A Heterogeneous Exchange Economy with Endogenous Probabilities and Risk Externality

1 A heterogeneous exchange economy

1.1 Set-up

At date 0, a single consumption good is exchanged and consumed, denoted by x. At date 1, the second period, two commodities are exchanged and consumed: commodity a and commodity b, with quantities denoted by x_a and x_b, respectively.

There are two types of individuals, $i = \alpha, \beta$, and each type consists of a continuum of individuals of unit mass.

Type β individuals: The intertemporal utility function of an individual of type β is given by:

$$u^\beta(x, x_a, x_b) = x - e + (1-\gamma)\ln x_a + \gamma \ln x_b,$$

where:

- $0 < \gamma < 1$ is the preference parameter,
- e represents a costly action chosen by type β individuals, which influences the endogenous probability $\pi(e)$,
- The endowment at date 1 consists of b units of commodity b.

Type α individuals: The intertemporal utility function of an individual of type α is given by:

$$u^\alpha(x, x_a, x_b) = x + \pi(e)\big(\gamma \ln x_a^1 + (1-\gamma)\ln x_b^1\big) + (1-\pi(e))\big(\gamma \ln x_a^2 + (1-\gamma)\ln x_b^2\big),$$

where:

- $\pi(e)$ is the endogenous probability of the favorable state, determined by the action e of type β individuals; $(1-\pi(e))$ is the probability of the unfavorable state,
- x_a^1 and x_a^2 denote the quantities of commodity a consumed by type α individuals in the favorable state $(+\epsilon)$ and unfavorable state $(-\epsilon)$, respectively,
- x_b^1 and x_b^2 denote the quantities of commodity b consumed by type α individuals in states $+\epsilon$ and $-\epsilon$, respectively.

The endowment of an α-type agent at date 1 consists only of commodity a, but it is subject to idiosyncratic shocks. Specifically:

$$a+\epsilon \quad \text{with probability } \pi(e), \quad \text{or} \quad a-\epsilon \quad \text{with probability } 1-\pi(e).$$

Trading structure:

- At date 0, individuals trade only the consumption good (x) and a risk-free bond that matures at date 1. The bond price is denoted by q.
- At date 1, individuals trade only the two commodities (a and b). Commodity a is chosen as the numéraire, and the relative price of commodity b is denoted by p.

1.2 Endogenous probabilities and equilibrium conditions

The action e of β-type agents determines the endogenous probability $\pi(e)$ of the favorable state. Let the equilibrium price of commodity b at date 1 be p. Then, by direct computation, the equilibrium price depends on the bond position y of α-type agents and the endogenous probability $\pi(e)$:

$$p(y) = \frac{(1-\gamma)a + (1-2\gamma)y + (1-\gamma)(2\pi(e)-1)\epsilon}{(1-\gamma)b}, \tag{1}$$

which depends non-trivially on asset holdings y, size of the idiosyncratic shock ϵ, and the probability of the favorable state $\pi(e)$, which itself depends endogeneously on the action e.

At date 1, the marginal utility of revenue for individuals of type β is:

$$\lambda^\beta = \frac{1}{pb-y},$$

while for individuals of type α, it varies with the realization of the idiosyncratic shock and is:

$$\lambda^\alpha(+\epsilon) = \frac{1}{a+\epsilon+y},$$

or

$$\lambda^\alpha(-\epsilon) = \frac{1}{a-\epsilon+y},$$

with probabilities $\pi(e)$ and $1-\pi(e)$, respectively.

1.3 Optimization and bond price

Type β agents. The optimization of individuals of type β at date 0 requires that:

$$q = \frac{1}{pb - y} = \frac{1-\gamma}{(1-\gamma)a - \gamma y + (1-\gamma)(2\pi(e) - 1)\epsilon},$$

where q reflects the marginal utility of revenue, accounting for β's bond position $y_\beta = -y$.

Type α agents. For type α agents, the optimization problem at date 0 determines q as:

$$q = \pi(e) \cdot \frac{1}{a + \epsilon + y} + (1 - \pi(e)) \cdot \frac{1}{a - \epsilon + y}.$$

1.4 Equilibrium bond position

At equilibrium, the bond position y is given by:

$$y = -\frac{a}{2} + \epsilon\pi(e) - \frac{\epsilon}{2} \pm \frac{\sqrt{a^2 - 4a\epsilon\pi(e) + 2a\epsilon + 16\epsilon^2\gamma\pi(e)^2 - 16\epsilon^2\gamma\pi(e) - 12\epsilon^2\pi(e)^2 + 12\epsilon^2\pi(e) + \epsilon^2}}{2}. \tag{2}$$

1.5 Optimal Effort e^* for Type β Agents

Type β individuals choose effort e to maximize their utility, balancing the direct cost of effort with its benefits through the endogenous probability $\pi(e)$. The first-order condition is:

$$-1 + \frac{\partial u^\beta}{\partial \pi(e)} \cdot \frac{\partial \pi(e)}{\partial e} = 0,$$

where:

$$\frac{\partial u^\beta}{\partial \pi(e)} = \frac{\partial u^\beta}{\partial p} \cdot \frac{\partial p}{\partial \pi(e)} = \lambda^\beta (x_b^\beta - b) \cdot \frac{2\epsilon}{b} = \frac{x_b^\beta - b}{pb - y} \cdot \frac{2\epsilon}{b}$$

Solving for e^*:

$$e^* : \frac{2\epsilon(x_b^\beta - b)\pi'(e)}{b(pb - y)} = 1.$$

The optimal e^* increases with ϵ (size of the shock), decreases with b (endowment of b), and increases with y (bond position).

1.6 Welfare Effects of Policy Perturbation

The welfare changes for type-α and type-β agents due to a policy perturbation (dx, dy) include direct bond effects, effort-induced probability effects, and price-induced reallocation effects.

The total welfare changes are expressed as:

$$\begin{bmatrix} du^\alpha \\ du^\beta \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & q - \Lambda^\alpha_{\text{price}} - \Lambda^\alpha_{\text{effort}} \\ -1 & -q - \Lambda^\beta_{\text{price}} - \Lambda^\beta_{\text{effort}} \end{bmatrix}}_{\text{Matrix of Coefficients}} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

Effort-Induced Probability Effects:

$$\Lambda^\alpha_{\text{effort}} = \Delta u^\alpha \cdot \pi'(e) \cdot \frac{\partial e}{\partial y},$$
$$\Lambda^\beta_{\text{effort}} = \Delta u^\beta \cdot \pi'(e) \cdot \frac{\partial e}{\partial y},$$

where:

$$\Delta u^\alpha = \gamma \ln \frac{x_a^1}{x_a^2} + (1-\gamma) \ln \frac{x_b^1}{x_b^2}, \quad \Delta u^\beta = (1-\gamma) \ln \frac{x_a^1}{x_a^2} + \gamma \ln \frac{x_b^1}{x_b^2}.$$

Price-Induced Reallocation Effects:

$$\Lambda^\alpha_{\text{price}} = \frac{\partial x_b^\alpha}{\partial p} \cdot [\pi(e)\lambda^\alpha(+\epsilon) + (1-\pi(e))\lambda^\alpha(-\epsilon)] \cdot \left(\frac{\partial p}{\partial y} + \frac{\partial p}{\partial \pi(e)} \cdot \pi'(e) \cdot \frac{\partial e}{\partial y} \right),$$
$$\Lambda^\beta_{\text{price}} = \lambda^\beta \cdot (x_b^\beta - b) \cdot \left(\frac{\partial p}{\partial y} + \frac{\partial p}{\partial \pi(e)} \cdot \pi'(e) \cdot \frac{\partial e}{\partial y} \right).$$

Price Derivatives:

$$\frac{\partial p}{\partial y} = \frac{1-2\gamma}{(1-\gamma)b},$$
$$\frac{\partial p}{\partial \pi(e)} = \frac{2\epsilon}{b}.$$

Total Welfare Change:

$$\begin{bmatrix} du^\alpha \\ du^\beta \end{bmatrix} = \begin{bmatrix} 1 & q - \frac{\partial x_b^\alpha}{\partial p} \cdot [\pi(e)\lambda^\alpha(+\epsilon) + (1-\pi(e))\lambda^\alpha(-\epsilon)] \cdot \left(\frac{\partial p}{\partial y} + \frac{\partial p}{\partial \pi(e)} \cdot \pi'(e) \cdot \frac{\partial e}{\partial y} \right) - \Delta u^\alpha \cdot \pi'(e) \cdot \frac{\partial e}{\partial y} \\ -1 & -q - \lambda^\beta \cdot (x_b^\beta - b) \cdot \left(\frac{\partial p}{\partial y} + \frac{\partial p}{\partial \pi(e)} \cdot \pi'(e) \cdot \frac{\partial e}{\partial y} \right) - \Delta u^\beta \cdot \pi'(e) \cdot \frac{\partial e}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$$

A policy perturbation (dx, dy) is Pareto-improving if $du^\alpha > 0$ and $du^\beta > 0$, with the matrix of coefficients nonsingular ($\det \neq 0$).