25 N

Frequency (f) = 100 Hz

amplitude (a) = 1.6 mm = 1.6×10^{-3} m

Speed of the wave (v) $= \sqrt{\frac{T}{\mu}} = \sqrt{\frac{25}{5.55 \times 10^{-3}}} = 67.082$ m/s

a. Average power (P) $= \frac{1}{2}\mu v \omega^2 a^2$

$= \frac{1}{2} \times 5.55 \times 10^{-3} \times 67.082 \times (2\pi \times 100)^2 \times (1.6 \times 10^{-3})^2 = 0.188$ watt

If amplitude is halved then, $a' = \frac{a}{2}$

Average power (P') $= \frac{1}{2}\mu v \omega^2 \left(\frac{a}{2}\right)^2 = \frac{1}{4}\left(\frac{1}{2}\mu v \omega^2 a^2\right) = \frac{1}{4} \times 0.188 = 0.0470$ watt.

Example 13: A source of sound has a frequency of 512 Hz and amplitude of 0.25 cm. What is the flow of energy across a square cm per second. If the velocity of sound in air is 340m/sec and density of air is 0.00129 gm/cm^3.

Solution:

Frequency, f = 512 Hz,

Amplitude, a = 0.25cm = 0.25×10^{-2}m,

velocity of sound in air,

v = 340m/sec, density of air,

iii. For an antinode amplitude is maximum

$\therefore \quad |\cos kx| = 1 = \cos n\pi$

$$x = \frac{p}{k} = \frac{p}{157} = 0.02 \text{ m.}$$

iv. The amplitude of vibration = $|A \cos kx|$

$= 10^{-3} \cos(157 \times 2.33 \times 10^{-2})$

$= 10^{-3} \cos 3\,(3.658 \text{ rad})$

$= 10^{-3} \cos\left(3.658 \times \frac{180}{\pi}\right)^{\circ}$

$= 10^{-3} \cos(209.59^{\circ})$

$= -10^{-3}(0.869)$

$= -8.69 \times 10^{-4}$ m

Example 15: A simple harmonic transverse wave is propagating along a string towards the left direction as shown in figure. The figure shows a plot of the displacement as a function of position at time t = 0. The string tension is 3.6 N and its linear density is 25 gm/m. Calculate

i. The amplitude $\mu = m/l$

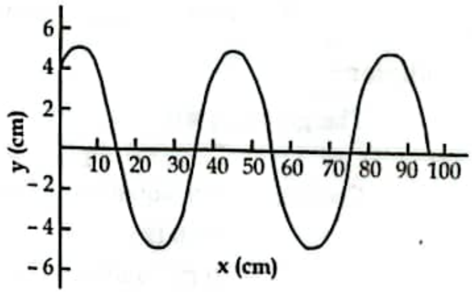
ii. The wave-length

iii. Wave speed

iv. The period

v. The maximum particle speed in the string

vi. Write an equation describing the travelling wave.



Solution:

T = 3.6 N, $\mu = 25$ gm/m $= 25 \times 10^{-3}$ kg/m

i. Amplitude (A) = 5 cm = 5×10^{-2} m

ii. Wavelength (λ) = 40 cm = 40×10^{-2} m

iii. wave speed $v = \sqrt{\frac{T}{m}} = \sqrt{\frac{3.6}{25 \times 10^{-3}}} = 12$ m/s

iv. Frequency (f) $= \frac{v}{\lambda} = \frac{12}{40 \times 10^{-2}} = 30$ Hz → $\omega = 2\pi f = 2\pi \times 30 = 60\pi t$

and Time period (T) $= \frac{1}{f} = \frac{1}{30} = 0.033$ sec.

v. Maximum particle speed (v_{max}) $= A\omega = A(2\pi f) = 5 \times 10^{-2} \times 2\pi \times 30 = 9.425$ m/s

vi. The equation of the travelling wave is

$$y = 5 \times 10^{-2} \text{ Sin}\left(60pt + \frac{2p}{40 \times 10^{-2}} x\right)$$ $(y = a\sin(\omega t + kx))$

$$= 5 \times 10^{-2} \sin(60\pi t + 5\pi x)$$

Example 16: A wave of frequency 500 cycles/sec. has a phase velocity of 350 m/sec. H... apart are two points 60° out of phase.

Solution:

Frequency f = 500 cycles/sec

Phase velocity v = 350 m/sec

Phase difference = 60°

Path difference = ?

Since $v = f\lambda \Rightarrow \lambda = \frac{v}{f} = \frac{350}{500}$

Since phase difference $2\pi \Rightarrow$ path difference λ.

Phase difference $1 \Rightarrow$ path difference $\frac{\lambda}{2\pi}$

Phase difference $60° \Rightarrow$ path difference $\frac{\lambda}{2\pi} \times 60°$

Therefore, path difference $= \frac{\lambda}{2\pi} \times 60° = \frac{\lambda}{6} = \frac{350}{500 \times 6} = 0.117\text{m}$

15/11/18, 2024 cnodo

Example 17: The equation of transverse wave on a string is $y = (2.0\text{mm}) \sin [(20\text{m}^{-1}) x - (600 \text{s}^{-1}) t]$. The tension on the string is 15 N. (i) What is the wave speed (ii) Find the linear density of the string in grams per meter.

Solution:

The given equation is,

$$y = (2.0 \text{ mm}) \sin [(20 \text{ m}^{-1}) x - (600 \text{ S}^{-1}) t]$$

Comparing this equation with general displacement equation

$$y = A \sin (kx - \omega t)$$

$$k = 20 \text{ m}^{-1} \text{ and } \omega = 600 \text{ S}^{-1}$$

Therefore, velocity, $v = \frac{\omega}{k} = \frac{600}{20} = 30 \text{ m/S}$

Here, tension on the string, T = 15 N

Since, $v = \sqrt{\frac{T}{\mu}} \Rightarrow v^2 = \frac{T}{\mu} \Rightarrow \mu = \frac{T}{v^2}$

$$\mu = \frac{15}{30^2} = \frac{1}{60} \text{ kg/m}$$

$$= \frac{1000}{60} \text{ gm/m} = 16.67 \text{ gm/m}$$

7-4

Example 18: 8T A stretched string has linear density 525 gm/m and is under tension of 45N. A sinusoidal wave with frequency 120 Hz and amplitude 8.5 mm is sent along the string from one end. At what average rate does the wave transport energy.

Solution:

Here, $\mu = 525$ gm/m = 0.525 kg/m, T = 45 N

Velocity of wave $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{45}{0.525}} = 9.26$ m/sec

f = 120 Hz, amplitude, a = 8.5 mm = 8.5×10^{-3}m

Solution:

$f = 2$ Hz, $T = 2N$, $x = 0.5$ m, $\mu = 0.005$ kg/m

We have, $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{2\ N}{0.005\ kg/m}}$

Speed (v) = 20 m/s

Also, $v = \omega\sqrt{A^2 - x^2}$

or, $20 = 2\pi f\sqrt{A^2 - (0.5)^2}$

or, $400 = 4\pi^2 f^2 [A^2 - (0.5)^2]$

or, $\frac{400}{4\pi^2 (2^2)} = A^2 - 0.25 \Rightarrow A^2 = 2.785$

$\Rightarrow$ Amplitude (A) = 1.67 m

Time period (T) $= \frac{1}{f} = \frac{1}{2} = 0.5$ sec

Wave length (λ) $= \frac{v}{f} = \frac{20}{2} = 10$ m

Example 24: A stretched string has a linear mass density of 5.0 gm/cm and a tension of 10 N. A wave on this string has an amplitude of 0.12 mm and a frequency of 100 Hz and is travelling in –ve x direction. At what average rate does the wave transport energy? Write the wave equation with appropriate units.

Solution:

Here, $\mu = 5gm/cm = \frac{5\times10^{-3}}{10^{-2}} = 0.5$ kg/m

Tension, T = 10 N

Velocity, $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{10}{0.5}} = 4.47$ m/sec

Here, amplitude, $a = 0.12$ mm $= 0.12 \times 10^{-3}$ m Frequency, f = 100 Hz

Since, $v = f\lambda \Rightarrow \lambda = \frac{v}{f} = \frac{4.47}{100} = 0.0447m$ Therefore, $k = \frac{2\pi}{\lambda} = 141\ m^{-1}$

Now, $P = \frac{1}{2}\mu v\omega^2 a^2 = \frac{1}{2} \times 0.5 \times 4.47 \times (2\pi \times 100)^2 \times (0.12 \times 10^{-3})^2 = 6.35 \times 10^{-3}$ watt.

The equation of wave travelling in –ve x direction is

$y = a \sin(\omega t + kx) = (0.12mm) \sin[(2\pi f) t + (141m^{-1}) x] = 0.12\ mm \sin[(628s^{-1}) t + (141\ m^{-1}) x]$

THEORETICAL ANSWER QUESTIONS

1. How standing waves are produced on a string? Write down the difference between travelling wave and standing wave.
2. Show that for a plane progressive wave, on the average, half the energy is kinetic and half-potential.
3. What is wave motion? Write the equation of sine wave travelling on the string; apply the theory of supper position to find the maxima and minima of interference pattern on the string.
4. Calculate amount of energy transmitted along a stretched string when a wave passes through it.
5. Differentiate between plane progressive wave and stationary wave. Show that intensity of a particular wave for a medium remains constant.
6. Deduce an expression for velocity of transverse waves in a stretched string.
7. What is super position of wave? Describe interference of wave.
8. In the progressive wave, show that the potential energy and kinetic energy of every particle will change with time but the average K.E per unit volume and P.E. per unit volume remains constant.
9. Distinguish between wave velocity and particle velocity. Obtain the relation for acceleration of the particle in wave motion.

(A) LASER

INTRODUCTION

In an ordinary source of light the billions and billions of atoms or molecules produce light of al... wave length in all possible direction. So due to superposition of different types of wave (di... phase, plane of polarization etc), the intensity decreases with distance. The light produced by... source of light is not unidirectional, not monochromatic and in out of phase.

The light from a source come as the sum of total radiations produced by all individual atoms or m... in the source. In laser, the radiations given out by all the emitters in the source are in mutual ag... not only in phase but also in the direction of emission and plane of polarization. Such monochroma... highly coherent sources are the laser sources. The word LASER stands for Light Amplifica... Stimulated Emission of Radiations.

PRINCIPLES OF GENERATION OF LASER LIGHT

Induced Absorption

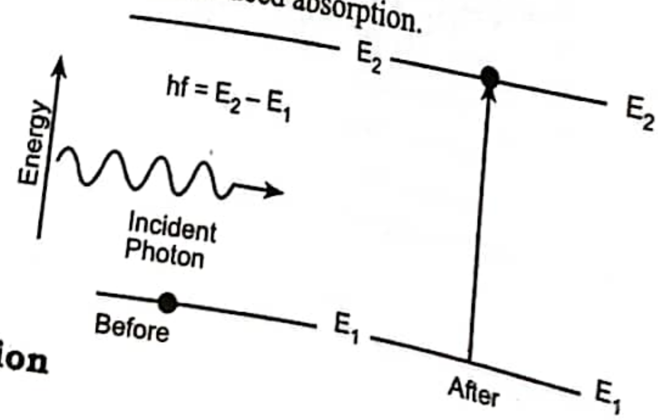
If the atom is initially in the lower state E_1, it can be raised to higher energy state E_2 by imparting it ... photon of energy $E_2 - E_1 = hf$, f = frequency of incident photon. The photon is absorbed by the atom... reached to higher energy state E_2. This phenomenon of absorption of photon by atom is called ind... absorption. The rate of induced absorption is given by,

$$R_{abs} = B_{12}\,\rho\, N_1$$

Where, N_1 = Population of atoms at E_1

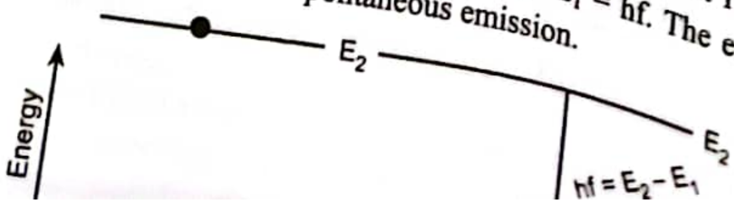
ρ = Energy density of incident beam

B_{12} = Einstein's coefficient for induced absorption.



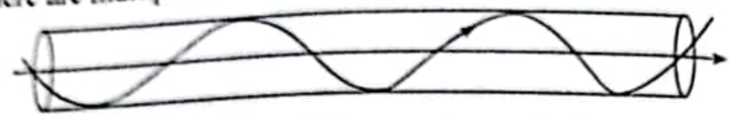
...ntaneous Emission

...n cannot stay in the excited state for a longer time. In a time of about 10^{-8} second the atom retu... ...wer energy state by releasing a photon of energy $E_2 - E_1 = hf$. The emission of a photon by a... ...hout any external impetus is called spontaneous emission.



3. Multimode Graded Index Fibre

i. In this fibre, there are multiple modes of propagation.



ii. The refractive index of core decreases with increasing radial distance from the axis.

iii. It has larger core diameter of 100 μm, which is very large as compared to the wave length of light.

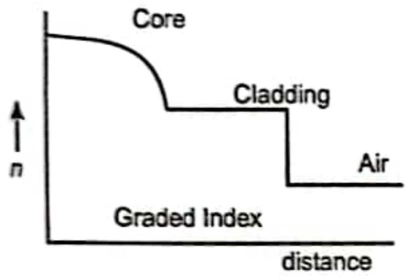
iv. Numerical aperture decreases with increasing radial distance from the axis.

v. The acceptance angle also deceases with radial distance from the axis.

vi. The number of modes in graded index fibre is about half of that in multimode step index fibre. So it has lower dispersion than in multimode step index fibre.

vii. It has low transmission loss because the continuous decrease in the refractive index causes bending of light towards the center of the core.

viii. It's manufacture is more complex than step index Multimode Fibre (MMF).



FRACTIONAL REFRACTIVE INDEX CHANGE

The ratio of difference in refractive indices of the core and the cladding to the refractive index of core is called fractional refractive index change. Fractional refractive index change, $\Delta = \frac{n_1 - n_2}{n_1}$. Since, $n_1 > n_2$ so Δ is always +ve.

Acceptance Angle

Consider an optical fibre through which light is launched at one end. Let n_1, n_2 and n_0 be the refractive indices of core, cladding, and the medium from which light is launched into the fibre.

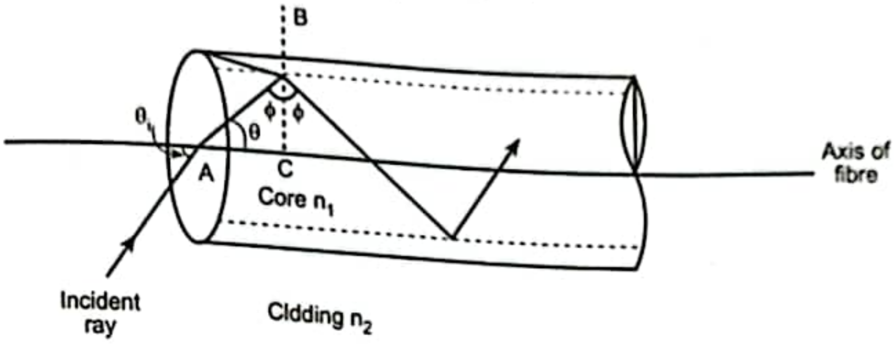
Assume a light ray enters the fibre at an angle θ_i to the axis of the fibre. The ray refracts at an angle θ_r and strikes the core-cladding interface at an angle ϕ. The angle ϕ is greater than critical angle ϕ_c, and total internal reflection occurs. So the light will stay within the fibre.

Applying Snell's law at the incident edge, we get,

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_1}{n_0} \qquad \ldots \text{(i)}$$

If θ_i is increased beyond a limit, ϕ will drop below the critical value ϕ_c and the ray escapes from the sidewall of the fibre. The largest value of θ_i occurs when $\phi = \phi_c$.

Acceptance angle is the maximum value of angle of incidence θ_i for which a light ray can propagate through fibre. It is the angle made by incident ray with fibre axis.



From figure in Δ ABC, $\theta_r = 180 - 90 - \phi = 90 - \phi$

$$\sin\theta_r = \sin(90-\phi) = \cos\phi \qquad \text{. . . (ii)}$$

From equation (1) and equation (2)

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_1}{n_0} \Rightarrow \frac{\sin\theta_i}{\cos\phi} = \frac{n_1}{n_0}$$

$$\sin\theta_i = \frac{n_1}{n_0}\cos\phi$$

For $\phi = \phi_c$, $\theta_i = \theta_{i\,max}$ [Here, θ_{imax} is the acceptance angle]

Therefore, $$\sin\theta_{i\,max} = \frac{n_1}{n_0}\cos\phi_c \qquad \text{. . . (iii)}$$

But at critical angle, total internal reflection just occurs

Applying Snell's law at core to cladding interface.

$$\frac{\sin\phi_c}{\sin 90} = \frac{n_2}{n_1}$$

$$\sin\phi_c = \frac{n_2}{n_1} \text{ and } \cos\phi_c = \sqrt{1-\sin^2\phi_c} = \sqrt{1-\frac{n_2^2}{n_1^2}} = \sqrt{\frac{n_1^2-n_2^2}{n_1^2}}$$

Substituting $\cos\phi_c$ for equation (3),

$$\sin\theta_{i\,max} = \frac{n_1}{n_0}\sqrt{\frac{n_1^2-n_2^2}{n_1^2}} = \frac{\sqrt{n_1^2-n_2^2}}{n_0} \qquad \text{. . . (iv)}$$

$$\theta_{i\,max} = \sin^{-1}\frac{\sqrt{n_1^2-n_2^2}}{n_0} \qquad \text{. . . (v)}$$

which is the expression for acceptance angle.

If the incident ray is launched from air medium,

$$\theta_0 = \theta_{i\,max} = \sin^{-1}\sqrt{n_1^2-n_2^2}.$$

The light ray incident at an angle less than $\theta_{i\,max}$ will propagate along the fibre and the light ray incident at an angle larger than $\theta_{i\,max}$ will escape out.

NUMERICAL APERTURE (N.A.)

Numerical aperture is the measure of the amount of light that can be accepted by a fibre. Mathematically it is defined as the sine of the acceptance angle i.e.

$$\text{N.A.} = \sin\theta_0$$

$$= \sqrt{n_1^2 - n_2^2}$$

$$= \{(n_1-n_2)(n_1+n_2)\}^{1/2} = \left(\frac{n_1-n_2}{n_1} \cdot \frac{n_1+n_2}{2} \cdot 2n_1\right)^{1/2}$$

Since fractional refractive index change $\Delta = \frac{n_1 - n_2}{n_1}$

$$\text{N.A.} = \left(\Delta \cdot \frac{n_1+n_2}{2} \cdot 2n_1\right)^{1/2}$$

Approximating $\frac{n_1+n_2}{2} \approx n_1$ as $n_1 > n_2$

$$\text{N.A.} = (\Delta . n_1 . 2n_1)^{1/2} = \sqrt{2n_1^2\Delta}$$

$$\text{N.A.} = n_1\sqrt{2\Delta}$$

It is seen that N.A. only dependent on the refractive indices of core and cladding. The value of N.A. ranges from 0.13 to 0.50. Larger value of N.A. means the fibre can accept more light from the source.

NORMALIZED FREQUENCY: (V–NUMBER)

The V-number determines the number of modes that can propagate through a fibre.
It is an important parameter that characterize an optical fibre. Mathematically it is give by.

$$V = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2}$$

Where a = radius of the core, λ = free space wave length, n_1 and n_2 are refractive indices for core and cladding.

- For $V < 2.405$, fibre can support only one mode and is called single mode fibre (SMF)
- For $V > 2.405$, fibre can support number of modes simultaneously and is called multimode fibre (MMF).

Application

Optical fibre has a wide variety of applications. Some of them are discussed below.

1. **Medical application**
   - The optical fibre technique is employed for endoscopy, which helps in the study of inaccessible part of human body.
   - This can be employed to join a detached retina or rectify other eye defects using laser.

b) **Smoke and pollution detector:** If foreign particle (due to smoke and pollution) are present in a region between two optical fibre the variation in emergent intensity determines the extent of foreign particle between the two fibre.

c) A loop of optical fibre can used to determine the level of liquid inside an unreachable place. The part of cladding at which optical fibre is going to touch liquid level is scraped. A bare (naked) core losses more light when it is immerged in liquid than when it is in air. Therefore sudden change of out coming intensity indicates liquid level.

d) The other sensors such as pressure sensor, current sensor, voltage sensor, magnetic field sensor are made on the basis of optical properties of optical fibre.

4. **Military Application**

Optical fibres have a lot of application in various warfare's and military operations. For control mechanism or communication, a lot of copper cables are to be transported by an aircraft, a ship or tank. However with fibre optics, the weight is enormously reduced and communication network is greatly improved. With fibre communication secrecy is also maintained.

Fibre guided missiles are used in recent wars. Sensors mounted on the missile transmit video information through the optical fibre to a ground control van and receives commands from the van again. The control van continuously monitors the missile, so that it can hit the target.

Solved Example

Example 1: A glass clad fibre is made with core glass of refractive index 1.5 and cladding is doped to give a fractional index difference of 5×10^{-4}. Determine i) the cladding index ii) The critical internal reflection angle iii) The external critical acceptance angle iv) numerical aperture.

Solution:

Here, refractive index of core, $n_1 = 1.5$

fractional refractive index change, $\Delta = 5 \times 10^{-4}$

i. We have, $\Delta = \dfrac{n_1 - n_2}{n_1} \Rightarrow n_1 - n_2 = n_1\Delta \Rightarrow n_2 = n_1 - n_1\Delta$

$\Rightarrow$ $n_2 = 1.5 - (1.5 \times 0.005) = 1.4925$

$\therefore$ $n_2 = 1.4925$

ii. Let ϕ_c be the critical internal reflection angle then,

$$\sin\phi_c = \frac{n_2}{n_1} \Rightarrow \phi_c = \sin^{-1}\left(\frac{1.4925}{1.5}\right) = 88.2^\circ$$

iii. The external critical acceptance angle is given by

$$\theta_0 = \sin^{-1}\sqrt{n_1^2 - n_2^2}$$

$$= \sin^{-1}\left(\sqrt{1.5^2 - 1.4925^2}\right)$$

$$= 112^\circ$$

iv. N.A. $= n_1\sqrt{2\Delta}$

$$= 1.5\sqrt{2 \times 0.005} = 0.0474$$

Example 2: An optical fibre has fractional index difference of 0.2 and a cladding refractive index 1.59. Determine acceptance angle for the fibre in water in which has refractive index of 1.33.

Solution:

Here, $\Delta = 0.2$, $n_2 = 1.59$, $n_o = 1.33$

Since, $\Delta = \frac{n_1 - n_2}{n_1} \Rightarrow n_1 - n_2 = n_1\Delta \Rightarrow n_1 - n_1\Delta = n_2$

$n_1(1-\Delta) = n_2 \Rightarrow n_1 = \frac{n_2}{1-\Delta} = \frac{1.59}{1-0.2} = 1.9875$

Again, Acceptance angle,

$$\theta_{i\,max} = \sin^{-1}\frac{\sqrt{n_1^2 - n_2^2}}{n_o} = \sin^{-1}\frac{\sqrt{1.9875^2 - 1.59^2}}{1.33}$$

$$\theta_{i\,max} = 63.72°$$

Example 3: Calculate the refractive indices of the core and cladding materials of a fiber from following data. Numerical aperture (NA) = 0.22 and fractional refractive index change $\Delta = 0.012$.

Solution:

Here, N.A = 0.22

$$\Rightarrow \quad n_1\sqrt{2\Delta} = 0.22$$

$$\Rightarrow \quad n_1 = \frac{0.22}{\sqrt{2\Delta}}$$

$$= \frac{0.22}{\sqrt{2 \times 0.012}}$$

$$= 1.42$$

Given, $\Delta = 0.012$

$$\Rightarrow \quad \frac{n_1 - n_2}{n_1} = 0.012$$

$$n_1 - n_2 = 0.012\, n_1$$

$$\Rightarrow \quad n_2 = n_1 - 0.012\, n_1$$

$$\Rightarrow \quad n_2 = 1.402$$

THEORETICAL ANSWER QUESTIONS

1. Write the principle of optical Fiber.
2. Write short notes on propagation of light wave through optical Fiber.
3. What is an optical fiber? How is it made? Write down main differences between step index and graded index multimode optical fibers with well diagrams.

5

Chapter

CAPACITOR AND DIELECTRIC

CHAPTER OUTLINE

After completion of this chapter, you will be able to know:

- Capacitor: Introduction, Types of capacitor, Charging and discharging of capacitor.
- Dielectric: Introduction, Dielectric constant, electric flux density, Polarization, Polarization in free space, Gauss law in dielectric, Electronic and Ionic polarization (Clausius-Mossotti equation).

Solved Example

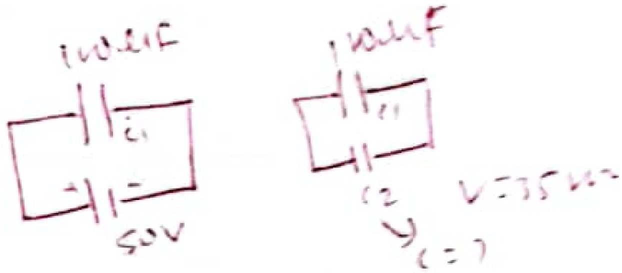
Example 1: A 100 μF capacitor is charged to a potential difference of 50 volts. The charging battery is then disconnected, the capacitor is then connected to second capacitor in parallel. If the measured potential drop to 35 volts. What is the capacitance of second capacitor.

Solution:

Before connection, V = 50 volts, C = 100 μF, Q = CV.

After connection let V_o be the common potential, Q_1 be the charge on first capacitor and Q_2 be the charge on second capacitor.

Therefore,



$$Q_1 = C_1 V_1 = CV_o$$

$$Q_2 = C_2 V_2 = C_2 V_o$$

Also,

$$Q = Q_1 + Q_2$$

$$CV = CV_o + C_2 V_o$$

$$\Rightarrow \quad C_2 = \left(\frac{V - V_o}{V_o}\right) . C$$

$$= \frac{(50 - 35) \times 100}{35} = 42.86\ \mu F$$

Example 2: A parallel plate capacitor with air as dielectric is charged to a potential V. It is then connected to an uncharged parallel plate capacitor filled with wax of dielectric constant K. Calculate common potential of both capacitor.

Solution:

Before connection let, Q and C are the charge and capacitance of capacitor.

Here K = 1 and Q = CV

After connection, let V_o be the common potential

For first capacitor, $Q_1 = C_1 V_1 = CV_o$. For second capacitor, $Q_2 = C_2 V_2 = C_2 V_o = KCV_o$ [Since, $\frac{C_2}{C_1} = K$]

Also,

$$Q = Q_1 + Q_2$$

$$CV = CV_o + KCV_o$$

$$V = V_o (1 + K)$$

$$V_o = \frac{V}{1 + K}$$

Example 3: An air-filled parallel plate capacitor has a capacitance of 10^{-12}F. The separation between the plates is doubled and wax is inserted between them. Which increases the capacitance to 2×10^{-12} F. Calculate the dielectric constant of wax.

Solution:

Here, $C_1 = 10^{-12}$ F, $d_1 = d$, $A_1 = A$ and $\epsilon_1 = \epsilon_o$

$$C_1 = \frac{\epsilon_1 A_1}{d_1} = \frac{\epsilon_o A}{d} \quad \ldots (i)$$

When d is increased to 2d and wax is introduced

$$C_2 = 2 \times 10^{-12}\text{F}, \quad d_2 = 2d, \quad A_2 = A \text{ and } \epsilon_2 = \epsilon \text{ (say)}$$

We have,

$$C_2 = \frac{\epsilon_2 A_2}{d_2} = \frac{\epsilon A}{2d} = \frac{K\epsilon_o A}{2d} \quad \ldots (ii)$$

$$\Rightarrow \quad \frac{C_2}{C_1} = \frac{K}{2} \Rightarrow K = \frac{2C_2}{C_1}$$

$$\Rightarrow \quad K = \frac{2 \times 2 \times 10^{-12}}{10^{-12}} = 4$$

Example 4: If the charge on a capacitor is increased by 2 coulomb, the energy stored in it increased by 21%. Find the original charge on the capacitor.

Solution:

Since, $U_1 = \frac{q_1^2}{2C}$ and $U_2 = \frac{q_2^2}{2C}$

Therefore, $\frac{U_2 - U_1}{U_1} = \frac{q_2^2 - q_1^2}{q_1^2}$

Given,

$$\frac{U_2 - U_1}{U_1} = 21\% = 0.21$$

$$\Rightarrow \quad \frac{q_2^2 - q_1^2}{q_1^2} = 0.21 \Rightarrow q_2^2 - q_1^2 = 0.21\, q_1^2$$

$$q_2^2 = 1.21\, q_1^2$$

$$q_2 = 1.1 \cdot q_1 \quad \ldots (1)$$

Also from question, $q_2 = q_1 + 2C$

$$\Rightarrow \quad 1.1q_1 = q_1 + 2$$

$$q_1 = \frac{2}{0.1} = 20 \text{ C}$$

Therefore, original charge on capacitor, $q_1 = 20$ C.

Example 5: The plates of a parallel plate capacitor are 10 cm apart and have area equal to $2m^2$. If the charge on each plate is 8.85×10^{-10}C. Find the electric field at a point between the plates and the capacitance of the capacitor.

Solution:

Here, $d = 10 \text{ cm} = 10^{-2}$m, $A = 2m^2$, $q = 8.85 \times 10^{-10}$ C.

We have, $E = \dfrac{q}{\epsilon_0 A} = \dfrac{8.85 \times 10^{-10}}{8.85 \times 10^{-12} \times 2} = 50$ N/C

and, $C = \dfrac{\epsilon_0 A}{d} = \dfrac{8.85 \times 10^{-12} \times 2}{10^{-2}} = 1.77 \times 10^{-9}$ F

Example 6: A leaky parallel plate capacitor is completely filled with a material having dielectric constant K and conductivity σ. Show that the time constant of capacitor is $\dfrac{K\epsilon_0}{\sigma}$.

Solution:

Let A be the plate area and 'd' the plate separation. The resistivity of material is given by, $\rho = \dfrac{RA}{d}$ where R is the resistance of the material.

The conductivity σ is, $\sigma = \dfrac{1}{\rho} = \dfrac{d}{RA}$ or $R = \dfrac{d}{\sigma A}$

The capacitance of capacitor, $C = \dfrac{\epsilon A}{d} = \dfrac{K\epsilon_0 A}{d}$

Therefore, time constant, $\tau = RC = \dfrac{d}{\sigma A} \cdot \dfrac{K\epsilon_0 A}{d} = \dfrac{K\epsilon_0}{\sigma}$.

Example 7: Show that capacitance due to a charged sphere of radius 'r' is $4\pi\epsilon_o r$. The total charge on the sphere is supposed to be concentrated at the centre.

Solution:

We have, $V = \dfrac{q}{4\pi\epsilon_o r}$

Since, $C = \dfrac{q}{V}$

$\Rightarrow \quad C = \dfrac{q}{q} \times 4\pi\epsilon_o r = 4\pi\epsilon_o r$

Example 8: If n drops, each of capacitance C, coalesce to form a single big drop. Find the capacitance of big drop.

Solution:

The mass of small drop, $M_1 = \dfrac{4\pi}{3} r^3 \times \rho$

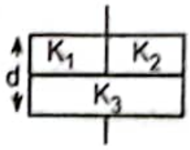
The mass of big drop, $M_2 = \dfrac{4\pi}{3} R^3 \times \rho$

Since $M_2 = nM_1$

$\Rightarrow \quad \dfrac{4\pi}{3} R^3 = n.\dfrac{4\pi}{3} r^3 \Rightarrow R^3 = nr^3$

$R = n^{1/3}$ r.

Example 12: A parallel plate capacitor of area A. plate separation d and capacitance C is filled with three different dielectric materials having dielectric constants K_1, K_2 and K_3 as shown in figure. Find C.



Solution:

We have,

$$C_1 = \frac{K_1 \epsilon_0 (A/2)}{(d/2)} = \frac{K_1 \epsilon_0 A}{d}$$

$$C_2 = \frac{K_2 \epsilon_0 (A/2)}{d/2} = \frac{K_2 \epsilon_0 A}{d}$$

$$\text{and, } C_3 = \frac{K_3 \epsilon_0 A}{(d/2)} = \frac{2 K_3 \epsilon_0 A}{d}$$

The capacitors, C_1 and C_2 are in parallel and their equivalent capacitance is

$$C' = C_1 + C_2 = \frac{(K_1 + K_2) \epsilon_0 A}{d}$$

This combination is in series with C_3, therefore, total capacitance C is given by

$$\frac{1}{C} = \frac{1}{C'} + \frac{1}{C_3} = \frac{d}{(K_1 + K_2) \epsilon_0 A} + \frac{d}{2K_3 \epsilon_0 A}$$

$$= \left(\frac{1}{K_1 + K_2} + \frac{1}{2K_3}\right)\frac{d}{\epsilon_0 A} = \frac{(K_1 + K_2 + 2K_3)}{2K_3(K_1 + K_2)} \frac{d}{\epsilon_0 A}$$

$$C = \frac{\epsilon_0 A}{d} \frac{2K_3 (K_1 + K_2)}{(K_1 + K_2 + 2K_3)} = \frac{2\epsilon_0 A K_3 (K_1 + K_2)}{d (K_1 + K_2 + 2K_3)}$$

Example 13: A parallel plate capacitor has a capacitance of 100×10^{-12} F, a plate area of 100 cm^2 mica is used as a dielectric. At 50 Volts potential difference, Calculate electric field intensity and magnitude of induced charge?

Solution:

Here, → known

$K = 5.4$, $C = 100 \times 10^{-12}$ F, $A = 100 \text{ cm}^2 = 100 \times 10^{-4} \text{ m}^2$

$V = 50$ Volt

$q = CV = 100 \times 10^{-12} \times 50 = 5 \times 10^{-9}$ C

$q' = q - \frac{q}{K}$

We have,

$q = q' - \frac{q}{K}$

$$E = \frac{q}{K \varepsilon_0 A}$$

$$= \frac{5 \times 10^{-9}}{5.4 \times 8.85 \times 10^{-12} \times 100 \times 10^{-4}}$$

$$= 1.05 \times 10^4 \text{ V/m}$$

Now, Induced charge, $q' = q\left(1 - \frac{1}{K}\right) = 5 \times 10^{-9}\left[1 - \frac{1}{5.4}\right] = 4.07 \times 10^{-9}$ C

Example 14: A parallel plate capacitor has circular plates of 8 cm radius and 1 mm separation. What charge will appear on the plates if a potential difference of 100 V is applied?

Solution:

$V = 100\ V,\ r = 8\ cm = 8 \times 10^{-2}\ m,$

$A = \pi r^2 = \pi (8 \times 10^{-2})^2 = 0.020\ m^2$

$d = 1\ mm = 1 \times 10^{-3}\ m$

Then,

$$C = \frac{\varepsilon_0 A}{d} = \frac{(8.85 \times 10^{-12}\ F/m) \times 0.020\ m^2}{1 \times 10^{-3}\ m} = 1.78 \times 10^{-10}\ F$$

Now,

$q = CV = 1.78 \times 10^{-10} \times 100 = 1.78 \times 10^{-8}\ C$

Example 15: A parallel plate capacitor has a capacitance of 100 μμF, a plate area of 100 cm² and mica as dielectric. At 50 volts potential difference calculate a) E in mica, b) the free charge on the plates and (c) the induced surface charge (Given K = 5.4 for mica).

Solution:

Here, $K = 5.4$, $C = 100\ \mu\mu F = 100 \times 10^{-12} F$, $A = 100 cm^2 = 10^{-2} m^2$ $V = 50$ volts.\

We have, $q = CV = 100 \times 10^{-12} \times 50 = 5 \times 10^{-9} C$

The induced surface charge is $q' = q\left(1 - \frac{1}{K}\right) = 5 \times 10^{-9}\left(1 - \frac{1}{5.4}\right) = 4.1 \times 10^{-9}\ C$

Electric field in mica is,

$$E = \frac{q}{K \epsilon_0 A}$$

$$= \frac{5 \times 10^{-9}}{5.4 \times 8.85 \times 10^{-12} \times 10^{-2}}$$

$$= 1.05 \times 10^4 V/m$$

Example 16: Two capacitors 2μF and 4 μF are connected in parallel across 300 volts pd. Calculate the total energy in the system.

Solution:

Here,

$C_1 = 2\mu F = 2 \times 10^{-6}\ F$

$C_2 = 4\ \mu F = 4 \times 10^{-6}\ F$

$V = 300$ volts.

Since C_1 and C_2 are connected in parallel, so their equivalent capacitance is given by.

$C \quad = C_1 + C_2 = 6 \times 10^{-6}\ F$

$U \quad = \frac{1}{2} CV^2$

$\quad = \frac{1}{2} \times 6 \times 10^{-6} \times 300^2 = 0.27$ Joule

Example 17: A capacitor of capacitance C is discharged through a resistor of resistance R. After how many time constant is the stored energy becomes one fourth of initial value.

Solution:

We have, $U = \frac{q^2}{2C} = \frac{q_o^{\,2}}{2C} e^{-2t/RC} = U_o e^{-2t/RC}$

According to question, $U = \frac{U_o}{4}$

$$\frac{U_o}{4} = U_o e^{-2t/RC}$$

$$e^{2t/RC} = 4$$

$$t = \frac{ln(4)}{2} RC = 0.693\ RC$$

$$\Rightarrow t = 0.693\ \tau$$

Example 18: A capacitor of capacitance C is charged through a resistor R. Calculate the time at which the potential across the resistor is equal to the potential across the capacitor.

Solution:

Here,

$$V_R = V_C \Rightarrow IR = \frac{q}{C}$$

$$RI_o e^{-t/RC} = \frac{q_o}{C}(1 - e^{-t/RC})$$

$$\Rightarrow \frac{R\varepsilon}{R} I_0 e^{\frac{-t}{RC}} = \frac{\varepsilon C}{C}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$\Rightarrow e^{-t/RC} = 1 - e^{-t/RC}$$

$$2e^{-t/RC} = 1$$

$$e^{t/RC} = 2$$

$$t = ln(2)\ .\ RC$$

$$t = 0.693\ RC$$

Example 19: Obtain the charging time constant of a capacitor in a RC circuit such that current through the resistor is decreased by 50% of its peak value in 5 seconds.

Solution:

$I = \frac{I_0}{2}$, $t = 5$ sec, $\tau = RC = ?$

We have,

$$I = I_0 e^{-t/RC}$$

or, $$\frac{I_0}{2} = I_0 e^{-t/\tau}$$

or $$e^{-5/\tau} = \frac{1}{2}$$

or $$e^{5/\tau} = 2$$

or $\frac{5}{\tau} = \ln 2$

$\therefore \quad \tau = \frac{5}{\ln 2} = 7.21$ sec.

Example 20: Obtain the charging time constant of a capacitor in a RC circuit such that current through the resistor is decreased by 50% of its peak value in 10 seconds.

Solution:

$I = \frac{I_o}{2} \qquad t = 10$ sec

We have, $I = I_o e^{-t/\tau}$

or, $e^{-t/\tau} = \frac{1}{2}$

or, $\frac{-t}{\tau} = \ln\left(\frac{1}{2}\right) = -0.693$

or, $\tau = \frac{10}{0.693} = 14.43$ sec.

Example 21: The electronic polarizibility of the Ar atom is 1.7×10^{-40} Fm^2. What is the static dielectric constant of solid Ar if it's density is 1.8 gcm^{-3}? Given $N_A = 6.02 \times 10^{23}$ mol^{-1}, Atomic mass of Ar = 39.95 g mol^{-1}.

Solution:

Here, $\alpha_e = 1.7 \times 10^{-40}$ Fm^2, density d = 1.8 gcm^{-3}

Number of atoms per unit volume, $N = \frac{dN_A}{Mat}$ $\Rightarrow N = \frac{N_A \times \rho}{M} = \frac{N_A}{V}$

$= \frac{6.02 \times 10^{23} \times 1.8}{39.95}$

$= 2.71 \times 10^{22}$ cm^3

$= 2.71 \times 10^{28}$ m^{-3}

From Calusius-Massotti equation,

$$\frac{N\alpha_e}{3\epsilon_o} = \frac{\epsilon_r - 1}{\epsilon_r + 2}$$

$$\Rightarrow \frac{N\alpha_e\epsilon_r}{3\epsilon_o} + \frac{2N\alpha_e}{3\epsilon_o} = \epsilon_r - 1$$

$$\epsilon_r\left(\frac{N\alpha_e}{3\epsilon_o} - 1\right) = -\left[1 + \frac{2N\alpha_e}{3\epsilon_o}\right]$$

$$\epsilon_r = \frac{1 + \frac{2N\alpha_e}{3\epsilon_o}}{1 - \frac{N\alpha_e}{3\epsilon_o}} = 1.63$$

Example 22: Consider a pure Si crystal that has $\epsilon_r = 11.9$ a). What the electronic polarizibility due to valence electrons per Si atom b). Suppose that a voltage is applied across Si crystal sample. By how much is the local field greater than the applied field? Given, $\epsilon_o = 8.85 \times 10^{-12}$ Fm^{-1}, $N = 5 \times 10^{28}$ atoms per m^3.

Solution:

Here, $\epsilon_r = 11.9$, $\epsilon_o = 8.85 \times 10^{-12}$ Fm^{-1}, $N = 5 \times 10^{28}$m^{-3}

a. From Clausius - Mossotti equation

$$\frac{N\alpha_e}{3\epsilon_o} = \frac{\epsilon_r - 1}{\epsilon_r + 2} \Rightarrow \alpha_e = \frac{3\epsilon_o}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$$

$$\alpha_e = \frac{3\,(8.85 \times 10^{-12})}{(5 \times 10^{28})}\left(\frac{11.9 - 1}{11.9 + 2}\right) = 4.17 \times 10^{-40}\ \text{Fm}^2$$

b. The local field is

$$E_{local} = E + \frac{P}{3\epsilon_o}$$

Also, $P = \epsilon_o \chi E = \epsilon_o (\epsilon_r - 1) E$

Substituting for P,

$$E_{local} = E + \frac{1}{3}(\epsilon_r - 1) E$$

$$\frac{E_{local}}{E} = \left(1 + \frac{\epsilon_r}{3} - \frac{1}{3}\right) = \frac{\epsilon_r + 2}{3} = 4.63$$

Example 23: The optical index of refraction and the dielectric constant for glass are 1.45 and 6.5 respectively. Calculate the percentage of ionic polarizibility.

Solution:

Here, optical index of refraction, $n = 1.45$

Dielectric constant, $\epsilon_r = 6.5$

We have from Clausius - Massotti equation

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{N(\alpha_e + \alpha_i)}{3\epsilon_o} \qquad \ldots(1)$$

At optical frequencies, ϵ_r in Clausius-Massotti relation for electronic polarization is replaced by n^2

$$\frac{n^2 - 1}{n^2 + 2} = \frac{N\alpha_e}{3\epsilon_o} \qquad \ldots(2)$$

Now, dividing equation (2) by (1)

$$\frac{n^2 - 1}{n^2 + 2} \times \frac{\epsilon_r + 2}{\epsilon_r - 1} = \frac{\alpha_e}{\alpha_e + \alpha_i}$$

So, percentage of ionic polarizibility is

$$\frac{\alpha_i}{\alpha_e+\alpha_i}\times 100 = \left[1-\frac{\alpha_e}{\alpha_e+\alpha_i}\right]\times 100$$

$$=\left[1-\left(\frac{n^2-1}{n^2+2}\right)\times\left(\frac{\epsilon_r+2}{\epsilon_r-1}\right)\right]\times 100$$

$$=\left[1-\left(\frac{1.45^2-1}{1.45^2+2}\right)\times\left(\frac{6.5+2}{6.5-1}\right)\right]\times 100$$

$$=58.47\%$$

Solutio

Example 24: Assuming that the polarizibility of an argon atom is equal to 1.43×10^{-40} Fm^2, calculate the relative dielectric constant at 0°C and 1 atmosphere.

Solution:

Here, $\alpha_e = 1.43\times10^{-40}\ Fm^2$, $\epsilon_r = ?$, $T = 0°C = 273\ K$

$P = 1\ atm = 1\times10^5\ N/m^2$

We have from Clausius-Mossotti equation

$$\frac{\epsilon_r-1}{\epsilon_r+2}=\frac{N\alpha_e}{3\epsilon_o}$$

$$3\epsilon_r\epsilon_o-3\epsilon_o=N\alpha_e\epsilon_r+2N\alpha_e$$

$$\epsilon_r(3\epsilon_o-N\alpha_e)=2N\alpha_e+3\epsilon_o$$

$$\epsilon_r=\frac{2N\alpha_e+3\epsilon_o}{3\epsilon_o-N\alpha_e}$$

From gas equation PV = nRT

$$P=\frac{\text{Number of atoms per unit volume (N)}}{\text{Number of atoms per mole }(N_A)}\times RT$$

$$P=\frac{N}{N_A}.RT$$

$$N=\frac{PN_A}{RT}$$

$$=\frac{1\times10^5\times6.02\times10^{23}}{8.31\times273}$$

$$=2.65\times10^{25}/m^3$$

Therefore, $$\epsilon_r=\frac{(2\times2.65\times10^{25}\times1.43\times10^{-40})+(3\times8.85\times10^{-12})}{(3\times8.85\times10^{-12})-(2.65\times10^{25}\times1.43\times10^{-40})}$$

$$\epsilon_r=1.00043$$

Example 25: If the refractive index of a medium with respect to free space, n, is defined as the ratio of the velocity of electromagnetic wave in free space to that in the medium, show that the Clausius - Mosotti equation becomes, $\frac{N\alpha}{3\epsilon_o}=\frac{n^2-1}{n^2+2}$

6

Chapter

ELECTROMAGNETISM

CHAPTER OUTLINE

After completion of this chapter, you will be able to know:

- EM Oscillation: LC oscillation, Damped LCR oscillation, Forced em oscillation, resonance and quality factor.
- EM waves: Maxwell equations in integral form, Conversion of Maxwell's equations in differential form, Continuity equation, Relation between electric field, magnetic field and speed of light, wave equations in free space, verification of light wave as an electromagnetic wave, Wave equation in dielectric medium.

We can now rewrite Ampere - Maxwell law as

$$\oint \vec{B}.\vec{dl} = \mu_o (I + I_d) \quad \ldots (ii)$$

$I_d = \frac{\epsilon_0 d\phi_E}{dt}$ is the displacement current

The charge stored in a parallel plate capacitor is q = CV.

or, $q = \frac{\epsilon_o A}{d} . V$ [since capacitance $C = \frac{\epsilon_o A}{d}$]

Where, A = area of each plate, d is distance between them

Therefore, $q = \epsilon_o A . \frac{V}{d} \quad = \epsilon_o A\, E$

$$\text{Now, the real current, } I = \frac{dq}{dt} = \epsilon_o A \frac{dE}{dt} \quad \ldots (iii)$$

Here, displacement current is:

$$I_d = \epsilon_o \frac{d\phi_E}{dt} = \epsilon_o \frac{d}{dt}(EA) \quad [\text{Since } E = \phi/A \Rightarrow \phi = EA]$$

$$I_d = \epsilon_o A \frac{dE}{dt} \quad \ldots (iv)$$

[Here, we assume, the Electric field $\vec{E}$ between the plates of capacitor is uniform, so $\phi_E = EA$]

From equation (3) and (4), it is seen that, the real current 'I' during charging and discharging of the capacitor is equal to displacement current 'I_d' between the plates.

So we can consider the fictitious displacement current simply to be a continuation of the real current through the capacitor.

Solved Example

Example 1: A radio tuner has a frequency range from 500 KHz to 5 MHz. If its LC circuit has an effective inductance of 400 μH, what is the range of its variable capacitor?

Solution:

The frequency of oscillation of an LC circuit is

$$f = \frac{1}{2\pi\sqrt{LC}} \Rightarrow f^2 = \frac{1}{4\pi^2 LC} \Rightarrow C = \frac{1}{4\pi^2 f^2 L}$$

Here, $L = 400\ \mu H = 400 \times 10^{-6}$ H

For $f = 500$ KHz $= 500 \times 10^3$ Hz

$$C = \frac{1}{4\pi^2 \times (500 \times 10^3)^2 \times (400 \times 10^{-6})} = 2.535 \times 10^{-10}\ F$$

For $f = 5$ mHz $= 5 \times 10^6$ Hz.

$$C = \frac{1}{4\pi^2 \times (5 \times 10^6)^2 \times (400 \times 10^{-6})} = 2.535 \times 10^{-12} F$$

Example 2: You are given an inductor of 1 mH. If you are asked to make it oscillate with a frequency of 1MHz, how can you make such an oscillatory device.

Solution:

Here, $L = 1\ mH = 1 \times 10^{-3}\ H$; $f = 1\ MHz = 1 \times 10^{6}\ Hz$

We have, $f = \dfrac{1}{2\pi\sqrt{LC}} \Rightarrow C = \dfrac{1}{4\pi^2 f^2 L}$

$$C = \frac{1}{4 \times 3.14^2 \times (1 \times 10^6)^2 \times 1 \times 10^{-3}} = 4 \times 10^{-10}\ F$$

By connecting a capacitor of capacitance 4×10^{-10} F we can make an oscillatory device of frequency 1MHz.

Example 3: What inductance must be connected to a 17 PF capacitor in an oscillator capable of generating 550 nm Electromagnetic wave?

Solution:

Here, $C = 17\ PF = 17 \times 10^{-12}\ F$, $\lambda = 550 \times 10^{-9}\ m$

We have, speed of e.m wave = $f\lambda$

$$3 \times 10^8 = f\lambda$$

$$f = \frac{3 \times 10^8}{550 \times 10^{-9}} = 5.45 \times 10^{14}\ Hz$$

Again, $f = \dfrac{1}{2\pi\sqrt{LC}} \Rightarrow L = \dfrac{1}{4\pi^2 f^2 C}$

$\therefore \quad L = 2.95 \times 10^{-10}\ H$

Example 4: A 40 mH inductor and 1000 μF capacitor form an oscillating circuit. What is the peak value of current if the initial charge is 40 μC?

Solution:

$L = 40\ mH = 40 \times 10^{-3}\ H$
$C = 1000\ \mu F = 1000 \times 10^{-6}\ F$
$Q_0 = 40\ \mu C = 40 \times 10^{-6}\ C$
We have,

$$\omega = \frac{1}{\sqrt{LC}}$$

The charge varies sinusoidally as,

$$Q = Q_0 \sin(\omega t + \phi)$$

Therefore, current,

$$I = \frac{dQ}{dt} = \frac{d}{dt}(Q_0 \sin(\omega t + \phi))$$
$$= Q_0\, \omega \cos(\omega t + \phi)$$

Therefore, $I_{peak} = Q_0\, \omega$

$$= \frac{Q_0}{\sqrt{LC}} = \frac{40 \times 10^{-6}}{\sqrt{40 \times 10^{-3} \times 1000 \times 10^{-6}}}$$

$I_{peak} = 6.32 \times 10^{-3}\ A = 6.32\ mA$

or $Q = Q_0 \sin(\omega t - \phi)$

$\dfrac{dq}{dt} = Q_0 \sin(\omega t - \phi) \times \omega$

$I_{peak} = \omega Q_0 \sin(\omega t - \phi)$

Example 5: If 10 mH inductor and two capacitors of 5 μF and 2 μF are given, find the two resona frequencies that can be obtained by connecting these elements in different ways.

Solution:

Here $L = 10$ mH, $C_1 = 5\ \mu F = 5 \times 10^{-6}$ F and $C_2 = 2\mu F = 2 \times 10^{-6}$ F

i. When the capacitors are connected in series. $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{5 \times 2}{5 + 2} = 1.43\ \mu F = 1.43 \times 10^{-6}\ F$$

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10 \times 10^{-3} \times 1.43 \times 10^{-6}}} = 1.33\ KHz$$

ii. When the capacitors are connected in parallel, $C = C_1 + C_2$

$$\Rightarrow C = 5 + 2 = 7\ \mu F = 7 \times 10^{-6}\ F$$

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10 \times 10^{-3} \times 7 \times 10^{-6}}} = 0.602\ KHz$$

Example 6: An alternating voltage (in Volts) varies with time t (in Seconds) as $V = 100 \sin (5\pi t)$ Calculate the peak value of voltage, the rms value of voltage and the frequency.

Solution:

Here, $V = 100 \sin (5\pi t)$

Comparing this equation with, $V = V_0 \sin \omega t$

i. The peak value of voltage, $V_0 = 100$ Volts.

ii. The rms value of voltage, $V_{rms} = \frac{V_0}{\sqrt{2}}$

$$= \frac{100}{\sqrt{2}}$$

$$= 70.71\ \text{Volts.}$$

$V_{rms} = \frac{\text{peak}}{\sqrt{2}}$

iii. Here, $\omega = 5\pi \Rightarrow f = \frac{5\pi}{2\pi} = 2.5$ Hz.

Example 7: A 1.5 μF capacitor is charged to 57 Volts. The charging battery is disconnected and a 12 mH coil is connected in series with capacitor so that LC oscillation occur. What is maximum current in the coil? Assuming that the circuit contains no resistance.

Solution:

Here, $C = 1.5\ \mu F = 1.5 \times 10^{-6}$ F, $V = 57$ volts, $L = 12$ mH $= 12 \times 10^{-3}$ H.

We have, $q = q_o \sin(\omega t + \phi)$

$$I = \frac{dq}{dt} = q_o\ \omega \cos (\omega t + \phi)$$

Therefore, $I_{max} = q_o\ \omega = \frac{CV}{\sqrt{LC}} = V\sqrt{\frac{C}{L}} = 57\sqrt{\frac{1.5 \times 10^{-6}}{12 \times 10^{-3}}} = 0.64$ A

$Q_{max} = Q_o e^{-Rt/2L}$, where Q_o is charge at t = 0

$e^{Rt/2L} = \frac{Q_o}{Q_{max}}$

$R = \frac{2L}{t} ln\left(\frac{Q_o}{Q_{max}}\right)$

From question, $Q_{max} = 99\%$ of $Q_o = 0.99\ Q_o$

Therefore,

$$R = \frac{2 \times 220 \times 10^{-3}}{0.5104} ln\left(\frac{Q_o}{0.99\ Q_o}\right)$$

$$= 8.66 \times 10^{-3}\ \Omega$$

Example 13: A 1200 PF capacitor is charged by a 500V battery. It is disconnected from the battery and is connected to a 75 mH inductor at t = 0. Determine i) the initial charge on the capacitor ii) the maximum current iii) the frequency and period of oscillation iv) the total energy oscillating in the system.

1 µF

do

Solution:

Here, C = 1200 PF = 1200×10^{-12} F, V = 500 V, L = 75×10^{-3} H

i. $Q_o = CV = 1200 \times 10^{-12} \times 500 = 6 \times 10^{-7}\ C = 0.6\ \mu C$.

ii. Since, $Q = Q_o \sin(\omega t + \phi)$

$$I = \frac{dQ}{dt} = Q_o\ \omega \cos(\omega t + \phi)$$

$$I_{max} = Q_o \omega = \frac{Q_o}{\sqrt{LC}} = 6.3 \times 10^{-2}\ A$$

iii. $f = \frac{1}{2\pi\sqrt{LC}} = 12{,}000\ Hz = 12 KHz$ and $T = \frac{1}{f} = 6 \times 10^{-5}$ sec

iv. Total Energy $= \frac{Q_o^2}{2C} = 1.5 \times 10^{-4}\ J$

Example 14: Calculate the resonating frequency and quality factor of a circuit having 0.02 µF capacitance, 8 mH inductance and 0.25 Ohm resistance.

Solution:

Here, C = 0.02 µF = 0.02×10^{-6} F, L = 8mH = 8×10^{-3}H and R = 0.25 Ω

We have, $\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

$$F = \frac{1}{2\pi}\sqrt{\frac{1}{8 \times 10^{-3} \times 0.02 \times 10^{-6}} - \frac{0.25^2}{4 \times (8 \times 10^{-3})^2}}$$

$$= \frac{1}{2\pi}\sqrt{6.25 \times 10^9 - 2.44 \times 10^2} = 12.59\ KHz.$$

and $Q = \frac{L\omega}{R} = \frac{2\pi f L}{R} = \frac{2\pi \times 12.59 \times 10^3 \times 8 \times 10^{-3}}{0.25} = 2530$

$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

Example 17: An LC circuit is converted into a LCR circuit inserting a resistance of 10 Ω. Calculate the percentage change in frequency in this conversion. Given inductance = 10 mH and capacitance = 10 μF.

Imp

Solution:

Here, $R = 10\ \Omega$

$L = 10\ mH = 10 \times 10^{-3}\ G$

$C = 10\ \mu F = 10 \times 10^{-6}\ F$

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{10 \times 10^{-3} \times 10 \times 10^{-6}}} = 503.2\ Hz$$

For LCR circuit the frequency is,

$$f' = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{10 \times 10^{-3} \times 10 \times 10^{-6}} - \left(\frac{10}{2 \times 10 \times 10^{-3}}\right)^2}$$

$$= \frac{1}{2\pi}\sqrt{10^7 - 2.5 \times 10^5}$$

$$= 496.96\ Hz$$

$$\%\ Change = \frac{f - f'}{f} \times 100\% = \frac{503.2\ Hz - 496.96\ Hz}{503.2\ Hz} \times 100\% = 1.23\%$$

Example 18: A circuit has L = 10 mH and C = 10 μF. How much resistance should be added to circuit so that frequency of oscillation will 1% less than that of LC oscillation.

Solution:

Here, $L = 10\ mH = 10 \times 10^{-3} H$, $C = 10\ \mu F = 10 \times 10^{-6}\ F$

The free oscillation frequency,

$$f_o = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2 \times 3.14\sqrt{1 \times 10^{-2} \times 1 \times 10^{-5}}} = 503.55\ HZ$$

The damped frequency is,

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$\Rightarrow \frac{1}{LC} - \frac{R^2}{4L^2} = 4\pi^2 f^2$$

$$\Rightarrow R^2 = 4L^2\left[\frac{1}{LC} - 4\pi^2 f^2\right]$$

According to question, $f = f_o - 1\%\ of\ f_o = f_o - \frac{f_o}{100} = 498.51\ Hz$

$$\Rightarrow R^2 = 4 \times (10^{-2})^2\left[\frac{1}{10^{-2} \times 10^{-5}} - 4\pi^2(498.51)^2\right]$$

$$\Rightarrow R^2 = 75.809$$

$$\therefore R = 8.706\Omega$$

Similarly, from equation (v)

$$-\omega E_o \cos(kx - \omega t) = -\frac{1}{\mu_o \epsilon_o}[k B_o \cos(kx - \omega t)]$$

$$\Rightarrow \quad \frac{E_o}{B_o} = \frac{1}{\mu_o \epsilon_o} \cdot \frac{k}{\omega}$$

$$\Rightarrow \quad \frac{E_o}{B_o} \cdot \frac{\omega}{k} = \frac{1}{\mu_o \epsilon_o}$$

$$\Rightarrow \quad \left[\text{Since } v = \frac{\omega}{k}, \text{ also } v = \frac{E_o}{B_o}\right]$$

Therefore, $v^2 = \dfrac{1}{\mu_o \epsilon_o}$

$$\Rightarrow \quad v = \frac{1}{\sqrt{\mu_o \epsilon_o}}$$

Therefore, velocity of E M wave is,

$$v = \frac{E}{B} = \frac{E_o}{B_o} = \frac{1}{\sqrt{\mu_o \epsilon_o}} = C \text{ (Speed of light)}$$

Solved Example

Example 1: What is the displacement current for a capacitor having radius 5 cm with variable electric field 8.9×10^{12} Volts/m.S?

Solution:

$d = \epsilon_o A$

Here,

$r = 5\text{cm} = 5 \times 10^{-2}\text{m}$,

$\dfrac{dE}{dt} = 8.9 \times 10^{12}$ Volts/m.S.

We have,

$$I_d = \epsilon_o \frac{d\phi_E}{dt} = \epsilon_o A \frac{dE}{dt} = \epsilon_o \pi r^2 \frac{dE}{dt}$$

$$I_d = 8.85 \times 10^{-12} \times \pi \times (5 \times 10^{-2})^2 \times 8.9 \times 10^{12} = 0.62 \text{ A}$$

Example 2: A parallel plate capacitor has capacitance 20 μF. At what rate the potential difference between the plates must be changed to produce a displacement current of 1.5 A?

Solution:

$\dfrac{dV}{dt} = ?$

Here,

$C = 20\ \mu F = 20 \times 10^{-6}$ F,

$I_d = 1.5$ A, $\dfrac{dV}{dt} = ?$

We have,

$$I_d = \epsilon_o \frac{d\phi_E}{dt} = \epsilon_o A \frac{dE}{dt} = \epsilon_o A \frac{d}{dt}\left(\frac{V}{d}\right) = \frac{\epsilon_o A}{d} \frac{dV}{dt}$$

$$I_d = C \frac{dV}{dt}$$ (4)

$$\frac{dV}{dt} = \frac{I_d}{C} = \frac{1.5}{20 \times 10^{-6}} = 7.5 \times 10^4 \text{ Volts/ sec}$$

Example 3: A sinusoidally varying voltage is applied across a 8 μF capacitor. The frequency of the voltage is 3 KHz and the voltage amplitude is 30 V. Find the maximum value of displacement current.

Solution:

Here $V = V_o \sin \omega t$

$V_o = 30$ V

$f = 3$ KHz $= 3 \times 10^3$ Hz and

$C = 8\ \mu F = 8 \times 10^{-6}$ F

from example-2

We have, $I_d = C\dfrac{dV}{dt} = C\,V_o\,\omega \cos \omega t$

$I_d\,(max) = C\,V_o\,\omega = CV_o \times 2\pi f$

$= 8 \times 10^{-6} \times 30 \times 2\pi \times 3 \times 10^3 = 4.52$ A

Example 4: Calculate the displacement current between the square plates of capacitor having one side 1.0 cm if the electric field between the plate is changing at the rate of 3×10^6 V/metre - S

same with 1

Solution:

Here, length of one side of plate, a = 1 cm

Area of the plate, $A = a^2 = 1\ cm^2 = 1 \times 10^{-4}\ m^2$

and $\dfrac{dE}{dt} = 3 \times 10^6$ V/mS

We have, $I_d = \epsilon_o A \dfrac{dE}{dt} = 8.85 \times 10^{-12} \times 1 \times 10^{-4} \times 3 \times 10^6 = 2.655 \times 10^{-9}$ A

Example 5: Using maxwell's equation, prove that $C = \dfrac{E_m}{B_m}$ where symbols carry their usual meaning.

Solution:

We have third Maxwell equation as

$$\nabla \times E = \frac{dB}{dt}$$

In one dimension it can be expressed as $\dfrac{dE}{dx} = -\dfrac{dB}{dt}$ . . . (i)

Since,

$$E = E_m \sin (kx - \omega t)$$

$$\frac{dE}{dx} = k.\ E_m \cos (kx - \omega t)$$

And

$$B = B_m \sin (kx - \omega t)$$

$$\frac{dB}{dt} = (-\omega)\ B_m \cos (kx - \omega t)$$

From equation (1)

$$k.\ E_m = \omega\ B_m$$

$$\frac{E_m}{B_m} = \frac{\omega}{k} = C$$

Therefore, $C = \dfrac{E}{B} = \dfrac{E_m}{B_m}$

Example 6: Prove the charge conservation theorem

OR - Derive continuity equation

OR - Show that $\nabla.\vec{J}+\frac{d\rho}{dt}=0$. Where symbols carry their usual meaning.

Solution:

We have fourth Maxwell equation as

$$\nabla\times\vec{B}=\mu_0(\vec{J}+\epsilon_0\frac{d\vec{E}}{dt})$$

Taking divergence on both sides

$$\nabla.(\nabla\times\vec{B})=\mu_0(\nabla.\vec{J}+\epsilon_0\frac{d(\nabla.\vec{E})}{dt})$$

Since, divergence of curl of any vector is zero, i.e. $\nabla.(\nabla\times\vec{B})=0$

and we know, $\nabla.\vec{E}=\frac{\rho}{\epsilon_0}$

$$0=\nabla.\vec{J}+\epsilon_0\frac{d}{dt}\left(\frac{\rho}{\epsilon_0}\right)$$

$$\nabla.\vec{J}+\epsilon_0\frac{d}{dt}\left(\frac{\rho}{\epsilon_0}\right)=0$$

$$\nabla.\vec{J}+\frac{d\rho}{dt}=0$$

Example 7: A parallel plate capacitor with circular plates 20 cm in diameter is being charged. The current density of the displacement current in the region between the plates is uniform and has a magnitude of 20 A/m². i) Calculate the curl of magnetic flux density in space between the plates ii) Calculate the rate of change of electric field in this region.

Solution:

Here,

$$d=20\text{ cm}\Rightarrow r=\frac{20}{2}=10\text{ cm}=0.1\text{m}$$

Current density,

$$J_d=20\text{ A/m}^2$$

Displacement current,

$$I_d=\frac{J_d}{\pi r^2}=\frac{20}{\pi\times0.1^2}=637\text{ A}$$

i. We have, $\nabla\times B=\mu_0\epsilon_0\frac{dE}{dt}$

$$\nabla\times B=\frac{\mu_0}{A}\epsilon_0 A\frac{dE}{dt}$$

$$=\frac{\mu_0 I_d}{A}=\mu_0 J_d=4\pi\times10^{-7}\times20=2.512\times10^{-5}\text{ Wb/m}^2$$

ii. Again,

$$I_d = \epsilon_0 A \frac{dE}{dt}$$

$$\Rightarrow \frac{dE}{dt} = \frac{I_d}{\epsilon_0 A} = \frac{J_d}{\epsilon_0} = \frac{20}{8.85 \times 10^{-12}} = 2.26 \times 10^{12} \text{ V/mS}$$

Example 8: **At some distance from transmitter of radio station, the magnitude of em wave emitted by radio station is found to be 1.6×10^{-4} T. If frequency of broadcast is 1020 KHz then find speed, wave length and maximum electric field of em wave.**

Solution:

Here, $f = 1020 \text{ KHz} = 1020 \times 10^3 \text{ Hz}$, $B_m = 1.6 \times 10^{-4}$ T

i. $$C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3 \times 10^8 \text{ m/s}$$

ii. $$\lambda = \frac{C}{f} = \frac{3 \times 10^8}{1020 \times 10^3} = 294.12 \text{ m}$$

iii. $$E_m = CB_m = 3 \times 10^8 \times 1.6 \times 10^{-4} = 4.8 \times 10^4 \text{ V/m}$$

THEORETICAL ANSWER QUESTIONS

1. What is displacement current? Compare the physical significance of Maxwell third and fourth equation.
2. Write Maxwell equations in free space. Prove that the ratio of magnitudes of electric and magnetic field is equal to the speed of light.
3. Write Maxwell equations in integral form and convert them into differential form.
4. Derive Maxwell's equation:

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} + \mu_0 \vec{J}.$$

5. Show that velocity of electromagnetic wave in free space is given by $C = \frac{1}{\sqrt{\mu_o \epsilon_o}}$.
6. Derive the velocity of electro magnetic wave using Maxwell's equation.
7. Prove the relation: $\nabla^2 \vec{E} = \mu_o \epsilon_o \frac{\partial^2 \vec{E}}{dt^2}$.
8. Explain the conduction current and displacement current.
9. Write Maxwell equations in differential form and convert them in to integral form.
10. Prove the charge conservation theorem $\frac{\partial \rho}{\partial t} + \nabla . \vec{J} = 0$
11. What are maxwell's equations? Using maxwell's equations derive wave equation of electromagnetic waves in a conducting medium in terms of electric and magnetic field
12. Write down Maxwell equations? Prove that $\nabla \times \vec{B} = \mu_o (\vec{J} + \epsilon_o \frac{d\vec{E}}{dt})$.
13. Starting from Maxwell's equation derive equation of continuity.

7

Chapter

QUANTUM MECHANICS

CHAPTER OUTLINE

After completion of this chapter, you will be able to know:

- Inadequacy of classical mechanics, Importance of quantum mechanics, Matter wave (de-Broglie equation), Wave function and its significance, Energy and momentum operator, Time independent and time dependent Schrodinger wave equations, Application of Schrodinger wave equation for the electron in metal, Normalized wave function describing the motion of an electron inside in an infinite potential well.

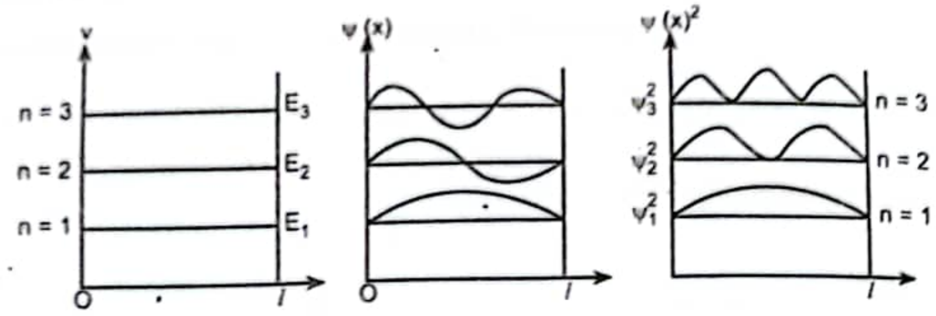


Figure: First three energy states, wave functions and probability densities for a particle in one dimensional box

Solved Example

Example 1: A ball of mass 10 gm has velocity 100 cm/sec. Calculate the wave length associated with it. Why does not this wave nature show up our daily observations. Given, $h = 6.62 \times 10^{-34}$ JS.

Solution:

Here, $m = 10 \text{ gm} = 10 \times 10^{-3}$ kg, $h = 6.62 \times 10^{-34}$ Js

We have,

$$\lambda = \frac{h}{mv}$$

$$= \frac{6.62 \times 10^{-34}}{10 \times 10^{-3} \times 1}$$

$$= 6.62 \times 10^{-32} \text{ m}$$

This wavelength is much smaller than the dimensions of the balls therefore in such cases wave - like properties of matter can not be observed in our daily observations.

Example 2: Calculate the wave length associated with an electron subjected to a potential difference of 1.25 KV.

Solution:

We have,

$$\frac{1}{2} mv^2 = eV$$

$$\Rightarrow \quad v = \sqrt{\frac{2eV}{m}}$$

Now,

$$\lambda = \frac{h}{mv}$$

$$= \frac{h}{\sqrt{2meV}}$$

$$= \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19} \times 1.25 \times 10^3}}$$

$$\lambda = 0.347 \text{Å}$$

Example 3: Show that wave velocity is greater than velocity of light

Solution:

We have, wave velocity, $u = f\lambda$

From Planck's law and Einstein's mass energy relation.

$hf = mc^2$

$f = \frac{mc^2}{h}$

Substituting f for u

$u = \frac{mc^2}{h}.\lambda = \frac{mc^2}{h}.\frac{h}{mv} = \frac{c^2}{v}$

$c^2 = uv$

Since particle velocity (v) must be less than velocity of light (c), so wave velocity (u) is greater than velocity of light.

Example 4: Find the energy of the neutron in units of electron - volt whose de - Broglie wave length is 1Å. Given mass of neutron $= 1.674\times10^{-27}$ kg, $h = 6.62\times10^{-34}$ joule-sec.

Solution:

Here, $\lambda = 1$ Å $= 1\times10^{-10}$ m, $m = 1.674\times10^{-27}$kg, $h = 6.62\times10^{-34}$ Joule sec.

We have, $E = \frac{1}{2}mv^2$

Also, $\lambda = \frac{h}{mv} \Rightarrow v = \frac{h}{m\lambda}$

$\lambda = \frac{h}{\sqrt{2mE_K}}$ uses

Therefore,

$E = \frac{1}{2}m.\frac{h^2}{m^2\lambda^2} = \frac{h^2}{2m\lambda^2}$

$\lambda^2 = \frac{h^2}{2mE_K}$

$= \frac{(6.62\times10^{-34})^2}{2\times1.674\times10^{-27}\times(1\times10^{-10})^2}$

$E_K = \frac{h^2}{2m\lambda^2}$

$= 1.3\times10^{-20}$ joule

$= \frac{1.3\times10^{-20}}{1.6\times10^{-19}} = 8.13\times10^{-2}$ eV

Example 5: Show that group velocity is equal to particle velocity.

Solution:

The group velocity is given by

$v_g = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{dE}{dP}$ $\left[\text{Since } E = hf = \frac{h}{2\pi}.2\pi f = \hbar\omega \text{ and } p = \frac{h}{\lambda} = \frac{h}{2\pi}.\frac{2\pi}{\lambda} = \hbar k\right]$

Since, $E = \frac{P^2}{2m}$ for free particle

Since, $v_g = \frac{d}{dp}\left(\frac{P^2}{2m}\right) = \frac{2P}{2m} = \frac{mv}{m} = v =$ particle velocity.

Example 6: An electron is confined to an infinite height box of size 0.1 nm. Calculate the ground state energy of the electron. How this electron can be put to the third energy level.

Solution:

The energy of the particle in one dimensional rigid box of side l is given by

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ml^2}$$

$$= \frac{n^2h^2}{8ml^2}$$

$$= \frac{(6.62\times10^{-34})^2 \times n^2}{8\times9.1\times10^{-31}\times(10^{-10})^2}$$

$$= 6.03 \times 10^{-18} n^2 \text{ joule}$$

$$= 37.7\ n^2 \text{ eV}$$

In the ground state, $n = 1$, $E_1 = 37.7$ eV

For third energy level, $n = 3$, $E_3 = 37.7\times3^2 = 37.7\times9$ eV

$\therefore \quad E_3 - E_1 = (9-1)\times 37.7 = 301.5$ eV

Hence to put the electron to third energy level an extra energy of 301.5 eV is to be given.

Example 7: What voltage must be applied to an electron microscope to produce electrons of wave length 0.50 Å? Given, $e = 1.6\times10^{-19}$ Coulomb, $m = 9.0\times10^{-31}$ kg, $h = 6.62\times10^{-34}$ Joule.sec.

Solution:

We have, $\frac{1}{2}mv^2 = eV$

$$V = \frac{mv^2}{2e}$$

The de-Broglie wavelength is given by, $\lambda = \frac{h}{mv} \Rightarrow v = \frac{h}{m\lambda}$

Therefore,

$$V = \frac{m}{2e}\cdot\frac{h^2}{m^2\lambda^2} = \frac{h^2}{2me\lambda^2}$$

$$= \frac{(6.62\times10^{-34})^2}{2\times9.1\times10^{-31}\times1.6\times10^{-19}\times(0.5\times10^{-10})^2}$$

$$= 602 \text{ Volts.}$$

Example 8: The wave function of a particle confined in a box of length l is $\psi(x) = \sqrt{\frac{2}{l}}\sin\frac{\pi x}{l}$.

Calculate the probability of finding the particle in the region $0 < x < \frac{l}{2}$.

Solution:

The probability of finding the particle in the length 0 to $\frac{l}{2}$ is

$$P = \int_0^{l/2} (\psi)^2\,dx = \frac{2}{l}\int_0^{l/2} \sin^2\frac{\pi x}{l}\,dx = \frac{1}{l}\int_0^{l/2}\left(1-\cos\frac{2\pi x}{l}\right)dx = \frac{1}{l}\cdot\frac{l}{2} = \frac{1}{2} = 0.5$$

...mple 9: **Normalize the one dimensional wave function**

$$\Psi = A \sin\left(\frac{\pi x}{a}\right), 0 < x < a$$

$$= 0 \qquad , \text{outside}$$

...ution:

We have normalizing condition

$$\int_{-\infty}^{\infty} |\Psi|^2 \, dx = 1$$

or $$\int_0^a A^2 \sin^2\frac{\pi x}{a} \, dx = 1$$

or $$\frac{A^2}{2}\int_0^a \left(1 - \cos\frac{2\pi x}{a}\right) dx = 1$$

or $$\frac{A^2}{2} . a = 1$$

∴ $$A = \sqrt{\frac{2}{a}}$$

Hence the normalized wave function is $\sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}$.

...ample 10: An electron moving is a wave has wave function $\Psi(x) = 2 \sin 2\pi x$. Find the probability of finding the electron in the region x = 0.25 to 0.5 m.

...lution:

The probability of finding the electron in given region is,

$$P = \int_{0.25}^{0.5} \Psi.\Psi^* \, dx = \int_{0.25}^{0.5} 4 \sin^2 2\pi x \, dx$$

$$P = 2\int_{0.25}^{0.5} 2 \sin^2 2\pi x \, dx = 2\int_{0.25}^{0.5} (1 - \cos 4\pi x) = 2\left[\int_{0.25}^{0.5} dx - \int_{0.25}^{0.5} \cos 4\pi x \, dx\right]$$

$$= 2\left[(x)\Big|_{0.25}^{0.5} - \frac{\sin 4\pi x}{4\pi}\Big|_{0.25}^{0.5}\right] = 2\,[(0.5 - 0.25) - 0] = 2\,[0.25 - 0] = 0.5$$

...mple 11: A particle is moving in 1–D box of infinite potential. Evaluate the probability of finding the particle within range 1 Å at the centre of box when it is in lowest energy state.

...lution:

$L = 1 \text{ Å} = 1 \times 10^{-10}$ m

We have, wave function for the particle in infinite potential well

$$\Psi(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$$

Solved Example

Example 1: A gas is contained is a piston cylinder device initially at a pressure of 150 kPa and a volume of 0.04 m^3 calculate the work done by the gas when it under goes the following process to a final volume of 0.1 m^3.

a. Constant pressure

b. Constant temperature

c. $PV^{1.35}$ = constant

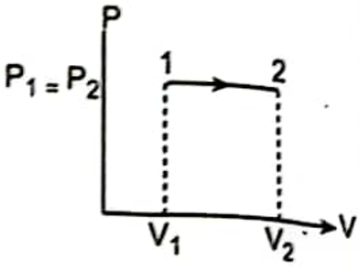
Solution:

Initial state $P_1 = 150$ kPa

$V_1 = 0.04 \text{ m}^3$

Final state $V_2 = 0.1 \text{ m}^3$

a. For constant pressure



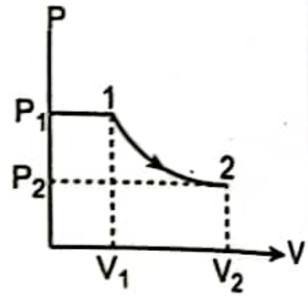
Work done

$$W = P_1 (V_2 - V_1)$$

$$= 150 \times 10^3 (0.1 - 0.04)$$

$$= 9000 \text{ J}$$

b. For constant temperature process



Work done, $W = P_1 V_1 \ln\left(\frac{V_2}{V_1}\right)$

$$= 150 \times 10^3 \times 0.04 \ln\left(\frac{0.1}{0.04}\right)$$

$$= 5497.7 \text{ J}$$

c. For polytropic process ($PV^{1.35}$ = constant)

Now,

$$P_1 V_1^{1.35} = P_2 V_2^{1.35}$$

or, $$P_2 = P_1 \left(\frac{V_1}{V_2}\right)^{1.35}$$

$$= 150 \times 10^3 \left(\frac{0.04}{0.1}\right)^{1.35}$$

$$= 43538.38 \text{ Pa}$$

Then, work done during the process

$$\frac{P_2V_2 - P_1V_1}{1-\gamma}.$$

$$W = \frac{P_1V_1 - P_2V_2}{(1-n)}$$

$$= \frac{150 \times 10^3 \times 0.04 - 4443538.38 \times 0.1}{1 - 1.35}$$

$$= \frac{-1646.16}{-0.35}$$

$$= 4703.3\text{J}$$

mple 2: A gas undergoes compression from an initial state of $V_1 = 0.1\ m^3$, $P_1 = 200$ kPa to final state of $V_2 = 0.04\ m^3$, $P_2 = 500$ kPa. If the pressure varies linearly with volume during the process, determine the work transfer.

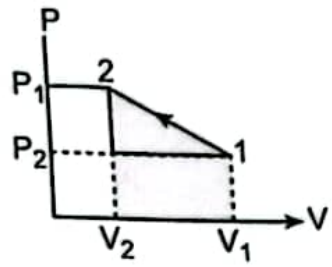
tion:

Initial state $V_1 = 0.1\ m^3$, $P_1 = 2300\ kPa = 200 \times 10^3$ Pa

Final state $V_2 = 0.04\ m^3$, $P_2 = 500\ kPa = 500 \times 10^3$ Pa.

Here,

The pressure varies linearly with volume during the process .



So,

Work transfer = Area under the curve

$= $ Area of trapezium

$$= \frac{1}{2}(P_1 + P_2)(V_1 - V_2)$$

$$= \frac{1}{2}(200 \times 10^3 + 500 \times 10^3)(0.1 - 0.04)$$

$$= 21000\ J = 21\ KJ$$

mple 3: A control mass containing 0.5 kg of a gas undergoes a process in which there is a heat transfer of 120 kJ from the system to the surroundings work done on the system is 60 kJ. If the initial specific internal energy of the system in 400 kJ/kg, determine its final specific energy.

lution:

Mass of gas (m) = 0.5 kg

Total heat transfer from the system (Q) = – 120 kJ

Work done on the system (w) = – 60 kJ

Initial specific energy (u_1) = 400 kJ/kg

Final specific energy (u_2) = ?

Total heat transfer is given by

$$Q = \Delta U + W$$

$$= (U_2 - U_1) + W$$

$$= m(u_2 - u_1) + W$$

$$-120 = 0.5(u_2 - u_1) - 60$$

$$\text{or,} \quad u_2 - u_1 = \frac{-120 + 60}{0.5}$$

$$\text{or,} \quad u_2 = -120 + 400 = 280\ kJ/kg$$

Example 4: A gas contained in a piston cylinder device undergoes a polytrophic process for which pressure volume relationship is given by $PV^{2.5}$ = constant. The initial pressure is 400 kPa, the initial volume is 0.2 m^3 and the final volume is 0.4 m^3. ~~The internal volume is 0.2 m^3 and the final volume is 0.4 m^3~~. The internal energy of the gas decrease by 20 kJ during the process. Determine the work transfer and heat transfer for the process.

Solution:

Initial state $P_1 = 400$ kPa,

$V_1 = 0.2 \text{ m}^3$

Final state $V_2 = 0.4 \text{ m}^3$

We have,

$PV^{2.5} = \text{constant}$

Change in internal energy $(\Delta U) = -20\text{kJ}$

Then,

$$P_1V_1^{2.5} = P_2V_2^{2.5}$$

$$P_2 = P_1\left(\frac{V_1}{V_2}\right)^{2.5}$$

$$= 400\left(\frac{0.2}{0.4}\right)^{2.5}$$

$$= 70.71 \text{ kPa.}$$

Work done during the polytropic process is

$$W = \frac{P_2V_2 - P_1V_1}{1-n}$$

$$= \frac{70.71 \times 0.4 - 400 \times 0.2}{1-2.5}$$

$$= 34.477 \text{ kJ}$$

$\therefore$ Heat transfer during the process $Q = \Delta U + W = -20 + 34.477 = 14.477$ kJ

Example 5: A piston cylinder arrangement contains 0.01 m^3 air at 150 kpa and 270° C. The air is now compressed in a process for which $PV^{1.25}$. To a final pressure 600 kPa Determine the work transfer and heat transfer for the process. [Take R = 287 J/kgK and C_v = 718 J/kgK]

Solution:

Initial state $V_1 = 0.01 \text{ m}^3$

$P_1 = 150$ kPa

$T_1 = 27°$ C

$27 + 273 = 300$ K

Final state $P_2 = 600$ kPa.

And process relation $PV^{1.25} = \text{Constant}$

So,

$$P_1V_{1.25}^{1} = P_2\,V_2^{1.25}$$

$$V_2^{1.25} = \left(\frac{P_1}{P_2}\right) V_{1.25}^{1}$$

$$V_2 = \left(\frac{P_1}{P_2}\right)^{\frac{1}{1.25}} V_1$$

$$= \left(\frac{150}{600}\right)^{0.8} \times 0.01$$

$$= 3.2987 \times 10^{-3}\ m^3$$

Work transfer during the process is

$$W = \frac{P_2V_2 - P_1V_1}{1-n}$$

$$= \frac{600 \times 3.2987 \times 10^{-3} - 150 \times 0.01}{1 - 1.25}$$

$$= \frac{0.4793}{-0.25}$$

$$= -1.917\ kJ$$

Also, $P_1V_1 = mRT_1$

Then, mass of air (m) $= \frac{P_1V_1}{RT_1}$

$$= \frac{150 \times 10^3 \times 0.01}{287 \times 300}$$

$$= 0.0174\ kg$$

Temperature at final state $T_2 = \frac{P_2V_2}{mR}$

$$= \frac{600 \times 10^3 \times 3.2987^{-3} \times 10}{0.0174 \times 287}$$

$$= 395.844\ K$$

Change in internal energy for the process is given by

$$\Delta U = mC_V (T_2 - T_1)$$

$$= 0.0174 \times 718\ (395.844 - 300)$$

$$= 1197.398\ J$$

$$= 1.1974\ kJ$$

Total beat transfer for the process is given by

$$Q = \Delta U + W$$

$$= 1.1974 - 1.917$$

$$= -0.7196\ kJ$$

ample 6: An ideal engine has an efficiency of 25%. If the source temperature is increased by 200° C its efficiency gets doubled, determine it's source and sink temperature.

lution:

Let, Source temperature $= T_H$

Sink temperature $= T_L$

Then, efficiency

$$\eta_1 = 1 - \frac{T_L}{T_H}$$

or, $0.25 = 1 - \frac{T_L}{T_H}$

or, $T_L = 0.75\ T_H$ ... (i)

When source temperature is increased by 200° C (200 K)

Its efficiency gets doubled i.e. $\eta_2 = 2\eta_1 = 0.5$

$$\eta_2 = 1 - \frac{T_L}{T_H + 200}$$

or, $0.5 = 1 - \frac{T_L}{T_H + 200}$

or, $\frac{T_2}{T_H + 200} = 0.5$ ... (ii)

$T_1 + 200 - T_2 = 0.5 \times (T_1 + 200)$

$T_1 + 200 - T_2 = 0.5T_1 + 100$

$T_1 - 0.5T_1 + 100 = T_2$

using equation (i) in (ii)

$$\frac{0.75\ T_H}{T_H + 200} = 0.5$$

or, $0.75\ T_H = 0.5\ T_H + 0.5 \times 200$

or, $0.25\ T_H = 100$

$$T_H = \frac{100}{0.25} = 400\ K$$

and $T_L = 0.75 \times 400 = 300\ K$

heat

Example 7: A heat pump has a coefficient of performance that is 80% of the theoretical maximum. It maintains hall at 20° C, which leaks energy 1kW per degree temperature difference to the ambient. For a maximum of 1.5kW power input, determine the minimum outside temperature for which the heat pump is sufficient.

Solution:

Higher Temperature (T_4) = 20° C = 293K

heating

Halting rate $\frac{dQ_H}{dt} = 1 \times (T_H - T_L)$

$= (T_H - T_L)$ kW

Power input $\frac{dw}{dt} = 1.5$ kW

Then,

$$(COP)_{actual, HP} = 0.8\ (COP)_{rev, HP}$$

or, $$\frac{\frac{dQ_H}{dt}}{\frac{dW}{dt}} = 0.8 \frac{T_H}{T_H - T_L}$$

or, $$\frac{T_H - T_L}{1.5} = 0.8 \frac{T_H}{T_H - T_L}$$

$\beta = 80\% = 0.8$

$T_1 = 20°$

$T_2 = ?$

$Q_2 = 1$ kw

$Q_1 = 1.5$ kw

$\beta = \frac{Q_1}{Q_1 - Q_2} = \frac{T_1}{T_1 - T_2}$

$(T_H - T_L)^2 = 1.2T_H = 1.2 \times 293 = 351.6$

or, $T_H - T_L = 18.751$

or, $T_L = T_H = -18.751$

or, $= 393 - 18.751$

$= 274.249$ K.

Example 8: During an experiment, a student claims that, a heat engine receives 300 kJ from a source at 500K converts 160kJ of it into work and rejects heat to the sink at 300K are these data reasonable?

Solution:

Higher temperature (T_H) = 500K

Lower temperature (T_L) = 300K

Heat input (Q_H) = 30 kJ

Work output (W) = 160 kJ

Then, maximum possible efficiency

We have,

$$\eta_{rev} = 1 - \frac{T_L}{T_H} = 1 - \frac{300}{500} = 40\%$$

$$\text{Also, } \eta_{student} = \frac{W}{Q_H} = \frac{160}{300} = 53.33\%$$

Here, $\eta_{student} > \eta_{rev}$, hence these data are not reasonable.

Example 9: A heat engine receives 400 kJ from a source at a temperature of 1000K. It rejects 150 kJ of heat to sink at a temperature of 300K. The engine produce 250 kJ of work output. Is this a reversible cycle, irreversible cycle or an impossible cycle?

Solution:

Higher temperature (T_H) = 1,000K

Lower temperature (T_L) = 300K

Heat rejected (Q_L) = 150kJ

Work output (W) = 250 kJ

Heat input (Q_H) = Q_L + W = 150 + 250 = 400 kJ

Then, maximum possible efficiency

Qdw = Q1 - Q2
= Idw + QL = Qh.

$$\eta_{rev} = 1 - \frac{T_L}{T_H}$$

$$= 1 - \frac{300}{1000}$$

$\Rightarrow \frac{700}{1000}$ $\frac{7}{10} = 0.7 = 70\%$

$$\text{And } \eta_{HE} = \frac{W}{Q_H} = \frac{250}{400} = 62.5\%$$

Hence, $\eta_{HE} < \eta_{rev}$, the cycle is irreversible.

...cept of Black Bodies

...bject that completely absorbs all radiant energy that hits its surface is referred to as a "black body" ...a black body, $\alpha = 1, \rho = 0, \tau = 0$). No real body is completely black; the idea of a black body is an ...lization that makes it easy to compare the radiation properties of real bodies.

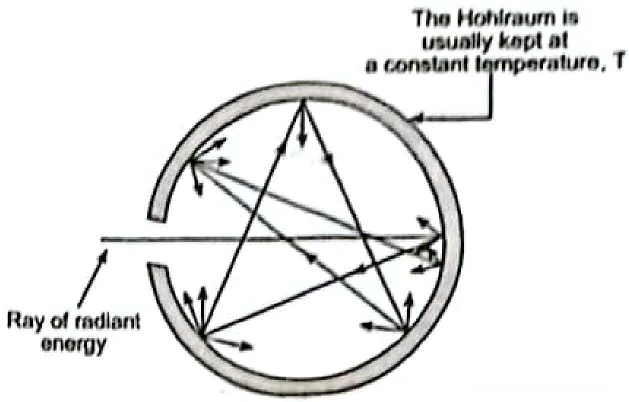
...ack body has the following properties :

i. It absorbs all the incident radiation falling on it and does not transmit or reflect regardless of wavelength and direction.

ii. It emits maximum amount of thermal radiations at all wavelengths at any specified temperature.

iii. It is a diffuse emitter (i.e., the radiation emitted by a black body is independent of direction).

...nsider a hollow enclosure with a very small hole for the passage of incident radiation as shown in ...ure.



...ident radiant energy travels through the tiny ...ening; some of it is absorbed and some of it is ...lected by the interior surface. The majority of this ...ergy is, however, absorbed during the second ...idence. Once more, only a tiny portion is ...lected. Only a very small portion of the original ...ident energy is reflected back out of the opening ...ter several of these reflections. Due to the fact that ... radiant energy entering through a small hole ...ading into a cavity (Hohlraum) is absorbed, it ...haves very similarly to a black body.

...othermal furnaces, with small apertures, approximate a black body and are frequently used to calibrate ...at flux gauges, thermometers and other radiometric devices.

Solved Example

...ample 1: An insulting material having a thermal conductive of 0.08 W/mk is used to limit the heat transfer of 80 W/m^2 for a temperature of 15°C across the opposite faces. Find the thickness of the material.

...olution:

Thermal conductivity (k) = 0.08 W/mK

Rate of heat transfer per unit Area

$$\frac{1}{A}\left(\frac{dQ}{dt}\right) = 80 \text{ W/m}^2$$

Temp. difference (ΔT) = 15° C = 15K

Thickness of material (L) = ?

$$\frac{1}{A}\frac{dQ}{dt} = \frac{k\Delta T}{L}$$

or, $$80 = \frac{0.8 \times 15}{L}$$

or, $$80 = \frac{0.8 \times 15}{L}$$

or, $L = 0.15m$

Example 2: A brick wall 12 cm thick and $5m^2$ surface area exposed to 50° C at one face and 20° C to another face. If the thermal conductivity of the material is 1.5 W/mK. Find the heat transfer rate.

Solution:

Thickness of brick wall (L) = 12 cm = 0.12m

Surface area of wall (A) = 5 m^2

Temp. of one face (T_1) = 50° C

Temp. of another face (T_2) = 20° C

Thermal conductivity (k) = 1.5 W/mK

We have,

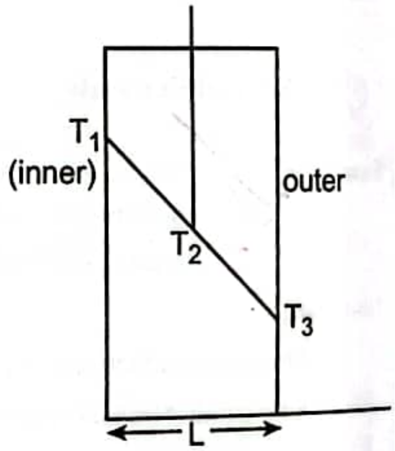
$$\frac{dQ}{dt} = \frac{kA\Delta T}{L}$$

$$= \frac{1.5 \times 5 \times (50-20)}{0.12}$$

= 1875 watt.

Example 3: Find the rate of heat loss form a brick wall (k = 0.7 W/mK) of length 5m, height 4 m and 0.25 m thick. The temperature of the inner surface 50° C and that of outer surface is 30°C. Also calculate the distance from the inner surface at which the temperature is 40° C.

Solution:



Thermal conductivity of brick (k) = 0.7 W/mK

Thickness of wall (L) = 0.25 m

Temp. of inner surface (T_1) = 50° C

Temp. of outer surface (T_3) = 30° C

Area of the wall (A) = $5 \times 4 = 20\ m^2$

We have,

Rate of heat loss $\frac{dQ}{dt} = \frac{kA(T_1 - T_3)}{L}$

$$= \frac{0.7 \times 20 \times (50-30)}{0.25}$$

= 1120 watt.

Now, for distance from the inner surface at which temperature is 40° C.

Again,

$$\frac{dQ}{dt} = \frac{kA(T_1 - T_2)}{L_1}$$

$$L_1 = \frac{0.7 \times 20 \times (50 - 40)}{1120}$$

$$L_1 = 0.125 \text{ m}$$

Example 4: A 1.2m long tube with outer diameter of 4cm having outside temperature of 120° C is exposed to the ambient air at 20° C. If the heat transfer coefficient between the tube surface and the air is 20 W/m²k. Find the rate of heat transfer from the tube to the air.

Solution:

Length of the tube (L) = 1.2m

Outer Radius (R) = 2 cm = 2×10^{-2} m

Outside temp of tube (T_A) = 120° c

Temp of Air (T_B) = 20° c

Heat transfer coeff (h) = 20 W/m²k

Area of tube (A) = 2 πRL

$$= 2\pi \times 2 \times 10^{-2} \times 1.2$$

$$= 0.150\,8 \text{m}^2$$

Then,

$$\text{Rate of heat transfer } \frac{dQ}{dt} = hA\,(T_A - T_B)$$

$$= 20 \times 0.1508 \times (120 - 20)$$

$$= 301.593 \text{ watt}$$

Example 5: A 2m long 0.35 cm diameter electrical wire in a room at 20° C generating heat and surface temperature of the wire found to be 150° C in steady cooperation. The voltage drop and electric current through the wire are 50 and 2A respectively. Find the convection h eat transfer coefficence. Between the outer surface of the wire and the air in the room.

Solution:

Length of the wire (ℓ) = 2m

Diameter of the wire (d) = 0.35 cm = 0.35×10^{-2} m

Room temp (T_B) = 20 C

Surface temp (T_A) = 150° C

Voltage drop (V) = 50 V

Current (I) = 2 A

Coeff. of convection

Heat transfer (b) = ?

Area (A) = πdl = $\pi \times 0.35 \times 10^{-2} \times 2$

Since, Electric power developed in the wire = 0.02199 m² = Rate of heat loss from the wire

$\frac{Q}{t} = kA\frac{dT}{dx}$

i.e. $h = \frac{IV}{A(T_A - T_B)}$

$= \frac{2 \times 50}{0.02199 \times (150 - 20)}$

$= 34.98 \text{ W/m}^2\text{K}$

Example 6: A room is maintained at 22° C by an air conditioning unit Determine the total rate of heat transfer from the person standing in the roof if the exposed surface area and the average outer surface temperature of the person area 1.5 m³ and 30° C, respectively, and the convection heat transfer coefficient is 10 W/m²K – take surface emissivity as 0.95.

Solution:

Room temperature (T_2) = 22° C = 295K

Expose surface area (A) = 1.5m²

Average outer surface temperature (T_1) = 30° C = 303K

Convection heat transfer coefficient (h) = 10 W/m²K

$h = \frac{k}{dx}$

Emissivity (ε) = 0.95

Now,

Total rate of heat transfer $\frac{dQ}{dt} = \frac{dQ_{convection}}{dt} + \frac{dQ_{radiation}}{dt}$

$$= hA(T_1 - T_2) + \sigma\varepsilon A\left(T_1^4 - T_2^4\right)$$

$$= 10 \times 1.5 \times (303 - 295) + 5.67 \times 10^{-8} \times 0.95$$

$$= 189.126\text{W}$$

Example 7: A thin metal plate is insulted on the back and exposed its front surface to solar radiation. The exposed surface of the plate has an emissivity of 0.7. If the solar radiation is including on the plate at the rate of 750 W/m² and the surrounding air temperature is 20°C, determine the surface temperature of the plate. Assume convection heat transfer coefficient 40 W/m²K.

Solution:

Emissivity of exposed surface of plate (ε) = 0.7

$\frac{1}{A}\left(\frac{dQ}{dt}\right) = 750 \text{ W/m}^2$

Air temperature (T_2) = 20° C = 293K

Coefficient of convection heat transfer (h) = 40 W/m²K $= \frac{k}{dx}$

Surface temperature of plate (T_1) = ?

Here,

Rate of heat incident per unit Area = Rate of heat dissipated per unit area by convection + Rate of heat dissipated per unit area by radiation

or, $$\frac{1}{A}\left(\frac{dQ}{dt}\right)_{incident} = h(T_1 - T_2) + \sigma\varepsilon\left(T_1^4 - T_2^4\right)$$

$$750 = 40(T_1 - 293) + 5.67 \times 10^{-8} \times 0.7\left(T_1^4 - 293^4\right)$$

…ple 8: A furnace inside temperature of 2250 K has a glass circular viewing of 6 cm diameter. If the transmissivity of glass is 0.08, make calculations for the heat loss from the glass window due to radiation.

…tion:

Temperature of furnace (T) $= 2250$ k

Diameter (d) $= 6$ cm $= 6 \times 10^{-2}$ m

Transmissivity (τ) $= 0.08$

The radiation heat loss from the glass window is given by

$$Q = \sigma_b A T^4 \times \tau$$

Where τ is the transmissivity of glass

$$Q = 5.67 \times 10^{-8} \times \frac{\pi}{4}(0.06)^2 \times 2250^4 \times 0.08$$

$$= 328.53 \text{ W}$$

…mple 9: **A thin metal plate of 4 cm diameter is suspended in atmospheric air whose temperature is 290 K. the plate attains a temperature of 295 K when one of its face receives radiant energy from a heat source at the rate of 2 W. If heat transfer coefficient on both surfaces of the plate is stated to be 87.5 W/m²K, workout the reflectivity of the plates.**

…lation:

Heat lost by convection from both sides of the plate $= 2hA\Delta T$

The factors 2 accounts for two sides of the plate $= 2 \times 87.5 \times \left\{\frac{\pi}{4}(0.04)^2\right\} \times (295 - 290)$

$$= 1.1\text{W}$$

For most of solids, the transmissivity is zero.

$\therefore$ Rae of energy lost by reflection $(E_r) = 2 - 1.1 = 0.9\text{W}$

Reflectivity $\rho = \frac{E_r}{E_i} = \frac{0.9}{2.0} = 0.45.$

THEORETICAL ANSWER QUESTIONS

Explain three different modes of heat transfer.

Derive an expression for the one dimensional steady state heat conduction through a plane wall.

What do you understand by conduction mode of heat transfer. State the Fourier law of thermal conduction also write the assumption of that law.

What is convection mode of heat transfer. State and explain Newton's law of cooling.

What is radiation mode of heat transfer. State and explain Stefan-Boltzmann law for two particle bodies at different temperatures.

What is black body. Write the properties of black body.