Chapter-2

Interpolation & Approximation.

Date
Page

Interpolation:

consider a given discrete set of data sample given as,

x	x_0	x_1	x_2	-----	x_n
y	y_0	y_1	y_2	----	y_n

→ Then the process of finding the functional value y_i at any arbitary point $x = x_i$ which lies betwen the range of $[x_0, x_n]$ is called interpolation and the process of determining y_i at any arbitary point $x = x_i$ lies outside the range of $[x_0, x_n]$ is called extrapolation.

The technique by which we determine the functional value is known as finite differences.

Finite difference:

consider a given discrete function given as :

x	x_0	x_1	x_2	---	x_n
y	y_0	y_1	y_2	---	y_n

Then, change in functional value y due to small finite change in independent variable x is called finite difference. The various finite differences are :

ⓐ forward difference.
ⓑ Backward difference.
ⓒ central difference.

a) Forward difference:

Consider a discrete function $y = f(x)$ which takes the values $y_0, y_1, y_2, \ldots y_n$ at points $x_0, x_1, x_2, \ldots x_n$. Assuming that values for x are equally spaced. Then first forward difference is given by:

$$\Delta y_0 = y_1 - y_0 \,, \quad \Delta y_1 = y_2 - y_1 \,, \quad \Delta y_2 = y_3 - y_2 \,, \ldots$$

$\Delta y_{n-1} = y_n - y_{n-1}$ where Δ is called forward difference operator. And the differences of first forward difference is called second forward differences, denoted by Δ^2 and defined by using relation,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1 \,, \quad \Delta^2 y_2 = \Delta y_3 - \Delta y_2, \ldots$$

and so on.

Similarly proceeding upto pth terms, we get:

$\Delta^p y_{r-1} = \Delta^{p-1} y_r - \Delta^{p-1} y_{r-1}$ is the pth forward differences are tabulated to given forward difference table as:

x	y	1st forward difference (Δ)	2nd forward diff (Δ^2)	3rd f/w diff (Δ^3)
x_0	y_0	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_1$
x_1	y_1	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
x_2	y_2	$\Delta y_2 = y_3 - y_1$		
x_3	y_3			

[Fig: Forward difference ~~difference~~ table]

b) Backward difference:

Consider a discrete function $y = f(x)$ which takes the values $y_0, y_1, y_2, \dots y_n$ at points $x_0, x_1, x_2 \dots x_n$. Assuming that values for x are equally spaced. Then first backward difference is given by:

$\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, $\nabla y_2 = y_3 - y_2$ ---- $\nabla y_n = y_n - y_{n-1}$

where, ∇ is called backward difference operator. And the difference of first backward difference is called second backward difference, denoted by ∇^2 and defined by using relation,

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$, $\nabla^2 y_3 = \nabla y_3 - \nabla y_2$. & so on---

Similarly proceeding upto pth term we get:

$\nabla^p y_r = \nabla^{p-1} y_r - \nabla^{p-1} y_{r-1}$ is the pth backward differences are tabulated to given forward difference table as:

x	y	first B/W diff (∇)	second B/W diff (∇^2)	third B/W diff (∇^3)
x_0	y_0	$\nabla y_1 = y_1 - y_0$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	$\nabla^3 y_3 = \nabla^2 y.$
x_1	y_1	$\nabla y_2 = y_2 - y_1$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	$\nabla^2 y_2$
x_2	y_2	$\nabla y_3 = y_3 - y_2$		
x_3	y_3			

[Fig: Backward difference table]

c) central difference:

consider a discrete function $y = f(x)$ which takes the values $y_0, y_1, y_2, \dots y_n$ at points $x_0, x_1, x_2, \dots x_n$. Assuming that values for x are equally spaced. This first forward difference is given by,

$$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1, \quad \delta y_{5/2} = y_3 - y_2 \; \dots$$

$\delta y_{\frac{2n-1}{2}} = y_n - y_{n-1}$ where δ is called central difference operator. And the difference of the first central difference is called second central differences denoted by δ^2 and defined by using relation.

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}, \quad \delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2} \text{ and so on} \dots$$

Similarly proceeding up to pth central difference. These difference are tabulate to give central difference table as,

x	y	1st central diff (δ)	2nd central diff (δ^2)	3rd central diff (δ^3)
x_0	y_0	$\delta y_1 = y_1 - y_0$	$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$	$\delta^3 y_{3/2} = \delta^2 y_2 -$
x_1	y_1	$\delta y_2 = y_2 - y_1$	$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$	$\delta^2 y_1$
x_2	y_2	$\delta y_3 = y_3 - y_1$		
x_3	y_3			

[Fig: central difference table]

↳ The various methods of interpolation are divided into 2 categories. according to their data distribution are:-

ⓐ Interpolation with equispaced data set
ⓑ Interpolation with unequally spaced data sets.

a) Interpolation with equispaced data set:

↳ The sample data taken at equal interval of time is called equispaced data set and various interpolation techniques applied for equiplaced data set are:

(i) Newton Interpolation formula →
- (i) forward interpolatio
- (ii) Backward "
- (iii) central " not in syllabus.

not in syllabus (ii) Gauss interpolation formula →
- (i) forward
- (ii) Backward
- (iii) central

b) Interpolation with unequally spaced data set:

↳ The sample data taken at unequal interval of time is called unequally spaced data set and various interpolation techniques applied for unequally spaced data set are:

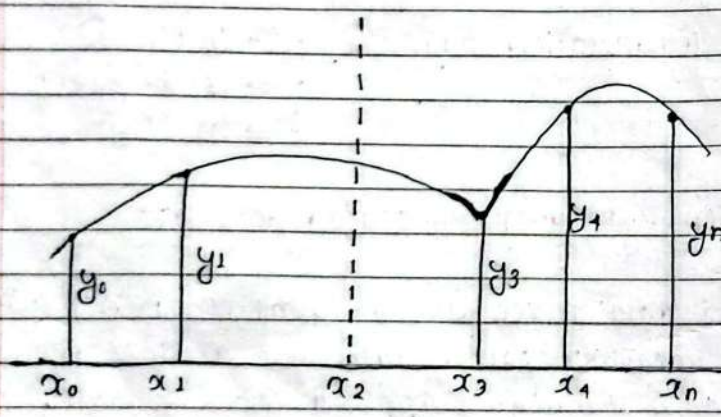
(i) langrange interpolation
(ii) Newton divided difference formula
(iii) Spline interpolation formula.

Newton Interpolation Formula:

a. Newton forward interpolation formula:

Consider $y = f(x)$ be a discrete function which takes the values $y_0, y_1, y_2, \ldots, y_n$ at point $x_0, x_1, x_2, \ldots, x_n$. Assuming that all the data is being equally spaced i.e. $x_i = x_0 + ih$ for $i = 0, 1, 2, \ldots n$. Let $y(x)$ be an n^{th} degree polynomial equation passing through all the data set such as $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2, \ldots$ $y(x_n) = y_n$. Then n^{th} degree polynomial eqn passing through all these points is given as,

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad -(1)$$



Now, substituting value of $x_0, x_1, x_2, \ldots x_n$ successively in eqn (1) we get,

Q(4)

At $x = x_0$,

$$y(x_0) = a_0$$
$$\Rightarrow a_0 = y_0$$

At $x = x_1$,

$$y(x_1) = a_0 + a_1(x_1 - x_0)$$
$$\Rightarrow y_1 - y_0 = a_1 h$$
$$\Rightarrow a_1 = \frac{\Delta y_0}{h}$$

Also,

At $x = x_2$;

$$y(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$\Rightarrow y_2 - y_0 - 2ha_1 = a_2 . 2h . h$$
$$\Rightarrow y_2 - 2\Delta y_0 - y_0 = 2h^2 a_2$$
$$\Rightarrow \frac{y_2 - y_1 - 2\Delta y_0 + y_1 - y_0}{2!\, h^2} = a_2$$
$$\Rightarrow \frac{\Delta y_1 - \Delta y_0}{2!\, h^2} = a_2$$
$$\Rightarrow a_2 = \frac{\Delta^2 y_0}{2!\, h^2}$$

Similarly,

$$a_3 = \frac{\Delta^3 y_0}{3!\, h^3}$$

Proceeding up to n^{th} term,

$$a_n = \frac{\Delta^n y_0}{n!\, h^n}$$

Putting all these coefficient value in eq^n(I), we get;

$$\therefore y(x) = y_0 + \frac{(x-x_0)}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 y_0 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{3!\,h^3}\Delta^3 y_0 + \dots +$$

$$\frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{n!\,h^n}\Delta^n y_0 \quad \text{--- (II)}$$

Consider that we have to determine value of y at any arbitary point $x = x_0 + p.h$; then,

$$x - x_0 = ph$$

and

$$x - x_1 = x - x_0 + x_0 - x_1$$
$$= ph - h$$
$$= h(p-1)$$

Also,

$$x - x_2 = x - x_1 + x_1 - x_2$$
$$= (p-1)h - h$$
$$= (p-2)h$$

Similarly,

$$x - x_3 = (p-3)h$$

Proceeding up to n^{th} term,

$$x - x_{n-1} = (p-n+1)h$$

Putting these values to eq^n(2), we get.

$$y(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 +$$

$$\dots + \frac{p(p-1)(p-2)\dots(p-n+1)}{n!}\Delta^n y_0 \quad \text{--- (III)}$$

which is known as newton forword interpolation formula where,

$$p = \frac{x - x_0}{h}$$

b. Newton backword interpolation:

Consider $y = f(x)$ be a discrete function which takes the values $y_0, y_1, y_2, \dots, y_n$ of points $x_0, x_1, x_2, \dots x_n$. Assuming that the data are sampled at equally spaced such that $x_i = x_0 + ph$, then newton ~~[illegible]~~ backward interpolation formula is given as,

$$y(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n +$$

$$\dots + \frac{p(p+1)(p+2)(p+3)\dots(p+n-1)}{n!}\nabla^n y^n \quad \text{--- (IV)}$$

which is known as newton backward interpolation formula.

$\frac{x-x_0}{h}$

Date

Page

Q. The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

length(x)	100	150	200	250	300	350	400
distance(y)	10.63	13.03	15.04	16.81	18.42	19.9	21.21

Find the value of y when,

(a) $x = 160$ feet (b) $x = 380$ feet

soln: The difference table for the given data set is obtained as,

x	y	1st diff	2nd diff	3rd diff	4th diff	5th diff	6th diff
100	10.63	2.4	-0.39	0.15	-0.07	0.02	0.02
150	13.03	2.01	-0.24	0.08	-0.05	0.09	
200	15.04	1.77	-0.1	0.03	-0.01		
250	16.81	1.61	-0.13	0.02			
300	18.42	1.48	-0.11				
350	19.9	1.37					
400	21.21						

(a) At $x = 160$,

$x_0 = 100$, $y_0 = 10.63$; $\Delta y_0 = 2.4$, $\Delta^2 y_0 = -0.39$, $\Delta^3 y_0 = 0.15$.
$\Delta^4 y_0 = -0.07$, $\Delta^5 y_0 = 0.02$, $\Delta^6 y_0 = 0.02$

$\therefore h = x_n - x_{n-1} = 50$

$$\therefore p = \frac{x - x_0}{h} = \frac{160-100}{50} = \frac{60}{50} = 1.2$$

Now, applying newton forward interpolation formula, we have,

$$y(x_0 + p\cdot h) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^4 y_0 +$$

$$\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^5 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{7!}\Delta^6 y_0$$

$$y(160) = 10.63 + 1.2 \times 2.4 + \frac{(1.2)\,0.2}{2} \times (-0.39) + \frac{(1.2)(0.2)(-0.8)(0.15)}{6} +$$

$$\frac{(1.2)(0.2)(-0.8)(-1.8)}{24} \times 0.07 + \frac{(1.2)(0.2)(-0.8)(-1.8)(-2.8)}{120} \times 0.2$$

$$+ \frac{1.2(0.2)(-0.8)(-1.8)(-2.8)(-3.8)}{720} \times 0.2$$

$y(160) = 13.45$

b) At $x = 380$,

$x_n = 400$, $y_n = 21.21$, $\nabla y_n = 1.37$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$,
$\nabla^4 y_n = -0.01$, $\nabla^5 y_n = 0.04$, $\nabla^6 y_n = 0.02$

$h = x_n - x_{n-1} = 50$

$\therefore p = \frac{x - x_n}{h} = \frac{380 - 400}{50} = \frac{-20}{50} = -0.4$

Now, Applying netwon B/W interpolation formula, we have:

$$y(x_n + p\cdot h) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)(p+5)}{6!}\nabla^6 y_n$$

$$y(380) = 21.27 + (-0.4)(1.37) + \frac{(-0.4)(0.6)}{2}(-0.11) + \frac{(-0.4)(0.6)(1.6)(0.02)}{6} + \frac{(-0.4)(0.6)(1.6)(2.6)}{24}(0.01) + \frac{(-0.4)(0.6)(1.6)(2.6)(3.6)}{120}(0.04) + \frac{(-0.4)(0.6)(1.6)(2.6)(3.6)(4.6)}{720} \times 0.02$$

$$\therefore y(380) = 21.27 - 0.548 + 0.0132 - 0.1664 + 4.16\times 10^{-4} + 1.19808\times 10^{-3} - 4.592\times 10^{-4}$$

$$= 20.569 \quad ans\#.$$

for unequaly ~~unsp.~~ spaced data

Langrange Interpolation:

↳ consider a discrete function $y = f(x)$ which takes the values $y_0, y_1, y_2 \ldots y_n$ at points $x_0, x_1, x_2 \ldots x_n$. Let $y(x)$ be the n^{th} degree of polynomial eqn passing through all these points (x_0, y_0) $(x_1, y_1) \ldots (x_n, y_n)$ such that $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$ ----- $y(x_n) = y_n$. Since there are $(n+1)$ different points, then n^{th} degree polynomial eqn passing through $n+1$ points are given as;

$$y(x) = a_0(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots(x-x_n) + a_2(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n) + \ldots + a_n(x-x_0)(x-x_1)(x-x_1)\ldots(x-x_{n-1}) \quad \text{---} (1)$$

Now, taking $x = x_0, x_1, x_2 \ldots x_n$ successively in eqn (1) we get:

at $x = x_0$,

$$y(x_0) = a_0(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)$$

$$\Rightarrow a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$

at $x = x_1$,

$$y(x_1) = a_1(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)$$

$$\Rightarrow a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_0)(x_1-x_3)\ldots(x_1-x_n)}$$

Date
Page

Similarly,
at $x = x_2$

$$y(x_1) = a_2 (x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \cdots (x_2 - x_n).$$

$$\Rightarrow a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \cdots (x_2 - x_n)}$$

Exceeding upto n^{th} term, we get:

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

Now, substituting all coefficient values to eqn (1) we get,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3) \cdots (x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3) \cdots (x_0-x_n)} * y_0 +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3) \cdots (x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3) \cdots (x_1-x_n)} * y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3) \cdots (x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3) \cdots (x_2-x_n)} * y_2 + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3) \cdots (x-x_n)}{(x_n-x_0)(x_n-x_1)(x_n-x_3) \cdots (x_n-x_{n-1})} * y_n \quad \cdots (2)$$

Which is known as langranges interpolation formula for unequally spaced data set.

$$\prod_{j=0}^{n} x_j = x_0 * x_1 * x_2 \cdots x_n$$

Date
Page

In general, it can also be written as;

$$\boxed{y(x) = \sum_{i=0}^{n} y_i * \prod_{j=0,\, i \neq j}^{n} \frac{x - x_j}{x_i - x_j}} \quad \cdots\cdots (3).$$

old qn

Q). Find the missing term in the following table using suitable interpolation method.

x	0	1	2	3	4
y	1	3	9	?	81

Solution: given:

x	0	1	2	4
y	1	3	9	81

$y(3) = ?$

Now, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 4$
$y_0 = 1$, $y_1 = 3$, $y_2 = 9$, $y_3 = 81$

and, $x = 3$
$y(x) = ?$

Now, applying langrage interpolation formula for $n = 3$, we have:

$$y(x) = \sum_{i=0}^{n=3} \prod_{j=0;\, i\neq j}^{3} \left(\frac{(x-x_j)}{(x_i-x_j)} * y_i \right) \quad \text{------} (1)$$

at $x = 3$.

$$y(3) = \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} \times 1 + \frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)} \times 3$$

$$+ \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} \times 9 + \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} \times 81$$

$$= 31.$$

Hence, $y(3) = 31$. ans #.

Q) from the following table, find the number of standard who obtained mark less than 45 marks.

marks	30-40	40-50	50-60	60-70	70-80
no. of students	31	42	51	35	37

Solution:

The c.f table for given data set is obtained as:

Marks less than (<)	No. of student
40	31
50	73
60	124
70	159
80	196

x	y	1st diff	Second diff	third diff	fourth diff
40	31 y_0	42 Δy_0	9 $\Delta^2 y_0$	-25 $\Delta^3 y_0$	37 $\Delta^4 y_0$
50	73	51	-16	12	
60	124	35	-4		
70	159	31			
80	196				

At $x = 45$,

$x_0 = 40$, $y_0 = 31$, $\Delta y_0 = 42$, $\Delta^2 y_0 = 9$, $\Delta^3 y_0 = -25$

$\Delta^4 y_0 = 37$

$h = x_n - x_{n-1} = 10$

$$p = \frac{x - x_0}{h} = \frac{45-40}{10}$$

Applying newton f/w interpolation formula, we get;

$$y(x) = y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(47) = 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{(0.5)(-0.5)(-1.5)}{6} \times (-25) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24} \times 37.$$

$\therefore y(47) = 47.86$

≈ 48 Students.

Q) The following table gives the viscosity of oil as function of temperature of tempr use langranges interpolation formula to find viscosity of oil at tempr of 140°c.

T°c	110	130	160	190
viscocity	10.8	8.1	5.5	4.8

$x_0 = 110$, $x_1 = 130$, $x_2 = 160$, $x_3 = 190$

$y_0 = 10.8$, $y_1 = 8.1$, $y_2 = 5.5$, $y_3 = 4.8$

$x = 140$, $y(x) = ?$

Now,

Applying langrage interpolation formula for $n=3$, we have:

0,1,2,3 davas 56.

$$y(x) = \sum_{i=0}^{n=3} \prod_{j=0;\, i\neq j}^{3} \left(\frac{(x-x_j)}{(x_i-x_j)} * y_i\right) \quad \text{----} (1)$$

at $x = 140$,

$$y(140) = \frac{(140-130)(140-160)(140-190)}{(110-130)(110-160)(110-190)} \times 10.8 + \frac{(140-110)(140-160)(140-190)}{(130-110)(130-160)(130-190)} \times 8.1 + \frac{(140-110)(140-130)(140-190)}{(160-110)(160-130)(160-190)} \times 5.5 + \frac{(140-110)(140-130)(140-160)}{(190-110)(190-130)(190-160)} \times 4.8$$

$$= -\frac{27}{20} + \frac{27}{4} - \frac{11}{6} - \frac{1}{5} = 3.367$$

SPLINE INTERPOLATION

- A spline fits a set of n^{th} degree polynomials $g_i(x)$ between each pair of points from x_i to x_{i+1}. The point at which spline joins are called knots.

- If the polynomials are all of 1^{st} degree we have a 'linear spline' and the curve would be discontinuous where the segments ~~used~~ joined as shown below:

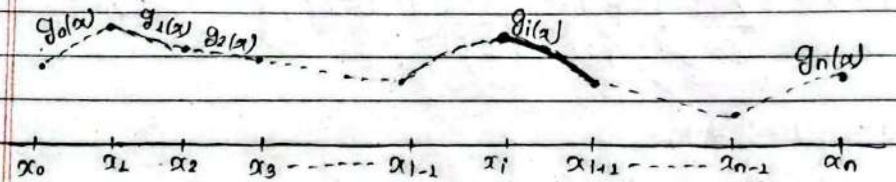


Fig: Linear splines

- Spline of degree greater than one do not have the problem of discontinuity most often cubic square are used.

- Let us consider a successive of cubic spline ~~are~~ over successive interval of the n given data. Each spline must join with it's neighbouring cubic spline polynomial at the knots where they joins with same slope and curvature.

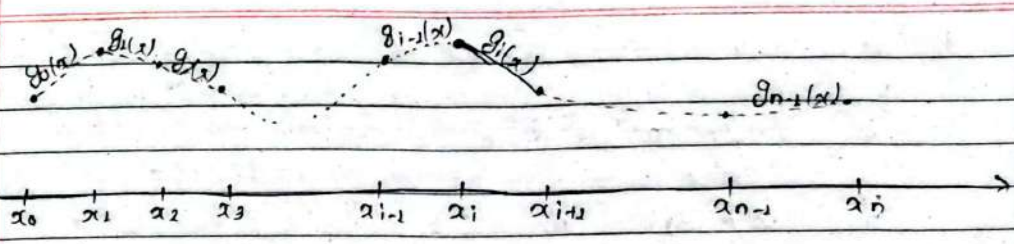


Fig: cubic splines

We can write eqn for a cubic ~~square~~ spline polynomial $g_i(x)$, in interval ~~because~~ between points (x_i, y_i) and (x_{i+1}, y_{i+1}). It looks like solid curve shown figure above. The dashed curves are other cubic polynomials. It has the eqn,

$$g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \quad \text{--- (1)}$$

where,

$$d_i = y_i$$

$$c_i = \frac{y_{i+1} - y_i}{h_i} + \frac{2h_i S_i + h_i S_{i+1}}{6}$$

$$b_i = \frac{S_i}{2}$$

$$a_i = \frac{S_{i+1} - S_i}{6 \cdot h_i}$$

and the simplified solution of (1), is given as,

$$h_{i-1} \cdot S_{i-1} + 2(h_{i-1} + h_i)S_i + h_i \cdot S_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_i}\right)$$

$$\cancel{= 6[f(x_i, x_{i+1})} -$$

$$= 6[f(x_i, x_{i+1}) - f(x_{i-1}, x_i)] \quad \text{---- (2)}$$

Taking $S_0 = 0$ and $S_n = 0$, we obtain natural spline, matches precisely with drafting device, if we write eq^n of $S_0, S_1, S_2, \ldots S_n$ in matrix form, we get

$$\begin{bmatrix} h_0 & 2(h_0+h_1) & h_1 & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & \\ & & h_2 & 2(h_2+h_3) & h_3 & \\ & & & \ddots & & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix} = 6 \begin{bmatrix} f(x_1,x_2) - f(x_0,x_1) \\ f(x_2,x_3) - f(x_1,x_2) \\ \vdots \\ f(x_{n-1},x_n) - f(x_{n-2},x_{n-1}) \end{bmatrix} \quad \text{---- (3)}$$

Taking $S_0 = S_n = 0$, the matrix forms,

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & \ddots & & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{n-1} \end{bmatrix}$$

$$= 6 \begin{bmatrix} f(x_1,x_2) - f(x_0,x_1) \\ f(x_2,x_3) - f(x_1,x_2) \\ \vdots \\ f(x_{n-1},x_n) - f(x_{n-2},x_{n-1}) \end{bmatrix} \quad \text{---- (4)}$$

On solving eq^n (4), we get the values of S_i. The using these value we ~~can~~ calculate the corresponding coefficient values from (1). Finally we get the required cubic spline polynomial eq^n for i^{th} interval.

Divided difference:

Let $y = f(x)$ be a discrete function which takes the value $y_0, y_1, y_2, \ldots y_n$ at points $x_0, x_1, x_2, \ldots x_n$ taken at random interval. Then first divided difference is denoted by, $[x_0, x_1]$ and defined as $[x_0, x_1] = \frac{y_1 - y_0}{(x_1 - x_0)}$, $[x_1, x_2] = \frac{y_2 - y_1}{(x_2 - x_1)}$, $[x_2, x_3] = \frac{y_3 - y_2}{(x_3 - x_2)}$

$\ldots\ldots$, $[x_n, x_{n-1}] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$.

And the differences of 1^{st} divided difference is called second divided difference denoted by $[x_0, x_1, x_2]$ and divided as,

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{(x_2 - x_0)}$$

$$[x_1, x_2, x_3] = \frac{[x_2, x_3] - [x_1, x_2]}{(x_3 - x_1)} \text{ and so on.}$$

The difference table for n given data set is obtained as:

x	y	1st divided diff $[x_0,x_1]$	2nd divided diff $[x_0,x_1,x_2]$	3rd divided diff (x_0,x_1,x_2,x_3)
x_0	y_0			
		$[x_0,x_1] = \frac{y_1-y_0}{x_1-x_0}$		
x_1	y_1		$[x_0,x_1,x_2] = \frac{[x_1,x_2]-[x_0,x_1]}{(x_2-x_0)}$	
		$[x_1,x_2] = y_2-y_1/x_2-x_1$		$[x_0,x_1,x_2,x_3] = \frac{[x_1,x_2,x_3]-[x_0,x_1,x_2]}{x_3-x_0}$
x_2	y_2		$[x_1,x_2,x_3] = \frac{[x_2,x_3]-[x_1,x_2]}{x_3-x_1}$	
		$[x_1,x_3] = \frac{y_3-y_2}{x_3-x_2}$		
x_3	y_3			

[Fig: divided difference table]

Q) Fit the given data with natural cubic spline and evaluate the spline value $g'(0.66)$, $g(1.75)$, and $g''(1.3)$.

x	0	1	1.5	2.25
y	2	4.4366	6.7134	13.9130

Solution:

Let the cubic spline function to be fitted for ith interval is given as,

$$g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \text{ ----- (1)}$$

where, $d_i = y_i$, $c_i = \frac{y_{i+1}-y_i}{h_i} - \frac{2h_iS_i + h_iS_{i+1}}{6}$

$b_i = \frac{S_i}{2}$, $a_i = \frac{S_{i+1}-S_i}{6h_i}$

From given data set, we have

i	x_i	h_i	y_i	$[x_i, x_{i+1}]$	$6([x_{i+2},x_i]-[x_i,x_{i+1}])$
0	0	1	2	2.4366	12.70
1	1	0.5	4.4366	4.5536	30.2754
2	1.5	0.75	6.7134	9.5995	
3	2.25		13.9130		

taking $n=3$, for natural cubic spline,

$S_0 = S_3 = 0$ and it's solution is given as:

$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} f(x_1,x_2) - f[x_0,x_1] \\ f(x_2,x_3) - f[x_1,x_2] \end{bmatrix}$$

$$\begin{bmatrix} 2(1+0.5) & 0.5 \\ 0.5 & 2(0.5+0.75) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 12.70 \\ 30.2754 \end{bmatrix}$$

solving,

$3S_1 + 0.5S_2 = 12.70$ — (i)

$0.5S_1 + 2.5S_2 = 30.2754$ — (ii)

$\therefore S_1 = 2.29$

$S_2 = 11.65$

Now, substituting these values to (1), we get.

i	interval	$g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$
0	[0,1]	$g_0(x) = 0.3816x^3 + 2.0549x + 2$
1	[1,1.5]	$g_1(x) = 3.12x^3 + 10.505x^2 - 8.449x + 5.5$
2	[1.5, 2.25]	$g_2(x) = -2.588x^3 - 5.821x^2 - 3.1345x + 30.3364$

When $i=0$,

$$a_0 = \frac{S_{i+1} - S_i}{6 \cdot h_i} = \frac{S_1 - S_0}{6 \cdot h_0} = \frac{2.29 - 0}{6 \cdot 1} = 0.3816$$

$$b_0 = \frac{S_i}{2} = 0$$

$$c_0 = \frac{y_{i+1} - y_i}{h_i} - \frac{2 h_i S_i + h_i S_{i+1}}{6}$$

$$= \frac{4.4366 - 2}{1} - \frac{2 \times 1 \times 0 + 1 \times 2.29}{6}$$

$$= 2.054933$$

$$d_0 = y_0 = 2$$

$$g_0(x) = 0.3816 \times (x-0)^3 + 0 + 2.0549(x-0) + 2$$

$$= 0.3816x^3 + 2.0549x + 2$$

When $i=1$,

$$a_1 = \frac{S_2 - S_1}{6h_1} = \frac{11.65 - 2.29}{6 \times 0.5} = 3.12$$

$$b_1 = \frac{S_1}{2} = \frac{2.29}{2} = 1.145$$

$$c_1 = \frac{y_2 - y_1}{h_1} - \frac{2h_1 S_1 + h_1 S_2}{6}$$

$$= \frac{6.7134 - 4.4366}{0.5} - \frac{2 \times 0.5 \times 2.29 + 0.5 \times 11.65}{6}$$

$$= 3.2011$$

$$d_1 = y_1 = 4.4366$$

$$g_1(x) = 3.12(x - x_1)^3 + 1.145(x-1)^2 + 3.201(x-1) + 4.436$$

$$= 3.12(x^3 + 3x^2 - 3x + 1) + 1.145(x^2 - 2x + 1) + 3.201(x-1) + 4.436$$

$$= 3.12x^3 + 10.505x^2 - 8.449x + 5.5$$

When $i=2$,

$$a_2 = \frac{0 - 11.65}{6 \times 0.75} = -2.588$$

$$b_2 = \frac{11.65}{2} = 5.825$$

$$c_2 = -11.8637$$

$$d_2 = y_2 = 6.7134$$

$$g_2(x) = -2.588(x-1.5)^3 + 5.825(x-1.5)^2 - 11.863(x-1.5) + 6.7134$$

$$= -2.588\{x^3 - 10.125x + 4.5x^2 + 4.5\} + 5.825\{x^2 - 3x + 3\} - 11.863x + 17.794 + 6.7134$$

$$= -2.588x^3 - 5.821x^2 - 3.1345x + 30.3364$$

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$g(1.75) = -2.588(1.75)^3 - 5.821x^2 - 3.1345x + 30.3364$

$= \cancel{-2.588x}$

$= 16.4663$

~~$g'(0.66) = 0.3816x^3 + 2.0549x + 2$~~

~~$= 0.3816(0.66)^3 + 2.0549(0.66) + 2$~~

~~$= 3.4659$~~

~~$g''(1/3) = 3.12x^3 + 10.505x^2 - 8.499x + 5.5$~~

~~$= 19.05$~~

$g'(0.66) = 1.1448x^2 + 2.0549$

$= 2.810468$

$g''(1.3) = 9.36x^2 + 21.01x - 8.499x$

$= 18.72x + 21.01$

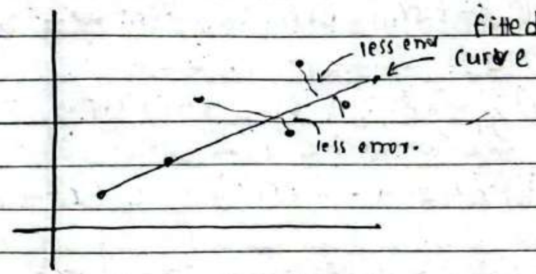
$= 45.346$

nth degree polynomial (tha navako) is interpola...

1st, 2nd, 3rd ... degree (tha vayeko) is curve fitting.

Curve fitting :

↳ The process of fitting a specific equation to the given set is called curve fitting. The various method of fitting a given are graphical method, group of average, least square method. All listed method except least square method fails to provide an unique curve which best-fit to a given data set.



least square method:

Let $y(x) = ax^2 + bx + c$ be the curve to be fitted over the given data set $(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots (x_n, y_n)$. It is assuming that $\eta_i = ax_i^2 + bx_i + c$ be exact curve to be fitted over the given data set. Then, the sum of square of error for every data sample is given as,

$\because e_i = y_i - \eta_i \quad \Rightarrow E = e_1^2 + e_2^2 + e_3^2 + \cdots + e_n^2$

$\therefore E = (y_1 - \eta_1)^2 + (y_2 - \eta_2)^2 + \cdots$

$E = \{y_1 - (ax_1^2 + bx_1 + c)\}^2 + \{y_2 - (ax_2^2 + bx_2 + c)\}^2 + \cdots$

Now,

For error to be minimum,

For a to be minimum,

$$\frac{\partial E}{\partial a} = 0$$

$\Rightarrow 2[y_1 - (ax_1^2 + bx_1 + c)] \times (-x_1^2) + 2[y_2 - (ax_2^2 + bx_2 + c)] \times -x_2^2 + \cdots,$

$2[y_n - (ax_n^2 + bx_n + c)] \times -x_n^2 = 0.$

$\Rightarrow x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3 + \cdots + x_n^2 y_n = a(x_1^4 + x_2^4 + \cdots + x_n^4) +$

$b(x_1^3 + x_2^3 + \cdots + x_n^3) + c(x_1^2 + x_2^2 + \cdots + x_4^2)$

$\Rightarrow \Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$ ——— (i)

For b to be minimum;

$$\frac{\partial E}{\partial b} = 0$$

$\Rightarrow 2[y_1 - (ax_1^2 + bx_1 + c)] \times (-x_1) + 2[y_2 - (ax_2^2 + bx_2 + c) \times (-x_2)$
$+ \cdots 2[y_n - (ax_n^2 + bx_n + c)] \times (-x_n)$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n = a(x_1^3 + x_2^3 + x_3^3 + \cdots x_n^3)$
$+ b(x_1^2 + x_2^2 + x_3^2 + \cdots x_n^2) + c(x_1 + x_2 + x_3 + \cdots x_n).$

$\Rightarrow \Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$. ------ (ii)

For c to be minimum,

$$\frac{\partial E}{\partial c} = 0.$$

$\Rightarrow 2[y_1 - (ax_1^2 + bx_1 + c)] \times (-1) + 2[y_2 - (ax_1^2 + bx_1 + c)] \times$
$+ \cdots + 2[y_n - (ax_n^2 + bx_n + c)] \times (-1)$

$\Rightarrow y_1 + y_2 + y_3 + \cdots y_n = a(x_1^2 + x_2^2 + \cdots + x_n^2) + b(x_1 + x_2 + x_3 + \cdots x_n$
$+ [c + c + c + c + \cdots + c] = 0$

$\Rightarrow \Sigma y = a\Sigma x^2 + b\Sigma x + nc$ ----- (iii).

eqn (i), (ii) and (iii) is known as normalized eqn on solving which simultaneously we get the value of coefficients a, b, c respectively. Finally, putting these values to (i), we get the required curve of best-fit.

Trick: $y = ax^2 + bx + c$

$x^2y = ax^4 + bx^3 + cx^2$
$\Sigma x^2y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$

$xy = ax^3 + bx^2 + cx$
$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$

$\Sigma y = a\Sigma x^2 + b\Sigma x + nc.$

Q) Fit a straight line to the given data set.

x	1.0	2.0	3.0	4.0
y	6	11	18	27

Solution: let, $y = ax + b$ ---- (1)
be the straight line ~~passing~~ to be fitted over the given data set and corresponding normalized eqn will be,

$$\Sigma xy = a\Sigma x^2 + b\Sigma x \quad --- \text{(i)}$$

$$\Sigma y = a\Sigma x + n \cdot b \quad ---- \text{(ii)}$$

Now, finding all Σ values for the given data set, we have;

x	y	xy	x^2
1	6	6	1
2	11	22	4
3	18	54	9
4	27	108	16
$\Sigma x = 10$	$\Sigma y = 62$	$\Sigma xy = 190$	$\Sigma x^2 = 30$

Substituting the value of Σ to normalized eqn we get,

$$190 = 30a + 10b \quad \text{——— (a)}$$
$$62 = 10a + 4b \quad \text{——— (b)}$$

n data

on solving,

$a = 7$
$b = -2$

$\therefore$ Eqn is $\boxed{y = 7x - 2}$ ans.

which is the required curve of best-fit.

Q) Fit the eqn of parabola to the given data set.

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Solution: let eqn of parabola to be fitted is given as,

$$y = ax^2 + bx + c \quad ----- \text{(A)}$$

and,
corresponding normalized eqn is given as:

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\,\Sigma x^2 \quad \text{—— (a)}$$
$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \quad \text{—— (b)}$$
$$\Sigma y = a\Sigma x^2 + b\,\Sigma x + nc \quad \text{—— (c)}$$

Now, finding all the Σ values for normalized Eqn using given data set, we have,

	x	y	x^2	x^2y	xy	x^3	x^4
	1	0.5	1	0.5	0.5	1	1
	2	2	4	8	4	8	16
	3	4.5	9	40.5	13.5	27	81
	4	8	16	128	32	64	256
	5	12.5	25	312.5	62.5	125	625
Σ=	15	27.5	55	489.5	112.5	225	979

Substituting all Σ values in (a), (b), (c).

$489.5 = 979a + 225b + 55c$ —— (i)
$112.5 = 225a + 55b + 15c$ —— (ii)
$27.5 = 55a + 15b + 5c$ —— (iii)

On solving,

$a = 0.5$
$b = 0$
$c = 0$

putting these variables in (A), we get:

$y = 0.5x^2 + 0 + 0$

$\boxed{\therefore y = 0.5x^2}$ ans.

Fitting special curve

(1) power eqn $y = ax^b$

(i) consider the curve to be fitted is given as,

$$y = ax^b \text{ - - - - - - - - - (1)}$$

taking $\log_{10}$ on both side, we get;

$$\log_{10} y = \log_{10} a + b\log_{10} b \text{ - - - - - - (2)}$$

which is of the form, $Y = A + BX$ - - - (3)

where,

$Y = \log_{10} y$
$A = \log_{10} a$
$B = b\log_{10} b$

and corresponding normalized eqn is given as,

$\Sigma Y = n.A + B\Sigma X$ - - - - - (iv)
$\Sigma XY = A\Sigma X + B\Sigma X^2$ - - - - - (v)

on solving which we get the value of A and B from which we get the value of a & b using relation; $A = \log_{10} a$ and $B = b$. Finally, putting the value of a and b in eqn (1), we get the required curve of best fit.

Date
Page

(ii) Exponential eqn $y = ae^{bx}$.

↳ consider the curve to be fitted is given as,

$$y = ae^{bx} \quad ---- (1)$$

taking $\log_{10} y = \log_{10} a + \cancel{\log_{10} e^{bx}}\ bx \log_{10} e$. — (2)

which is of the form $Y = A + BX$ ---- (3)

wher,

$$Y = \log_{10} y,\ A = \log_{10} a,\ B = b \cdot \log_{10} e,\ X = x$$

and corresponding Normalized eqn is given as,

$$\Sigma Y = n \cdot A + B \Sigma X \quad — (I)$$
$$\Sigma XY = A \Sigma X + B \Sigma X^2 \quad — (II)$$

on solving which, we get the value of A & B from which we get the value of a & b. using relation, $A = \log_{10} a$ and $B = b \log_{10} e$. finally putting value of a & b in eqn (1), we get the required curve of best fit.

Date
Page

(iii) Gas eqn $pv^{\gamma} = k$ ($xy^a = b$).

consider the curve to be fitted in given as, $pv^{\gamma} = k$ --- (1)

taking $\log_{10}$ on both side, we get:

$$\log_{10} p + \gamma \log_{10} V = \log_{10} k$$

$$\Rightarrow \log_{10} V = \frac{\log_{10} k}{\gamma} - 1/\gamma \cdot \log_{10} p \quad — (2)$$

which is of the form, $Y = A + BX$ ----- (3)

wher,

$$Y = \log_{10} V,\ A = \frac{\log_{10} k}{\gamma},\ B = -1/\gamma$$

$$X = \log_{10} P$$

and corresponding normalized eqn is given as,

$$\Sigma Y = nA + B \Sigma X \quad — (I)$$
$$\Sigma XY = A \Sigma X + B \Sigma X^2 \quad — (II)$$

on solving which, we get the value of A & B. from which we get the value of a and b using relation, $A = \frac{\log_{10} P}{\gamma}$, and $B = -1/\gamma$.

finally, putting the value of a and b in eqn (1), we get the required curve of best fit.

Q) Fit the given eqn $y = ax^b$ to the given data set.

x	5	7	10	20	25	30
y	12.6	6.92	4.3	5.26	7.21	3.2

Solution:

The standard equation to be fitted is,

$$y = ax^b \text{ ------ } (1)$$

taking $\log_{10}$ on both sides, we get.

$$\log_{10} y = \log_{10} a + b \log_{10} x \text{ --- } (2)$$

which is of form,

$$Y = A + BX \text{ --- } (3)$$

where,

$$Y = \log_{10} y, \; A = \log_{10} a, \; B = b, \; X = \log_{10} x$$

and corresponding normalized eqn is given by.

$$\Sigma Y = n \cdot A + B \Sigma X \text{ ——— } (\text{i})$$

$$\Sigma XY = A \Sigma X + B \Sigma X^2 \text{ ——— } (\text{ii})$$

Now, finding all Σ values for normalized eqn using table for L.S.M, we get:

X : Y : $A = \log_{10} X$: $B = \log_{10} Y$: AB : A^2

	x	y	$X = \log_{10} x$	$Y = \log_{10} y$	XY	X^2	
	5	12.6	0.698	1.1003	0.7691	0.4885	
	7	6.92	0.8450	0.8401	0.7099	0.7141	
	10	4.31	1	0.6344	0.6344	1	
	20	5.26	1.301	0.7269	0.9380	1.0926	
	25	7.21	1.3979	0.8579	1.199	~~2.181~~	1.951
	30	3.2	1.4771	0.5051	0.7461		2.181
Σ			6.7199	4.6587	4.9965	8.0310	

putting all Σ values to normalized eqn, we get

$$4.6587 = 6A + 6.7199B \text{ —— } (1)$$

$$4.9965 = 6.7199A + 8.6310B \text{ — } (\text{ii})$$

on solving which, we get:

$A = 1.2671$ $\quad$ $B = -0.4381$

$n = 6.$

Now,

finding value of a & b using relation,

$$A = \log_{10} a \Rightarrow a = 10^A$$

$$= 10^{1.2671}$$

$$= 18.49$$

and

$$B = b \Rightarrow b = -0.4381$$

putting these value to (1), we get:

$$\boxed{y = 18.49 \, x^{-0.4381}}$$

which is the required curve of best fitting.

Assignment

Q) Fit the eqn $y = a \cdot e^{bx}$ to the given data set.

Solution:

The standard equation to be fitted is;

$$y = a \cdot e^{bx} \quad \text{------------ ①}$$

taking $\log_{10}$ on both sides, we get;

$$\log_{10} y = \log_{10} a + bx \log_{10} e \quad \text{-------- ②}$$

Which is of form,

$$Y = A + BX \quad \text{------------ ③}$$

Where,

$Y = \log_{10} y$, $A = \log_{10} a$, $B = b$, $X = x\log_{10} e$

And corresponding normalized equation is given by;

$$\Sigma Y = nA + B\Sigma X \quad \text{-------- (i)}$$

$$\Sigma XY = \Sigma X \cdot A + B\,\Sigma X^2 \quad \text{----- (ii)}$$

Now, finding all Σ values for normalized eqn using table for L.S.M, we get;

X: Y: $A = \log_{10} Y$: $B = X \log_{10} e$ AB B^2

x	y	$Y = \log_{10} y$	$X = x\log_{10} e$	XY	X^2
5	12.6	1.0969	2.1714	2.381	4.715
7	6.92	0.8401	3.0400	2.5539	9.2419
10	4.3	0.633	4.342	2.751	18.861
20	5.26	0.720	8.685	6.262	75.449
25	7.21	0.857	10.857	9.314	117.882
30	3.2	0.505	13.028	6.581	169.7505
Σ		4.652	42.123	29.842	395.8935

Now,

putting all Σ values to normalized Eqn, we get.

$$4.652 = 6 \cdot A + B \cdot 42.123 \quad \text{------ ④}$$

$$29.842 = 42.123\,A + 395.893\,B \quad \text{---- ⑤}$$

on solving,

$$A = 0.9727$$
$$B = -0.0281$$

Now, Finding the value of a & b,

$A = \log_{10} a$	$B = b$
or, $0.9727 = \log_{10} a$	$= -0.0281$
or, $a = 10^{0.9727}$	
or, $a = 9.3907$	

So,

eqn ① becomes,

$$\boxed{y = 9.39\, e^{-0.028x}}$$

Which is the required curve of best fitting.

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Chapter-3 Numerical Differentration and Integration.

Numerical Differentiation:

↳ The process of finding higher order derivatives of a discrete function is called Numerical Differentation. The basic idea to find higher order derivatives is to determine an appropriate interpolation function $\phi(x)$ for a discrete function $y = f(x)$. then find its derivative as many times as we require.

① Higher order derivatives using Newton interpolation formula:

i) Using Newton formula interpolation formula:

↳ The Newton forward interpolation formula is given as,

$$y(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \quad \text{--- ①}$$

Differentiating ① w.r.to. p we get,

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 y_0 + \cdots \quad \text{--- ②}$$

we know that,

$$p = \frac{x-x_0}{h} \quad \text{--- ③}$$

Diff. (3) w.r.to. x we get,

$$\frac{dp}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

$$= \frac{1}{h}\left(\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 y_0 + \cdots\right) \quad \text{------ (4)}$$

At $x = x_0$, $p = 0$ then,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \frac{1}{7}\Delta^7 y_0 + \cdots\right) \text{ — (5)}$$

We know that,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right) \times \frac{dp}{dx}$$

$$= \frac{1}{h^2}\left(\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{12p^2-36p+22}{24}\Delta^4 y_0 + \cdots\right) \text{ — (6)}$$

At $x = x_0$, $p = 0$ then

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right) \text{ — (7)}$$

Also,

$$\frac{d^3y}{dx^3} = \frac{d}{dp}\left(\frac{d^2y}{dx^2}\right) \times \frac{dp}{dx}$$

$$\Rightarrow \frac{d^3y}{dx^3} = \frac{1}{h^3}\left(\Delta^3 y_0 + \frac{24p-36}{24}\Delta^4 y_0 + \cdots\right) \text{ — (8)}$$

At; $x = x_0$, $p = 0$ then;

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right) \text{ --- (9)}$$

$\therefore$ eqn (5), (7) and (9) are the formula for higher order derivatives at $x = x_0$ using newton forward formula.

ii) Using Newton backward interpolation formula:

↳ Consider Newton backward interpolation formula given as

$$y = \left(y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \cdots\right) \text{ — (1)}$$

Then it's higher order derivatives are given as,

$$\frac{dy}{dx} = \frac{1}{h}\left(\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\nabla^4 y_n\right) \text{ ----- } (2)$$

at $x = x_n$, $p = 0$ then.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right) \text{ —— } (3)$$

So,

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right) \times \frac{dp}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{h^2}\left(\nabla^2 y_n + \frac{6p+6}{6}\nabla^3 y_n + \frac{12p^2+36p+22}{24}\nabla^4 y_n + \text{----}\right) \text{ —— } (4)$$

At $x = x_n$, $p = 0$, then,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right) \text{ — } (5)$$

Also,

$$\frac{d^3y}{dx^3} = \frac{d}{dp}\left(\frac{d^2y}{dx^2}\right) \times \frac{dp}{dx}$$

$$\Rightarrow \frac{d^3y}{dx^3} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{24p+36p^4 y_n}{24} + \cdots\right) \text{ — } (6)$$

At $x = x_n$, $p = 0$ then,

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right) \text{ — } (7)$$

∴ eqn (3), (5), (6) and (7) are formula for higher order derivatives at $x = x_n$, using Newton backward interpolation formula.

Q) given that:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at

(a) $x = 1.1$

(b) $x = 1.6$

Solution:

The difference table for given data set is obtained as.

x	Y	1st diff.	2nd diff	3rd
1.0	7.989	0.414	−0.036	0.006
$x = x_0 =$ 1.1	8.403	0.378	−0.030	0.004
1.2	8.781	0.348	−0.026	0.003
1.3	9.129	0.322	−0.023	0.005
1.4	9.451	0.299	−0.018	
1.5	9.750	0.281		
1.6	10.031			

4th	5th	6th
−0.002	0.001	0.002
−0.001	0.003	
0.002		

Now,

① at $x = 1.1$,

$x_0 = 1.1$, $y_0 = 8.403$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.030$,

$\Delta^3 y_0 = 0.004$, $\Delta^4 y_0 = -0.001$, $\Delta^5 y_0 = 0.003$, $h = 0.1$

$$\therefore \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \tfrac{1}{2}\Delta^2 y_0 + \tfrac{1}{3}\Delta^3 y_0 - \tfrac{1}{4}\Delta^4 y_0 + \tfrac{1}{5}\Delta^5 y_0\right)$$

$$p = \frac{x - x_0}{h}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{0.1}\left(0.0378 - \tfrac{1}{2}\,0.030 + \tfrac{1}{3}\,0.004 + \tfrac{1}{4}\,0.001 + \tfrac{1}{5}\,0.003\right)$$

$$= 3.951$$

And,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \tfrac{5}{6}\Delta^5 y_0\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.1} = \frac{1}{(0.1)^2}\left(0.030 - 0.004 + \frac{11}{2}(-0.001) - \frac{5}{6}(0.003)\right)$$

$$= 3.74$$

② at $x = 1.6$

$x_n = 1.6$, $y_n = 10.031$, $\Delta y_n = 0.281$, $\Delta^2 y_n = -0.018$,
$\Delta^3 y_n = 0.005$, $\Delta^4 y_n = 0.002$, $\Delta^5 y_n = 0.003$
$\nabla^6 y_n = 0.002$, $h = 0.1$

$$\because \left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \tfrac{1}{2}\nabla^2 y_n + \tfrac{1}{3}\nabla^3 y_n + \tfrac{1}{4}\nabla^4 y_n + \tfrac{1}{5}\nabla^5 y_n + \tfrac{1}{6}\nabla^6 y_n\right)$$

$$\left(\frac{dy}{dx}\right)_{x=1.6} = \frac{1}{0.1}\left(0.281 + \frac{-0.018}{2} + \frac{0.005}{3} + \frac{0.002}{4} + \frac{0.003}{5} + \frac{0.002}{6}\right)$$

$$= 2.75$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + 5/6\,\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.00\ldots) + \frac{137}{180}(0.002)\right)$$

$$= -0.715 \text{ ans.}$$

Numerical Integration:

↳ The process of evaluating a definite integral from a set of tabulated values of the integrated $f(x)$ is called numerical integration. This process when applied to a function of a single variable is called quadrature.

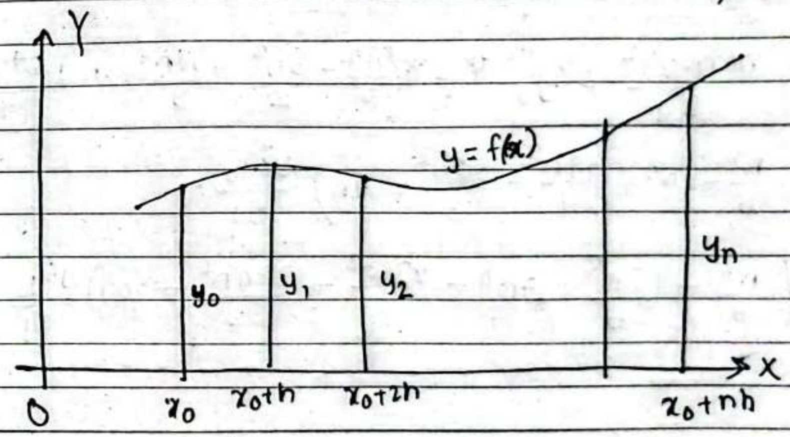
a) Newton-Cotes Quadrature Formula:

Let $I = \int_a^b f(x)\,dx$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots y_n$ for $x = x_0, x_1, x_2, \ldots x_n$.

Let us divide the interval (a,b) into n sub-interva of width h so that,

$x_0 = a$, $x_1 = x_0 + h$; $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$ ---
$x_n = x_0 + nh = b$. Then,



$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = h\int_0^n f(x_0+rh)\,dr,$$

putting $x = x_0 + rh$, $dx = h\,dr$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0\right.$$

$$+\frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0$$

$$+\frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \cdots \Big] dr$$

[by Newton's forward interpolation formula]

Integrating term by term we obtain.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\Big[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0$$

$$+ \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} +$$

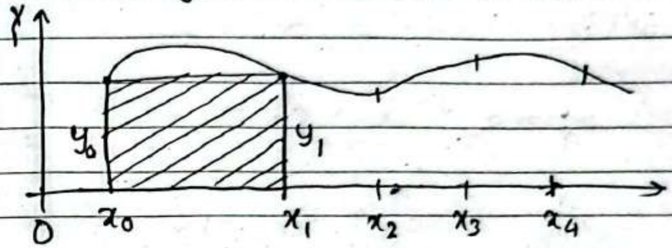
$$\left(\frac{n^5}{6} - 2n^4 + \frac{34n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} +$$

$$\left(\frac{n^6}{7} - \frac{15n^3}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \cdots\Big] \quad \text{--- (1)}$$

This is known as Newton-Cotes quadrature formula. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3$.

① Trapezoidal rule:

Putting $n=1$ in ① and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line. i.e a polynomial of first order so that differences of order higher than first becomes zero, we get:



$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \tfrac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Similarly,

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1 + \tfrac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

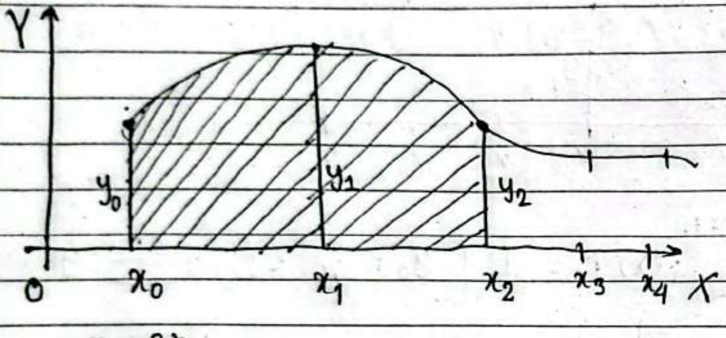
Adding these integrals, we obtain:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right] \quad \text{--- (2)}$$

This is known as the trapezoidal rule.

(ii) Simpson's one-third rule:

↳ putting $n=2$ in (1) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola i.e a polynomial of the second order so that differences of order higher than the second vanish, we get:



$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Similarly, $$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

- - - - - - - - - - - - - - - -

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$, n being even

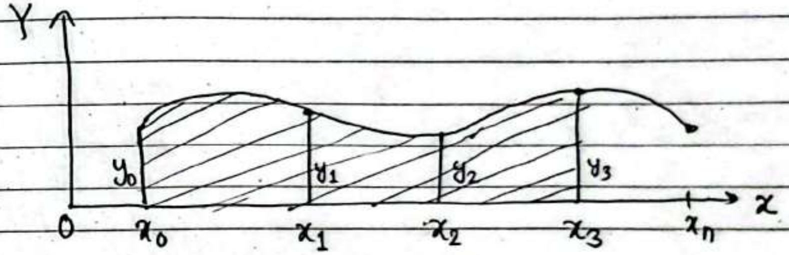
Adding all these integrals, we have when n is even.

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\right]$$

This is known as Simpson's one-third rule or simply Simpson's rule and is most commonly used.

(iii) Simpson's three-eighth rule.

↳ putting $n=3$ in (1) above and taking the curve through (x_i, y_i) : $i = 0, 1, 2, 3$ as a polynomial of the third order so that differences above third order vanish, we get.



$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$ and so on..

Adding all such expressions from x_0 to x_0+nh, where n is a multiple of 3. we obtain,

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})\right] \quad \text{—— (4)}$$

Date
Page

Tips:

for traperzoidal rule $n>=2$
for simpson's $1/3^{rd}$ rule $n\%2==0$
for simpson's $3/8^{th}$ rule $n\%3==0$

for all these we can take $n=6$.

Date
Page

Q) Integrate, $I = \int_0^6 \frac{1}{1+x^2}\,dx$ · Using :

a) Trapezoidal rule
b) simpson's 1/3
c) simpson's 3/8

solution:

Given;

$$I = \int_0^6 \frac{1}{1+x^2}\,dx$$

Here,

$a=0,\ b=6,\ f(x) = 1/1+x^2$

Taking $n=6$, $h=(b-a)/n = 6/6 = 1$

Now, The discrete value for the function is given as,

mod 7, start, end, h=steps

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0384	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, Applying trapezoidal rule;

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027) + 2(0.5+0.2+0.1+0.0588+0.0384)\right]$$

$$= 1.4107$$

Applying simpson's 1/3 rule,

$$I = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027) + 4(0.5+0.1+0.0384) + 2(0.2+0.\right.$$

$$= 1.36667$$

Applying simpson's 3/8th rule,

$$I = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2*y_3\right]$$

$$= \frac{3}{8}\left[(1+0.027) + 3(0.5+0.2+0.0588+0.0384) + 2*0.1\right]$$

$$= 1.35697.$$

Gaussian Integration:

↳ Gaussian integration yields better accuracy over the cote's quadrature for[redacted] same number of n but with different spacing.

↳ Gauss integration formula for a discrete function,

$$I = \int_{-1}^{1} f(x)\,dx \text{ is given as:}$$

$$\boxed{I = \int_{-1}^{+1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)} \quad (1)$$

where w_i & x_i are the values of for weights and abscissae of the given function respectively. The values for weights and abscissae for different value of n is listed below:-

x-axis

n	w_i	x_i
2	1	$-1/\sqrt{3}$
	1	$1/\sqrt{3}$
3	5/9	$-\sqrt{3/5}$
	8/9	0
	5/9	$\sqrt{3/5}$
4 (nasodhia)	$\frac{18-\sqrt{30}}{36}$	$-\sqrt{(15+2\sqrt{30})/35}$
	$\frac{18+\sqrt{30}}{36}$	$-\sqrt{(15-2\sqrt{30})/35}$

$\frac{18+\sqrt{30}}{36}$	$\sqrt{(15-2\sqrt{30})/35}$
$\frac{18-\sqrt{30}}{36}$	$\sqrt{(15-2\sqrt{30})/35}$

[table: Gauss legendre nodes and coefficient table]

↳ Gauss integration have the limitation that it should only be applicable within the range of [-1,1].

Hence, to integrate a function within the range [a,b] we have to change the range of limit to [-1,1] using transformation.

$$\boxed{x = \frac{(b-a)}{2}u + \frac{(b+a)}{2}}$$

and then gaussian integration is being applied over it.

Q) Integrate the given function $I = \int_{-1}^{1} \frac{1}{1+x^2} dx$ using Gauss-legendre formula for.

(i) $n=2$

(ii) $n=3$

Solution:

$$I = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{1}{1+x^2}\,dx.$$

so,

$$f(x) = \frac{1}{1+x^2}$$

Then, Gaussian integration formula is given as,

$$I = \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} \omega_i f(x_i)$$

(i) For $n=2$,

$$I = \sum_{i=1}^{2} \omega_i f(x_i)$$

$$= \omega_1 f(x_1) + \omega_2 f(x_2)$$

where, $\omega_1 = 1$, $\omega_2 = 1$, $x_1 = -1/\sqrt{3}$, $x_2 = 1/\sqrt{3}$

then,

$$I = 1 \times f(-1/\sqrt{3}) + 1 \times f(1/\sqrt{3})$$

$$= \frac{1}{1+(-1/\sqrt{3})^2} + \frac{1}{1+(1/\sqrt{3})^2}$$

$$= \frac{3}{3+1} + \frac{3}{3+1}$$

$$= 6/4$$

$$= 3/2$$

$$= 1.5$$

Page

(i) for n=3,

$$I = \sum_{i=1}^{3} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

Here,

$w1 = 5/9 \quad w2 = 8/9 \quad w3 = 5/9$

$x1 = -\sqrt{3/5} \quad x2 = 0 \quad x3 = \sqrt{3/5}$

So,

$$I = \frac{5}{9} \times \frac{1}{1+3/5} + \frac{5}{9} \times \frac{1}{1+3/5} + \frac{8}{9} \times \frac{1}{1+0}$$

$$= 0.6944 + 8/9$$

$$= 1.583$$

Q) Integrate the given function $I = \int_{0.2}^{1.5} e^{-x^2} dx$ using Gauss-legendre formula for:

(i) n=2

(ii) n=3.

Solution:

$$I = \int_a^b f(x)\,dx = \int_{0.2}^{1.5} e^{-x^2} dx$$

where,

$f(x) = e^{-x^2}$, $a = 0.2$, $b = 1.5$.

Now, changing the limit $[a,b]$ to $[-1,1]$ using transformation.

Date

Page

$$x = \left(\frac{b-a}{2}\right)u + \frac{b+a}{2}$$

$$\Rightarrow x = \left(\frac{1.5-0.2}{2}\right)u + \frac{1.5+0.2}{2}$$

$$\Rightarrow x = \frac{1.3u + 1.7}{2}$$

i.e. $\frac{dx}{du} = 0.65$

$$\Rightarrow dx = 0.65\,du.$$

So,

$$I = \int_{0.2}^{1} e^{-x^2} dx = \int_{0.2}^{1} e^{-(0.65u+0.85)^2} \times 0.65\,du$$

where, $f(u) = 0.65 \times e^{-(0.65u+0.85)^2}$

Now, applying gauss legendre formula, we get,

$$I = \int_{-1}^{1} f(u)\,du = \sum_{i=1}^{n} w_i f(u_i)$$

(a) For n=2,

$$I = w_1 f(u_1) + w_2 f(u_2)$$

$w_1 = 1$, $w_2 = 1$, $u_1 = -1/\sqrt{3}$, $u_2 = 1/\sqrt{3}$

finally,

$$I = 1 \times f(-1/\sqrt{3}) + 1 \times f(1/\sqrt{3})$$

$$= 0.65\, e^{-(0.65 \times -1/\sqrt{3} + 0.85)^2} + 0.65\, e^{-(0.65 \times 1/\sqrt{3} + 0.85)^2}$$

$$= 0.6636$$

(b) For n=3.

$$I = w_1 f(u_1) + w_2 f(u_2) + w_3 f(u_3).$$

$w_1 = 5/9$ $\quad w_2 = 8/9$ $\quad w_3 = 5/9$

$u_1 = -\sqrt{3/5}$ $\quad u_2 = 0$ $\quad u_3 = \sqrt{3/5}$

finally,

$$I = 5/9\, f(-\sqrt{3/5}) + 8/9\, f(0) + 5/9\, f(\sqrt{3/5})$$

$$= \frac{5}{9} \times 0.65\, e^{-(0.65 \times -\sqrt{3/5} + 0.85)^2} + 8/9 \times 0.65\, e^{-(0.85)^2} +$$

$$\frac{5}{9} \times 0.65\, e^{-(0.65 \times \sqrt{3/5} + 0.85)^2}$$

$$= \cancel{0.6628} \quad 0.65859$$

$\frac{b-a}{n}$; $\frac{2}{6} = 0.33$

(Q) Integrate $I = \int_0^2 \frac{1}{1+x}\, dx$ using,

(a) Trapezoidal rule

(b) Simpson's rule

(c) Simpson's 3/8 rule

Also, integrate using Gauss integration formula for

(i) n = 2

(ii) n = 3

solution; Given,

$$I = \int_a^b f(x)\, dx = \int_0^2 \frac{1}{1+x}\, dx.$$

where,

$a = 0 \qquad b = 2$

taking $n = 6$,

$$h = \frac{b-a}{n} = 2/6 = 0.333$$

then,

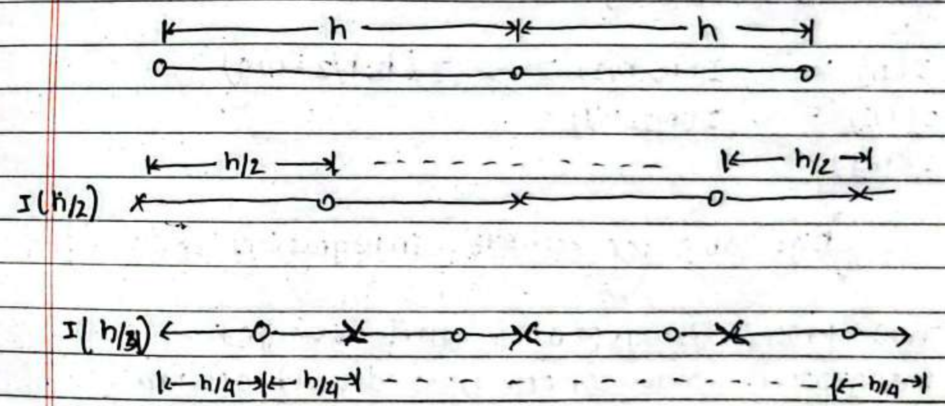
x	0	2/6	0.666	0.999	1.333	1.666	~~1.99~~ 2
y	1	0.7501	0.6682	0.5002	0.4288	0.3752	0.3333
	y_0	y_1	y_2	y_3	y_4	y_5	y_6.

Page

Date
Page

Romberg's integration:

It is the improvement of trapezoidal rule. firstly, Integration of $I = \int_a^b f(x)\,dx$ is obtained by taking $n=2$ using trapezoidal rule as $I(h)$. Then, further integration values ~~of~~ for given integration is obtained by having the ~~internal~~ interval as $\frac{h}{2}, \frac{h}{4}, \frac{h}{8}, \frac{h}{16}, \dots$ using trapezoidal rule given as $I(h/2), I(h/4) \dots$ successively as follows:-



[Fig: demonstration of Romberg's integration Method.]

Where, $O \longrightarrow$ New sample point

$X \longrightarrow$ Old sample point

Then the new improved value by Romberg's integration formula is given by.

$$\boxed{I(h, h/2) = \frac{4 * I(h/2) - I(h)}{3}}$$

i.e. $\boxed{I' = \frac{4B - A}{3}}$

↳ The successive Romberg's integration value is obtained as below,

$$\begin{array}{ccccc} I(h) & \searrow & & & \\ & & I(h, h/2) & \searrow & \\ I(h/2) & \nearrow\searrow & & & I(h, h/2, h/4) \\ & & I(h/2, h/4) & \nearrow & \\ I(h/4) & \nearrow & & & \end{array}$$

[Fig: Table for Romberg's integration method].

↳ The process is repeated until we get the integration value correct upto desired accuracy.

Q) Use Romberg's integration to find the solution correct upto 3 decimal place.

$$I = \int_0^1 \frac{1}{1+x^2} dx$$

Given,

$$I = \int_a^b f(x)\,dx = \int_0^1 \frac{1}{1+x^2} dx$$

where,

$a = 0$, $f(x) = 1/1+x^2$

$b = 1.$

Now, taking $n = 2$ (Always).

$$h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$$

then,

x	0	0.5	1
y	1	0.8	0.5

y_0 y_1 y_2

Applying trapezoidal rule,

$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_0 + y_2 + 2 \cdot y_1\right]$$

$$\therefore I(h) = \frac{0.5}{2}\left[(1 + 0.5) + 2 \times 0.8\right]$$

$$= 0.775$$

Again, taking $h = \frac{h}{2} = 0.25$, then,

x	0	0.25	0.5	0.75	1
y	1	0.941	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

Applying trapezoidal rule, we get:

$$I = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{0.25}{2}\left[(1+0.5) + 2(0.941+0.8+0.64)\right]$$

$$I\left(\frac{h}{2}\right) =$$

Similarly,

taking $h = h/4 = \frac{0.5}{4} = 0.125$, then;

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.941	0.8767	0.8	0.7191	0.64	0.5663	0.5

Applying trapezoidal rule,

$$I = \frac{h}{2}\left[(y_0+y_8) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \frac{0.125}{2}\left[(1+0.5) + 2(0.9846+0.941+0.876+0.8+0.7191+0.64+0.5663)\right]$$

$$I\left(\frac{h}{4}\right) = 0.7847125$$

Now, Applying Romberg's integration formula, we get,

$$I(h, h/2) = \frac{4\,I(h/2) - I(h)}{3}$$

i.e. $I' = \frac{4B - A}{3}$

$I(h) = 0.775$

$I(h, h/2) = 0.785$

$I(h/2) = 0.7825$

$I(h, h/2, h/4) = 0.7856$

$I(h/2, h/4) = 0.78545$

$I(h/4) = 0.7847125$

$$\therefore I = \int_0^1 \frac{1}{1+x^2}\,dx = 0.7856$$

correct upto 3 decimal place.

2022

(Q) Use romberg's integration method, to compute $\int_0^2 \frac{e^x + \sin x}{1+x^2}\,dx$ correct upto 3 decimal place.

Solution:

$$I = \int_a^b f(x)\,dx = \int_0^2 \frac{e^x + \sin x}{1+x^2}\,dx$$

Where,

$a = 0$, $b = 2$, $f(x) = \frac{e^x + \sin x}{1+x^2}$

Now taking, $n = 2$

$$h = \frac{b-a}{n} = \frac{2-0}{2} = 1$$

then,

x	0	1	2
y	1	1.7799	1.6596
	y_0	y_1	y_2

Applying trapezoidal rule,

$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_0 + y_2 + 2y_1\right]$$

$$= \frac{1}{2}\left[1 + 1.7799 + 1.6596\right]$$

$$I(h) = 2.21975$$

Again taking $h = h/2 = 0.5$ then.

x	0	0.5	1	1.5	2.0
y	1	1.7025	1.7798	1.6859	1.6596
	y_0	y_1	y_2	y_3	y_4

Applying trapezoidal rule,

$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.5}{2}\left[(1 + 1.6596) + 2(1.7025 + 1.7798 + 1.6859)\right]$$

$$I(h/2) = 3.249$$

Again taking $h = h/4 = 0.125$ then.

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875
y	1	1.2384	1.4413	1.5967	1.7025	1.7642	1.7911	1.7933
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
1.7798	1.7577	1.7324	1.7075	1.6859	1.6692	1.6587	1.6553	1.65
y_8	y_9	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}

Applying trapezoidal rule,

$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_{16}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14} + y_{15})\right]$$

$$= \frac{0.125}{2}\left[(1 + 1.6596) + 2(1.2384 + 1.4413 + 1.5967 + 1.7642 + 1.7911 + 1.7933 + 1.7798 + 1.7577 + 1.7324 + 1.7075 + 1.6859 + 1.6692 + 1.6587 + 1.6553)\right]$$

$$= 2.2409 + 1.254525$$

$$I(h/4) = 3.495425$$

Now, Applying Romberg's integration formula, we get,

$$I(h, h/2) = \frac{4\,I(h/2) - I(h)}{3} = \frac{4B - A}{3}$$

$I(h) = 2.21975$

$I(h/2) = 3.249$

$I(h/4) = 3.495425$

$I(h), I(h/2) \to 2.21975$

$I(h/2), I(h/4) \to 3.5775$

$2.21975, 3.5775 \to 4.030172222$

Solution to the System of linear algebraic Equation.

System of linear algebric equation c

① Gauss Elimination method:

↳ Consider the system of linear eqn:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad -(1)$$

↳ Then its solution is obtained through 2 step processing:

ⓐ Forward Elimination:-

↳ In this step the unknown variables are eliminated from their successive equation to reduce the system of eqn in the form of upper triangular matrix as follows:-

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \quad -(2)$$

The coefficient with respect to which unknown variable in successive lower equations are eliminated is called pivot point and corresponding equation is called pivot equation.

↳ For convergence of the method, the eqn having largest coefficient for the first variable is taken as first pivot equation and the largest coefficient for second variable leaving first equation is choosen as second pivot eqn and so on.
The reducing the system of linear eqn to upper triangular matrix is called partial pivoting. while rearranging these equation after each step of elimination to reduce the system of eqn to upper triangular matrix is called full pivoting.

② Backward Substitution:

↳ From the reduced system of matrix value for unknown variables are determined from back side as given below:

~~From the reduced system of~~

From (iii) of (1),

$$z = \frac{d_3''}{c_3''}$$

Substituting value of z to (ii) of (2), we get,

$$y = \frac{1}{b_2'}(d_2' - c_2'z)$$

and putting value of y & z to (i) of 2

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

∴ The solution is obtained by using Gauss elimination Method.

(2) Gauss- Jordan Method :

↳ It is an improvement of Gauss elimination method, as it reduces the system of eqn to the diagonal matrix and then values for unknown are determined radily from reduced system of eqn. Thus it removes overhead of backward substitution at the cost of additional computation.

Q) Solve the system of linear eqn using gauss elimination method.

$$20x+y-2z=17$$
$$2x-3y+20z=25$$
$$3x+20y-z=-18$$

Solution: By re-arranging given system of eqn, we get;

$$20x+y-2z=17$$
$$3x+20y-z=-18$$
$$2x-3y+20z=25$$

Now, reducing the given system of eqn to upper triangular matrix, we get;

Applying, $R_2 \leftarrow 20R_2-3R_1$, $R_3 \leftarrow 20R_3-2R_1$.

$$20x+y-2z=17$$
$$0+397y-14z=-411$$
$$0-62y+404z=466$$

Applying $R_3 \leftarrow 397R_3+62R_2$,

$$20x+y-2z=17 \quad \text{——— (i)}$$
$$0+397y-14z=-411 \quad \text{——— (ii)}$$
$$0+0+159520z=159520 \quad \text{——— (iii)}$$

Now, finding the value of unknown variables using backward substitution we get.

from (iii), $z=1$,

putting value of z in (ii),

$$y=\frac{1}{397}(-411+14)$$
$$=\frac{-397}{397}=-1$$

Substituting value of z and y in (i).

$$x=\frac{1}{20}(17+1+2)$$
$$=1$$

$\therefore$ $x=1$, $y=-1$, $z=1$ } ans #.

Factorization Method / L-U Decomposition Method.

↳ This method is based on the fact that every square matrix can be decomposed into lower & upper triangular matrix provided that it's principal minors are singular matrix.

i.e. $|a_{11}| \neq 0$; $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$;

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0. \text{ etc.}$$

↳ Consider the given system of equation,

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad \text{—— (1)}$$

which can be written as . $A * X = B$ —— (2).

where,

$$A = \begin{bmatrix} a_{11} & a_{13} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Now, the square matrix A is decomposed into lower & upper triangular matrix using any of following method.

(I) DO-Little method:-

↳ In this method A is factorized into L and U where lower triangular matrix have unit diagonal value.

i.e.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{21} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(II) Crout's decomposition:

↳ In this method square matrix A is factorized in L & U where upper triangular matrix having unit diagonal value,

i.e.

$$A = L \cdot U = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(III) Cholesky decomposition:

↳ This method of factorization is only applicable for symmetric matrix given as.

$$A = L \cdot L^T = U^T \cdot U.$$

where,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \; \& \; U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now, factorizing A into L & U, we have,

$$A = LU \text{ ——— } (3)$$

from (2) & (3), we get.

$$L \cdot U \cdot X = B \text{ ——— } (4)$$

putting,

$U \cdot X = V$ ——— (5), eqn (4) becomes,

$$L \cdot V = B \text{ ——— } (6)$$

From which, we get the value of V putting which in eqn (5) we determine value of unknown variable X by using backward substitution.

Q) Solve the given system of eqn using factorization metho

$$x + y + z = 4 \text{ ——— } (1)$$
$$x + 4y + 3z = 8 \text{ ——— } (2)$$
$$x + 6y + 2z = 6 \text{ ——— } (2)$$

Solution:

Writing the given system of eqn in terms of,

$A \cdot X = B$ ——— (1), we get.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 2 \\ 1 & 4 & 3 \end{bmatrix} ; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \; B = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

Now, factorizing A into lower & upper triangular matrix using DO-little method of decomposition, we ge

$$A = L \cdot U.$$

i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

or, Multiplying L & U and equating with elements of A, we gets

(i) $u_{11} = a_{11} = 1$, $u_{12} = a_{12} = 1$, $u_{13} = a_{13} = 1$.

(ii) $l_{21} \cdot u_{11} = a_{21}$

$l_{21} = \frac{a_{21}}{u_{11}} = 1/1 = 1$

$l_{31} \cdot u_{11} = a_{31}$

$l_{31} = \frac{a_{31}}{u_{11}} = 1/1 = 1$.

(iii) $l_{21} \cdot u_{12} + u_{22} = a_{22}$

$u_{22} = a_{22} - l_{21} \cdot u_{12}$

$= 6 - 1*1$

$= 5$

$l_{21} \cdot u_{13} + u_{23} = a_{23}$

$u_{23} = a_{23} - l_{21} \cdot u_{13}$

$= 2 - 1\times1$

$= 1$

(iv) $l_{31} \cdot u_{12} + l_{32} \cdot u_{22} = a_{32}$

$\Rightarrow l_{32} = \frac{1}{u_{22}}\left[a_{32} - l_{31} \cdot u_{12}\right]$

$= \frac{1}{5}\left[4 - 1\times1\right] = 3/5$

(v) $l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + u_{33} = a_{33}$

$u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23}$

$= 3 - 1\cdot1 - 3/5 \cdot 1$

$= 2 - 3/5$

$= 7/5$

3.8 cm

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3/5 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 7/5 \end{bmatrix}$

$\therefore A = LU$ —— (2)

from (2) & (3), $L \cdot \underbrace{U \cdot X}_{V} = B$ —— (4)

putting, $U \times X = V$ —— (5)

we get,

$L \cdot V = B$ —— (6)

from eqn (6), we get;

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3/5 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

on solving which, we get.

$\therefore V_1 = 4$

& $V_1 + V_2 = 6$

$\therefore V_2 = 2$.

Also,

$$V_1 + \frac{3}{5}V_2 + V_3 = 8$$

or, $V_3 = 8 - 4 - \frac{3}{5} \times 2$

$$\therefore V_3 = \frac{14}{6}$$

$$\therefore V = \begin{bmatrix} 4 \\ 2 \\ 14/5 \end{bmatrix}$$

Substituting V in (5) and solving by backward substitution

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 7/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 14/5 \end{bmatrix}$$

From (3),

$$\frac{7}{5}z = \frac{14}{5}$$

$$\therefore z = 2.$$

putting value at z in (ii),

$$5y + z = 2$$

$$\Rightarrow y = \frac{2-2}{5} = 0$$

putting value of y & z in (i),

$$x + y + z = 4$$

$$\therefore x = 4 - 0 - 2$$

$$= 2$$

$\therefore x = 2$
$y = 0$
$z = 2.$

(6) Factorizing A into L & U by using crout's method we get,

$$A = L \cdot U \quad \text{—— (2)}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

Multiplying L & U and equating with elements of A, w

Date
Page

(i) $l_{11} = a_{11} = 1, \quad l_{21} = a_{21} = 1, \quad l_{31} = a_{31} = 1$

(ii) $l_{11} \cdot u_{12} = a_{12}$

$u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{1} = 1$

$l_{11} \cdot u_{13} = a_{13}$

$u_{13} = \frac{a_{13}}{a_{11}} = 1$

(iii) $l_{21} \cdot u_{12} + l_{22} = a_{22}$

$l_{22} = a_{22} - l_{21} \cdot u_{12}$
$= 6 - 1 \times 1$
$= 5$

$l_{31} \cdot u_{12} + l_{32} = a_{32}$

$l_{32} = a_{32} - l_{31} \cdot u_{12}$
$= 4 - 1 \times 1$
$= 3$

(iv) $l_{21} \cdot u_{13} + l_{22} \cdot u_{23} = a_{23}$

$\Rightarrow u_{23} = \frac{a_{23} - l_{21} \cdot u_{13}}{l_{22}}$

$= \frac{1}{5}[2 - 1 \cdot 1]$

$= 1/5$

(v) $l_{31} \cdot u_{13} + l_{21} \cdot u_{23} + l_{33} = a_{33}$

$l_{33} = a_{33} - l_{31} u_{13} - l_{21} u_{23}$
$= 3 - 1 \cdot 1 - 1/5 \cdot 3$

$= 7/5$

Date
Page

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 3 & 7/5 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & 1 \end{bmatrix}$$

~~A = LU~~

Now, we know that, $AX = B$ & $A = LU$

So, $L \cdot U \cdot X = B$ —— (3)

So,
putting $U \cdot X = V$ —— (4),

Then (3) becomes,

$$L \cdot V = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 3 & 7/5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

$$\left.\begin{aligned} V_1 &= 4 \\ V_1 + 5V_2 &= 6 \\ V_1 + 3V_2 + \tfrac{7}{5} V_3 &= 8 \end{aligned}\right\}$$

So,
$V_1 = 4, \quad V_2 = 2/5, \quad V_3 = 2.$

$$V = \begin{bmatrix} 4 \\ 2/5 \\ 2 \end{bmatrix}$$

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Substituting (V) in (4)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2/5 \\ 2 \end{bmatrix}$$

$x+y+z=4$ —— (a)

$y+\frac{z}{5}=2/5$ —— (b)

$z=2$ —— (c)

from (c),

$z=2$

from (b),

$y=\frac{2}{5}-2/5=0$

from (a), $x=2$

$\therefore X = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ ans.

Q) Decompose the given matrix by using Cholesky decomposition method.

So $A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$

Solution.

Decomposing given matrix by using cholesky decomposition we have,

$$A = L\cdot L^T = U^T\cdot U$$

i.e. $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$= \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$

Now, multiplying L & L^T and equating with elements of A, we get:

(i) $l_{11}^2 = a_{11}$
$\Rightarrow l_{11} = \sqrt{a_{11}}$
$= \sqrt{4} = 2$

$l_{21}\cdot l_{11} = a_{21}$
$l_{21} = \frac{a_{21}}{l_{11}}$
$= 12/2$
$= 6$

$l_{31}\cdot l_{11} = a_{31}$
$l_{31} = \frac{a_{31}}{l_{11}}$
$= -16/2$
$= -8$

(ii) $l_{21}^2 + l_{22}^2 = a_{22}$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{37-36} = 1$$

$l_{31} l_{21} + l_{32} l_{22} = a_{32}$.

$$l_{32} = \frac{1}{l_{22}}\left(a_{23} - l_{31} l_{21}\right) = \frac{1}{1}\left(-43 - (-8)\cdot 6\right) = 5$$

(iii) $l_{31}^2 + l_{32}^2 + l_{33}^2 = a_{33}$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{98-64-25} = \sqrt{9} = 3$$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}, \quad \& \quad L^T = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

so.

Iterative method:

① Jacobi-iteration method:

↳ Consider the given system of eqn;

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\} \quad \text{—— ①}$$

Assuming a_1, b_2 and c_3 is largest coefficients among all respective variable then, the residual system of eqn is obtained as,

$$\left.\begin{aligned} x &= \frac{1}{a_1}\left(d_1 - b_1 y - c_1 z\right) \\ y &= \frac{1}{a_2}\left(d_2 - b_2 y - c_2 z\right) \\ z &= \frac{1}{a_3}\left(d_3 - b_3 y - z\right) \end{aligned}\right\} \quad \text{—— ②}$$

$$x = \frac{1}{a_1}\left(d_1 - b_1 y - c_1 z\right)$$

$$y = \frac{1}{b_2}\left(d_2 - a_1 x - c_2 z\right)$$

$$z = \frac{1}{c_3}\left(d_3 - a_3 x - b_3 y\right)$$

Now, starting with $x = x_0$, $y = y_0$ & $z = z_0$ then, putting these values to eqn ②, we get new value of x, y, z computed as;

$$\left.\begin{aligned} x_1 &= \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0) \\ y_1 &= \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0) \\ z_1 &= \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0) \end{aligned}\right\} \text{———} ③$$

Again, putting the value of x_1, y_1 & z_1 to 2nd eqn, we get,

$$\left.\begin{aligned} x_2 &= \frac{1}{a_1}(d_1 - b_2y_1 - c_2z_1) \\ y_2 &= \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1) \\ z_2 &= \frac{1}{c_3}(d_3 - a_3x_1 - c_3y_1) \end{aligned}\right\} \text{———} ④$$

This process is repeat until we get the value of x, y, z correct upto desired accuracy i.e $|x^n - x^{n-1}| \leq \varepsilon$. Hence the process is called jacobi iteration method.

Q) Solve the given system of eqn using Jacobi iteration method:-

$$2x_1 + x_2 + x_3 = 5$$
$$3x_1 + 5x_2 + 2x_3 = 15$$
$$2x_1 + x_2 + 4x_3 = 8$$

Solution: Writing the given system of eqn in terms of x_1, x_2 and x_3, we get,

$$\left.\begin{aligned} x_1 &= \tfrac{1}{2}(5 - x_2 - x_3) \\ x_2 &= \tfrac{1}{5}(15 - 3x_1 - 2x_3) \\ x_3 &= \tfrac{1}{4}(8 - 2x_1 + x_2) \end{aligned}\right\} \text{———} ①$$

Now,
Assuming $x_1 = x_2 = x_3 = 0$ as initial values of variables and solving the system of eqn by Jacobi-iteration method, we get.

no. of iteration	$x_1 = \frac{1}{2}(5-x_2-x_3)$	$x_2 = \frac{1}{5}(15-3x_1-2x_3)$	$x_3 = 1/4(8-2x_1+x_2)$
0	0	0	0
1	2.5	3	2
2	0	0.7	0
3.	2.15	3	1.825
4.	0.0875	0.98	0.175
5.	1.9225	2.8775	1.71125
6.	0.2056	1.162	0.3193

7.	1.7593	2.7488	1.6067 ~~0.9193~~
8.	0.3225	1.3017	~~1.6067~~ 0.4331
9.	1.6325	2.6339	1.5134
10.	0.4625	1.4151	0.5297
11.	1.5257	2.533	1.4329
12.	0.5165	1.509	0.601
13.	1.444	2.4494	1.3644
14.	0.5930	1.5874	0.6653
15.	1.3743	2.3787	1.3066.

Q) Solve the given system of eqn by Jacobi-iteration method.

$20x + y - 2z = 17$

$2x - 3y + 20z = 25$

$3x + 20y - z = 18.$

Now, writing the given system of eqn in terms of x, y, z, we have;

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{-1}{3}(25 - 2x - 20z) \\ z &= 3x + 20y - 18 \end{aligned}\right\} \quad \text{--- (1)}$$

Now, starting with $x = y = z = 0$, and solving by using jacobi-iteration method, we get;

no. of iteration	$x = \frac{1}{20}(17 - y + 2z)$	$y = \frac{-1}{3}(25 - 2x - 20z)$	$z = 3x + 20y - 1$
0	0	0	0
1	0.85	-0.9	1.25
2	1.02	-0.965	1.03
3.	1.00125	~~[illegible]~~	
~~4~~	~~1.0042~~	-1.0015	1.00325
4	1.0042	-1.014	0.9897
5	0.9997	-1.0005	0.9977
6.	0.9997	-1.00067	0.9999.

At 6th iteration, $|x^6 - x^5| \approx 0.$

$\therefore x = 0.9997 \approx 1$

$y = -1.00067 \approx -1$

$z = 0.9999 \approx 1.$

Gauss Seidal method:

↳ It is the improvement over Jacobi iteration method such that as soon as new value for unknown variable is calculated it is used immediately to next equation same iteration to get improved result. Thus this method is faster than previous one.

↳ Consider the given system of eqn,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{———} (1)$$

Assuming a_1, b_2 & c_3 being largest coefficient values among all consecutive variables, then writing these eqn in terms of x, y, z, we get;

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_1}(d_2 - a_2x - c_2z) \\ z &= 1/c_3(d_3 - a_2x - b_3y) \end{aligned}\right\} \text{———} (2)$$

Now, putting $y = y_0$ & $z = z_0$ in 1^{st} of eqn (2), we get.

$$x_1 = 1/a_1 (d_1 - b_1y_0 - c_1z_0)$$

and putting $x = x_1$, $z = z_0$ in 2^{nd} of (2), we get

$$y_1 = 1/b_2 (d_2 - a_2x_1 - c_2z_0)$$

Also, putting $x = x_1$, $y = y_1$ in 3^{rd} of (2), we get:

$$z_1 = 1/c_3 (d_3 - a_3x_1 - b_3y_1)$$

i.e. as soon as new value are found it is immediat used in next step of same iteration.

↳ This process is repeated until we get the values fo all variable correct upto desired accuracy.

i.e. $|x^{(n)} - x^{(n-1)}| \approx 0$.

Q) Solve the given system of eqn using Gauss seidal method:-

$$2x_1 + x_2 + x_3 = 5$$
$$3x_1 + 5x_2 + 2x_3 = 15$$
$$2x_1 + x_2 + 4x_3 = 8$$

Solution:

Writing the given system of eqn in terms of x, y, z we get:

$$x_1 = 1/2 (5 - x_2 - x_3)$$
$$x_2 = 1/5 (15 - 3x_1 - 2x_3)$$
$$x_3 = 1/4 (8 - 2x_1 - x_2)$$

Starting with $x_1 = x_2 = x_3 = 0$ and solving by using Gauss Seidal method, we get:

no. of iteration	$x_1 = \frac{1}{2}(5 - x_2 - x_3)$	$x_2 = \frac{1}{5}(15 - 3x_1 - 2x_3)$	$x_3 = \frac{1}{4}(8 - 2x_1 - x_2)$
0	0	0	0
1	2.5	1.5	0.375
2	1.5625	1.9125	0.7400
3	1.17343	1.9996	0.9133
4	1.0434	2.0085	0.97611
5	1.0076	2.0049	0.9949
6	1.00005	2.0019	0.9994

At 6th iteration $|x^{(6)} - x^{(5)}| \approx 0$

$$\left.\begin{aligned} \therefore\ x_1 &= 1.00005 \approx 1 \\ x_2 &= 2.0019 \approx 2 \\ x_3 &= 0.9994 \approx 1 \end{aligned}\right\} \text{ans.}$$

Eigen value and Eigen vector

Let A be a square matrix of order $n \times n$ & non-zero vector V is an eigen vector of A if:

$$AV = \lambda V$$

Where,

$\lambda \in R$ is the eigen value,

and $V \in R^n$, $V \neq 0$ is the eigen vector.

For largest eigen value, we have

$$\boxed{A * X = \lambda * X} \quad \text{——— (1)}$$

Where,

$A \longrightarrow$ given matrix

$\lambda \longrightarrow$ Eigen value

$X \longrightarrow$ corresponding eigen vector

Now, multiplying (1) by A^{-1} on both side, we get;

$$A \cdot A^{-1} \cdot X = \lambda \cdot A^{-1} \cdot X$$

$$\Rightarrow \boxed{A^{-1} \cdot X = \frac{1}{\lambda} \cdot X} \quad \text{——— (2)}$$

Where,

$A \longrightarrow$ given matrix, $A^{-1} \longrightarrow$ increase of A.

$\frac{1}{\lambda} \longrightarrow$ smallest eigen value & $X \longrightarrow$ corresponding eigen vector.

Q) Find the largest eigon value & corresponding vector using power method for the given matrix.

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}$$

Solution:

Given,

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix}$$

let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial eigen vector. ~~be~~ 1

Then, finding the largest value using power method iteratively, we have,

$$A * X = \lambda * X$$

Iteration 1,

$$A * x^{(0)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 1 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \underset{\lambda^{(1)}}{\overset{\text{highest}}{5}} \underset{x^{(1)}}{\begin{bmatrix} 0.4 \\ 1 \\ 0.2 \end{bmatrix}}$$

$\therefore \lambda^{(1)} = 5$

$\therefore x^{(1)} = \begin{bmatrix} 0.4 \\ 1 \\ 0.2 \end{bmatrix}$

Iteration 2,

$$A * x^{(1)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.4 \\ 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 6.6 \\ 0.6 \\ 5.4 \end{bmatrix}$$

$$= \underset{\lambda^{(2)}}{6.6} \underset{x^{(2)}}{\begin{bmatrix} 1 \\ 0.0909 \\ 0.818 \end{bmatrix}}$$

$\therefore \lambda^{(2)} = 6.6$

$x^{(2)} = \begin{bmatrix} 1 \\ 0.0909 \\ 0.818 \end{bmatrix}$

Iteration 3,

$$A * x^{(2)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0909 \\ 0.818 \end{bmatrix} = \begin{bmatrix} 5.727 \\ 7.27 \\ 9.25 \end{bmatrix}$$

$$= \underset{\lambda^{(3)}}{9.4545} \underset{x^{(3)}}{\begin{bmatrix} 0.606 \\ 0.7696 \\ 1 \end{bmatrix}}$$

Iteration 4,

$$A * x^{(3)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.606 \\ 0.7696 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.062 \\ 4.4925 \\ 12.919 \end{bmatrix}$$

$$= \underset{\lambda^{(4)}}{12.9135} \underbrace{\begin{bmatrix} 0.7014 \\ 0.3477 \\ 1 \end{bmatrix}}_{x^{(4)}}$$

$$\therefore \lambda^{(4)} = 12.9135 \qquad x^{(4)} = \begin{bmatrix} 0.7014 \\ 0.3477 \\ 1 \end{bmatrix}$$

<u>iteration 5,</u>

$$A * x^{(4)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.7014 \\ 0.3477 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.144 \\ 5.8116 \\ 11.744 \end{bmatrix}$$

$$= 11.744 \begin{bmatrix} 0.608 \\ 0.4948 \\ 1 \end{bmatrix}$$

$$\therefore \lambda^{(5)} = 11.744 \quad , x^{(5)} = \begin{bmatrix} 0.608 \\ 0.4948 \\ 1 \end{bmatrix}$$

<u>iteration 6,</u>

$$A * x^{(5)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.608 \\ 0.4948 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.6907 \\ 5.0509 \\ 12.093 \end{bmatrix}$$

$$= 12.093 \begin{bmatrix} 0.6359 \\ 0.4176 \\ 1 \end{bmatrix}$$

$$\lambda^{(6)} = 12.093 \qquad x^{6} = \begin{bmatrix} 0.6359 \\ 0.4176 \end{bmatrix}$$

<u>iteration 7,</u>

$$A * x^{(6)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.6359 \\ 0.4176 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7.3603 \\ 5.3445 \\ 11.889 \end{bmatrix} = \underset{\lambda^{(7)}}{11.889} \underbrace{\begin{bmatrix} 0.619 \\ 0.4495 \\ 1 \end{bmatrix}}_{x^{(7)}}$$

<u>iteratron 8,</u>

$$A * x^{(7)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.619 \\ 0.4495 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4859 \\ 5.1964 \\ 11.967 \end{bmatrix}$$

$$= \underset{\lambda^{(8)}}{11.967} \underbrace{\begin{bmatrix} 0.6253 \\ 0.4342 \\ 1 \end{bmatrix}}_{x^{8)}}$$

<u>iteratron 9,</u>

$$A * x^{(8)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.6253 \\ 0.4342 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4 \\ 5.2 \\ 11. \end{bmatrix}$$

$$= \underset{\lambda^{(9)}}{11.928} \underbrace{\begin{bmatrix} 0.6222 \\ 0.4409 \\ 1 \end{bmatrix}}_{x^{(9)}}$$

iteration 10,

$$A * x^{(9)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.6222 \\ 0.4409 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4495 \\ 5.2293 \\ 11.945 \end{bmatrix}$$

$$= \underset{\lambda^{(10)}}{11.945} \underset{x^{(10)}}{\begin{bmatrix} 0.6236 \\ 0.4378 \\ 1 \end{bmatrix}}$$

iteration 11,

$$A * x^{(10)} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & -2 & 3 \\ 1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.6236 \\ 0.4378 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.4366 \\ 5.2428 \\ 11.937 \end{bmatrix}$$

$$= \underset{\lambda^{(11)}}{11.937} \underset{x^{(11)}}{\begin{bmatrix} 0.6229 \\ 0.4392 \\ 1 \end{bmatrix}}$$

At 11^{th} iteration,

$$|x^{(11)} - x^{(10)}| \approx 0.$$

∴ Largest eigen value $(\lambda) = 11.937$

Eigen vector ; $x = \begin{bmatrix} 0.6229 \\ 0.4392 \\ 1 \end{bmatrix}$

Or, $x = \begin{bmatrix} 0.6229 & 0.4392 & 1 \end{bmatrix}^T$

(Q) Find the largest eigen value of the corresponding Eigen vector of the matrix using power method.

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution:

Let $x^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial eigen vector.

Then, finding the largest value using power method iteratively, we have,

$$A * x = \lambda * x$$

Iteration 1,

$$A * x^{(0)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

$$= \underset{\lambda^{(1)}}{3} \underset{x^{(1)}}{\begin{bmatrix} 1 \\ -0.667 \\ 0 \end{bmatrix}}$$

Iteration 2,

$$A * x^{(1)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.667 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.6667 \\ -4.6666 \\ 0.6667 \end{bmatrix}$$

$$= \underset{\lambda^{(2)}}{3.6667} \begin{bmatrix} 1 \\ -1.272 \\ 0.1818 \end{bmatrix} x^{(2)}.$$

Iteration 3,

$$A * x^{(2)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1.272 \\ 0.1818 \end{bmatrix}$$

$$= \begin{bmatrix} 4.272 \\ -7.6334 \\ 1.4538 \end{bmatrix} = \underset{\lambda^{(3)}}{4.272} \underset{x^{(3)}}{\begin{bmatrix} 1 \\ -1.786 \\ 0.3403 \end{bmatrix}}$$

Iteration 4,

$$A * x^{(3)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1.786 \\ 0.3403 \end{bmatrix} = \begin{bmatrix} 4.786 \\ -10.16 \\ 2.1263 \end{bmatrix}$$

$$= 4.786 \begin{bmatrix} 1 \\ -2.123 \\ 0.4442 \end{bmatrix}$$

Iteration 5,

$$A * \vec{x}^{(4)} =$$

Chapter-5

Solution to the ordinary differential Equation.

The solution of an ordinary differential eqn means finding an explicit expression for y in terms of a finite number of elementary function of x. Such a solution of a differential equation is known as the closed or finite form of solution. If the absence of such a solution, we have re course to numerial method of solution.

↳ Let us consider the first order differential equation,

$$\frac{dy}{dx} = f(x,y), \text{ given } y(x_0) = y_0 \quad \text{--- (i)}$$

↳ Then the ~~proble~~ problem of ordinary differential eqn is divided into 2 categories.

(i) Initial value problem (ii) Boundary value problem

(i) Initial value problem:-

↳ when an ordinary differential equation $\frac{dy}{dx} = f(x,y)$ is defined at single point $f(x_0) = y_0$, then it is known as initial value problems. The solution to the 1st order ordinary differential eqn is:

(i) Picard's method (method of integration).
(ii) Taylor's series method (method of differentiation)
(iii) Cauler's method (R-K 1st order method).

(iv) modified Euler's method / Heun's method / R-K 2nd order method

(v) Runge method (R-K 3rd order method)

(vi) Runge-Kutta / classical R-K method / R-K 4th order method.

② Boundary value problem:

↳ When an ordinary differential eqn is defined over more than one point i.e. $f(a) = A$ and $f(b) = B$, then it is known as boundary value problem. The various techniques for solving boundary value problem are:

① finite difference method
② Shooting method.

Solution to the 1st order ordinary differential eqn (initial value problem):

↳ Consider the first order differential eqn, $\frac{dy}{dx} = f(x,y)$. defined at any initial point $f(x_0) = y_0$. then it can be solved by using either of method described below.

① Euler's method (R-K 1st order):

↳ The solution to initial value problem of 1st order differential eqn using R-K 1st order given as,

$$\boxed{y_1 = y_0 + h * f(x_0, y_0)} \quad \text{—} ①$$

Assuming h, k be the small increments in variables x, y respectively.

(ii) Modified Euler's / Heun's method / R-K 2nd order.

↳ The solution to initial value problem of 1st order differential eqn using R-K 2nd order method is given as,

$$\boxed{y_1 = y_0 + 1/2\,(K_1 + K_2)} \quad \text{—} ⑪$$

where,

$K_1 = h \cdot f(x_0, y_0)$
$K_2 = h \cdot f(x_0 + h, y_0 + K_1)$

Assuming h, k be the small increments in variable x & y respectively.

(iii) Runge method (R-K 3rd order)

$$y_1 = y_0 + \frac{1}{4}(K_1 + 2K_2 + K_3)$$

where,

$K_1 = h \cdot f(x_0, y_0)$
$K_2 = h \cdot f(x_0 + h/2, y_0 + K_1/2)$
$K_3 = h \cdot f(x_0 + h, y_0 + K_2)$

Assuming h, k be the increment is x & y respectively.

(iv) Runge-Kutta method / R-K 4th order.

↳ The solution to initial value problem using R-K 4th order for solving 1st order differential eqn is given as,

$$y_1 = y_0 + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)$$

where,

$K_1 = h \cdot f(x_0, y_0)$
$K_2 = h \cdot f(x_0 + h/2, y_0 + K_1/2)$
$K_3 = h \cdot f(x_0 + h/2, y_0 + K_2/2)$
$K_4 = h \cdot f(x_0 + h, y_0 + K_3)$

Assuming h, k be the small increments in variable x & y respectively.

Aim: $n \geq 5$
$h \leq 0.3$

Q) Apply Euler's method (R-K 1st order) to approximate value of $y(0.3)$ for the differential eqn.

$$\frac{dy}{dx} = x+y, \quad y(0)=1.$$

Given,

$$\frac{dy}{dx} = x+y = f(x,y)$$

$$\therefore f(x,y) = x+y$$

With initial condition,

$$y(0)=1$$
i.e. $y(x_0) = y_0$
$\therefore x_0 = 0$
$\therefore y_0 = 1$

And,
$x_n = 0.3$
$y(x_n) = y(0.3) = ?$

Now, Taking $n=5$ (Assume), $h = \dfrac{x_n - x_0}{n} = \dfrac{0.3-0}{5} = 0.06$

Again taking $h=0.06$ and applying Euler's successively for solving it, we get,

$$y_1 = y_0 + h \cdot f(x_0, y_0) \quad \text{——— (1)}$$

	A	B	C	
no. of iteration	x_0	y_0	$f(x_0,y_0) = x_0+y_0$	$y_1 = y_0 + h \cdot f(x_0, y_0)$
0	0	1	1	$1+0.06\times 1 = 1.06$
1	0.06	1.06	1.12	$1.06+0.06\times 1.12 = 1.12$
2	0.12	1.272	1.2472	$1.272+0.06\times 1.2472 = 1.202$
3	0.18	1.20203	1.38203	$1.20203+0.06\times 1.38203 = 1.284$
4	0.24	1.2849	1.5249	
5	3	~~1.34705~~ 1.3764	——	$1.2845+0.06\times 1.5249 = 1.3076$

Hence, $y(0.3) = 1.3764$ ans.

Replace $A = x_0 + h$
$B = y$.

Q) Apply modified Euler's method (R-K 2nd order) to approximate value of y at $x=0.3$ for the differential eqn.

$$\frac{dy}{dx} = x^2+2y, \quad y(0)=1, \text{ taking } h=0.1.$$

Solution,

$$\frac{dy}{dx} = f(x,y) = x^2+2y$$

$x_0 = 0, \; y_0 = 1$

$x_n = 0.3$

$h = \cancel{\dfrac{x_n - x_0}{n}} = 0.1$

Date
Page

iteration 1,

taking $h=0.1$ and finding solution by R-K order

Heur's method,

$k_1 = h \cdot f(x_0, y_0)$
$= 0.1 * f(0,1)$
$= 0.1 (0^2 + 2\times1)$
$= 0.2$

$k_2 = h \cdot f(x_0+h, y_0+k_1)$
$= 0.1 * f(0.1, 1.2)$
$= 0.1 \times (0.1^2 + 2\times1.2)$
$= 0.241$

$\therefore y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$y(x_0+h) = 1 + 1/2 (0.2 + 0.241)$

$y(0.1) = 1.2205$

iteration 2,

$x_0 = 0.1$, $y_0 = 1.2205$

taking $h=0.1$ and finding solution by R-K 2nd order, we have,

$k_1 = h \cdot f(x_0, y_0)$
$= 0.1 \, f(0.1, 1.2205)$
$= 0.1 (0.1^2 + 2*1.2205)$
$= 0.2451$

$k_2 = h \cdot f(x_0+h, y_0+k_1)$
$= 0.1 \, f(0.2, 1.4656)$
$= 0.1 (0.2^2 + 2*1.4656)$
$= 0.2971$

$\therefore y_1 = y_0 + 1/2 (k_1+k_2)$

$y(x_0+h) = 1.2205 + 1/2 (0.2451 + 0.2971)$

$y(0.2) = 1.4916$ #

Date
Page

iteration 3,

$x_0 = 0.2$, $y_0 = 1.4916$

Taking $h=0.1$ and finding solution by applying hean's method, we get:

$k_1 = h \cdot f(x_0, y_0)$
$= 0.1 \, f(0.2, 1.4916)$
$= 0.1 (0.2^2 + 2\times1.4916)$
$= 0.30232$

$k_2 = h \cdot f(x_0+h, y_0+k_1)$
$= 0.1 \, f(0.3, 1.79392)$
$= 0.1 (0.3^2 + 2\times1.793)$
$= 0.367784$

$\therefore y_1 = y_0 + 1/2 (k_1+k_2)$

$y(x_0+h) = 1.4916 + \frac{1}{2}(0.30232 + 0.367784)$

$y(0.3) = 1.826652$ ans.

p.t.o.

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2024.

Q) Solve the following differential eqn within $0 \le x \le 0.$ using R-K ~~[illegible]~~ 4th order method.

$$10\frac{dy}{dx} = x^2+y^2, \; y(0) = 1 \text{ with } h = 0.15$$

Solution: Given,

$$10\frac{dy}{dx} = x^2+y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{10} = f(x,y)$$

$x_0 = 0$
$y_0 = 1$
$h = 0.15$

$y(0.15) = ?$
$y(0.3) = ?$

Iteration 1,

taking $h = 0.15$ and finding soln by applying R-K 4th order, we have

$\cancel{K_1 = h \cdot f(x_0, y_0)}$
$\cancel{= 0.15 f(0.15, 1.01533)}$
$\cancel{= 0.15\left(\frac{0.15^2 + 1.01533^2}{10}\right)}$
$\cancel{= 0.01580}$

$\cancel{K_2 =}$

$$K_1 = h \cdot f(x_0, y_0)$$
$$= 0.15 * f(0,1)$$
$$= 0.15 * \left(\frac{0^2+1^2}{10}\right)$$
$$= 0.015$$

$$K_2 = h \cdot f(x_0 + h/2, \; y_0 + K_1/2)$$
$$= 0.15 f(0.075, 1.0075)$$
$$= 0.15\left(\frac{0.075^2 + 1.0075^2}{10}\right)$$
$$= 0.0153$$

$$K_3 = h f(x_0 + h/2, \; y_0 + K_2/2)$$
$$= 0.15 f(0.075, 1.0076)$$
$$= 0.15\left(\frac{0.075^2 + 1.0076^2}{10}\right)$$
$$= 0.01531$$

$$K_4 = h \cdot f(x_0 + h, \; y_0 + K_3)$$
$$= 0.15 f(0.15, 1.01531)$$
$$= 0.15\left(\frac{0.15^2 + 1.01531^2}{10}\right)$$
$$= 0.01580$$

$$\therefore y_1 = y_0 + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)$$

$$y(x_0+h) = 1 + 1/6\,(0.015 + 2\times0.0153 + 2\times0.01531 + 0.0158$$
$$y(0.15) = 1.01533$$

Iteration 2,

$x_0 = 0.15$
$y_0 = 1.01533$

taking $h = 0.15$ and finding soln by applying R-K 4th order, we have,

$$K_1 = h \cdot f(x_0, y_0)$$
$$= 0.15 f(0.15, 1.01533)$$
$$= 0.15\left(\frac{0.15^2 + 1.01533^2}{10}\right)$$
$$= 0.01580$$

$$K_2 = h \cdot f\left(x_0 + h/2, \; y_0 + \frac{K_1}{2}\right)$$
$$= 0.15 f(0.225, 1.02323)$$
$$= 0.15\left(\frac{0.225^2 + 1.02323^2}{10}\right)$$
$$= 0.01646$$

$k_3 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= 0.15 f(0.225, 1.02356)$

$= 0.15 \left(\frac{0.225^2 + 1.02356^2}{10}\right)$

$= 0.01647$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$

$= 0.15(0.3, 1.03180)$

$= 0.15 \left(\frac{0.3^2 + 1.03180^2}{10}\right)$

$= 0.01731$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(x_0 + h) = 1.01533 + \frac{1}{6}(0.01580 + 2 \times 0.01646 + 2 \times 0.01647 + 0.01731)$

$y(0.3) = 1.031825$ ans#.

$\left\{ \begin{array}{l} \therefore y(0.15) = 1.01533 \\ y(0.3) = 1.031825 \end{array} \right\}$ ans#.

Solution to simultaneous differential equation:

↳ The simultaneous differentiable eqn is of the type

$$\frac{dy}{dx} = f(x, y, z) \quad \text{--- (i)}$$

and,

$$\frac{dz}{dx} = \phi(x, y, z) \quad \text{--- (ii)}$$

with the initial condition $y(x_0) = y_0$ and $z(x_0) = z_0$. Then it can be solved by either taylor's series method or R-K methods.

↳ Assuming h, k, l be the small increments in variable x, y, z respectively, then it's solution by R-K methods are given as below:-

(I) Euler's method:

$$y_1 = y_0 + h \cdot f(x_0, y_0, z_0)$$

$$z_1 = z_0 + h \cdot \phi(x_0, y_0, z_0)$$

(II) modified Euler's method (R-K 2nd order):

$k_1 = h \cdot f(x_0, y_0, z_0)$

$k_2 = h \cdot f(x_0 + h, y_0 + k_1, z_0 + l_1)$

$l_1 = h \cdot \phi(x_0, y_0, z_0)$

$l_2 = h \cdot \phi(x_0 + h, y_0 + k_1, z_0 + l_1)$

$\therefore y_1 = y_0 + 1/2(k_1 + k_2)$

$\therefore z_1 = z_0 + 1/2(l_1 + l_2)$

(III) R-K 4th order (classical Runge-Kutta).

$k_1 = h \cdot f(x_0, y_0, z_0)$

$k_2 = h \cdot f(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$k_3 = h \cdot f(x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + l_2/2)$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3, z_0 + l_3)$

where,

$l_1 = h \cdot \phi(x_0, y_0, z_0)$

$l_2 = h \cdot \phi(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$l_3 = h \cdot \phi(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$l_4 = h \cdot \phi(x_0 + h, y_0 + k_3, z_0 + l_3)$

$\therefore y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore z_1 = z_0 + 1/6(l_1 + 2l_2 + 2l_3 + l_4)$

Solution to second order differential eqn.

↳ Consider the second order differential eqn,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \quad \text{—— (1)}$$

by writing $\frac{dy}{dx} = z$, equation can be reduced to two simultaneous first order differential eqn.

$$\frac{dy}{dx} = z = f(x, y, z) \quad \text{—— (i)}$$

and,

$$\frac{dz}{dx} = \phi(x, y, z) \quad \text{—— (ii)}$$

Now, these simultaneous eqn can be solve by any of the method explained above provided with their initial condition.

Q) Solve $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y$ for $x = 1.5$, given that $y = z = 1$ for $x = 1$ by using modified Euler's method.

Solution;

$$\frac{dy}{dx} = x + z = f(x, y, z)$$

and,

$$\frac{dz}{dx} = x - y = \phi(x, y, z)$$

$x_0 = 1$
$y_0 = 1$
$z_0 = 1$ } $y = z = 1$ at $x = 1$ (initial condition)
$x_n = 1.5$

Trick: Here, h is not given so,
$n \geq 2$
$h \leq 0.3$

$$h = \frac{x_n - x_0}{n} = \frac{1.5 - 1}{2} = 0.25$$

Taking $n = 2$,

$$h = \frac{x_n - x_0}{n} = \frac{1.5 - 1}{2} = 0.25$$

Date
Page

iteration 1,

Taking $0.25 = h$ and applying R-K 2nd order, we get,

$K_1 = h \cdot f(x_0, y_0, z_0)$
$= 0.25 f(1,1,1)$
$= 0.25 \times (1+1)$
$= 0.25 \times 2$
$= 0.5$

$l_1 = h \cdot \phi(x_0, y_0, z_0)$
$= 0.25 * \phi(1,1,1)$
$= 0.25 \times (1-1)$
$= 0$

So,
$K_2 = h \cdot f(x_0 + h, y_0 + K_1 + z_0 + l_1)$
$= 0.25 \times f(1.25, 1.5, 1)$
$= 0.25 (1.25 + 1)$
$= 0.5625$

$l_2 = h \cdot \phi(x_0 + h, y_0 + K_1, z_0 + l_1)$
$= 0.25 \times \phi(1.25, 1.5, 1)$
$= -0.0625)$

$\therefore y_1 = y_0 + 1/2 (K_1 + K_2)$

$y(x_0 + h) = 1 + 1/2 (0.5 + 0.5625)$

$y(1.25) = 1.53125$

Date
Page

And,
$z_1 = z_0 + 1/2 (l_1 + l_2)$

$z(x_0 + h) = 1 + 1/2 (0 - 0.0625)$
$z(1.25) = 0.96875.$

Iteration 2,

$x_0 = 1.25$
$y_0 = 1.53125$
$z_0 = 0.96875$

Taking $h = 0.25$ and applying R-K 2nd order formula,

$K_1 = h \cdot f(x_0, y_0, z_0)$
$= 0.25 * f(1.25, 1.53125, 0.96875)$
$= 0.25 * (1.25 + 0.96875)$
$= 0.5546.$

$l_1 = h \cdot \phi(x_0, y_0, z_0)$
$= 0.25 * \phi(1.25, 1.531, 0.968$
$= 0.25 (1.25 - 1.53125)$
$= -0.0703$

$K_2 = h \cdot f(x_0 + h, y_0 + K_1, z_0 + l_1)$
$= 0.25 * f(1.5, 2.08593, 0.89843)$
$= 0.25 (1.5 + 0.89843)$
$= 0.59960$

$l_2 = h \cdot \phi(x_0 + h, y_0 + K_1, z_0 + l_1)$
$= 0.25 * (1.5 - 2.08593)$
$= -0.1464$

$\therefore y_1 = y_0 + 1/2 (K_1 + K_2)$
$y(x_0 + h) = 1.53125 + 1/2 (0.5546 + 0.59960)$
$y(1.5) = 2.10835.$

CamScanner

Date
Page

$\therefore z_1 = z_0 + 1/2\,(l_1 + l_2)$

$z(x_0+h) = 0.96875 + 1/2(-0.0703 - 0.1464)$

$z(1.5) = (0.8604)$

$\therefore y(1.25) = 1.53125$ & $z(1.25) = 0.96875$

$y(1.5) = 2.10835$ $z(1.5) = 0.8604$.

24 spring

Q) Solve the following differential eqn within $0 \le x \le 0.5$ using R-K 4th order method.

$$10\frac{d^2y}{dx^2} + \frac{2dy}{dx} - 3y = 5,\ y(0) = 0,\ y'(0) = 0.$$

Given,

$$10\frac{d^2y}{dx^2} + \frac{2dy}{dx} - 3y = 5$$

$$\Rightarrow \frac{d^2y}{dx^2} = 0.5 - 0.2\frac{dy}{dx} + 0.3y \quad \text{--- (I)}$$

let,

$$\frac{dy}{dx} = z = f(x,y,z) \quad \text{--- (II)}$$

then eqn (I) becomes,

$$\frac{dz}{dx} = 0.5 - 0.2z + 0.3y = \phi(x,y,z)$$

Date
Page

$x_0 = 0,\ y_0 = 0,\ z_0 = 0$

taking $n = 2$

$x_n = 0.5$

$$h = \frac{x_n - x_0}{n} = 0.25$$

iteration-1:

Taking $h = 0.25$ and applying R-K 4th order, we ge

$K_1 = h \cdot f(x_0, y_0, z_0)$
$= 0.25 * f(0,0,0)$
$= 0$

$l_1 = h \cdot \phi(x_0, y_0, z_0)$
$= 0.25 \cdot \phi(0,0,0)$
$= 0.25 * (0.5)$
$= 0.125$

$K_2 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2} + z_0 + \frac{l_1}{2})$
$= 0.25 * f(0.125, 0, 0.0625)$
$= 0.25 * 0.0625$
$= 0.015625$

$l_2 = h \cdot \phi(x_0 + h/2, y_0 + K_1/2 + z_0 + l_1/2)$
$= 0.25\,\phi(0.125, 0, 0.0625)$
$= 0.121875$

$K_3 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2})$
$= 0.25 * f(0.125, 0.00753125, 0.0609)$
$= 0.25 * 0.0609$
$= 0.01523$

$l_3 = h \cdot \phi(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$
$= 0.25 * \phi(0.125, 0.00753125, 0.0609)$
$= 0.100198$

Date
Page

$k_4 = h \cdot f(x_0+h, y_0+k_3, z_0+l_3)$
$= 0.25 f(0.25, 0.01523, 0.1001981)$
$= 0.02504$

$l_4 = h \cdot \phi(x_0+h, y_0+k_3+z_0+l_3)$
$= 0.25\ \phi(0.25, 0.01523, 0.1001981)$
$= 0.25 * (-0.5 - 0.2 * 0.1001981 + 0.3 * 0.01523)$
$= 0.131412$

$\therefore y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\Rightarrow y(x_0+h) = 0 + 1/6\ (0 + 2 \times 0.015625 + 2 \times 0.01523$
$+ 0.02504)$

$\Rightarrow y(0.25) = 0.014458$

and,

$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$z(x_0+h) = 0 + 1/6\ (0.125 + 2 \times 0.1218 + 2 \times 0.10019 + 0.131412)$

$\therefore z(0.25) = 0.11673$

iteration 2:

$x_0 = 0.25,\ y_0 = 0.014458,\ z_0 = 0.11673$

Now, taking $h = 0.25$ and solving by using R-K 4th order, we get;

Date
Page

$k_1 = h \cdot f(x_0, y_0, z_0)$
$= 0.25 * f(0.25, 0.0144, 0.1167)$
$= 0.25 * 0.11673$
$= 0.029118.$

$l_1 = h \cdot \phi(x_0, y_0, z_0)$
$= 0.25 * (0.5 - 0.2 \times 0.1167 + 0.3 \times 0.0144)$
$= 0.12024$

$k_2 = h \cdot f(x_0+h, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$= 0.25 * f(0.375, 0.02904, 0.176853)$
$= 0.25 * 0.176853$
$= 0.04421$

$l_2 = h \cdot \phi(x_0+h, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$= 0.25 * \phi(0.375, 0.02904, 0.176853)$
$= 0.25 * (0.5 - 0.2 \times 0.176853 + 0.3 * 0.02904)$
$= 0.11833$

$k_3 = h \cdot f(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$
$= 0.25 * 0.17589$
$= 0.04397$

$l_3 = h \cdot \phi(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$
$= 0.25 * (0.5 - 2 * 0.17589 + 3 \times 0.036563)$
$=$

$k_4 = h \cdot f(x_0+h, y_0+k_3, z_0+l_3)$
$= 0.25 * f(\cancel{0.75}\ 0.5, \quad , 0.156527)$
$= 0.25 * 0.156527$
$= 0.039131$

$l_4 = h \cdot \phi(x_0+h, y_0+k_3+z_0+l_3)$
$= 0.25\ \phi(0.5, 0.058429, 0.156527)$

$= 0.25 * (0.5 - 2 * 0.156527 + 0.3 * 0.058429)$
$= 0.05111$

$\therefore y_1 = y_0 + 1/6(\cancel{y_1+24}\ k_1 + 2(k_2+k_3) + k_4)$

$y(x_0+h) = 0.014458 + \frac{1}{6}(0.29118 + \cdots)$

$y(0.5) = 0.0552261$

Also,
$z_1 = z_0 + 1/6(\cancel{k_1}\, l_1 + 2(l_2 + \cancel{k_3}\, l_3) + l_4)$

$z(x_0+h) = 0.11693 + 1/6(0.12024 + 2(\cdots))$

$z(0.5) =$ // ans.

Boundary value problem:

Shooting method:

↳ In this method, the given boundary value problem is first transformed to an initial value problem. Then this initial value problem is solved by taylor's series R-K methods, etc. finally boundary value problem's solve

↳ consider the boundary value problem,

$$y''(x) = y(x), \quad y(a) = A \text{ and } y(b) = B \quad \text{— (1)}$$

↳ one condition is $y(a) = A$ and let us assume that $y'(a) = m$ which represents the slope. we start with two initial guess for m, then find corresponding values for $y(b)$ using any initial value method.

↳ Let the two guesses be m_0, m_1 so that the corresponding values of $y(b)$ are $y(m_0, b)$ and $y(m_1, b)$. Assuming that the values of m and $y(b)$ are linearly related, we obtain a better approximation m_2 for m from the relation,

$$\Rightarrow \frac{m_2 - m_1}{y(b) - y(m_1, b)} = \frac{m_1 - m_0}{y(m_1, b) - y(m_0, b)}$$

This gives,

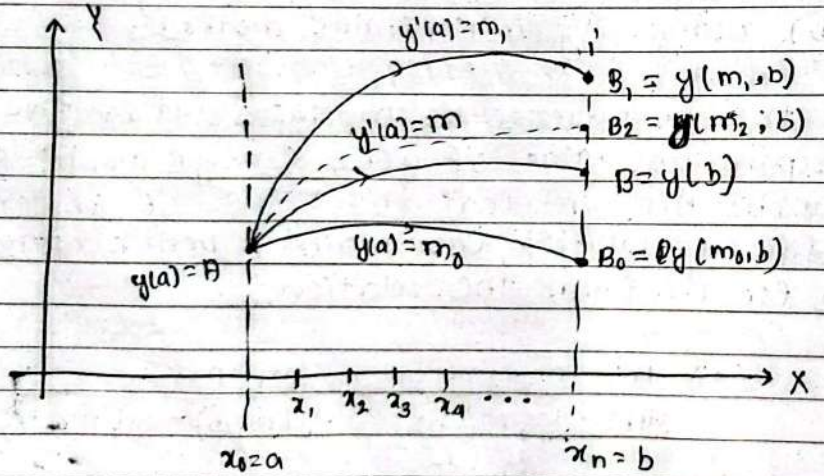
$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, b) - y(b)}{y(m_1, b) - y(m_0, b)} \quad \text{—— (2)}$$

↳ we now solve the initial value problem,

$$y''(x) = y(x),\ y(a) = A,\ y'(a) = m_2$$

and obtain the solution $y(m_2, b)$.

↳ To obtain a better approximation m_3 for m, we again use the linear relation (2) with $[m_1, y(m_1, b)]$ and $[m_2, y(m_2, b)]$. This process is repeated until the value of $y(m_i, b)$ agrees with $y(b)$ to desired accuracy.



[Fig: shooting method]

Q) Using shooting method, solve the boundary value problem:

$$y''(x) = y(x),\ y(0) = 0,\ y(1) = 1.17$$

Solution:

Given,

$y''(x) = y(x)$

$x_0 = a = 0$

$y_0 = A = 0$

$x_n = b = 1$

$y_n = B = 1.17$

Now,

$$y(x_0) = y_0$$

let, $y'(x) = M$ Assuming initial value, for m be $m_0 = 0.5$ and $m_1 = 0.7$, then,

$\Rightarrow \quad y'(0) = m$

and $\quad y''(x) = y(x) \Rightarrow y''(0) = y(0) = 0$

Differentiating successively,

$y'''(x) = y'(x) \Rightarrow y'''(0) = y'(0) = m$

or, $y^{IV}(x) = y''(x) \Rightarrow y^{IV}(0) = y''(0) = 0$

or $y^{V}(x) = y'''(x) \Rightarrow y^{V}(0) = y'''(0) = m$

and so on ——

Date
Page

Now, putting the values to taylor's series,

we have,

$$y(x) = y(0) + x \cdot y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + \cdots$$

$$\Rightarrow y(x) = m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)$$

$$\Rightarrow y(1) = m\left(1 + \frac{1}{6} + 1/120 + 1/5040 + \cdots\right)$$

for $m_0 = 0.5$, $y(m_0, 1) = 0.5\left(1 + 1/6 + 1/120 + \frac{1}{5040} + \cdots\right) =$

for $m_0 = 0.7$, $y(m_1, 1) = 0.7\left(1 + 1/6 + 1/120 + 1/5040 + \cdots\right)$
$=$

Hence, Better approximation for m i.e. m_2 is given as,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, 1) - y(1)}{y(m_1, 1) - y(m_0, 1)}$$

$$= 0.7 - (0.7 - 0.5)\frac{0.8226 - 1.17}{0.8226 - 0.5875}$$

$$= 0.9955$$

Date
Page

Now, solving the initial value problem,

$$y''(x) = y(x),\ y(0) = 0,\ y'(0) = m_2$$

Taylor's series soln is given by,

$$y(m_2, 1) = 0.9955\left(1 + 1/6 + 1/120 + 1/5040 + \cdots\right)$$

$$\approx 1.1699 \approx 1.17$$

$\therefore$ The solution at $x = 1$ is $y = 1.1699$ which is close to exact value $y(1) = 1.17$ at $m = 0.9955$.

Chapter-6

Solution to the partial differential equation:

Classification of partial differential equation:

↳ Concider the second order partial differential equation,

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, \frac{du}{dx}, \frac{\partial u}{\partial y}\right) \quad \text{--- (1)}$$

is said to be :

(i) elliptical, if : $B^2 - 4AC < 0$
(ii) parabolic, if : $B^2 - 4AC = 0$
(iii) hyperbolic, if : $B^2 - 4AC > 0$.

(1) Elliptical equations:

↳ Those equations satisfying the characteristics $B^2 - 4AC <$ is called elliptic equations. The elliptic equations are:

(a) Laplace equation:

↳ The eqn of the form,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or, $u_{xx} + u_{yy} = 0$

or, $\Delta^2 u = 0$

~~poss~~ poisson Equation:

The eqn of the form,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

or, $U_{xx} + U_{yy} = f(x,y)$

$$\Delta^2 U = f(x,y)$$

The laplace equation arises in study of steady state flow and potential problem while poisson's equation arises dealing with fluid mechanics, electricity & magnetism, etc.

Solution to Laplace Equation:

Consider a laplace eqn, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —— (1)

applied over the given rectangular region as shown in figure below.

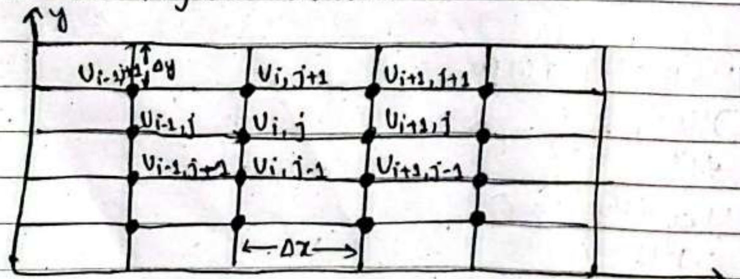


Fig: Graphical representation for laplace eqn.

↳ Now dividing this region into subregions having width $\Delta x = h$ & $\Delta y = K$ respectively. Then the point at which the subregions meet are called mesh point, grid point, nodal point.

↳ The finite differences at point $U_{i,j}$ is;

$$\frac{\partial u}{\partial x} = \frac{U_{i+1,j} - U_{i,j}}{h} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{U_{i,j} - U_{i-1,j}}{h}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\frac{U_{i+1,j} - U_{i,j}}{h} - \frac{U_{i,j} - U_{i-1,j}}{h}}{h^2}$$

$$= \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$$

Also,

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{U_{i,j+1} - U_{i,j}}{h} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{U_{i,j} - U_{i,j-1}}{h}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\frac{U_{i,j+1} - U_{i,j}}{K} - \frac{U_{i,j} - U_{i,j-1}}{K}}{K^2}$$

$$= \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{K^2}$$

putting these values to eqn ①,

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = 0$$

Assuming $h = k = 1$,

$$\boxed{\therefore U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right]} \quad \text{②}$$

which is known as standard five point formula.

↳ In absence of any corresponding non-diagonal ~~vohitte~~ values, we use diagonal five point formula to evaluate eqn at internal mesh point given as,

$$\boxed{U_{i,j} = \frac{1}{4}\left[U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1}\right]} \quad \text{③}$$

After finding nodal equation at each mesh points we solve these simultaneous eqn using gauss-seidal method.

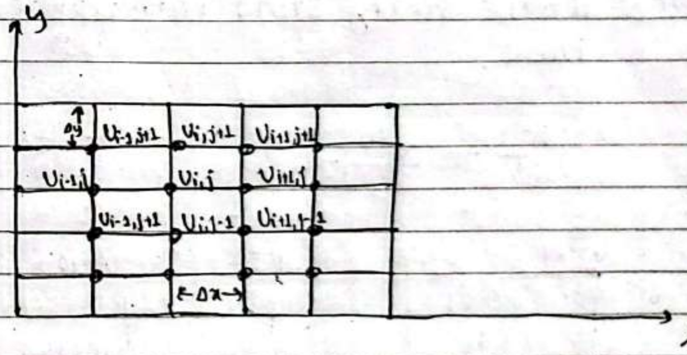
Solution to the poisson equation:

↳ Consider the poisson's eqn, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ — ①

is applied over the region below:-

↳ Dividing the region into n subregion having width $\Delta x = h$, and $\Delta y = k$ and substituting the values of corresponding finite difference to ①, we get;

$\cancel{U_{i+1,j} - 2U_{i,j}}$



$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = f(x,y)$$

Assuming $h = k$, we get,

$$\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4 \cdot U_{i,j}\right] = h^2 f(x,y) \quad \text{②}$$

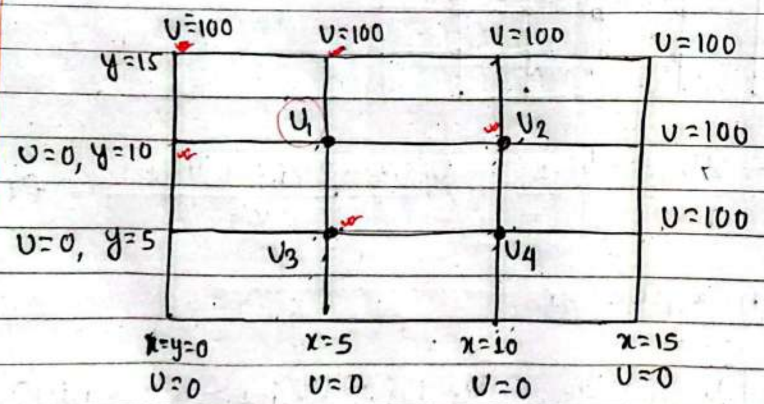
↳ Which is the soln of poisson's eqn at any interval mesh point.

↳ After finding model eqn at each mesh points we solve these simultaneous eqn using gauss-Seidal method.

Q) Heat on a square bar of size 15X15 cm. If two side are held at 100°c and other two side at 0°c, calculate the steady state temperature at interior points. Assume grid of size 5X5 cm.

Solution: let the steady state eqn applied over the bar is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$



If, U_1, U_2, U_3, U_4 be the internal mesh points. Then soln of laplace eqn at the corresponding point's using standard-five point formula is given as,

$$\boxed{U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right]} \quad \text{—— (2)}$$

At point U_1,

$$U_1 = 1/4\,(0 + 100 + U_2 + U_3)$$

$$U_1 = \frac{U_2 + U_3 + 100}{4} \quad \text{—— (a)}$$

At point U_2,

$$U_2 = 1/4\,[U_1 + 100 + 100 + U_4]$$

$$U_2 = \frac{U_1 + U_4 + 200}{4} \quad \text{—— (b)}$$

At point U_3,

$$U_3 = 1/4\,[0 + 0 + U_1 + U_4]$$

$$U_3 = \frac{U_1 + U_4}{4} \quad \text{—— (c)}$$

At point U_4,

$$U_4 = \left(\frac{0 + U_3 + U_2 + 100}{4} \right)$$

$$\therefore U_4 = \frac{U_2 + U_3 + 100}{4} \quad \text{——— (d)}$$

Now,

no. of iteration	$U_1 = \frac{U_2+U_3+100}{4}$	$U_2 = \frac{U_1+U_4+200}{4}$	$U_3 = \frac{U_1+U_4}{4}$	$U_4 = \frac{U_2+U_3+100}{4}$
0	0	0	0	0
1	25	56.25	6.25	40.625
2.	40.625	70.312	20.3125	47.6562
3.	47.65625	73.8281	23.8281	49.4140
4.	49.4140	74.7070	24.7070	49.8535
5.	49.8535	74.9267	24.9267	49.9633
6.	49.96337	74.9816	24.9816	49.9908
7	49.9908	74.9954	24.9954	49.9977
8	49.9977	74.9988	24.9988	49.9994

At 8th iteration, $|x^{(8)} - x^{(7)}| \approx 0$

So,

$U_1 = 49.9977 \approx 50$

$U_2 = 74.9988 \approx 75$

$U_3 = 24.9988 \approx 25$

$U_4 = 49.9994 \approx 50$.