BSC O214

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ANNAMALAI UNIVERSITY

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B.Sc. Computer Science

FIRST YEAR

SCIENTIFIC COMPUTING

Scientific Computing

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NUMERICAL SOLUTION OF ALGEBRAIC, TRANSCENDENTAL AND SIMULTANEOUS EQUATION

1.1 INTRODUCTION

With the use of digital computers is increasing day by day, the computational methods have been successfully applied to study problems in Mathematics, Engineering, Computer Science and Physical Sciences.

1.2 APPROXIMATIONS

A computer has a finite word length and so only a fixed number of digits are stored and used during computation. This means that even in storing an exact decimal number in its converted form in the computer memory an error is introduced. This error is machine dependent and called machine epsilon.

After the computation is over the result in the machine form is again converted to the decimal form and some more error may be introduced at this stage.

ERROR is defined as

ERROR = TRUE VALUE - APPROXIMATE VALUE

RELATIVE ERROR = $\dfrac{|ERROR|}{|TRUEVALUE|}$

ABSOLUTE ERROR = | ERROR |

The errors may be classified into the following types.

1. The Inherent Error:

It is the error which is already present in the statement of the problem before solution.

2. The Round-off error:

It is the quantity R which must be added to the finite representation of a computed number in order to make it the true representation of that number.

Computer Arithmetic

The basic arithmetic operations performed by the computer are addition, subtraction, multiplication and division. The decimal numbers are first converted to the machine numbers consisting of the digits 0 and 1 with a base depending on the computer.

1) If the base is Two it is called Binary System
2) If the base is Eight it is called Octal System.
3) If the base is Sixteen it is Hexa Decimal System.

The Decimal System has base – 10

The decimal integer number 4987 means

$(4987)_{10} = 4 \times 10^3 + 9 \times 10^2 + 8 \times 10^1 + 7 \times 10^0$ which represents polynomial in the base 10. Similarly a fractional decimal number 0.6251 means

$(0.6251)_{10} = 6 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3} + 1 \times 10^{-4}$ which is a polynomial in 10^{-1}

Combining

$$(4987.6251)_{10} = 4 \times 10^3 + 9 \times 10^2 + 8 \times 10^1 + 7 \times 10^0 + 6 \times 10^{-1}$$
$$+ 2 \times 10^{-2} + 5 \times 10^{-3} + 10^{-4}$$

Thus a number N = $(d_{n-1}d_{n-2} \ldots d_0 d_{-1} d_{-2} \ldots . d_{-m})$ in the decimal system is always represented by

$$(N)_{10} = d_{n-1}10^{n-1} + d_{n-2}10^{n-2} + \ldots + d_1 10^1 + d_0 10^0 + d_{-1}10^{-1} + d_{-2}10^{-2} + \ldots + d_0\ 10^{-m}$$

where $d_{n-x}, d_{n-2}, \ldots\ldots d_{-m}$ are any digit between 0 and 9.

1.3 BINARY NUMBER SYSTEM

It has Base 2 with 0 and 1 called ***bits*** and any number N can be written as $(N)_2 = b_{n-1}\ b_{n-2} \ldots b_1\ b_0\ b_{-1}\ b_{-2} \ldots b_{-m}$ when b's are binary bits 0 or 1 and the point is called binary point.

The corresponding decimal can be calculated using the formula

$$(N)_{10} = b_{n-l}\ 2^{n-1} + b_{n-2}\ 2^{n-2} + \ldots + b_l\ 2^1 + b_0\ 2^0 + b_{-1}\ 2^{-1} + b_{-2}\ 2^{-2} + \ldots + b_{-m}\ 2^{-m}$$

Example

Find the decimal number corresponding to the binary number $(111.011)_2$

$$(111.011)_2 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} = (7.375)_{10}$$

Now let us consider the conversion of an integer N in decimal system in to binary number.

Let N = $b_{n-1}\ 2^{n-1} + b_{n-2}\ 2^{n-2} + \ldots + b_1 2^1 + b_0\ 2^0$

b_0 = 0 if and only if N is even

b_1 = 0 if and only if $\frac{N - b_0}{2}$ is even and soon.

Thus we have N_0 = N

$N_{k+1} = \frac{N_k - b_k}{2}$, k = 0, 1, 2 . . . . . until $N_k = 0$

$$b_n = \begin{cases} i\ if\ N_k\ is\ odd \\ 0\ is\ N_k\ is\ even \end{cases}$$

Example

Convert $(58)_{10}$ to binary

$N_0 = 58 \Longrightarrow b_0 = 0$

$N_1 = \frac{58-0}{2} = 29 \Longrightarrow b_1 = 1$

$N_2 = \frac{29-1}{2} = 14 \Longrightarrow b_2 = 0$

$N_3 = \frac{14-0}{2} = 7 \Longrightarrow b_3 = 1$

$N_4 = \frac{7-1}{2} = 3 \Longrightarrow b_4 = 1$

$N_5 = \frac{3-1}{2} = 1 \Longrightarrow b_5 = 1$

$N_6 = \frac{1-1}{2} = 0$

Thus $(58)_{10} = (111010)_2$

Next we convert the fraction N in the decimal system into binary.

Let N = $b_{-1}2^{-1} + b_{-2}2^{-2} + b_{-m}2^{-m}$, $0 < N < 1$

b_{-1} = 1 if the only if 2 $N \geq 1$ and

= 0 if and only if 2N < 1

Thus we have

$N_1 = N$

$$b_{-k} = \begin{cases} 1\ if\ 2\,N_k \geq 1 \\ 0\ if\ 2\,N_k < 1 \end{cases}$$

$N_{k+1} = 2N_k - b_{-k}$, k = 1, 2

Example

Covert $(0.859375\ )_{10}$ in to binary fraction

K	b-k	N k+l
0		.859375
1	1	.718750
2	1	.437500
3	0	.875000
4	1	.750000
5	1	.500000
6	1	.000000

Hence $(.859375)_{10} - (0110111)_2$

Floating Point Arithmetic

The first step in the computation with digital computers is to convert the decimal numbers to another number system with base (3 (say) understandable to that particular computer and then store in the computer memory. The memory of the digital computer is divided into separate cells called ***WORDS.*** Each word can hold the same number of digits called ***BITS*** with respect to its base plus a sign. Negative numbers are stored as absolute values plus a sign or in complement form. The number of digits which can be stored in a computer word is called its ***wordlength.*** This varies from computer to computer.

The numbers in the computer word can be stored in two forms.

1) Fixed Point Form
2) Floating Point Form.

In ***Fixed Point form*** a t digit number is assumed to have its decimal point at the left-hand end of the word. This means that all numbers are assumed to be less than 1 in magnitude.

A t digit number with base β in fixed point form may be written as

$$\pm \sum_{k=1}^{t} a_k \quad \beta^k \text{ where } 0 \leq a_k < \beta$$

If y is any real number and y* is its machine representation then the error in y* is almost $a_{t+1}\,\beta^{-(t+1)}$ or

$$/\ y - y^*\ |\ \leq \beta^{-t}$$

To avoid the difficulty of keeping every number lessthan 1in magnitude during computation most computers use ***FLOATING POINT*** Representation.

A floating point number is characterised by four parameters.

(i) The base β (ii) The number of digits t (iii) Exponent range (m, M)

Definition

A floating point number in represented as $.\, d_1\, d_2 \,.....\, d_t \times \beta^e$ where $d_1 , d_2 .....$ d_t are digits with $0 \leq d_1 < \beta$ and exponent e is such that $m \leq e \leq M$.

The fractional part. $d_1\, d_2\ .\ .\ .\ d_t$ is called the ***MANTISSA*** and it lies between + 1 and -1

The number O is written as

$+ .000 \ldots 0 \times \beta^e$

Definition

A non zero floating point number is in ***NORMAL FORM*** if the value of the mantissa lies in the interval $\left(\left[-1,-.\frac{1}{\beta}\right] \text{ or in the interval} \left[.\frac{1}{\beta}.1\right]\right)$

Subtract the following floating point numbers 0.36143447×10^7 and 0.36132346×10^7 We have

0.36143447×10^7

$\underline{-0.36132346 \times 10^7}$

0.00011101×10^7

The result is a floating point number but not a normalised one due to the presence of three leading O's

It can be written as 0.11101×10^4 in the normalised form.

Definition

A non zero floating point number is in t - digit mantissa standard form if it is

i) normalized

ii) mantissa consists of exactly t digits

If $x = .\, d_2\, d_2 \ldots d_t\, d_{t+1} \ldots \times \beta^e$ then fl(x) in t - digit matissa standard form can be obtained in the following two ways

Chopping

Here we neglect the digits d_{t+1} , d_{t+2} …………

and obtain f 1 (x) = $d_1\ d_2 \ldots d_t \times \beta^e$

ii) Rounding

Here fractopmal part is written as $d_1\ d_2 \ldots d_{t+1}\ \beta$ and the first t - digits are taken to write the floating point number.

Example

Find the sum of $.123 \times 10^3$ and $.456 \times 10^2$ and write the result in 3 - digits mantissa.

We have

$.1230 \times 10^3$

$.0456 \times 10^3$

$$.1686 \times 10^3 = \begin{cases} .168 \times 10^3 & \text{for chopping} \\ .169 \times 10^3 & \text{for rounding} \end{cases}$$

1.4 NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

An important Problem in Science and Engineering is to find the roots of an equation f(x) = 0. This problem occurs frequently in modern computing. The functions f(x) may be algebraic or transcendental. It may not always be possible to find the roots of the equation f (x) = 0 exactly. However there are various methods to compute the roots to any desired degree of accuracy.

The following fundamental theorem enables us to locate the real root of an equation f(x) = 0.

Theorem

If f (x) is continuous in the closed intervel [a,b] and f(a) and f(b) are of opposite signs, the equation f(x) = 0 will have atleast one real root between a and b.

Bisection method

Let the equation f (x) = 0 has a root between a and b.

As a first approximation we take

$x_0 = \dfrac{a+b}{2}$ as a root of f(x) = 0

If f (x_0) = 0 then x_0 is the exact root. Otherwise the root lies between a and x_0 or x_0 and b according to the signs of f (a), f (b) and f (x_0). Then we bisect the interval and continue the process until the root is known to the desired accuracy.

Example

Find a root of the equation x^3 -5x + 1 = O by the method of bisection.

f (x) = x^3 - 5x + 1

f(0) > O and f(l) < 0

∴ The root lies between 0 and 1.

First approximation to the root is $x_0 = \frac{0+1}{2}$

$f(x_0)$ = f(0.5) < 0

∴The root lies between 0 and 0.5

Second approximation is $x_1 = \frac{0+0.5}{2}$ 0.25

f (0.25) <0

∴The root lies between o and 0.25

Third approximation is $x_2 = \frac{0+0.25}{2} = 0.125$

$f(x_2)$ = f (0.125) > 0

∴The root lies between 0.125 and o.25

Fourth approximation is $x_3 = \frac{0.125+0.25}{2} = 0.1875$

$f(x_3)$ = f(0.1875) > 0

The root lies between 0.1875 and 0.25

Fifth approximation is $x_4 = \frac{0.1875+0.25}{2} = 0.21875$

$f(x_3)$ = f (0.21875) < 0

∴ The root lies between 0.1875 and 0.21875

Sixth approximation is $x_5 = \frac{0.1875+0.21875}{2} = 0.203125$

The approximate root may be taken as= 0.203125.

Note The bisection method requires a large number of iterations to achieve a reasonable degree of accuracy for the root. It requires one function evaluation for each root.

Exercise

1) Find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places.
2) $\cos x = 3x - 2$
3) $e^x - 3x = 0$

Regula - Falsi Method

Let a root of $f(x) = 0$ lies between x_1 and x_2. Then the first approximation to the root is

$$x^{(1)} x_1 - \frac{f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1)$$

Then the root lies between x_1 and $x^{(1)}$ or x_2 and $x^{(1)}$ according to the signs of $f(x_1)$, $f(x^{(1)})$ and $f(f_2)$. The process is continued until the desired accuracy is achieved.

Example

Find a root of x tan x + 1 = 0 lying between 2.5 and 3 correct to three decimal places.

$f(x) = x \tan x + 1$

$f(2.5) = -0.8675$

$f(3) = 0.5724$

$$x^{(1)} = x_1 - \frac{f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1)$$

$$x^{(1)} = 2.5 - \frac{(-0.8675).(3 - 2.5)}{0.5724 - (0.8675)} = 2.8012$$

$f(2.8012) = 0.00787$

The root lies between 2.5 and 2.8012

The second approximation is

$$x^{(2)} = 2.5 - \frac{f(2.5)}{f(2.8012) - f(2.5)}(2.8015 - 2.5)$$

$= 2.7984$

$\therefore$ The root lies between 2.5 and 2.7984

Then $x^{(2)} = 2.5 - \dfrac{f(2.5)}{f(2.7984) - f(2.5)}$ (2.7984 – 2.5)

$= 2.798$

Hence the required root is 2.798

Example

Find the root of x e^x = 3 correct to three decimal places.

$f(x) = x\ e^{x\ -\ 3}$

f(l) =-0.28172

f (1.5) = 3.72253

$\therefore$The root lies between land 1.5

$x^{(1)} = x_1 - \dfrac{f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1)$

Substituting we get $x^{(1)}$ = 1.035

f(1.035) = -0.0864

$\therefore$ The root lies between 1.035 and 1.5

Substituting in the formula we get $x^{(1)}$ = 1.045

f (1.045) = -0.0864

The root lies between 1.045 and 1.5 substituting in the formula

$x^{(3)}$ = 1.049

f (1.049) = -0.00532

$\therefore$ The root lies between 1.049 and 1.5 .

The next approximation

$x^{(4)}$ = 1.0496

Hence the root may taken as 1.050

Exercise

1) Find a root of the equation $x^3 - 2x - 5 = 0$ between 2 and 3 correct to three places decimal.
2) Find the root of the equation $x^3 - 9x + 1 = 0$ correct to the three decimal places.
3) $3x - \log_{10}x = 6$

4) $x e^x - 2 = 0$ between 0 and 1

Method of Successive approximations

Let $f(x) = 0$ be the given equation

Express the equation in the form

$x = \phi(x)$

Let x_0 be the first approximation to the actual root

Then $x_1 = \phi(x_0)$

$x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$

are the successive approximations. This process is to be continued until the root converges.

Conditions for Convergence

Let a be a root of $f(x) = 0$. Let the equation be expressed in the form $x = \phi(x)$. If $|\phi^1(x)| < |$, for all x in the interval I contanining α, provided the initial approximation x_0 is chosen in the intervel I, then convergence takes place.

Example

Find a real root of the equation $x^3 + x^2 - 100 = 0$

$f(4) < 0$ and $f(5) > 0$

There is a root between 4 and 5

$x^2(x+1) = 100$

$$x = \frac{10}{\sqrt{x+1}} = \phi(x)$$

$$\phi^1(x) = \frac{5}{(x+1)^{3/2}}$$

$|\phi^1(x)| <$ in the interval (4,5)

Take $x_0 = 4.2$ as the initial approximation

$$x_1 = \phi(x_0) = \frac{10}{\sqrt{4.2+1}} = 4.38529$$

$$x_2 = \phi(x_1) = \frac{10}{\sqrt{4.38529+1}} = 4.30919$$

$x_3 = \phi(x_2) = 4.33996$

$x_4 = \phi\ (x_3) = 4.32744$

$x_5 = \phi\ (x_4) = 4.33252$

$x_6 = \phi\ (x_5) = 4.33046$

$x_7 = \phi\ (x_6) = 4.33129$

$x_8 = \phi\ (x_7) = 4.33096$

$x_9 = \phi\ (x_8) = 4.33109$

$x_{10} = \phi\ (x_9) = 4.33104$

$x_{11} = \phi\ (x_{10}) = 4.33106$

$x_{12} = \phi\ (x_{11}) = 4.33105$

$x_{13} = \phi\ (x_{12}) = 4.33105$

The root is 4.33105

Example

$\cos x = 3x - 1$

$f(x) = \cos x - 3x + 1$

$f(0) > 0,\ f\left(\frac{\pi}{2}\right) < 0$

The root lies between 0 and $\frac{\pi}{2}$

We write $x = \frac{1 + \cos x}{3} = \phi\ (x)$

$\phi^1\ (x)\ \frac{-\sin x}{3}$

$|\phi^1\ (x)| < 1$ in $\left(0, \frac{\pi}{2}\right)$

Let $x_0 = 0$

$x_1 = \phi\ (x_0) = 0.66667$

$x_2 = \phi\ (x_1) = 0.59530$

$x_3 = \phi\ (x_2) = 0.60933$

$x_4 = \phi\ (x_3) = 0.60668$

$x_5 = \phi\ (x_4) = 0.60718$

$x_6 = \phi\ (x_5) = 0.60709$

$x_6 = \phi\ (x_6) = 0.60710$

$x_7 = \phi\ (x_6) = 0.60710$

The root is 0.60710

Exercise

1. $x_3 - 2x + 5 = 0$
2. $\cos x + 3 = 2x$
3. $3x - \log_{10}x = 6$
4. $e^x - 3x = 10$
5. $x^3 - 2x^2 - 4 = 0$

Newton - Raphson method

Let f(x) = 0 be the equation. Let a root of the equation lies between a and b. If the root is nearer to a than to b we take.

x_0 = a as the initial approximation and if the root is nearer to b then x_0 = b is to be taken as the initial approximation.

$$x_1 = x_0 - \frac{f(x_o)}{f^1(x_o)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

Note

The convergence in this method is very fast.

Example

Find the root of $\sin x = 1 + x^3$ between -2 and -1.

$f(x) = x^3 - \sin x + 1$

$f(-1) = 0.8415$

$f(-2) = -6.0907$

The root lies between -1 and -2

Further

The root is nearer to -1

Take x_0 = - 1 as the initial approximation.

f^1 (x) = $3x^2$ - cos x

$x_1 = - x_0 = \frac{f(x_0)}{f^1(x_0)} = -1 - \frac{f(-1)}{f^1(-1)} = -1.3421$

$x_2 = - x_1 = \frac{f(x_1)}{f^1(x_1)} = -1.2565$

$x_3 = - x_2 = \frac{f(x_2)}{f^1(x_2)} = -1.2490$

$x_4 = - x_3 = \frac{f(x_3)}{f^1(x_3)} = -1.2491$

$\therefore$The root is -1.2491.

Example

3x = cos x + 1

f (x) = 3x - cos x -1

f(O) = -2

f(l) = 1.45969

The root lies between 0 and 1 nearer to 1.

Take x_Q = 0.6 as the initial approximation.

f^1 (x) = 3 + sin x

$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 0.6071$

$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 0.607101$

Hence the root is 0.6071 correct to 4 places.

Exercise

1) $x^3 - x - 1 = 0$
2) $x^2 + 4\sin x = 0$
3) x tan x = 1.28

$x e^x = \cos x$

Find $3 \sqrt{17}$

Graffe Root Squaring Method

This method is used to find all the roots of a polynomial equations.

The underlying principle in this method is

i. construct an equation whose roots are negative squares of the roots of the given equation.
ii. Repeat step (i), m times.
iii. Estimate the roots Xi of the final equation.
iv. Determine the roots of the given equation from λ_1.

Let the given equation be

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0 \ldots. (1)$

Let $\alpha_1, \alpha_2, \ldots$ an be the roots

The working rule is given in tabular form for a 4th degree equation.

	a_0	a_1	a_2	a_3	a_4
	a_0^2	a_1^2	a_2^2	a_3^2	a_3^2
		$-2\, a_0 a_2$	$-2\, a_1 a_3$	$-2\, a_2 a_4$	
			$-2\, a_0 a_4$		
Add	b_0	b_1	b_2	b_3	b_4

This process is called one squaring. Do m squarings and get the equation

$B_0 u^4 + B_1 u^3 + B_2 u^2 + B_3 u + B_4 = 0 \ldots\ldots (2)$

Let λ_1 are the roots of equation (2). Then λ_2 and λ_3 are related by

$\lambda_1 = -\alpha_1 - 2^m$

Let α_1 be real $|\alpha_1|\ |\alpha_2| < |\alpha_3| < |\alpha_4|$

Also from (2) we have

$$\lambda_1 = \frac{-B_1}{B_0}, \lambda_2 = \frac{B_2}{B_1}, \lambda_3 = \frac{B_3}{B_2},$$

$$\lambda_4 = \frac{B_4}{B3}$$

The roots of the original equation are obtained from

$|\alpha_1| = (-\lambda_1)^{\frac{1}{2^m}}$

The sign of α_1 can be determined by actual substitution in the given equation. The Descarte's Rule of signs can be helpful in this respect.

Descarte's rule of signs

Let f(x) = 0 be a polynomial equation with real coefficients. Then

i) The number of positive roots of f(x) = 0 is either equal to the number of sign changes in f(x) or less than that number by a positive even integer.

ii) The number of negative roots of f(x) = 0 is either equal to the number of sign changes in f(-x) or less than that number by a positive even integer.

Example

Solve $3x^3 - 9x^2 + 8 = 0$ by 3 Squarings

3	-9	0	8	
9	81 0	0 144	64	
9	81	144	64	First Squaring
81	6561 -2592	20736 –10368	4096	
81	3969	10368	4096	Second Squaring
6561	15752961 - 1679616	107495424 - 32514048	16777216	
6561	14073345	74981376	16777216	Third Squaring

Hence $|\alpha_1| = \left(\frac{14073345}{6561}\right)^{\frac{1}{2^3}} = 2.6087$

$|\alpha_2| = \left(\frac{74981376}{14073345}\right)^{\frac{1}{2^3}} = 1,2326$

$|\alpha_3| = \left(\frac{16777216}{74981376}\right)^{\frac{1}{2^3}} = 0.8293$

By substitution we find the signs

$\alpha_1 = 2.6087$, $\alpha_2 = 1.2326$, $\alpha_3 = 0.8293$

Example

Solve $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ by two squarings.

1	-10	35	-50	24	
1	100	1225	2500	576	
	-70	- 1000	-1680		
		48			
1	30.	273	820	576	First Squaring
1	900	74529	672400	331776	
	546	-49200	314496		
		1152			
1	354	26481	357904	331776	Second Squaring

$$|\alpha_1| = \left(\frac{354}{1}\right)^{\frac{1}{2^2}} = 4.3375$$

$$|\alpha_2| = \left(\frac{26481}{354}\right)^{\frac{1}{2^2}} = 2.9409$$

$$|\alpha_3| = \left(\frac{357904}{26481}\right)^{\frac{1}{2^2}} = 1.9174$$

$$|\alpha_4| = \left(\frac{331776}{357904}\right)^{\frac{1}{2^2}} = 0.9812$$

We find all the roots are positive.

Note. The exact roots are 4, 3, 2, 1

Exercise

1) $x^3-2x-1=0$
2) $x^3-2x^2-x+1=0$
3) $x^3-9x^2+23x+14=0$
4) $x^4-15x^2-10x+24=0$

1.5 SYSTEM OF LINEAR EQUATIONS

Consider the system of Linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

It can be written in the matrix form

AX = B where

A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ x = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and B = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Gauss Elimination Method

This is direct method

Step 1

Form the augumented Matrix

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Step 2

Eliminate x_1 from the second and third equations by performing the elementary row transformations $R_2 - \frac{a_{21}}{a_{11}} R_1$ and $R_3 - \frac{a_{31}}{a_{11}} R_1$

Where R_1 represents the ith row. The equations are rearranged in such a manner that $a_{11} \neq 0$

Step 3

Let transformed matrix be

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Here it is assumed that $a_{22} \neq 0$

Eliminate x_2 from the third equation by performing $R_3 - \frac{a_{32}}{a_{32}} R_2$

Then we got the upper triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & & a_{33} & b_3 \end{bmatrix}$$

Then x_1 , x_2 , x_3 can be found by back substitution

Note

It is better to rearrange the equations such that the first equation has the largest coefficient of x_1 second equation has the largest coefficient of x_2 etc. Here a_{11} and a_{22} are called pivotols.

Gauss Jordaon Method

This is a slightly modified method. Here the coefficient matrix A is transformed into a diagonal matrix by performing suitable elementary transformations and values x_x, x_2, x_3 are determined discreetly.

Example

Solve

$5x_1 - 2x_2 + 3x_3 = 18$

$x_1 + 7x_2 - 3x_3 = -22$

$2x_1 - x_2 + 6x_3 = 22$ by the Gauss method

$$[A\,|\,B] = \begin{bmatrix} 5 & -2 & 3 & 18 \\ 1 & 7 & -3 & -22 \\ 2 & -1 & 6 & 22 \end{bmatrix}$$

Perform $R_2 - \frac{1}{5}$ and $R_3 - \frac{2}{5} R_1$ The matrix reduces to

$$\begin{bmatrix} 5 & -2 & 3 & 18 \\ 0 & \frac{37}{5} & \frac{18}{5} & \frac{-128}{5} \\ 0 & \frac{-1}{5} & \frac{24}{5} & \frac{74}{5} \end{bmatrix}$$

For convenience perform $5R_2$ and $5R_3$

$$\begin{bmatrix} 5 & -2 & 3 & 18 \\ 0 & 37 & -18 & -128 \\ 0 & -1 & 24 & 74 \end{bmatrix}$$

Now perform $R_3 + \frac{1}{37} R_2$

By back Substitution we get

$$\begin{bmatrix} 5 & -2 & 3 & 18 \\ 0 & 37 & -18 & -128 \\ 0 & 0 & \frac{870}{37} & \frac{2610}{37} \end{bmatrix}$$

By back substitution we get

$$\frac{870}{37} x_3 = \frac{2610}{37}$$

i.e. $x_3 = 3$

$37x_2 - 18x_3 = -128 \implies x_2 = -2$

$5x_1 - 2x_2 + 3x_3 = 18 \implies x_1 = 1$

So we set $x_1 = 1$, $x_2 = -2$, $x_3 = 3$

Example

So the above problem using Gauss – Jordan Method
We have already

$$\begin{bmatrix} 5 & -2 & 3 & 18 \\ 0 & 37 & -18 & -128 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

In the resulting matrix do suitable transformations using R_3 as pivotal row to make it diagonal matrix.

$R_1 + \frac{2}{37} R_2$

Then we get

$$\begin{bmatrix} 5 & 0 & 0 & 5 \\ 0 & 37 & 0 & -74 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Hence $x_2 = 3$, $x_2 = -2$, $x_1 = 1$

Exercise

Gauss Elimination Method

1) $3x - y + 2z = 12$, $x + 2y + 3z = 11$, $2x - 2y - z = 2$
2) $5x - y - 2z = 142$, $x - 3y - z = -30$, $2x - y - 3z = 5$
3) $2x + y + 4z = 12$, $8x - 3y + 2z = 20$, $4x + 1 ly - z = 33$

Gauss - Jordon Method

1) $2x - 3y + x = -1$, $x + 4y + 5z = 25$, $3x \sim 4y + z = 2$
2) $2x + y + 4z = 12$, $8x - 3y + 2z = 20$, $4x + lly - z = 33$

Grouts Method

This is also direct method consider the system of equations

A X = B Here A is written as A = L U where

Let U X = K

From the above equations all the unknowns can be found.

Working Rule

The augmented matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$ ..... (1)

The derived matrix is $\begin{bmatrix} l_{11} & u_{12} & u_{13} & k_1 \\ l_{21} & l_{22} & l_{23} & k_2 \\ l_{31} & l_{32} & l_{33} & k_3 \end{bmatrix}$ ..... (2)

Step 1

First column of (2) is same as first column (1)

Step 2

The first row to the right of the first column

Each element $\dfrac{Correspondng\ elements in (1)}{Leading element in (2)}$

Step 3

Remaining second column.

$I_{22} = a_{22} - I_{21}\, u_{12}$

$L_{32} = a_{32} - I_{31}\, u_{12}$

Step 4

Remaining Second row

$u_{23} = \dfrac{a_{32} - I_{21}\, u_{13}}{I_{22}}$

$k_2 = \dfrac{b_2 - I_{21}\, k_1}{I_{22}}$

Step 5

Remaining third column

$l_{33} = a_{33} - \left(l_{31}\, u_{13} + l_{32}\, k_2\right)$

Step 6

Remaining third row

k3 = $\dfrac{b_3-(b_{31}k_1+I_{32}k_2)}{I_{33}}$

Step 7

Consider U X = K

By back substitution we get the values of x_3, x_2 and x_1

Example

2x - 6y + 8z = 24

5x + 4y - 3z = 2

3x + y + 2z= 16

$$\begin{bmatrix} 2 & -6 & 8 & 24 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{bmatrix} \qquad \text{....(1)}$$

The derived matrix is $\begin{bmatrix} l_{11} & u_{12} & u_{13} & k_1 \\ l_{21} & l_{22} & u_{23} & k_2 \\ l_{31} & l_{32} & l_{33} & k_3 \end{bmatrix}$

The first column of (2) is same as first column (1)

$\therefore I_{11} = 2, I_{21} = 5, I_{31} = 3$

Remaining elements of first row

u_{12} = $\dfrac{-6}{2}$ -3, u_{13} = $\dfrac{8}{2}$ = 4, k_1 = $\dfrac{24}{2}$ = 12

Remaining elements of second column

$I_{22} = a_{22} - I_{21}\, u_{12} = 4 - 5\,(-3) = 19$

$I_{32} = a_{32} - I_{31}\, u_{12} = 1 - 5\,(-3) = 10$

Remaining elements of second row

$u_{23} = \dfrac{a_{23}-I_{21}u_{13}}{I_{22}} = \dfrac{-3-5\times 4}{19} = -\dfrac{23}{19}$

$k_2 = \dfrac{b_2 - I_{21}k_1}{I_{22}} = \dfrac{2-5\times 12}{19} = \dfrac{-58}{19}$

Remaining third column

$I_{33} = a_{33} = \left(I_{31}u_{13}+1_{32}u_{23}\right)$

$= 2 - \left(3\times 4+10\left(\dfrac{-23}{19}\right)\right) = \dfrac{40}{19}$

Remaining third row

$$k_3 = \frac{b_3 - \left(I_{31}\,k_1 + I_{32}\,k_2\right)}{I_{33}}$$

Thus we get from UX = K

$$= \frac{16 + \left\{3 \times 12 + 10\left(-\frac{58}{19}\right)\right\}}{\frac{40}{19}} = 5$$

Thus we get from UX = K

$$A = \begin{bmatrix} 1 & -3 & \\ 0 & 1 & -\frac{23}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -\frac{58}{19} \\ 5 \end{bmatrix}$$

By back substitution

z = 5, y = 3, x = 1.

Exercise

1) 10x + y + z=12, 2x+10y + z = 13, 2x + 2y + 10z = 14
2) x + y + z = 3, 2x- y + 3z = 16, 3x + y - z = -3
3) 3x + y + z = 8, -x + y - 2z = -5, -2x + 2y-3z = -7

Gauss-siedal method

This is an iterative method.

Condition for convergence

In each equation of the system, the absolute value of the leading coefficient is greater than the sum of the absolute values of all the remaining coefficients. i.e.

$$|a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

In other words matrix A is diagonally dominant. Let the system of equation be

$a_1\,x + b_1\,y + C_1\,z = d_1$

$a_2\,x + b_2\,y + c_2\,z = d_2$

$a_3\,x + b_3\,y + c_3\,z = d_3$

If the convergent conditions are not satisfied interchange the equations suitably so that larger coefficients are in the leading diagonal.

Write the equations as

$$x = \frac{1}{a_1}\left(d_1 - b_1\, y\, c_1\, z\right)$$

$$y = \frac{1}{b_2}\left(d_2 - a_2\, x\ \ c_2\, z\right)$$

$$z = \frac{1}{c_2}\left(d_3 - a_3\, x\ \ b_3\, y\right)$$

Start with initial values 0, 0, 0 for x, y, z substitute y = 0. z = 0. in (1) and get the value of x say $x^{(1)}$

Now substitutes in 2 $x = x^{(1)}$ and z = 0 and find $y^{(1)}$

Now substitute in (3) $x = x^{(1)}$ and $y = y^{(1)}$ and find $z^{(1)}$

As soon as a new value for a variable is found, it is used immediately in the following equations.

Repeat the process until the roots converge. Example

Example

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110.$

Here the convergence conditions are satisfied

$$x = \frac{1}{27}(85 - 6y + z) \qquad \text{---(1)}$$

$$y = \frac{1}{15}(72 - 6x - 2z) \qquad \text{---(2)}$$

$$z = \frac{1}{54}(110 - x - y) \qquad \text{---(3)}$$

Put y = 0, z = o in (1) $x^{(1)} = \frac{85}{27} = 3.14815$

Put x = 3.14815, z = 0 in (2) then y(1) = 3.54074

Put x = 3.14815, y = 3.54074 in (3) then $z^{(1)} = 1.91317$

Repeat the process and tabulate the values.

X	y	Z
0	0	0
3.14815	3.54074	1.91317
2.43218	3.57204	1.92585
2.42569	3.57294	1.92595
2.42549	3.57301	1.92595
2.42548	3.57301	1.92595

The roots are

x = 2.4255

y = 3.5730

z = 1.9269

Example 2

X + 17y-2y = 48

30x - 2y + 3z = 75

2x + 2y + 18z = 30

We rearrange the equation as

30x - 2y + 3z = 75

X + 17y - 2z = 48

2x + 2y + 18z = 30

Now the leading diagonal has largest coefficients. We can processed similarly as in Example 1.

Exercise

1) 8x - 3y + 2z = 30, 4x + 11y - z = 33, 6x + 3y + 12z = 35
2) 8x - y + z = 18, x + y - 3z = -6, 2x + 5y - 2z = 3
3) 10x - 2y + z = 2, x + 9y - z 10, 2x - y + llz = 20

Dr. Rm. Somasundaram
Reader in Engineering Mathematics
Annamalai University

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INTERPOLATION

2.1 INTRODUCTION

Suppose the following table represents a set of corresponding values of two quantities x and y

x : x_0 x_1 x_2 . . . x_n

y : y_0 y_1 y_2 . . . y_n

Then the process of finding the value of y corresponding to any value of x = x i between x_0 and x_n is called interpolation. Thus interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation. However, the term interpolation is applied to both processes.

2.2 NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function y = f(x) take the values y_0 y_1 y_2 . . . y_n corresponding to the values x_0 x_1 x_2 ... x_n of x. Let these values of x be equi-spaced such that $x_1 = x_0 + i_h$ (i = 0 , 1 .... n). If it is required to evaluate y for x_0 = x + ph.

$$y(x) = y\left(x_0 + ph\right) = y_p = y_0 + \left(P\Delta y_0\right) + \frac{p\left(p+1\right)}{2!}\Delta^2 y_0 + \frac{p\left(p-1\right)\left(p-2\right)}{3!}\Delta^3 y_0 + \ldots$$

$$+ \ldots \frac{p(p-1)\ldots\left(p-\overline{n-1}\right)}{n!}\Delta^n y_0 + .$$

It is called Newton's forward interpolation formula. This formula is used for interpolating the values of y near the beginning of a set of tabulated values.

2.3 NEWTON'S BACKWARD INTERPOLATION FORMULA

This formula is used for interpolating the values of y near the end of a set of tabulated values.

Suppose it is required to evaluate y(x) for x = x_n + ph where p is any real number

$$y_p = y\left(x_n + ph\right) = y_n + p\,\nabla y_n + \frac{p\left(p+1\right)}{2!}\nabla^2 y_{n} + \frac{p\left(p+1\right)\left(p+2\right)}{3!}\nabla^3 y_{n} + .$$

It is called Newton's backward interpolation formula,

Worked Examples

Ex. 1 Using Newton's forward interpolation formula, find the value of y when x = 21 from the following tabulated values of the function

x:	20	23	26	29
y:	0.3420	0.3907	0.4384	0.4848

∴ 21 is near the beginning of the table. Hence we use Newton's forward formula.

x	y	Δ	$\Delta^2 y$	$\Delta^3 y$
20	0.3420			
		.0487		
23	0.3907		-.001	
		.0477		-.0003
26	.4384		-.0013	
		.0464		
29	0.4848			

We have $y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots.$

Here $x_0 = 20$, $y_0 = 0.3420$, $\Delta y_0 = 0.0487$, $\Delta^2 y_0 = -0.0010$

$\Delta^3 y_0 = -0.0003$. Let y_0 be the values of y where x = 21.

Then $p = \frac{x - x_0}{n} = \frac{21-20}{3} = 0.33333$ (n = increment on x = 3) = 0.3

$y_p = 0.3420 + 0.3(0.0487) + \frac{0.3(-0.7)}{2}(-0.0010) + \frac{.3(-.7)(-1.7)}{6}$ (-.0003_

$= 0.3420 + 0.01461 + 0.000105 – 0.00001785 = 0.3567$

Ex 2: Using Newton's formula find y when x = 27, from the following data

x:	10	15	20	25	30
y:	35.4	32.2	29.1	26.0	23.1

Since the value 27 of x is near the end of the table, to get the corresponding value of y, we use Newton's backward formula.

x	y	∇_y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	35.4				
		-3.2			
15	32.2		0.1		
		-3.1		-0.1	
20	29.1		0.		0.3
		-3.1		-0.2	
25	26.0		0.2		
		-2.9			
30	23.1				

Here

$x_n = 30$, $y_n = 32.1$, $\nabla\ y_n = -2.9$, $2y_n = 0.2$ $\nabla^3 y_n = 2.$ $\nabla^4 y_n = 0.3$

Then $p = \frac{x - x_n}{n} = \frac{27-30}{5} = -0.6$

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\ \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

$$= 23.1 \quad + (-0.6)\ (-2.9) + \frac{(-0.6)(0.4)}{2}\ (0.2)$$

$$+ \frac{(-0.6)(0.4)(1.4)}{6}(0.2) + \frac{(-0.6)(0.4)(1.4)(2.4)}{24}\ (0.3)$$

$$= 23.1 \quad + \ 1.74 - 0.024 \ - 0.0112 - 0.01008$$

$$= 24.79472 \quad 24.8$$

Exercise

1) Estimate exp (1.85) from the following table

x:	1.7	1.8	1.9	2.0	2.1	2.2	2.3
exp (x):	5.474	6.050	6.686	7.389	8.166	9.025	9.974

2) If f (1.15) - 1.0723, f (1.20) = 1.0954, f (1.25) = 1.1180 and f (1.30) > 1. 1401, find f (1.28).

3) Given sin 45 ° = 0.7071, sin 50 ° = 0.7660 , sin 55 ° = 0.8192 and sin 60° = .8660 find sin 52 ° using Newton's forward formula.

4) The following table gives the values of f(x) for five values of x.

x:	0	1	2	3	4
for:	4	12	32	76	156

Find f(5)

Answers (1) 6.36 (2) 1.1313 (3) 0.788 (4) 284

Ex. 3 Construct Newton's forward interpolation polynomial for the following data:

x :	4	6	8	10
y:	1	3	8	16

Use it to find the value of y for x = 5.

x	y	Δ_y	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

By Newton's forward Interpolation formula, we have

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\,\Delta^2 y_0$$

Here $x_o = 4$, $y_0 = 1$, 1 $\Delta y_0 = 2$, $\Delta^2 y_0 = 3$

$$p = \frac{x - x_0}{h} = \frac{x-4}{2}$$

$$\therefore\ y = 1 + \left(\frac{x-4}{2}\right) x^2 + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2} x^3$$

$$y = 1 + x - 4 + \frac{(x-4)(x-6)}{2}\ x^3 = \frac{3x^2}{8} - \frac{11x}{4} + 6$$

Exercise

1) Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x.

   x: 1 2 3 4 5

   y: 1 1-1 -1 1

2) A third degree polynomial passes through the points (0,-1), (1,1), (2,1) and (3,-2). Find the polynomial.

Answers (1) $y = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56 \times 31$

(2) $y = \frac{-1}{6}(x^3 + 3x^2 - 16x +. 6)$

Equidistant terms with one or more missing values

When one or more of the values of f(x) corresponding to the equidistant values of x are missing. We can find the missing values, by using the symbolic finite difference operators E and A. The method is illustrated in the following examples

Ex 1. Given $\mu_0 = -4$, $\mu_1 = 2$, $\mu_4 = 220$, $\mu_5 = 546$, $\mu_6 = 1148$, find μ_2 and μ_3

Since five values of μ_x are given, it is possible to express μ_x as a polynomial of the 4^{th} degree. In that case, the fourth differences of μ_x will be constant and all higher differences will vanish ie $\Delta^5\,\mu_{x=0}$, $\Delta^6\,\mu_{x=0}$ etc for all values of x. considering the leading differences, we have

$\Delta^5\,\mu_0 = 0$...... (1) and $\Delta^6\,\mu_0 = 0$ ......(2)

Since the $\Delta = E - 1$ equation (1) becomes

$(E-l)^5 \ \mu_0 = 0$

i.e. $(E^5 - 5c_1 E^4 + 5c_2 E^3 - 5c_3 E^2 + 5c_4 E - 1) \mu_0 = 0$

i.e. $\mu_6 - 5\mu_4 + 10 \mu_3 - 10\mu_2 - 5 \mu_1 - \mu_0 = 0$

Substituting the given values

$546 - 1100 + 10 \mu_3 - 10\mu_2 - 10 + 4 = 0$

$10 \mu_3 - 10\mu_2 = 560$

$\mu_3 - \mu_2 = 56$ ... (3)

|||ly Equation (2) becomes $- 4 \mu_3 + 3 \mu_2 = - 236$ ... (4)

Solving (3) , (4) $\mu_3 = 68$ and $\mu_2 = 12$

Exercise

1. Find the missing values in the following data

x:	0	5	10	15	20	25
y:	6	10	-	17	-	31

2. Using a polynomial of the 3rd degree, complete the record given below of the export of a certain commodity during five years.

Year	: 1917	1918	1914	1920	1921
Export in tonns)	: 443	384	-	397	467

3. Find the missing value from the following data

x :	2	4	6	8	10
y:	5.6	8.6	13.9	-	35.6

4. Answers (l), y 2 = 13.15, $y_4 = 22.50$ (2) $y_2 = 369$ (3)22.55

2.4 CENTRAL DIFFERENCE TABLES

Let y = f(x) be the functional relation between x and y. Suppose we give consecutive values to x, differing by h, i.e. x takes the values a, a + h, a + 2h Let y_0, y_1, y_2, ... be the corresponding values of y. When x = a - h, a - 2h, a-3h, the corresponding values of y are y_{-1}, y_{-2} , y_{-3}. We can form a difference table with the values of f(x) on either side of x = a.

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
a – 3h	y_{-3}						
		Δy_{-3}					
a – 2h	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
a – h	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
a	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
a + h	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
a + 2h	y_2		$\Delta^2 y_1$				
		Δy_2					
a + 3h	y_3						

Central Difference Interpolation Formulae

We shall develop central difference formulae which are best suited for interpolation near the middle of the table.

Gauss's forward interpolation formula :

$$y_p = y_0 + p\Delta y_0 + \frac{P(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \text{where } \frac{p = x - x_0}{h}$$

The above formula involves odd differences below the central line and even differences on the line.

$$y_0 \qquad \Delta^2 y_{-1} \qquad \Delta^4 y_{-2} \qquad \Delta^4 y_{-3}$$

$$\Delta y_0 \qquad \Delta^3 y_{-1} \qquad \Delta^5 y_{-2}$$

This formula is called Gauss's forward formula of interpolation, as it is used to interpolate the values of y of p (o < p < 1) measured forwardly from the origin.

Gauss's backward interpolation formula.

$$y_p = y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)\,p\,(p+1)(p-1)}{4!}\,\Delta^4 y_{-1} + \ldots \text{ where } p = \frac{x-x_0}{h}$$

This formula involves odd differences above the central line and even differences on the line.

$$\begin{array}{ccccccccccccc} & & \Delta y_{-1} & & & & \Delta^3 y_{-2} & & & & \Delta^5 y_{-2} & & \\ & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & \\ y_0 & & & & \Delta^2 y_{-1} & & & & \Delta^4 y_{-2} & & & & \Delta^6 y_{-3} \end{array}$$

This formula is called Gauss's backward formula of interpolation, as it is used to interpolate the value of the function for a negative value of P (- 1 < P < 0)

Ex 1 Apply Gauss's forward formula to find the value of f(x) at x = 3.75 from the table

x:	2.5	3.0	3.5	4.0	4.5	5.0
f(x) :	24.145	22.043	20.225	18.644	17.262	16.047

Taking 3.5 as the origin, h = .5 and 8 = $\frac{x-3.5}{.5}$ we have the following central difference table.

p	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	24.145					
		-2.102				
-1	22.043		0.284			
		-1.818		0.047		
0	20.225		0.237		0.009	
		1.581		0.038		-0.003
1	18.664		0.199		0.006	
		-1.382		0.032		
2	17.262		0.167			
		-1.215				
3	16.047					

Gauss's forward formula is

$$y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p+1)}{2!}\,\Delta^2 y_{-1} + \frac{(p+2)\,p\,(p-1)}{5!}\,\Delta^3 y_{-1}$$

$$+ \frac{(p+2)\,p\,(p-1)(p-2)}{4!}\,\Delta^4 y_{-1} + \frac{(p+2)(p+1)p\,(p-1)(p-2)}{5!}\,\Delta^5 y_{-1} + \ldots \text{ (1)}$$

Where x = 3.75 , p = $\frac{3.75-3.5}{.5}$ = .5

$y_0 = 20.225$, $\Delta^2 y_{-1} = 0.237$, $\Delta^4 y_{-2} = 0.009$

$\Delta y_0 = -1.581 \; \Delta^3 y_{-1} = -0.038 \; \Delta^5 y_{-2} = -0.003$

Hence substituting in (1)

$$Yp = 20.225 + 9.5)\,(-1.581) + \frac{(.5)(-.5)}{2}(.237)$$

$$+\frac{1.5(.5)(-.5)}{6}\,(-0.038) + \frac{1.5(.5)(-5)(-1.5)}{24}(0.009)$$

$$\frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{120}\,(-0.003)$$

$= 20.225 – 0.7905 – 0.002375$

$+ 0.00022109375 – 0.0000315425 = 19.407$

Ex 2 Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that

Year :	1939	1949	1959	1969	1979	1989
Population (in thousand) :	12	15	20	27	39	52

We choose the origin at 1969, h = 10 , $p = \frac{x - x_0}{h} = \frac{1974 - 1969}{10} = .5$

The central difference table is

p	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-3	12					
		3				
-2	15		2			
		5		0		
-1	20		2		3	
		$\boxed{7}$		$\boxed{3}$		-10
0	$\boxed{27}$		$\boxed{5}$		$\boxed{-7}$	
		12		-4		
1	39		1			
		13				
2	52					

Gauss’s backward formula is

$$y_p = y_0 + p\,\Delta\; y_{-1} + \frac{(p+1)\,p}{2!}\;\Delta^2\; y_{-1} + \frac{(p+2)\,p\;(p-1)}{3!}\;\Delta^2\, y_{-2}$$

$$+\frac{(p+2)\;(p+1)\,p\,(p-1)}{4!}\;\Delta^4\, y_{-2} + \frac{(p+2)(p+1)p\;(p-1)(p-2)}{5!}\;\Delta^5\, y_{-3} + \ldots$$

$$= 27 + (0.5)\,7 + \frac{(1.5)(.5)}{2}(5)\;\frac{(1.5)(.5)(-.5)}{6}\;(3)$$

$$+ \frac{(2.5)(1.5)(0.5)(-.5)}{120} (-.7) + \frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{120} (-10)$$

= 27 +3.5 1.875 – 0.1875 +0.2743 – 0.1172 = 32.345 thousands

Exercise

1) Using Gauss's forward formula, evaluate f (2.7) from the table.

x	:	1.5	2.0	2.5	3.0	3.5	4.0
f(x)	:	37.9	246.2	409.3	537.2	636.3	715.9

2) Find the value of cos 51° 42 ' by Gauss's backward formula given that

x	:	50°	51°	52°	53°	54°
Cosx	:	0.6428	0.6293	0.6157	0.6018	0.5878

3) Given that $\sqrt{12500}$ = 111 .8033999 , $\sqrt{12510}$ = 111.848111,

$\sqrt{12520}$ = 111. 892806 , $\sqrt{12530}$ = 111.937483

show by Gauss's backward formula that V12516 = 111.874930

Answers (1) 464.38 (2) 0.6198

Stirling's Formula

$$y_p = y_0 + \frac{p}{1}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2_{y-1}$$

$$\frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4_{y-2}$$

$$+ \frac{p(p^2-1)(p^2-4)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \frac{p^2(p^2-1)(p^2-4)}{6!}\Delta^6_{y-3} +$$

This formula involves the means of the odd differences just above and just below the central line and even differences on the line

$$\begin{array}{ccccc} & \Delta y_{-1} & & \Delta^4 y_{-2} & \\ y_0 & & \Delta^2 y_{-1} & & \Delta^4 y_{-2} \\ & \Delta y_{-2} & & \Delta^3 y_{-1} & \end{array}$$

y0 A2y_x A4y_2

Ay-2 A3y-i

In using this formula, we should have - 1 /2 < p < 1/2 Good estimates will be got if - 1/4 < p < 1/4.

Ex 1 : Given θ:		0°	5°	10°	15°	20°	25 °	30°
tan θ	:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Show that tan 16 ° = 0.2867. Use Stirling's formula

Taking 15 ° as the origin, h = 5° , p = $\dfrac{x-15}{5}$

p	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
-3	0.0000						
		0.0875					
-2	0.0875		0.0013				
		0.0888		0.0015			
-1	0.1763		0.0028		0.0002		
		$\boxed{0.0916}$		$\boxed{0.0017}$		$\boxed{-0.0002}$	
0	$\boxed{0.2679}$		$\boxed{0.0045}$		$\boxed{0.0000}$		$\boxed{0.0011}$
		$\boxed{0.0916}$		$\boxed{0.0017}$		$\boxed{0.0009}$	
1	0.3640				0.0009		
		0.1023	0.0062	0.0026			
2	0.4664						
		0.1111	0.0088				
3	0.5778						

Stirling's formula is

$$y_b = y_0 + \frac{p}{1}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p2}{2!}\Delta^2 y_{-1}$$

$$+ \frac{p\left(p^2-1\right)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2\left(p^2-1\right)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{p\left(p^2-1\right)\left(p^2-4\right)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \frac{p^2\left(p^2-1\right)\left(p^2-4\right)}{6!}\Delta^6 y_{-3} + \ldots$$

At x = 16°, p = .2

$y_0 = 0.2679$ $\Delta^2 y_{-1} = 0.0045$ $\Delta^4 y_{-2} = 0.0000$ $\Delta^6 y_{-3} = 0.0011$

$\Delta y_{-1} = 0.0916$ $\Delta^3 y_{-2} = 0.0017$ $\Delta^5 y_{-3} = 0.0002$

$\Delta y_0 = 0.0961$ $\Delta^3 y_{-1} = 0.0017$ $\Delta^5 y_{-2} = 0.0009$

Substituting these values in (1)

$$y_p = 0.2679 + .2\,\frac{(0.0916+0.0961)}{2} + \frac{.2\times.2}{2}(0.0045)$$

$$+ \frac{(.2)\,(.04-1)}{6}\;\frac{(0.0017+0.0017)}{2} + \frac{0.04\,(0.04-1)}{24}\,(0.0000)$$

$$\frac{(.2)(.04-1)(.04-4)}{120}\left(\frac{0.0009-0.3002}{2}\right)+\frac{.04(.04-1)(.04-4)}{720}\ (0.0011)$$

= .2679 + 0.01877 + 0.0009 – 0.0000544 + 0.0000022176 + 0.0000002323

= 0.2867 neraly

Exercise

1. Employ Stirling's formula to compute $u_{12.2}$ from the following table (u_x = 1 +log sin x)

x^0	:	10	11	12	13	14
$10^5 U_X$	:	23.967	28.060	31.788	32.209	355.368
Ans	:	0.32495				

2.5 INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formula derived so far possess the disadvantage of being applicable only to equally spaced values of the arguments. For unequally spaced values of x, we use Lagrange, interpolation formula.

Lagrange's interpolation formula: If y = f (x) takes the value $y_0, y_1\ y_2, \ \dots\ y_n$ corresponding to $x_0\ x_1, \ \dots\ x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_n)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}\ y_1$$

$$+ \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}\ y_n$$

This is known as Lagrange's interpolation formula for unequal intervals.

Ex 1 Given the values

x	:	14	17	31	35
f(x)	:	68.7	64.0	44.0	39.1

Find the value of f (x) corresponding x = 27

Since there are only four corresponding pairs of values given the polynomial representing the data is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\ y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\ y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \qquad (1)$$

Here $x_0 = 14$ $x_1 = 17$ $x_2 = 31$ $x_3 = 35$

$y_0 = 68.7$ $y = 64.0$ $y_2 = 44.0$ $y_3 = 39.1$

Putting x = 27 in (1)

$$\therefore y_{27} = \frac{(27-17)(27-31)(27-34)}{(31-14)(31-17)(31-35)} 68.7 + \frac{(27-14)(27-31)(27-35)}{(17-14)(17-31)(17-35)} 64.0$$

$$+ \frac{(27-14)(27-17)(27-34)}{(31-14)(31-17)(31-35)} 44.0 + \frac{(27-14)(27-17)(27-31)}{(35-14)(35-17)(35-31)} 39.1$$

= - 20.52 + 35.22 + 48.07 - 13.45 = 49.3

Exercise

1. Use Lagrange's interpolation formula to find the value of y when x = 10 if the following values x and y are given

x :	5	6	9	11
y :	12	13	14	16

2. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ $\log_{10} 661 = 2.8202$.
3. Given f(0) = - 18, f (1) = 0, f (3) = 0, f (5) = - 248, f (6) = 0 f(9)= 13 105 find f(x)
4. Given $y_0 = - 12$, $y_1 -= 0$, $y_3 = 6$, $y_4 = 12$ find y_2
5. The observed values of a function f (x) for different values of x are given in the following table

x	:	30	35	45	55
f(x)	:	148	96	68	34

Answers : (1) 14.63 (2) 2.8168 (3) $x^5 - 9x^4 + 18x^3 - x^2 + 9x - 18$ (4) 4 (5) 75

2.6 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of" values (x_1, y_4)

Formulae for derivatives: Consider the function y = f (x) which is tabulated for the values $x_1(= x_0 + ih)$, i = 1, 2 n. If the value of $\frac{dy}{dx}$ is required at a point near the beginning of the table, we use Newton - Gregory forward formula

At $x = x_0$,p = 0

$$\left(\frac{dy}{dx}\right) x_0 = \frac{1}{h}\left(\Delta y_0 - 1/2\ \Delta^2 y_0 \frac{1}{3}\Delta^3 y_0 - 1/4\,\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right)$$

$$\left(\frac{d^2y}{dx^3}\right) x_0 = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_3 + \ldots\right)$$

$$\left(\frac{d^3y}{dx^3}\right) x_0 = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right)$$

If the value of $\frac{dy}{dx}$ is required at a point near the end of the table, we use Newton-Gregory backward formula.

At $x = x_n$, $p = 0$

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \dots\right)$$

$$\left(\frac{d^3y}{dx^3}\right) x_n = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \dots\right)$$

$$\left(\frac{d^3y}{dx^3}\right) x_n = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^3 y_n + \dots\right)$$

If the value of $\frac{dy}{dx}$ is required at a point near the middle of the table, we should use Stirling's formula. At $x = x_0$, $p = 0$

$$\left(\frac{dy}{dx}\right) x_0 = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta 5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right]$$

$$\left(\frac{d^2y}{dx^2}\right) x_0 = \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \dots\right)$$

Ex 1 Given that

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9,750	10.031

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) x = 1.1 (b) x = 1.6

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		.414					
1.1	8.403		-.036				
		.378		.006			
1.2	8.781		-.030		-.002		
		.348		.004		.002	
1.3	9.129		-.026		.000		-.003
		.322		.004		-.001	
14	9.451		-.023		-.001		
		.299		.005			
1.5	9.750		-.018				
		.281					
1.6	10.051						

We have

$$\left(\frac{dy}{dx}\right) x_0 = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right)$$

Hence h = 0.1 x_0 = 1.1 Δy_0= .378, $\Delta^2 y_0$ = -0.03 , $\Delta^3 y_0$ = .004

$\Delta^4 y_0$ = 0 , $\Delta^5 y_0$ = -0.001

$$\left(\frac{dy}{dx}\right) 1.1 = \frac{1}{.1}\left(.378 - \frac{1}{2}(-.03) + \frac{1}{3}(.004) - \frac{1}{4}0 + \frac{1}{5}(-.001)\right)$$

= 10 (.378 + .015 + .0013 - .0010)

= 3.933

$$\left(\frac{d^2y}{dx^2}\right) x_0 = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 - \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots..\right)$$

$$= \frac{1}{.1^2}\left(-.03 - .004 + \frac{11}{12}.0 - \frac{5}{6}(-.001)\right)$$

= -3.317

For x = 1.6, we use the above difference table and the backward difference operator ∇ instead of Δ.

$$\left(\frac{dy}{dx}\right) x_n = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right)$$

Here h = .1 , x_n + 1.6 , ∇y_n = .281 , $\nabla^2 y_n$ = -.018 , $\nabla^3 y_n$ = .005

$\nabla^4 y_n$ = -.001 , $\nabla^5 y_n$ = -.001 , $\nabla^6 y_n$ = -.003

$$\left(\frac{dy}{dx}\right) 1.6 = \frac{1}{.1}\left(.281 + \frac{1}{2}(-.018) + \frac{1}{3}(.005) + \frac{1}{4}(-.001) + \frac{1}{5}(-.001) + \frac{1}{6}(.-003)\right)$$

= 2.727

$$\left(\frac{d^2y}{dx^2}\right) x_n = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right)$$

$$\left(\frac{d^2y}{dx^2}\right) 1.6 = \frac{1}{(.1)^2}\left(-.018 + .005 + \frac{11}{12}(-.001) + \frac{5}{6}(-.001) - \frac{137}{180} + (-.003)\right)$$

= −1.703

Ex 2: A slider in a machine moves along a fixed straight rod. It is distance x cm along the rod is given below for 5 various values of the time t seconds. Find the velocity of the slider and its acceleration when t = .3 seconds

t :	0	.1	.2	.3	.4	.5	.6
x :	30.13	31.62	32.87	33.64	33.95	33.81	33.24

The difference table is

t	y	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$	$\Delta^5 x$	$\Delta^6 x$
0	30.13						
		1.49					
.1	31.62		-.24				
		1.25		-.24			
.2	32.87		-.48		-.26		
		0.77		.-2		-.27	
.3	33.64		-.46		-.01		.29
		0.31		-.01		.02	
.4	33.45		-.45		-.01		
		-0.14		-.02			
.5	33.81		-.43				
		-0.57					
.6	33.24						

As the derivative are required near the middle of the table, we use Stirling's formula

$$\left(\frac{dx}{d\,t}\right) t_0 = \frac{1}{h}\left[\frac{\Delta x_0 + \Delta_x + 1}{2} - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) \ldots\ldots\right]$$

Here h $= .1$, $t_0 = 3$, $\Delta x_0 = 31$, $\Delta x_{-1} = .77$, $\Delta^3 x_{-2} = 0.02$

$\Delta^3 x_{-1} = .01$ $\Delta^5 x_{-2} = .02$, $\Delta^5 x_{-2} = .27$ etc.

$$\left(\frac{dx}{dt}\right) .3 = \frac{1}{.1}\left(\frac{.31+77}{2} - \frac{1}{6}\left(\frac{.01+.02}{2}\right) + \frac{1}{30}\left(\frac{.02+.27}{2}\right)\right) \qquad = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right) t_0 = \frac{1}{h^2}\left(\Delta^2 x_{-1} - \frac{1}{2}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \ldots\ldots\right)$$

$$= \frac{1}{(.1)^2}\left(-.46 - \frac{1}{12}(-.01) + \frac{1}{90}(.29) -\right)$$

$$= -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is - 45.6 cm/sec ^{2}

Ex. 3 From the table below, for what value of x, y is minimum? Also find this value of y.

x :	3	4	5	6	7	8
y :	.205	.240	.259	.262	.250	.224

The difference tables as

X	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
3	.205					
		.035				
4	.240		-.016			
		.019		.000		
5	.259		-016		.0001	
		.003		.001		-.00
6	.202		-.015		0	
		-.012		.001		
7	.250		-.014			
		-.026				
8	.224					

Taking $x_0 = 3$, $y_0 = .205$, $\Delta y_0 = .035$, $\Delta^2 y_0 = -.016$. $\Delta^3 y_0 = 0$

Newton's forward difference formula gives

$$y = .205 + p(.035) + \frac{p(p-1)}{2}(-.016) \quad (1)$$

Differentiating it w.r.t p

$$\frac{dy}{dp} = .035 - .008\,(2p-1)$$

For y to be minimum $\frac{dy}{dp} = 0$

$\therefore$ .35 - .008 (2 p – 1) = 0 which gives p 2.6875

$\therefore$ $x = x_0 + ph = 3 + 2.6878 \times 1 - 5.6875$

Hence y is minimum when x = 5.6875

Putting p = 2.6875 in (1), the minimum value of y

$= .205 + 2.6875 \times .035 + 1/2\ (2.6875 + 1.6875)\ (-.016) = .2628$

Exercise

1) Find the first and second derivative of f(x), at x = 1.5 if

x :	1.5	2.0	2.5	3.0	3.5	4.5
f(x) :	3.375	7.000	13.625	24.000	38.875	59.000

2) Given the following table of values of x and y

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y :	1.0000	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) x = 1.00 (b) x = 1.25 (c) x = 1.15

3. Find the value of cos 1.74 using the values given in the table below:

x	:	1.70	1.74	1.78	1.82	1.86
sinx	:	.9916	.9857	.9781	.9691	.9584

4. The population of a certain town is shown in the following table

Year	:	1951	1961	1971	1981	1991
Population (in thousand)	:	19.96	39.65	58.81	77.21	94.61

Find the rate of growth of population in 1981.

Answers (1) 4.75, 9 (2) (a) .5005, -.2752 (b) .4473, -.583 (c) .4662, -.2043.

(3) -.175 (c) 1.8 thousands/yr

2.7 NUMERICAL INTEGRATION

The exact value of the definite integral J $\int_a^b f(x)\,dx$ can be a computed only when the function f(x) i.e. integrable in finite terms. Whenever the function f (x), cannot be exactly integrated in finite terms or the evaluation of its integral is to cumber some, integration can be more conveniently performed by numerical method. The definite integral $\int_a^b f(x)\,dx$ represents the area between the curve y = f(x), the x -axis and the ordinates at x = a and x = b.

The Trapezoidal Rule

Let I = $\int_a^b f(x)\,dx$ where f(x) takes the value $y_0, y_1, y_2 \ldots y_n$ for

$a = x_0, x_1, x_2 \ldots x_n = b$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + \frac{1}{2}(y_1 + y_2 + y_3 \ldots y_{n-1})\right]$$

This is known as the trapezoidal rule.

Simpson's one-third rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\frac{1}{2}(y_1 +_3 \ldots y_{n-1}) + 2(y_2 + y_4 + \ldots . + y_{n-2})\right]$$

This is known as Simpson's one-third rule while applying this rule, the given interval must be divided into even number of equal sub-intervals.

Simpson's three eight rule

$$\int_a^b f(x)\,dx = \frac{3h}{4}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_5 \ldots y_{n-1}) + 2(y_3 + y_6 \ldots y_{n-3})\right]$$

Which is known as Simpson's three - eighth rule while applying this rule, the number of sub intervals should be taken as multiple of 3.

Ex 1

i) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (1), Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's 3/8 rule. Divide the interval (0, 6) into six parts each of width h = 1. The values of f (x) = $\frac{1}{1+x^2}$ are given below:

x	0	1	2	3	4	5	6
f(x)	1	.5	.2	.1	.0588	.0385	.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_5

i) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[1+.027+2(.5+.2+.1+.588+.0385)\right]$$

= 1.4108

ii) By Simpon's 1/3 rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{2}\left[1+.027+4(.5+.1+.0385)+2(.2+.0588)\right]$$

= 1.3662

iii) By Simpson's 3/8 rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_3+y_5)+2y_3\right]$$

$$= \frac{1}{2}\left[1+.027+3(.5+.2+.0588+.0385)+2(.1)\right]$$

= 1.3571

Ex 2 The velocity v (km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:

t :	2	4	6	8	10	12	14	16	18	20
v :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

If s(km) be the distance covered in t (min), then

$\frac{ds}{dt}$ = v

The distance covered

$$\int_0^{20} vd\ t = \frac{h}{3}\left[(v_0 + v_n) + 4(v_1 + v_3 + v_5 + v_7 + v_9) + 2(v_2 + v_4 + v_6 + v_8)\right]$$

= 309.33 km

Exercise

1) Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's rule. Hence obtain approximate value of $\log_e 2$

2) Calculate the value of $\int_0^{\pi/2}$ sinxdx by Simpon's 1/3 rule using 11 ordinates.

3) Given that

x	:	4.0	4.2	4.4	4.6	4.8	5.0	5.2
logx	:	1.3863	1.4351	1.4816	1.5261	1.5786	1.6094	1.6487

evaluate $\int_4^{5.2}$ log x dx by Trapezoidal rule.

Answers (1) .6932 (2) .9985 (3) 1.8276551

Mr. S. Sriram
Lecturer in Mathematics
Faculty of Engineering and Technology
Annamalai University

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (FIRST ORDER)

3.1 INTRODUCTION

A number of problems in science and technology can be formulated into differential equations. For example, Newtons's law f = ma = $\frac{d^2x}{dt^2}$ m is a 2nd order differential equation expressing the fact that force is proportional to acceleration. Thus the behaviour of many physical processes particularly those in systems undergoing time-dependent changes can be described by differential equations. Hence the solution of those equations is a matter of great importance to engineers and scientists. Now we will discuss various methods for finding, (to any desired degree of accuracy) the numerical solution of any ordinary differential equation with given initial conditions.

Suppose the first order differential equation is given $\frac{dy}{dx}$ = f (x,y) .... (1) with the initial condition $y(x_0)$ = yo. If we can obtain a formula for the solution, we may evaluate it numerically, either directly or by the use of tables. If that formula is too complicated or if no formula for the solution is available, we may apply step-by-step method. In this method we start from $y(x_0) = y_0$ and proceed with the approximate value of y_1 of the solution of (1) at $x = x_1 = x_0 + h$. In the 2nd step we compute an approximate value of y of the solution at $x = x_2 = x_0 + 2h$ etc. Here h is a fixed number called step size.

3.2 TAYLOR SERIES METHOD

If y = f(x),. then its Taylor series about the point

$x = x_0$ is $f(x) = f(x_o) + (x - x_0) f^1(x_0) + \frac{(x-x_0)^2}{2!} f^{11}(x_0)$

This formula can also be written as

$$y(x) = y_0 + (x - x_0)\ y_0^1 + \frac{(x-x_0)^2}{2!}\ y_0^{11} + \ldots \quad (1)$$

Where x_0 and y_0 denote the initial values of x and y.

To find out the values of y's by Taylor formula it is desirable to keep $|x-x_0|$ numerically small in order to have rapid convergence of the series and therefore higher accuracy in the values of y's. Using the notation $y_1 = y(x_0 + h)$, $y_2 = y(x_0 + 2h)$ and etc. and $y_{-1} = y(x_0 - h)$, $y_2 = y(x_0 - 2h)$ etc. We have by Taylor's formula.

$y(x_0 + h) = y_1 = y_0 + h\ y_0^1 + \frac{h^2}{2!} y_0^{11} + \frac{h^3}{3!} y_0^{111} + \ldots$

In general $y_{n+1} = y_n + h\ y_n^1 + \frac{h^2}{2!} y_n^{11} + \frac{h^3}{3!} y_n^{111} + ..$

$y(x_0 + h) = y_{-1} = y_0 - hy_0^1 + \frac{h^2}{2!} y_0^{11} - \frac{h^3}{3!}\ {}_0^{111} + \ldots$

$y(x_0 + 2h) = y_{-2} = y_1 + h\ y_1^1 + \frac{h^2}{2!} y_1^{11} - \frac{h^3}{3!}\ y_1^{111} + \ldots$

Note 1

If we increase the terms in Taylors formula then we get more accurate answers

Note 2

y_0^1 means the value of y^l at $x = x_0$ and $y = y_0$ y, ${}_0^{11}$ means the value of y^{11} at $x = x_0$ and $y = y_0$ and so on. Similarly y_1^1 means the value of y^l at $x = x_1$ and $y = y_1$, y_1^{11} means the value of y^{11} at $x = x_1$ and $y = y_1$

Note 3

It will be convenient if we find the higher derivatives say y^{n} , y^{ni} etc before applying Taylors formula.

Example 1

Using Taylor's series method compute y(0.1) correct to 4 decimal places if y(x) satisfies $y^l = x + y$, $y(0) = 1$

Given $y^1 = x + y$ and $x_0 = 0$ and $y_0 = 1$, we know that the Taylor series formula for y_1 is

$y_1 = y_0 + \frac{h}{1!}\ y_0^1 + \frac{h^2}{1!}\ {}_0^{11} + \frac{h^3}{3!} y_0^{111} +$ ... (1)

Take h = 0.1

$y^1 = x + y$: $y_0^1 = (x_0 + y_0) = 0 + 1 = 1$ ... (2)

$y^{11} = 1 + y^1$; $y_0^{11} = (1 + y_0^1) = 1+1 = 2$ ... (3)

$y^{111} = y^{11}$; $y_0^{111} = y_0^{11} = 2$ ... (4)

Substituting (2), (3), (4) and h = 0.1 , y_0 = 1(1) we get

$y_1 = y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2!} 2 + \frac{(0.1)^3}{3!}\ .\ 2$

= 1.1103

$\therefore y(0.1) = 1.1103$

Example 2

Given $\frac{dy}{dx}$ = 3x + $\frac{y}{2}$ and y (0) = 1. Find the values of y(0.1) and y(0.2) using Taylors series method.

Given yl = 3x + $\frac{y}{2}$ and x_0 = 0, y_0=1 We know that the Taylor series formula for y_1 is

$$y_1 = y_0 + hy_0^1 + \frac{h^2}{2!}{}_0^{11} + \frac{h^3}{3!} + \ldots \qquad \ldots (1)$$

$$y^I = 3x + \frac{y}{2},\ y_0^1 = 3x_0 = 0 + ½ = 0.5 \qquad \ldots (2)$$

$$y^{II} = 3 + \frac{y^1}{2}\ .\ y_0^{11} = 3 + \frac{y_0^1}{2} = 2.25 \qquad \ldots (3)$$

$$y^{III} = \frac{y^{11}}{2}\ ,\ y_0^{111} = \frac{y_0^{11}}{2} = \frac{3.25}{2} = 1.625 \qquad \ldots (4)$$

$$y^{IV} = \frac{y^{111}}{2}\ ,\ y_0^{IV} = \frac{y_0^{111}}{2} = \frac{1.625}{2} = 0.8125 \qquad \ldots (5)$$

Substituting (2), (3), (4), (5) and h=0.1, y_0 = 1 in (1) we get

$$y_1 = y\ (0.1) = 1 + (0.1)\ (0.5) + \frac{(0.1)^2}{2!}\ 3.25 + \frac{(0.1)^3}{3!}\ 1.625 + \frac{(0.1)^4}{4!}\ (0.8125)$$

y(0.1) = 1.0665

We know that the Taylor series formula for y_2 is

$$Y_2 = y_1 + hy_1^1 + \frac{h^2}{2!}y_1^{11} + \frac{h^3}{3!}\ y_1^{111} + \ \ldots \qquad \ldots (6)$$

$$y^I = 3x + \frac{y}{2},\ y_1^1 = 3x_1 = \frac{y_1}{2}\ 3(0.1) + \frac{1.0665}{2} = 0.83325 \qquad \ldots (7)$$

$$y^{II} = 3 + \frac{y^1}{2}\ .\ y_1^{11} = 3 + \frac{y_1}{2} = 3 + \frac{0.85825}{2} = 3.416625 \qquad \ldots (8)$$

$$y^{III} = \frac{y^{11}}{2}\ ,\ y_1^{111} = \frac{y_1^{11}}{2} = \frac{3.416625}{2} = 1.7083125 \qquad \ldots (9)$$

$$y^{IV} = \frac{y^{111}}{2}\ ,\ y_1^{IV} = \frac{y_1^{111}}{2} = \frac{1.7083125}{2} = 0.8541562 \qquad \ldots (10)$$

Substituting (7), (8), (9), (10) and y_1 = 1.0665 in (6) we get

$$y_2 = y\,(0.2) = 1.0665 + (0.1)\ (0.83325) + \frac{(0.1)^2}{2!}\ (3.416615) + \frac{(0.1)^3}{3!}(1.708312)\ + \frac{(0.1)^4}{4!} = (0.854156)$$

y (0.2) = 1.167196

Exercise

1) Using Taylor series method, compute the value of y(0.2), correct to 3 decimal places from dy/dx = 1 - 2xy given that y(0) = 0.
2) Evaluate by means of a Taylor series expansion, the following problem at x = 0.2 to three significant figures $y^1 - 2y = 3e^{x}$, y(0) = 0
3) Using Taylor series method solve $\frac{dy}{dx} = x^2 - y$, y (0) = 1 at x = 0.1, 0.2, 0.3 and 0.4
4) Using Taylor's series method solve $\frac{dy}{dx} = x - y^2$, y (0) = 1 at x = 0.1

Answers (1) 0.19467, (2) 0.811, (3) 0.9052, 0.8213, 0.7492, 0.6897,
(4) 0.9138.

3.3 EULER'S METHOD

Consider the initial value problem $y^1 = f(x, y), y(x_0) = y_0$ As per Taylor's series.

$$y_1 = y_0 + \frac{h}{1!}\ y_0^1 + \frac{h^2}{2!}\ {}_0^{11} + \ldots.$$

But y_1 f(x, y) $y_0^1 = f\,(x_0\,, y_0)$

Therefore the Taylor's series upto the first order term gives

$y_1 = y_0 + hf\,(x_0\,, y_0)$

Similarly we can derive $y_2 = y_1 + hf\,(x_1\,, y_1)$

In general $y_{n+1} = y_n + h(x_n\,, y_n)$ This is called Euler's formula to solve an initial value problem.

Improved Euler's method:

In this method

$y_1 = y_0 + h\ \left[f\,(x_0, y_0) + f\left(x_0, y_1^{(1)}\right)\right]$ where $y_1^{(1)} = y_0 + hf\,(x_0\,, y_0)$

In general $y_{n+1} = y_h\ \frac{1}{2}h\ \left[f\,(x_n, y_n) + f\,(x_n + h\,, y_n) + hf\,(x_n\ y_n)\right]$

Modified Euler's method:

In this method y_1 , = y_0 + hf $\left[x_0 \frac{h}{2}, y_0 + \frac{h}{2} \; f(x_0, y_0)\right]$

In general $y_{n+1} = y_n + h \left(f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\right)(x_n \; y_n)\right)$

Example

Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with y(0) = 1 find y (0.1) by Euler's method.

The Euler's formula is $y_{n+1} = y_n + hf(x_n \; y_n)$ ... (1)

Putting n = 0 in (1) $y_1 = y_2 + h\, f(x_0, y_0)$

Here $x_0 = 0$, $y_0 = 1$, f(x, y) = $\frac{(y - x)}{y + x}$

$\therefore f(x_0, y_0) = \frac{y_0 - x_0}{y_0 + x_0} = 1$

$\therefore y_1 = 1 + (0.1) = 1.1$

$\therefore y(0.1) = 1.1$

Example 2

Using modified Euler's method find y at x = 0.1 and x = 0.2 given $\frac{dy}{dx} = y - \frac{2x}{y}$, y(0) = 1

The modified Euler's algorithm is

$y_{n+1} = y_n + \frac{h}{2} \; [f(x_n, y_n) + f(x_n + h, y_n) + hf(x_n \; y_n)]$ . . . (1)

Putting n = 0 in (1)

$y_1 = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_0 + h, y_0) + hf(x_0 \; y_0)]$ . . . (2)

Here $x_0 = 0$, $y_0 = 1$, f (x, y) = y - $\frac{2x}{y}$

$\therefore f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - 0 = 1$ . . . (3)

Substituting (3) in (2) we get

$y_1 = y_0 + \frac{h}{2} \; (1 + f(x_0 + h, h\, y_0 + h.1))$

$= + \frac{0.1}{2}\left[1 + f(0+0.1, 1+0.1)\right]$

$= 1 + \frac{0.1}{2}\left[1 + f(0.1, 1.1)\right]$ . . . (4)

Now f(0.1, 1 .1) = 1.1 - $\frac{2(0.1)}{1.1}$ = 0.9182 . . . (5)

Substituting (5) in (4)

$y_1 = 1 + \frac{0.1}{2}$ (1 + 0.9182)

y (0.1) = 1.09591

Putting n = 1 in (1)

$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_1 + h, y_1) + h\, f(x_1 - y_1)\right]$ . . . (6)

Hence $x_1 = 0.1$, $y_1 = 1.0959$

Now $f(x_1, y_1) = 1.05959 - \frac{2(0.1)}{1.0959} = 0.9134$ . . . (7)

Substituting (7) and (6)

$y_2 = y_1 + \frac{h}{2}\left[0.9135 + f(0.2, y_1) + h(0.9135)\right]$

$= 1.0959 + \frac{0.1}{2}$ [0.9135 + f (0.2 , 1.0959 + (0.1) 0.9135]] = 1.1941

i.e. y (0.2) = 1.1841

Exercise

1) Using Euler's improved method, find the value of y when x=0.1. Given that y(0) = 1 and $y^1 = x^2 + y$
2) Using modified methods of Euler, solve $\frac{dy}{dx}$ = 1- y , y (0) = 0 in the range $0 \le x \le 0.3$ taking h = 0.1.
3) Using modified Euler's method, obtain the solution of the following at x = 0.2 and 0.4 taking h = 0.2 $\frac{dy}{dx} = x - y^2$, y (o) = 1
4) Using modified Euler's method solve $\frac{dy}{dx} = -xy^2$, y(0) = 2 at x = .2 taking h = .1

Answers

(1)1.1055, (2) 0.0945, 0.180975, 0.25878, (3) 0.858,0.78859, (4) 1.9227

3.4 PICARD'S METHOD

Suppose we want to solve the differential equation $\frac{dy}{dx}$ = f (x, y) subject to the condition that y = y_0 when x = x_0

In this method, we get a first approximation for y , let this be y (1)

$$y^{(1)} = y_0 + \int_{x_0}^{x} f\left(x, y_0\right) dx$$

A second approximation $y^{(2)}$ for y

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right) dx$$

In general $y^{(n)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(n-1)}\right) dx$

Example

Given $\frac{dy}{dx}$ = x + y with boundary condition y = 1 when x = 0 find approximately the value of y for x = 0.1, x = 0.2

$y^{(1)} = 1 + \int_{0}^{x} f\left(x+y\right) dx$ put y = 1 in y + x

$$= 1 + \int_{0}^{x} \left(x+y\right) dx = 1 + x \frac{x^2}{2}$$

For the 2nd approximation

$$y^{(2)} = 1 + \int_{0}^{x} \left(x+y^{(1)}\right) dx$$

$$= 1 + \int_{0}^{x} \left(x+1+x\frac{x^2}{2}\right) dx$$

$$= 1 + x^2 + x + \frac{x^3}{6}$$

For the 3rd approximation.

$$y^{(1)} = 1 + \int_{x_0}^{x} \left(x + \frac{x^3}{3} + x^2 + x + 1 \right) dx$$

$$= \frac{x^4}{24} + \frac{x^3}{3} + x^2 + x + 1$$

Thus y is found as a power series in x.

For x = 0.1 , $y = \frac{.0001}{24} + \frac{.001}{3} + 0.1 + .1 + 1 = 1.1103$

For x = 0.2 , $y = \frac{.0016}{24} + \frac{.008}{3} + 04 + .2 + 1 = 1.2437$

We can get a better value by continuing the approximations to $y^{(4)}$, $y^{(5)}$ etc.

Example

Find the value of a y for x =.1 by Picard's method, given that $\frac{dy}{dx} = \frac{y - x}{y + x}$,y(0) = 1

For the first approximation,

$$y^{(1)} = 1 + \int_{0}^{x} \frac{1-x}{(1+x)} dx$$

$$= 1 + \int_{0}^{x} \left(-1 + \frac{2}{1+x} \right) dx$$

$$= 1 + (-x + 2\log(1 + x))_{0}^{x}$$

$$= 1 + (-x + 2\log(1 + x) \qquad \dots (1)$$

For the second approximation,

$$y^{(2)} = 1 + \int_{0}^{x} \frac{1 - x + 2\log(1+x) - x}{1 - x\, 2\log(1+x) + 2} \, dx$$

$$= 1 + \int_{0}^{x} \frac{1 - 2x + 2\log(1+x)}{1 + 2\log(1+x)} dx$$

$$= 1 + \int_{0}^{x} \left(1 - \frac{2x}{1 + 2\log(1+x)} \right) dx$$

$$= 1 + x - 2 \int_{0}^{x} \frac{x}{1 + 2\log(1+x)} \, dx$$

and this is difficult to integrate. Hence we use the first approximation (1) itself and putting x =.l. We get y=l-.l + 2 log (l + .1) =-.9828.

Exercise

1) Find an approximate value of y when x = 0.1, if $\frac{dy}{dx} = x - y^2$ and y = 1 at x = 0 using Picard's method. Answer (1) 0.9138.

2) Integrate the differential equation $\frac{dy}{dx} = 2x - y$ with $x_0 = 1$, $y_0 = 3$ by Picard's method.

3.5 FOURTH ORDER RUNGE-KUTTA METHOD

The use of Taylor's series to solve numerically differential equations is restricted by the labour involved in the determination of higher order derivatives. Runge-Kutta method does not require the calculation of the higher order derivatives.

Let $\frac{dy}{dx} = f(x, y)$ be a given differential equation to be solved under the condition $y(x_0) = y_0$ be the length of the interval between equidistant values, then the first increment in y is computed from the formula.

$$K_1 = hf\left(x_0, y_0\right)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = hf\left(x_0 + h, y_0 + K_3\right)$$

$$\Delta 4 = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

Now $x_1 = x_0 + h_1$, $y_1 = y_0 + \Delta y$

The increment in y for the 2nd interval is computed in a similar manner by means of the formula.

$$K_1 = hf\left(x_1, y_1\right)$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$K_4 = hf\left(x_1 + h, y_1 + K_3\right)$$

$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

and so on for succeeding intervals.

Example

Apply the fourth order Runge-Kutta method, to find an approximate value of y when x=.2 given that $y^l = x + y$, y (0) = 1.

(Here f (x , y) = x + y, x_0 = 0 , y_0 = l , h = 0.1 For the 1st interval),

K_1 = hf (x_0, y_0) = .1 $(x_0 + y_0)$ = .1(0+1) = .1

K_2 = hf $\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$ = .1 f(.05.1.05) = .1 (.05 + 1.05) = .11

K_3 = hf $\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$ = .1 $\left(x_0 + \frac{h}{2} + y_0 + \frac{K_2}{2}\right)$

= (.1) $\left(0 + \frac{1}{2} + 1 + \frac{11}{2}\right)$ = .1105

K_4 = hf $(x_0 + h, y_0 + K_3)$ = .1 $(x_0 + h + y_0 + K_3)$

= .1 (0 + .1 + 1 + .1105

= .12105

Δy = $\frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ = $\frac{1}{6}$ $(.1 + 2(.11) + 2(.1105) + .12150 = .111034)$ = .111034

y_1 = y (0.1) = y_0 + Δy = 1 + .11034 = 1.11034

We calculate new values of K_1, K_2 K_3 K_4 replacing (x_0, y_0) in the first set by (x_1, y_1) respectively.

Hence K_1 = hf (x_1, y_1) = .1 (x_1, y_1) = .1 (.1 + 1.11034) = .12103

K_2 = hf $\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$ = .1 $\left(x_1 + \frac{h}{2} + y_1 + \frac{K_1}{2}\right)$ (.1) $\left(x_1 + \frac{h_1}{2} + y_1 + \frac{K_1}{2}\right)$

= (.1) $\left(.1 + \frac{1}{2} + 1.11034 + \frac{.12103}{2}\right)$

= .13208

K_3 = hf $\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$ = (.) $\left(x_1 + \frac{h_1}{2} + y_1 + \frac{K_2}{2}\right)$

= (.1) $\left(.1 + \frac{1}{2} + 1.11034 + \frac{.13208}{2}\right)$

$= .13264$

$K_3 = hf\left(x_1 + h,\, y_1 + K_3\right) = 1\left(x_1 + h + y_1 + K_3\right)$

$= .1\ (.1+.1+ 1.10+34 + .13263)$

$= .144298$

$$\Delta y = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$= \frac{1}{6}\ (.121103 + 2\ (.13208) + 2\ (.13263) + .144298)$$

$= .13246$

$y_2 = y(.2) = y_1 + \Delta y = 1.11034 + .13246 = 1.2428$

So the value of y when x = .2 is 1.2428.

Exercise

1) Using Runge - Kutta method of fourth order determine correct to 3 decimal places the value of y at x = . 1, .2 if y satisfies the equation

   $\frac{dy}{dx} - x^2 y = x,\ y(0) = 1$

2) Using Runge-Kutta method of order 4 find y(o.2) given that

   $\frac{dy}{dx} = 3x + \frac{1}{2}\ y,\ y(0) = 1$ taking h = .l.

3) Using Runge-Kutta method of order 4, solve $\frac{dy}{dx}$ with y (0) = 1 at x = .2, .4.

4) Using Runge-Kutta method of order 4, solve

   $\frac{dy}{dx} = x^3 + \frac{y}{2}$,y (1) = 2 forx = 1.1,1.2

Answers

(1) 1.1103 - 1.2428, (2) 2.5005, (3) 1.196 - 1.3752, (4) 2.2213 - 2.4914

3.6 MILNE'S PREDICTOR CORRECTOR FORMULAE

Suppose we want to solve numerically the differential equation $= \frac{dy}{dx}$ f(x,y) subject to the condition that $y(x_0) = y_0$

Milne's Predictor formula is

$$y_{n+1,\,p} = y_{n-3} + \frac{4h}{3}\left(2y^1_{n-1} + y^1_n\right)$$

Milne's corrector formula is

$y_{n+1} + c = y_{n-1} + \frac{h}{3}\left(y^1_{n-1} + 4y^1 + y^1_{n+1}\right)$

To use Milne's Predictor- Corrector formula we need atleast four values prior to the required value. If the initial four values are not given we can obtain these values by using Taylor's Series, Euler's or Runge-Kutta method,

Example

Solve the differential equation $y^1 = xy + y^2$ using Milne's Predictor Corrector method for x = .4, given y(0) = 1. The values of y for x - .1, .2 and .3 should be computed by a Taylor series expansion.

Given $y^1 = xy + y^2$, $x_0 = 0$, $y_0 = l$ and h = . 1

Now$y^{11} = xy^1 + y + 2yy1$

$y^{111} = xy^{11} + 2y^1 + 2yy^{11} + 2\left(y^1\right)^2$

To find y(O.l)

$y(0.1) = y1 = h\, y^1_0 + \frac{h^2}{2!}\; y^{11}_0 + \frac{h^3}{3!}\; y^{111}_0 + \ldots$ ...(1)

$y^1_0 = \left(xy + y^2\right)\left(x_0\, y^1_0\right) = \left(x_0\, y_0 + y^2_0\right) = 1 \quad \ldots$ ...(2)

$y^{11}_0 = \left(x_0\, y^1_0 + y_0 - 2y_0\, y^1_0\right) = 3 \quad \ldots$ ...(3)

$y^{111}_0 = \left(x_0\, y^{11}_0 + 2y^1_0 + 2y_0\, y^{11}_0 + 2\left(y^1_0\right)^2\right) = 10 \ldots$ ...(4)

Substituting (2) , (3) and (4) in (1) we get

$y\ (0,\ 1) = 1 + .1 + \frac{(0.1)^2}{2} \times 3 + \frac{(0.1)^3}{3!} \times 10 = 1.11667$

$\therefore$ y(0.1) = 1.11667

By Taylor Series = $y_2 = y_1 + h\, y^1_1 + \frac{h^2}{2!}\; y^{11}_1 + \frac{h^3}{3!}\; {}^{111}_{1} + \ldots$ ...(5)

$y^1_1 = x_1\, y_1 + y^2_1 = (.1)\ (1.11666) + (1.1166)^2 = 1.3586\ldots$ ...(6)

$y^{11}_1 = x_1\, y^1_1 + y_1 + 2y_1\, y^1_1 = 4.2867\ldots$ ...(7)

$y^{111}_1 = x_1\, y^{11}_1 + 2y_1\, y^1_1 + 2\left(y^1\right)^2 = 16.411\ \ldots$ ...(8)

Substituting (6), (7), (8) in (5)

$y(0.2) = 1.11667 + (.1)\ (1.3586) + \frac{(.1)^2}{2}\ (4.2867) + \frac{(.1)^2}{2}\ (16.411)$

$= 1.27669$

By Taylor series to find

$$y(0.3) : y_3 = y_2 + h\, y_2^1 + \frac{h^2}{2!}\, y_2^{11} + \frac{h^3}{3!}\, y_2^{111} + \ldots \qquad \ldots (9)$$

$$\text{Now } y_2^1 = \left(x_2\, y_2 + y_2^2\right) = 1.8852 + \ldots \qquad \ldots (10)$$

$$y_2^{11} = \left(x_2\, y_2^1 + y_2 + 2y_2 y_2^1\right) = 6.4674 \qquad \ldots (11)$$

$$y_2^{111} = x_2\, y_2^{11} + 2y_2^1 + 2y_2 y_2^{11} + 2y_2^{12} = 28.6855 \qquad \ldots (12)$$

Substituting (10), (11), and (12) in (9)

$$y(0.3) = 1.27668 + (.1)\,(1.8852) + \frac{(.1)^2}{2}\,(6.4675)$$

$$+ \frac{(.1)^3}{6}\,(28.6855)$$

$$= 1.50233$$

∴ We have the following values

$x_0 = 0$ $\quad y_0 = 1$

$x_1 = .1$ $\quad y_1 = 1.11666$

$x_2 = .2$ $\quad y_2 = 1.27668$

$x_3 = .3$ $\quad y_3 = 1.50233$

To find y (0.4) by Milne's predictor formula

$$y_{n+1}\,,\, p = y_{n-3} + \frac{4h}{3}\left(2y^1 - y_2^1 + 2y_3^1\right)$$

$$\text{Putting } n = 3\,,\ y_4\, p = y_0 + \frac{4h}{3}\left(2y_1^1 - y_2^1 + 2y_3^1\right)$$

$$y_3^1 = x_3\, y_3 + y_3^2 = (.3)\,(1.50233) + (1.50233)^2 = 2.70769$$

$$y_4\,,\, p = 1 + \frac{4(.1)}{3}(2(1.35857) - 1.8852 + 2(2.70769)) = 1.8329$$

To find y (0.4) by Milne's corrector formula

Milne's corrector formula is

$$y_{n+1}\,,\, C = y_{n-1} + \frac{h}{3}\left(y_{n-1}^1 + 4y_n^1 + y_n^1 + 1\right)$$

Putting n = 3

y_4 C = y_2 + $\frac{h}{3}$ $\left(y_2^1 + 4y_3^1 + y_4^1\right)$

Now y_4^1 = $\left(x_4\, y_4 + y_4^2\right)$ = (0.1) (1.8329) + $(1.8329)^2$ = 4.09268

y_4 .C = 1.27668 + $\frac{0.1}{3}$(1.8852+ 4(2.70761) + 4.09268)

= 1.83696

Exercise

1) Solve numerically, using Milne's method $y^1 = \frac{1}{x+y}$, y (0) = 2. Take the starting values y (0.2) = 2.0933, y (0.4) = 2.1755 Y (0.6) = 2.2493. Find the values of y (0.8)
2) Use the Milne's method to find y(0.3) from $y^1 = x^2 + y^2$, y (0) = 1. Find the initial values y (–.1), y (.1) and y (.2) from the Taylor's series method.
3) Given $y_1 = \frac{1}{2}(1 + x^2)y^2$ and y (0) = 1 . y (.1) - 1.06 y (0.2) = 1.12 , y(0.3) = 1.21 find y (0.4) Using Milne's predictor corrector formula.

Answers: (1), y (0.8) = 2.3161 y (0.8) = 2.3164

(2) 1.4392

(3) y_4 , p = 1.2772 , y_4 c = 1.2797

4) If $\frac{dy}{dx} = 2e^x - y$, y(0) = 2 ,y(0.1) = 2.010 , y (0.2) = 2.040 and y(3) - 2.090 find y (0.4) and y (0.5) correct to three decimals applying Milne's predictor method.

Ans: y (0.4) = 2.1623, y (0.4) = 2.1621

y (0.5) = 2.2551, y (0.5) = 2.2546

5) Using Runge Kutta method of order 4 find y for x = 0, 1, 0.2, 0.3 given that. $\frac{dy}{dx} = xy + y^2$, y (0) = 1 continue the solution at x = 0.4 using Milne's method

Ans: y (0.1) = 1.11686, y (0.2) = 1.2773, y (0.3) = 1.5039,

$y(.4)_p$ = 1.8360. p $y(.4)_c$ = 1.838

6) Given the differential equation $\frac{dy}{dx} = x^2 y + x^2$ and the data y (1) = 1, y (1.1) = 1.233, y (1.2) = 1.548488, y (1.3) = 1.978921 determine y (1.4) by Milne's method.

Ans: 2.5751

Mr. S. Sriram
Lecturer in Mathematics
Faculty of Engineering and Technology
Annamalai University

❑

DIFFERENCE EQUATION AND CURVE FITTING

4.1 FACTORIAL POLYNOMIAL

If n is a positive integer, we define

$x^{(1)} = x$ (4.1)

$X^{(2)} = x(x-1)$ (4.2)

$x^{(3)} = x(x-1)$ (4.3)

:

:

$x^{(n)} = x (x - 1) (x - 2) \ldots x - (x - 1)$ (4.4)

Any given polynomial can be expressed in terms of factorial polynomial

For example

$x = x^{(1)}$

$x^2 = x^{(2)} + x^{(1)}$

$x^3 = x^{(3)} + 3x^{(2)} + x^{(1)}$

$x^3 = 5x^2 + 7x - 6 = \left[x^{(3)} + 3x^{(2)} + x^{(1)}\right] -4 \left[x^{(2)} + x^{(1)}\right] + 7x^{(1)} - 6$

$= x^{(3)} + 2x^{(2)} + 3x^{(1)} - 6$

To express $x^3 - 5x^2 + 7x - 6$ in factorial polynormial we can also proceed as follows

Let $x^3 - 5x^2 + 7x - 6 = A x^{(2)} + Bx^{(2)} + Cx^{(1)} + D$

$= A x (x - 1) (x - 2) + B x (x - 1) + (x + 1)$

Putting x = 0 we get $-6 = D \therefore D = -6$

Putting x = 1 we get $-3 = C + D \therefore D = 3$

Putting x = 2 we get $-4 = 2B + 2C + D$.-. $B = -2$

Putting x = 3 we get $-3 = 6A + 6B + 3C + D \therefore A = 1$

Hence $x^3 - 5x^2 + 7x - 6 = x^{(3)} - 2x^{(2)} + 3x^{(1)} - 6$

We can also use the method of synthetic division, as illustrated below

0	1	-5	5	-6
		0	0	0
1	1	-5	7	-6
		1	-4	
2	1	-4	3	
		2		
	1	-2		

$\therefore\ x^3 - 5x^2 + 7x - 6 = x^{(3)} - 2x^{(2)} + 3x^{(1)} - 6$

When a polynomial is expressed in the factorial notation, then the difference operator Δ behaves like the differential operator D and Δ^{-1} behaves like D^{-1} (i.e. integration) as illustrated below:

It can be easily verified that $\Delta x^{(n)} = n\, x^{(n-1)}$ … (4.5)

i.e. $\Delta x^{(1)} = 1$

$\Delta x^{(2)} = 2x^{(1)}$

$\Delta x^{(3)} = 2x^{(2)}$

$\Delta x^{(4)} = 2x^{(3)}$

Similarly $\Delta^{-1} = x^{(n)} = \dfrac{x^{(n+1)}}{n+1}$ … (4.6)

i.e. $\Delta^{-1} k = kx^{(1)}$ where k is a constant

$\Delta^{-1} x^{(1)} = \dfrac{x^{(2)}}{3}$

$\Delta^{-1} x^{(2)} = \dfrac{x^{(3)}}{3}$

$\Delta^{-1} x^{(3)} = \dfrac{x^{(4)}}{4}$

4.2 DIFFERENCE EQUATION

A difference equation is an equation involving one independent variable, one dependent variable and the successive differences of the dependent variable such as Δy, $\Delta^2 y$, $\Delta^3 y$ …..

The following are examples of difference equation

$\Delta^2 y_x + 3\Delta y_x - 4y_x = x^2$ … (1)

$\Delta^3 y_x - 2\Delta^2 y_x - D\, y_x + 2\, y_x = 2^X$ … … (2)

Since Δ = E - 1, we can write

$\Delta y_x = (E–l)\, y_x = E\, y_x - y_n = y_{x+1} - y_x$

$\Delta 2 y_x = (E-l)^2\, y_x = (E^2 - 2E + 1)\, y_x = yx + 2 - 2y_{x+1} + y_x$

$\Delta^3 y_x = (E-l)^3 y_x = (E^3 - 3E^2 + 3E - 1)\, y_x = y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x$

Hence the difference equations (1) and (2) can be written in terms of E as follows

$E^2 y_x + E y_x - 6 y_x = x^2$

$E^3 y_x + 5E^2 y_x - 6\, E\, y_x = 2^X$

or can be written in terms of successive values of y_x as follows:

$y_{x+} 2 + y_{x+1} - 6\, y_x + x^2$

$y_{x+3} + 5y_{x+2} - 6y_{x+1} = 2^X$

The order of a difference equation is the difference between the highest and the lowest subscripts of y (when the difference equation is written in the form free from Δ). Thus the order of difference equations (1) and (2) are both equal to 2.

Note: The order of a difference equation need not be equal to the highest power of A or E. In example (1) it maybe turn but in example (2) it is not true i.e. order of the difference eqn in (2) is not equal to the highest power of Δ or E. In first this difference equation can be wirtten as $(E^2 - 5E + 6)\, y_{x+1} = 2^x$

The degree of a difference equation, (written in a form free from Δ) is the highest power of the y's. The degree of the difference equation in examples (1) and (2) are both equal to 1, whereas the degree of the difference equation $y_{X+3} + 3y_{x+2}\, y_x - 4y_{x+1}\, y_x = \text{Cos } x$ is 3

The general solution of a difference equation of order n is a solution which contains a arbitrary constants.

4.3 LINEAR DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

The general form of a linear difference equation (with constant coefficients) of order n is $a_0\, y_{x+n} + a_1 y_{x+n-1} + \ldots a_{n-1}\, y_{x+1} + a_n y_x = \phi\,(x)$

or $\left(a_0 E^n + 2_1 E^{n-1} + \ldots \quad \ldots. + a_{n-1} E + a_n\right)\, y_x = \phi\,(x)$ ………….. (4.7)

Where a_0 , a_x , a_n are constants and ϕ (x) is a known function of x.

The above difference equation can be written as

$f(E)\, y_x = \phi(x)$ ... ... (4.8)

Where f(E) is a polynomial expression in E.

The general solution of the above difference equation is y_x = C.F. + P.I where C.F. (complementary function) is the solution of the homogeneous difference equation f (E) $y_x = 0$ and P.I. (Particular integral) is a particular solution of the non-homogenous difference equation (4.8)

To find the complementary function

Where auxiliary equation is obtained by replacing E by m in f(E) and equating it to zero.

(i.e.) f(m) = 0

Solving we get n roots m_1, m_2 , ... m_n

Case (1) Roots are all distinct (unequal)

Then f(E) can be written as

$a_0 (E - m_1)(E - m_2) \ldots (E - m_n)\, y_x = 0$

$\therefore$ C.F. is the linear combination of the solutions of the component equations $(E - m_1)\, y_x = 0$, $(E - m2)\, y_x = 0$ .... $(E - m_n)\, y_x = 0$

Consider the equation $(E - m_1)\, y_{x=0}$

multiplying both sides by m_1^{-x-1}

$m_1^{-x-1}\, y_{x+1} - m_{-1}^{-x}\, y_x = 0$

$\Delta \left(m_{-1}^{-x}\, y_x\right) = 0$

$\therefore\ m_1^{-x}\, y_1 = c_1$

$\therefore\ y_1 = C_1\, m_1^x$

Hence the C.F. = $\sum_{i=1}^{n} C_1 m_1^x$

$= C_1\, m_1^x + C_2\, m_2^x + C_3 m_3^x + \ldots\ldots + C_3 m_n^x$

Case (ii) Some of the roots are equal

Suppose $m_1 = m_2$

Then we can show that

C.F. = $(C_1 + C_2 x)\ m_1^x\ + C_3\ C_3 x + \dots \dots +\ C_n\ C_n^x$

Suppose $m_1 = m_2 = m_3$

Then C.F. = $(C_1 + C_{2\,0} x + C_3 x^2)\ m_1^x\ + C_4\ m_4^x\ + \dots + C_n\ m_n^x$

Case (iii) Some of the roots are complex.

Suppose two of the roots one $\alpha \pm\ 1\ \beta$

Then we can show that

$C.F. = r^x\ (C_x\ Cos\ \theta\ x + C_2\ sin\ \theta x) + C_3\ m_3^x\ + \dots\dots\ + c_n\ m_n^x$

Where r = $|\alpha + \beta|\ =\ \sqrt{\alpha^2 + \beta^2}$

q = $\tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Suppose the complex roots are repeated twice

$(i.e. m_1 = m_2 = \alpha + 1\,\beta\ , m_3 = m_4 = a\text{ - }1\beta)$ than

C.F. $r^x\left[(C_1 + C_2\ x) Cos\,\theta + (C_3 + C_4\ x) sin\,\theta\,x\ \right]\ +\ C_5\ m_5^x + \dots\dots + C_n\ m_n^x$

where r = $\sqrt{\alpha^2 + \beta^2}$

θ = $tan\ -1\left(\frac{\beta}{x}\right)$

To find the particular integral

P.I. = $\frac{1}{f\,(E)}\ \phi\ (x)$

We consider four cases according as the RHS function ϕ (x) is of the form *(i)* x^m *(ii)* a^n *(iii)* $a^x\ x^m$ and (iv) sin ax or cos ax

Type I: RHS = x^m

P.I = $\frac{1}{f\,(E)}\ \phi\ (x)\ =\ \frac{1}{f\,(E)}\ x^m$

$= \left[f(1+\Delta)\right]^{-1} x^m$

We expand $\left[f(1+\Delta)\right]^{-1}$ in ascending powers of Δ upto Δ^m and then operate on x^m term by term.

Type II: RHS = a^x

P.I = $\frac{1}{f\,(E)}$

$= \frac{1}{f(a)} a^x$

i.e. replace E by a, provided Dr $\neq$ 0 i.e f (a) = 0. If f (a) = 0, apply the shift rule given in the next type.

Type III : RHS = $a^x x^m$

P.I. = $\frac{1}{f(E)} a^x x^m = a^x \frac{1}{f(aE)} x^m$

i.e shift a^x to the left of the operator and replace E by aE in f(E)

Then it will reduce to type I.

Note: The shift rule can be applied to type II when the Dr. becomes zero.

If the Dr = 0 i.e. f(x) = 0, then E-a is a (repeated or non-repeated) factor of f(E). Suppose $(E - a)^r$ is a factor of f(E). Let f (E) = $(E - a)^r = \phi$ (E)

$$\text{P.I. } \frac{1}{f(E)} a^x = \frac{1}{(E-a)^r \phi(E)} a^x$$

$$= \frac{1}{(E-a)^r \phi(a)} a^x \text{ (replacing E by a)}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{(E-a)^r} a^x \cdot 1$$

$$= \frac{1}{\phi(a)} a^x \frac{1}{a^r (a\ E-a)} \text{ (by the shift rule)}$$

$$= \frac{a^x}{\phi(a)} \cdot \frac{1}{a^r (E-1)^r} \cdot 1$$

$$= \frac{a^{x-r}}{\phi(a)} \cdot \frac{1}{\Delta r} \cdot 1$$

$$= \frac{a^{x-r}}{\phi(a)} \cdot \frac{x^r}{r!}$$

Type IV :

RHS = sin ax or cos ax we cos ax = R.P. of e^{iax} & sin ax = I.P of e^{tax} and proceed as in Type II $\left[since\, c^{1ax} = \left(e^{1a}\right)^x \right]$

Problem (1)

From the difference equation of lowest order by eliminating the arbitrary constants from

$y_x = A2^x + B3^x$

$y_x = A2^x + B3^x$

$y_{x+1} = A2^{x+2} + B3^{x+2} = 2A.\ 2^x + 3B.3^x$

$y_{x+2} = A2^{x+2} + B3^{x+2} = 4A.2^X + 9B.3^X$

We have $(A.2^x) + (B.3^x) - y_x = 0$

$$2\ (A\ 2^x) + 3\ (B\ .\ 3^x) - y_{x+1} = 0$$

$$4(A2^x) + 9(B.3^x) - y_{x+2} = 0$$

Eliminating $A\ .\ 2^x$ and $B\ .\ 3^x$ from the above 3 equations we get

$$\begin{vmatrix} 1 & 1 & -y_x \\ 2 & 3 & -y_{x+1} \\ 4 & 9 & -y_{x+2} \end{vmatrix} = 0$$

$1(-3y_{x+2} + 9y_{x+1}) - 1(-2y_{x+2} + 4y_{x+1}) - y_x\ (18-12) = 0$

$-3\ y_{x+2} + 9_{x+1} + 2y_{x+2} - {}^4y_{x+1} - 6y_x = 0$

$-y_{x+2} + 5y_{x+2} - 6y_x = 0$

$y_{x+2} - 5y_{x+1} + 6y_x = 0$

Problem: (2)

Form the difference equation of the lowest order by eliminating the arbitrary constants from $y_x = (A + Bx)2^x$

$y_x = A2^x + Bx2^x$ ....(1)

$y_{x+1} = A2^{x+1} + B(x+1)2^{x+1}$

$= 2A2^x + 2Bx2^x + 2B2^x$ ....(2)

$y_x + 2 = A2^{x+2} + B(x+2)2^{x+2}$

$= 4A2^x + 4Bx2^x + 8B2$ ....(3)

(2) – 2 × (1) ==> $2y_{x+1} - 2y_x = 2B2^x$ ....(4)

(3) – 2 × (4) ==> $(y_{x+2} + 2y_{x+1}) - 2\ (y_{x+2} + 2y_x) = 0$

$y_{x+2} = 4y_{x+1} + 4y_x = 0$

is the required difference equation

Problem: (3)

Find the difference equation satisfied by $y_x = ax^2 + bx$

$y_x = ax^2 + bx$ ....(1)

$$\Delta y_x = a\left[(x+1)^2 - x^2\right] + b\left[(x+1) - x\right]$$

$$= a\,(2x + 1) + b$$

$$\Delta^2 \; y_x = a\;\left[2(x+1) + (2x+1)\right] + 0$$

$$= 2a$$

$$(1) - x \times (2) \Longrightarrow (y_x - x\Delta y_x) = a\left[x^2 - x(12x+1)\right]$$

$$= a\left(x^2 + x\right)$$

$$= \frac{\Delta^2 y_x}{2}\left(x^2 + x\right), \text{ using } (3)$$

$$\therefore 2\;(y_x - x\Delta y_x) = -\left(x^2 + x\right)\Delta^2\; y_x$$

$\therefore \left(x^2 + x\right)\;\Delta^2\; y_x - 2x\;\Delta y_x + 2y_x = 0$ is the required difference equation

Problem: (4)

Solve $y_{x+3} - 2y_{x+2} - 5y_{x+1} + 6y_x = 0$

Writing the given difference equation in terms of E we have

$$\left(E^3 - 2E^2 - 5E + 6\right)\, y_x = 0$$

aux eqn is $m^3 - 2m^2 - 5m + 6 = 0$

By inspection we find m = 1 is a root

1	-2	-5	6
0	1	- 1	-6
1	- 1	-6	$\boxed{0}$

$\therefore\;(m-1)(m^2-m-6) = 0$

$(m - 1)\,(m + 2)\,(m - 3) = 0$

$\therefore\; m = 1\,,\,-2,\,3\,.$

$\therefore$ C.F. $= A(1)^x + B(-2)^x + C(3)^x$

P. I = 0

$\therefore\; y_x =$ C.F. + P.I

$= A + B\,(-2)^x + C\,3^x$

Problem: (5)

Solve $y_{x+3} - 3y_{x+2} + 4y_x = 0$

The given difference equation can be written as

$\left(E^3 - 3E^2 - 4\right) y_x = 0$

aux eqn. is $m^3 - 3m^2 + 4 = 0$

By inspection we find m = - 1 is a root

1	-3	0	4
0	- 1	4	-4
1	-4	4	0

$\therefore$ *(m+ l)(m²-4m + 4) = 0*

$(m+1)(m-1)^2 = 0$

$\therefore$ m = - 1 , 2, 2

$\therefore$ C.F. = $A(1)^x + (Bx+c)2^x$

P. 1 = 0

$\therefore y_x$ = C.F. + P.I

= A $(-1)^2$ $(Bx+c)2^x$

Problem: (6)

Solve $y_{x+3} - 3y_{x+2} + 4y_x = 0$

The given difference equation is $(E^3 - 6E^2 + 12E - 8)\, y_x = 0$

aux eqn in $m^3 - 6m^2 + 12m - 8 = 0$

By inspection we find m = 2 is a root

1	-6	12	-8
0	2	-8	8
1	-4	4	0

$(m - 2)(m^2 - 4m + 4) = 0$

$(m-2)(m-2)^2 = 0$

$(m - 1)^3 = 0$

$\therefore$ m = 2, 2, 2

$\therefore$C.F. = $(A + Bx + Cx^2)2^x$

P. I = 0

$\therefore y_x$ = C.F. + P.I

= $\left(A + bx + cx^2\right) 2^x$

Problem: (7)

Solve $y_{x+3} - 3y_{x+2} + 4y_x = 0$

The given difference equation is $(E^2 - 2E^2 + 2)\, y_x = 0$

aux eqn in $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = 1 + I \qquad \alpha = 1,\ \beta = 1$$

$\therefore$C.F. = r^x (A cos θ x + B sin θ x)

Where $r = \sqrt{\alpha^2 + \beta^2} = \sqrt{1+1} = \sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$$

$$\left[r\cos\theta = 1 \text{ or } \cos\theta = -\frac{1}{\sqrt{2}} \right.$$

$$r\sin\theta = 1 \text{ or } \sin\theta = \frac{1}{\sqrt{2}}$$

$\therefore$ θ lies is the second quadrant

$$\sin(\pi - \theta) = \sin\theta = \frac{1}{\sqrt{2}}$$

$$\cos(\pi - \theta) = \cos\theta = \frac{1}{\sqrt{2}}$$

$$\left. \therefore\ \pi - \theta = \frac{3\pi}{4} \text{ or } \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \right]$$

$$\therefore \text{ C.F } = \left(\sqrt{2}\right)^x \left[A\,Cos\frac{3\pi}{4} x + B\sin\frac{3\pi}{4} \right]$$

P.I. = 0

$\therefore$ y_x = C.F. + P.I

$$= \left(A\cos\frac{3\pi x}{4} + B\sin\frac{3\pi x}{4} \right)$$

Problem: (8)

$y_{x+3} - y_{x+2} + y_{x+1} = 0$

The given difference equation can be written as $(E^2 - E + 1)\, y_{x+1} = 0$.

(Note that the order of the difference equation is 2)

aux. eqn is $m^2 - m + 1 = 0$

$$m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$\therefore \quad \alpha = \frac{1}{2} . \ \beta = \frac{\sqrt{3}}{2}$

$\therefore$ Solution is

[A Cos θ (x + 1) + B Sin θ (x + l)]

Where r = $\sqrt{\alpha^2 + \beta^2}$ = $\sqrt{\frac{1}{4} + \frac{3}{4}}$ = 1

$$\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right) tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$\therefore \ y_{x+1} = (1)^{x+1}\left[A\cos\frac{\pi}{3}(n+1) + B\sin\frac{\pi}{3}x\right]$$

$$\therefore y_x = (1)^x \left[A\cos\frac{\pi}{3}x + B\sin\frac{\pi}{3}x\right]$$

$$= A\cos\frac{\pi}{3}x + B\sin\frac{\pi}{3}x$$

Problem: (9)

Solve y_{n+} 3 - $3y_{n+1}$ + $2y_n$ = 0

given that y_1 = 0 y_2 = 8 and y_3 = - 2

The given difference equation is $\left(E^3 - 3E + 2\right)$ y_n = 0 aux. equation is n^3 - 3m + 2 = 0

By inspection we find one root to be m = 1

1	0	-3	2
0	1	1	-2
1	1	-2	0

$\therefore$ (m -1) (m^2 + m – 2) = 0

(m - 1) (m - 1) (m + 2) = 0

$\therefore$ m = 1 , 1 , –2

$\therefore$ Solution is y_n = (A + B_n) l^n + c $(-2)^n$

y_n = A + Bn + c $(-2)^n$

Put n = 1 $\therefore y_1$ = 0 A + B – 2c . . . . (1)

Put n = 2 $\therefore y_2$ = 8 = A + 2B + 4c . . . . (2)

Put n = 3 $\therefore y_3$ = –2 = A + 3B – 8c . . . . (3)

(2) - (1) 8 = B + 6C

(3) - (2) -10 = B - 12C

(4) - (5) 18 = 18c + ∴ 6C

From (4) we get B = 2

From (1) we get A = 0

∴ Required solution is $y_n = 2n + (-2)^n$

Problem (10) Solve the difference equation $y_{n+3} - 2y_{n+2} \cos\alpha + y = o$, given that

$y_0 = 0$ and $y_1 = 1$

Then given difference equation centre written as

$\left(E^2 - 2E\cos\alpha + 1\right) y_{n+1} = 0$

(Note that the order of the difference equation is 2) aux eqn. is
$m^2 - 2\cos\alpha \cdot m + 1 = 0$

$\therefore m = \dfrac{2\cos\alpha\sqrt{4\cos^2\alpha + \sin^2\alpha + 1}}{2}$

$= \cos\alpha \pm i\sin\alpha$

Hence $y_{n-1} = r^{n-1} \quad \left(A\cos\Theta(n-1) + \beta\sin\Theta(n-1)\right)$

Where $r = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

$\Theta = \tan^{-1} \pm \left(\dfrac{\sin\alpha}{\cos\alpha}\right) = \tan^{-1}(\tan\alpha) = \alpha$

$\therefore\ y_{n+1} = (1)^{n-1} \quad \left[A\cos(n-1)\alpha + B\sin(n-1)\alpha\right]$

$\therefore\ y_n = 1^n \ (A\cos n\alpha + B\sin n\alpha)$

Put n = 0: $y_0 = 0 = A + 0$ ∴ A = 0

Put n = 1: $y_1 = 1 = A\cos\alpha + B\sin\alpha$ $\therefore B = \dfrac{1}{\sin\alpha}$

Hence $y_n = \dfrac{\sin n\alpha}{\sin\alpha}$

Problem: 12

Solve $y_{x+1} - 4y_x = 9x^2$

The given difference equation can be written as

$(E^2 - 4)\, y_x = 9x^2$

Aux eqn is $m^2 - 4 = 0$

$(m + 2)(m - 2) = 0 \quad \therefore\ m = 2\ .\ -2$

$\therefore$ C.F. $= A\,2^x + B(-2)^x$

$$\text{P.I.} = \frac{1}{\left(E^2-4\right)} 9x^2$$

$$= \frac{1}{(1-\Delta)^2-4} 9x^2 \quad \text{since } E = 1 + \Delta$$

$$= \frac{1}{-3+2\Delta+\Delta^2}\ 9x^2$$

$$= \frac{1}{-3\left(1-\frac{2}{3}\Delta-\frac{1}{3}\Delta^2\right)}\ 9x^2$$

$$= \frac{1}{3}\left[1-\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)\right]^{-1}\ 9x^2$$

$$= \frac{1}{3}\left[1-\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)+\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)^2+\ldots\right]\ 9x^2$$

$$= \frac{1}{3}\left[1+\frac{2}{3}\Delta+\frac{\Delta^2}{3}+\frac{4\Delta^2}{9}\right] 9x^2\ .\ \text{omitting higher powers}$$

$$= \frac{1}{3}\left[1+\frac{2}{3}\Delta+\frac{7}{9}\ \Delta^2\right]\ 9\left[x^{(2)}+x^{(1)}\right]$$

$$= -3\left[x^{(2)}+x^{(1)}+\frac{2}{3}2x^{(1)}+\frac{2}{3}+\frac{7}{9}(2)\right]$$

$$= -3\left[x(x-1)+x+\frac{4}{3}x+\frac{2}{3}+\frac{14}{9}\right]$$

$$= -3\left[x^2+\frac{4}{3}x+\frac{20}{9}\right]$$

$$= -3x^2 - 4x - \frac{20}{3}$$

$\therefore$Required solution is

$$= A2^x + B(-2)^x - 3x^2 - 4x - \frac{20}{3}$$

Note: We can also obtain the P.I as follows:

Let P.I. = $\frac{1}{\left(E^2-4\right)}$ $9x^2 = ax^2 + bx + c$

$\therefore$ $(E^2 - 4)(ax^2 + bx + c) = 9x^2$

$a(x+2)^2 + b(x+2) + c - 4(ax^2 + bx + c) = 9x^2$

$a(x^2 + 4x + 4) + b(x+2) + c - 4ax^2 - 4bx - 4c = 9x^2$

$-3ax^2 + x(4a - 3b) + (4a + 2b - 3c) = 9x^2$

Equating the coefficient we get

$-3a = 9 \Longrightarrow a = -3$

$4a - 32 = 0 \Longrightarrow b = -4$

$4a + 2b - 3c = 0 \Longrightarrow b = -4$

$\therefore$ P.I. = $3x^2 - 4x - \frac{20}{3}$

Problem (13)

Solve $yx + 2\ 4y_{x+1} + 4y_x = x^2 + x + 4$

The given difference equation is $\left(E^2 - 4E + 4\right)yx = x^2 + x + 4$

aux eqn is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \qquad \therefore\ m = 2\ ,3$

$\therefore$ C.F. = $(A + Bx)\ 2^x$

$$\text{P.I.} = \frac{1}{E^2 - 4E + 4}\, x^2 + x + 4$$

$$= \frac{1}{(1+\Delta)^2 - 4(1+\Delta) + 4}\, x^2 + x + 4$$

$$= \frac{1}{1 - 2\Delta + \Delta^2}\, x^2 + x + 4$$

$$= \left[1 - 2\Delta + \Delta^2\right]^{-1} \left(x^2 + x + 4\right)$$

$$= \left[1 - \left(2\Delta + \Delta^2\right)\right]^{-1} \left(x^{(2)} + 2x^{(1)} + 4\right)$$

$$= \left[\ 1 + \left(2\Delta - \Delta^2\right) + \left(2\Delta - \Delta^2\right)^2 + \ldots\right] \left(x^{(2)} + 2x^{(1)} + 4\right)$$

$$= \left(1+2\Delta - \Delta^2 + 3\Delta^2\right)\left(x^{(2)}+2x^{(1)}+4\right)$$

$$= \left(1+2\Delta + 3\Delta^2\right)\left(x^{(2)}+2x^{(1)}+4\right)$$

$$= x^{(2)}+6x^{(1)}+14$$

$$= x^2+5x+14$$

$\therefore$ Required solution is

y_x = C.F. + P. I

= (A + Bx) 2x + x^2 + 5x +14

Note : Let P.I.= $\frac{1}{E^2-4E+4}$ $x^2 + x + 4 = ax^2 + bx + c$

$\left(E^2-4E+4\right)\left(ax^2+bx+c\right) = x^2 + x + 4$

$\left[a(x+2)^2+b(x+2)+c\right] - 4\left[a(a+1)^2+b(x+1)+c\right] + 4\left[ax^2+bx+c\right] = x^2 + x + 4$

$ax^2 + (-4a + b)x + (-2x+c) = x^2 + x + 4$

Equating coefficient we get

a = 1

-4a + b = 1

-2b + c = 4+

Hence a=l, b = 5, c+14

$\therefore$ P.I. = $x^2 + 5x + 14$

Problem: (14)

Solve $\Delta^2 y_x - 2\Delta y_x + y_x = 3x + 2$

The given difference equation can be written as

$\left[\Delta^2 - 2\Delta + 1\right] y_x = 3x + 2$

$(\Delta - 1)^2 y_x = 3x + 2$

$(E - l - l)^2 y_x = 3x+2$ ($\therefore \Delta = E - 1$)

$(E - 2)^2 y_x = 3x + 2$

aux. eqn in $(m - 2)^2 = 0$ $\therefore$ m = 2, 2

C.F. = $(Ax + B)2^x$

P.I. = $\frac{1}{(E-2)^2}$ 3x + 2

= $\frac{1}{(1+\Delta-2)^2}$ 3x + 2

= $\frac{1}{(\Delta-1)^2}$ 3x + 2

= $\frac{1}{(1-\Delta)^2}$ 3x + 2

= $(1-\Delta)^2$ 3x + 2

= $\left[1+2\Delta + 3\Delta^2 + ...\right]$ $3x^{(1)}$ + 2

= $\left[3x^{(1)}+2\right]$ + 2 (3) + 0

= 3x + 8

$\therefore$ Required solution is y_x = C.F. P.I

= $(Ax + B) 2^x + 3x + 8)$

Note: Let P.I. = $\frac{1}{(E-2)^2}$ (3x + 2) = ax + b

$\therefore$ $(E - 2)^2$ ax + b = 3n + 2

$\left(E^2-4E+4\right)$ (ax + b) = 2n + 2

$\left[a(x+2)+b\right] - 4\left[a(x+1)+b\right] + 4\left[ax+b\right]$ = 3x + 2

ax + (-2a + b) = 3x + 2

Equating coefficient we get

a = 3

– 2a + b = 2 $\therefore$ b = 8

$\therefore$ P.I. = 3x + 8

Problem: (15)

Solve $2y_{x+2} - 5y_{x+1} + 2y_x = 3^x$

The difference equation can be written as

$\left(2E^2-5E+2\right) y_x = 3^x$

aux. eqn is $2m^2 - 5m + 2 = 0$

(2m- 1) (m-2) = 0

m = 2 . $\frac{1}{2}$

$\therefore$ C.F. = A2^x + B $\left(\frac{1}{2}\right)^x$

P.I = $\frac{1}{(2E^2 - 5E + 2)}3^x$

= $\frac{1}{2(9)-5(3)+2}$ (Replacing E = a = 3)

= Required solution is yx = A2^x + B . $\frac{1}{2^x}$

Problem : (16)

Solve y_x - $7y_{x-1}$ + $12y_{x+2}$ = 2^x

The given difference equation can be written as $(E^2 - 7E + 12)$ y_{x-2} = 2^x

aux. eqn is m^2 - 7m + 12 = 0

(m-3) (m-4) = 0

m = 3, 4

$\therefore$ C.F. A B^{x-2} + B 4^{x-2}

P.I. = $\frac{1}{E^2\,7E + 12}2^x$

= $\frac{1}{(4) - 7(2) + 12}2^x$ (Replacing E = a =2)

= $\frac{1}{2}2^x = 2^{x-1}$

$\therefore$ Required solution is

$y_{x-2} = A3^{x-2} + B.4^{x-2} + 2^{x-1}$

Replacing x by x + 2 we get

$y_x = A3^x + B\ 4^{x-2} + 2^{x-1}$

Note: In the given difference equation itself we can replace x by x + 2

$\therefore$ $(E^2 - 7E + 12)$ y_x = 2^{x+2} = 4.2^x

C.F. A 3^x + B 4^x

P.I. $= \frac{1}{(E^2 - 7E + 12)} 4.2^x$

$= \frac{1}{(4)-7(2)+12} 4.2x$

$= \frac{1}{2} 4.2^x + B.4^x + 2^{x+1}$

Problem: (17)

Solve $y_{x+2} - 3y_{x+1} - 4y_x = 4^x$

The given difference equation is $(E^2 - 3E - 4)y_x = 4^x$

aux. eqn in $m^2 - 3m - 4 = 0$

$(m - 4)(m + 1) = 0$

$m = 4 , -1$

$\therefore$C.F. $= A4^x + B(-1)^x$

P.I. $= \frac{1}{E^2 - 3E - 4} 4^x$

$= \frac{1}{(16) - 3(4)a - 4} 4^x$ (E = a = 4)

The Dr becomes zero

$\therefore$ P.I. $= \frac{1}{(E-4)(E+1)} 4^x$

$= \frac{1}{(E-4)} \left[\frac{1}{(4+1)} .4x \right]$

$= \frac{1}{(E-4)} \cdot \frac{1}{(4+1)} 4x$

$= \frac{1}{5} \cdot \frac{1}{(E-4)} \cdot 4^x \cdot 1$

$= \frac{1}{5} \cdot 4^x \cdot \frac{1}{(4E-4)}$ 1 (applying shift rule Replace E by aE by 4E)

$= \frac{4^x}{5} \frac{1}{4(E-1)} . 1$

$= \frac{4^{x-1}}{5} . x$

$\therefore y_x = A\,4^x + (-1)^x + \frac{4^{x-1}}{4} \cdot x$

Note: We can also apply the formula

$\frac{1}{E-a} a^x = a^{x-1} x$

$\therefore \frac{1}{E-4} 4^x = 4^{x-1} x$

Problem: (18)

$U_{x+2} - 4U_{x+1} + 4U_x = 2^x$

The given difference equation is

aux. eqn is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2.2$

$\therefore$ C.F. $(Ax + B)\,2^x$

P.I. $= \frac{1}{(E^2 - 4E + 4)} 2^x$

$= \frac{1}{(4) - 4(2) + 4} 2^x$ (E = a = 2)

Dr becomes zero

$= 2^x \cdot \frac{1}{2^2 (2E-2)^2}$ (applying shift rule replacing E by aE. i.e. 2E)

$= 2^x \frac{1}{2^2 (E-1)^2}$

$= 2^{x-2} \cdot \frac{x^{(2)}}{2}$

$= 2^{x-3}\, x\,(x-1)$

$= 2^{x-3}\, x\,(x^2 - 1)$

$\therefore y_x = (Ax + B) + 2^{x-3}(x^2 - x)$

Note: We can also apply the formula

$\frac{1}{(E-a)^2} a^x = a^{x-2} \cdot \frac{x^{(2)}}{2}$

$\therefore \frac{1}{(E-a)^2} \cdot a^x = R^{x-2} \frac{x^{(2)}}{2} = 2^{x-3} \left(x^2 - x\right)$

Problem: (19)

Solve $y_{x+2} - 2y_{x+1} + y_x = 2^x x^2$

The difference equation can be wirtten as

$(E^2 - 2E + 1)\, y_x + 2^x \,.\, x^2$

aux. eqn is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \therefore m = 1, 1$

$\therefore$ C.F. $= (Ax + B)\, 1^x$

$= Ax + B$

$$\text{P.L.} = \frac{1}{\left(E^2 - 2E + 1\right)} 2^x x^2$$

$$= 2^x \frac{1}{(2E)^2 - 2(2E) + 1} x^2 \text{ (applying shift rule replacing E by 2E)}$$

$$= 2^x \frac{1}{4(1+\Delta)^2 - 4(1+\Delta) + 1}$$

$$= 2^x \frac{1}{1 + 4\Delta + 4\Delta^2} x^2$$

$$= 2^x \frac{1}{(1 + 2\Delta)^2} x^2$$

$$= 2^x (1 + 2\Delta)\ x^2$$

$$= 2^x \left[1 - 2(2\Delta) + 3(2\Delta)^2 + \ldots\ldots.\right] x^2$$

$$= 2^x \left[1 - 4\Delta + 12\Delta^2\right] x^{(2)} + x^{(1)} \quad \left[\therefore x^2 = x^{(2)} + x^{(1)}\right]$$

$$= 2^x \left[x^{(2)} + x^{(1)} - 4\ \left(2x^{(1)} + 1\right) + 12\ (2)\right]$$

$$= 2^x \left[x(x-1) + x - 8x - 4 + 24\right]$$

$$= 2^x \left(x^2 - 8x + 20\right)$$

$$\therefore y_x = Ax + B\ 2x \left(x^2 - 8x + 20\right)$$

Problem : (20)

Solve $y_{n+2} - 4y_{n+1} + 3y_n = n^4$

The given difference equation is

$\left(E^2 - 4E + 3\right) 3y_n = n^4$

aux eqn is $m^2 - 4m + 3 = 0$

$(m-1)(m-3) = 0$

$m = 1, 3$

$\therefore$ C.F. = A , 1^n + B. 3^n

= A + B 3^n

P.I $= \dfrac{1}{(E^2 - 4E + 3)} 4^n n$

$= 4^n \dfrac{1}{(4E)^2 - 4(4E) + 3}$ n (applying shift rule replace E by 4E)

$= 4^n \dfrac{1}{16E^2 - 16E + 3}$ n

$= 4^n \dfrac{1}{16(1+\Delta)^2 - 16(1+\Delta) + 3}$ n

$= 4^n \dfrac{1}{3 + 16\Delta + 16\Delta^2}$. n

$= 4^n \dfrac{1}{3\left(1 + \frac{16}{3}\Delta + \frac{16}{3}\Delta^2\right)}$ n

$= \dfrac{4^n}{3}\left[1 - \left(\frac{16}{3}\Delta + \frac{16}{3}\Delta^2\right) + \left(\frac{16}{3}\Delta + \frac{16}{3}\Delta^2\right)^2 + \ldots\ldots\right]$ n

$= \dfrac{4^n}{3}\left[1 - \frac{16}{3}\Delta\right]$ n

$= \dfrac{4^n}{3}\left[1 - \frac{16}{3}\Delta(1)\right]$

y = A + B . $3^n + \dfrac{4^n}{3}(3n - 16)$

Problem: (21)

Solve $y_{n+2} - 2y_{n+1} + 2y_n = \text{Cos}\left(\dfrac{n\pi}{2}\right)$

The given difference equation is $(E^2 - 2E + 2)\, y_n = \text{Cos}\left(\dfrac{n\pi}{2}\right)$

aux eqn is $m^2 - 2m + 2 = 0$

$m = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i \qquad \alpha = 1, \; \beta = 1$

C.F. = r^n (A Cos n θ + B sin n θ)

Where r = $\sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$ (1) = $\tan^{-1}\frac{\pi}{4}$

$\therefore$ C.F. = $\left(\sqrt{2}\right)^n \left(A\,Cos\frac{n\pi}{4} + B\,Sin\frac{n\pi}{4}\right)$

P.I. = $\frac{1}{E^2\ 2E+2}$ Cos $\left(\frac{n\pi}{2}\right)$

$= \text{R.P. of } \frac{1}{E^2\ 2E+2}\left(e^{i\pi/2}\right)^n$

$= \text{R.P. of } \frac{1}{\left(e^{i\pi/2}\right)^2 - 2\left(e^{i\pi/2}\right) + 2}\left(e^{i\pi/2}\right)^n$

$= \text{R.P. of } \frac{1}{e^{i\pi} - 2e^{i\pi/2} + 2}\left(e^{i\pi/2}\right)^n$

$= \text{R.P. of } \frac{1}{\cos\pi + i\sin\pi - 2\left[\cos\pi/2 + i\sin\pi/2\right] + 2}\ e\frac{in\pi}{2}$

$= \text{R.P. of } \frac{1}{-1-2i+2}\ e^{in\pi/2}$

$= \text{R.P. of } \frac{1}{(1-2!)\,(1+2!)}\ e^{in\pi/2}$

$= \text{R.P. of } \frac{(1+2!)}{(1-2!)\ (1+2!)}\ e^{in\pi/2}$

$= \text{R.P. of } \frac{(1-2!)\left(Cos\frac{n\pi}{2} = i\,Sin\,\frac{n\pi}{2}\right)}{5}$

$= \frac{1}{5}\left(Cos\frac{n\pi}{2} - 2\,Sin\frac{n\pi}{2}\right)$

$\therefore$ y = $2^{n/2}\left(A\,Cos\frac{n\pi}{4} + B\,Sin\frac{n\pi}{4}\right) + \frac{1}{5}\left(Cos\frac{n\pi}{2} + 2\,Sin\frac{n\pi}{2}\right)$

Problem: (22)

Solve $U_{n+2} + U_n = \sin\left(\frac{n\pi}{6}\right)$

The given difference eqn in $\left(E^2+1\right)$ u_n = sin $\left(\frac{n\pi}{6}\right)$

aux eqn in $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm 1$

$\therefore$ C. F. r^n (A Cos n θ + sin nθ)

Where r = $\sqrt{\alpha^2+\beta^2} = \sqrt{0+1} = 1$

$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$\therefore$ C.F. 1n $\left[A\,Cos\left(\frac{n\ \pi}{2}\right) + B\,Sin\left(\frac{n\ \pi}{2}\right)\right]$

P.I. = $\frac{1}{(E^2+1)} \sin\left(\frac{n\ \pi}{2}\right)$

$= \text{I.P of } \frac{1}{(E^2+1)} e^{\frac{in\,x}{6}}$

$= \text{I.P of } \frac{1}{(E^2+1)} \left(e^{i\,\pi/6}\right)^n$

$= \text{I.P of } \frac{1}{\left(e^{i\,\pi/6}\right)^2+1} \left(e^{i\,\pi/6}\right)^n . \left(Put\,E = a = e^{i\,\pi/6}\right)$

$= \text{I.P of } \frac{1}{e^{i\,\pi/3}+1} e^{i\,\pi\,n/6}$

$= \text{I.P of } \frac{1}{Cos\frac{\pi}{3} + i\,Sin\frac{\pi}{3} + 1} e^{i\,\pi\,n/6}$

$= \text{I.P of } \frac{1}{\frac{3}{2} + i\frac{\sqrt{3}}{2} + 1} e^{i\,\pi\,n/6}$

$= \text{I.P of } \frac{1}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} e^{i\,\pi\,n/6}$

$= \text{I.P of } \frac{\left(3-\sqrt{3}\right)}{\frac{1}{2}\left(3+i\sqrt{3}\right)\left(3-i\sqrt{3}\right)} \left(Cos\frac{n\,\pi}{6} + i\,Sin\frac{n\,\pi}{6}\right)$

$= \frac{1}{6}\left(3\,sin\,\frac{n\,\pi}{6} - \sqrt{3}\,cos\frac{n\,\pi}{6}\right)$

$= \text{A Cos}\left(\frac{n\,\pi}{2}\right) + \text{B in} + \frac{1}{6}\left[3\,sin\,\frac{n\,\pi}{6} - \sqrt{3}\,Cos\frac{n\,\pi}{6}\right]$

Problem : (23)

Solve A $U_x + \Delta 2\ U_x = \text{Cos } x$

The given difference equation is $\left(\Delta+\Delta^2\right)$ U_x = Cos x

i.e. $\left[(E-1)+(E-1)^2\right]$ U_x = Cos x

$\left[E-1+E^2-2E+1\right]$ U_x = Cos x

$\left(E^2-E\right)$ U_x = Cos x

(E – 1) U_{x+1} = Cos x

(E – 1) V_x = Cos x Were $V_x = U_{x+1}$

(Note that the order of the difference equation in 1 and not 2)

aux eqn in m – 1 = 0

m = 1

$\therefore$ C.F. = A $(1)^x$ = A

$$\text{P.I.} = \frac{1}{(E-1)} \text{Cos } x$$

$$= \text{RP of } \frac{i}{(E-1)} e^{ix}$$

$$= \text{RP of } \frac{1}{(E-1)} \left(e^i\right)^x$$

$$= \text{RP of } \frac{1}{\left(e^1-1\right)} \left(e^i\right)^x \quad \left(E = a = e^i\right)$$

$$= \text{RP of } \frac{\left(e^{-1}-1\right)}{\left(e^1-1\right)\left(e^{-1}-1\right)} e^{ix}$$

$$= \text{RP of } \frac{e^{i(x-1)}}{1 - e^i - e^{-1} + 1}$$

$$= \text{RP of } \frac{Cos(x-1)+i\,Sin(x-1)-Cos\,x\,i\,Sin\,x}{2-2\,Cos\,1}$$

$$= \frac{Cos(x-1)-Cos\,x}{2(1-Cos\,1)}$$

$$U_x = \text{C.F.} + \text{P.I.} = A + \frac{Cos(x-1)-Cos\,x}{2(1-Cos\,1)}$$

$$\therefore U_x = A + \frac{Cos(x-2)-Cos\,(x-1)}{2(1-Cos\,1)}$$

Problem : (24)

Solve is consistence of the every face is a triangle. Show that the number of faces of such as solid having n vertices is (2n 4)

Let y_x be the number of faces of solid with n vertices. Adding one more vertex we gain 3 faces and lose on face i.e. as a result we gain 2 faces.

$\therefore y_{n+1} = y_{n+2}$

$\therefore y_{n+1} - y_{n+2}$

$(E - 1)\, y_n = 2$

aux eqn is $m - 1 = 0$

$m = 1$

$\therefore$ C.F. $A\,(1)^n = A$

$$P.I = \frac{1}{(E-1)}\, 2$$

$$= \frac{1}{\Delta}\, 2$$

$$= 2x^{(1)} = 2n$$

$\therefore y_n = A + 2n$

But when n = 4 (tetraheden) number of faces $= y_4 = B^4$

$\therefore y_4 = 4 = A + 2(4) \therefore A = -4$

$\therefore y_n = 2n - 4\ (n \geq 4)$

Problem: (25)

Evaluate the following nth order determinant by forming the difference equation.

$$\begin{vmatrix} 2\cos\theta & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\theta & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2\cos\theta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2\cos\theta \end{vmatrix}$$

Expanding the determinant along the first column we see that $A_n = 2\cos\theta\, A_{n-1} - 1\, A_{n-2}$

$\therefore A_n - 2\cos\theta\, A_{n-1} + A_{n-2} = 0$

$(E^2 - 2\cos\theta\, E + 1)\, A_{n-2} = 0$

aux. eqn in $m2 - (2\cos\theta)\, m + 1 = 0$

m = $\frac{2\,Cos\,\theta \pm \sqrt{4\,Cos^2\,\theta - 4}}{2}$

= Cos θ ± I Sin θ α= Cos θ

β= Sin θ

∴ C.F. r^{n-2} $\left[A\,cos\,(n-2)^{\phi} + B\,sin(n-2)^{\phi}\right]$

Where = $\sqrt{\alpha^2+\beta^2}$ = $\sqrt{Cos^2\,\theta + Sin^2\,\theta}$ = 1

φ = $\tan^{-1}\left(\frac{\beta}{\alpha}\right)$= $\tan^{-1}$ $\left[A\,Cos(n-2)\theta + B\,Sin\,(n-2)\theta\right]$

P.I. = 0

A_{n-2} = A Cos (n – 2) θ + B sin (n – 2) θ

∴ A_n = A Cos n θ + B Sin n θ

We find that when n = 1 A_1 = |2 Cos θ| = 2 Cos θ – 1 ...... (1)

Actual when n 2, A_2 = $\begin{vmatrix} 2\,Cos\,\theta & 1 \\ 1 & 2\,Cos\,\theta \end{vmatrix}$ = 4 Cos^2 θ – 1

Putting n = 1 and n = 2 in (1) we get

A_1 = A Cos θ + B Sin θ = 2 Cos θ ...... (2)

A_2 = A Cos 2 θ + B Sin θ = 4 Cos θ^2 = 1 ...... (3)

(2) × 2 Cos θ – (3) gives

A$\left[2\,Cos^2\,\theta - Cos\,2\,\theta\right]$ = 1

A (1 + Cos 2 θ – Cos 2 θ) = 1 ∴ A = 1

∴ B = Cot θ

∴ An = Cos n θ + Cot θ Sin n θ

= Cos n θ + $\frac{Cos\,\theta}{Sin\,\theta}$ Sin n θ

= $\frac{Cos\,n\,\theta\;\;Sin\,\theta\; + Sin\,n\,\theta\;\;Cos\,\theta}{Sin\,\theta}$

= $\frac{Sin(n+1)\theta}{Sin\,\theta}$

Exercise

1) Form the difference equation of the least order by eliminating the arbitrary constants from y = A2^x + B

$$Ans: \left[y_{n+2} - 3y_{x+1} + 2y_x = 0\right]$$

2) Form the difference equation satisfied by y_n = (An +B) 3^n

$$Ans: \left[y_{n+2} - 3y_{x+1} + 9y_n = 0\right]$$

Solve the following difference equations (3 to 18)

3) $\Delta^2 - 7\,\Delta\, y_n + 12\ y_n = 0$

$$Ans: \left[y_n = A4^n + B5^n\right]$$

4) $4y_{x+2} - 4y_{x+1} + y_n = 0$

$$Ans: \left[y_x = (Ax + B)\frac{1}{2^x}\right]$$

5) $U_{n+2} - 2U_{n+1} = 4U_n = 0$

$$Ans: \left[U_n = 2^n\left(A\,Cos\frac{n\pi}{3} + B\,Sin\,\frac{n\pi}{3}\right)\right]$$

6) $2y_{n+2} - 5\,y_{n+1} + 2y_n = 0$ given than y_0 = and $y_1 = 1$

$$Ans:\left[y_n = \frac{2}{3}\left(2^n - \frac{1}{2^n}\right)\right]$$

7) $y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4_n = 0$ given than $y_0 = 3$. $y_1 = 2$ and $y_4 = 22$

$$Ans: \left[y_n = 6 + (n-3)2^n\right]$$

8) $\Delta^2 y_x + \Delta_x + y_x + x^2$

$$Ans: \left[y_n = A\,Cos\frac{\pi x}{3} + B\,Sin\,\frac{\pi x}{3} + x^2 - 2x - 1\right]$$

9) $U_{n+2} - 2U_{n+1} + U_n = 3n + 4$

$$Ans: \left[U_n = An + B + \frac{1}{2}n(n-1)(n+2)\right]$$

10) $y_{x+2} - 4y_{x+1} + 3y_x = 2^x$

$$Ans: \left[y_x = A + B.3^x - 2^x\right]$$

11) $y_{x+2} - 6y_{x+1} + 8y_x = 4^x$

$$Ans: \left[y_x = A.2^x + B.4^x + \frac{x}{2}4^{x-1}\right]$$

12) $y_{x+2} - 6y_{x+1} + 9y_x = 3^x$

$$Ans: \left[y_x = (Ax + B)3x + 3^{x-2}\,\frac{(x^2 + x)}{2}\right]$$

13) $U_{x+2} + U_{n+} = 5.2^n$ given that $U_0 = 1$ and $U_1 = 0$

$$Ans: \left[U_n = 2^n - 2\sin\left(\frac{n\pi}{2}\right) \right]$$

14) $y_{x+2} - 8y_{x+1} + 16y_x = 2^x \left(x^2 + 1\right)$

$$Ans: \left[y_x = (Ax + B)4^2 + \frac{2^x}{4}\left(x^2 + 4x + 9\right) \right]$$

15) $y_{n+2} - 7y_{n+1} + 8y_n = 2^n (3n + 4)$

$$Ans: \left[y_n = A.8^n + B(-1)^n - \frac{2^{n-1}}{3}(n+1) \right]$$

16) $y_{n+2} - 7y_{n+1} + 6y_n = \text{Sin}\left(\frac{n\pi}{2}\right)$

$$Ans: \left[y_n = A3^n + B(-2)^2 + \frac{1}{50}\left(7\,Sin.\frac{n\pi}{2} + Cos\frac{n\pi}{2}\right) \right]$$

17) $y_{n+2} - y_n = \sin\left(\frac{n\pi}{6}\right)$

$$Ans: \left[y_n = A + B(-1)^n + \frac{1}{2}\left(\sqrt{3}\,Cos\frac{n\pi}{6} + Sin\frac{n\pi}{6}\right) \right]$$

18) $y_{n+2} + 7y_n = \sin\left(\frac{n\pi}{3}\right)$

$$Ans: \left[y_x = A\,Cos\left(\frac{n\pi}{2}\right) + B\,Sin\left(\frac{n\pi}{2}\right) + \frac{1}{2}\;Cos\left(\frac{n\pi}{3}\right) + \frac{\sqrt{3}}{2}\,Sin\left(\frac{n\pi}{3}\right) \right]$$

4.4 CURVE FITTING

The data involving two variables, say x and y (y dependent on x) from experimental observations is required to be expressed in the form of a law connecting the two variables x and y. In such cases the corresponding values $(x_1 . y_1)$ of the given data are plotted on graph paper and a smooth curve is drawn passing through the plotted points. Such a curve is called an approximating curve. Its equation, say y = f (x) is known as an 'empirical equation'. Since it is possible to draw a number of such curves through or near the points, different empirical equations can be obtained to express the data. Now the problem is to find the equation of the curve which is best suited to estimating the unknown values. The process of finding such a equation of the 'best fit' is known as 'curve fitting'.

Method of Group Averages

Suppose given n observations $(x_1 . y_1)$ i = 1, 2 ...... n. to fit a straight line of the form y = a + bx by group averages method.

When $x = x_1$, the observed value of $y = y_1$

The expected value is $y = a + b\,x_1$

Then $d_1 = y_1\ (a + bx_1)$

Which is called the residual at x_1 Similarly, the residuals

$d_2 = y_2 - (a + bx_2)$

$d_n = y_n\ (a + bx_n)$

Here some of these residuals may be positive while other are negative. The method of group of averages is based on the assumption that the sum of the residuals is zero.

(i.e.) $\sum_{i=`}^{k} di = 0$

To find the constants a and b, divide the given data into two groups, (both contain approximately equal number of observations). Suppose the first group containing k observations and the second group containing the remaining n - k observations. Assuming that the sum of the residuals in each group is zero.

i.e. $\sum_{i=`}^{k} di = \sum_{i=`}^{k} [yi - (a + bx_1)] = 0$

and $\sum_{i=k+1}^{n} di = \sum_{i=k+n}^{n} [yi - (a + b_i)] = 0$

Which gives on simplification

$\bar{y}_1 = a + b\,\bar{x}_1$ .... (1)

$\bar{y}_2 = a + b\,\bar{x}_2$ ....(2)

$\bar{x}_1$. $\bar{y}_1$ are the averages of x's and y's of the first group.

$\bar{x}_2$. $\bar{y}_2$ are the averages of x's and y's of the second group solving (1) and (2) we get a and b.

Problem – 1:

Obtain an equation of the form y = a + bx for the following data using the method of grouping.

x	0	5	10	15	20	25
y	12	15	17	22	24	30

Solution

Let us divide the given data into two groups as follows:

Group I		Group II	
X	y	x	y
0	12	15	22
5	15	20	24
10	17	25	30
Σx = 15	Σy = 44	Σx = 60	Σy = 76

$\bar{x}_1 = \frac{15}{8}$ = 15 $\qquad\qquad$ $\bar{x}_2 = \frac{60}{3}$ = 20

$\bar{y}_1 = \frac{44}{3}$ = 14.667 $\qquad\qquad$ $y_2 = \frac{76}{3}$ = 25.333

Substituting the average of x's y's of the two groups in the required line y = a + bx

14.667 = a +5b

25.333 = a + 20b

Solving these two equations, we get a = 11.1117 and b = 0. 7111.

Hence the equation of the best fit is y = 11.1117 + 0.711IX.

Problem – 2 :

Fit a curve of the form $y = ax^b$ for the following data

x	1.2	1.4	1.6	1.8	2.0	2.4	2.6
y	4.2	6.1	8.5	11.5	14.9	23.5	27.1

Solution

The given low can be written in linear form as

$y = ax^b$

$\log_{10} y = \log_{10a} + b\log_{10}x$

Y = A + BX

Where $Y = \log_{10} y$. $A = \log_{10}a$

$B = b$ and $X = \log_{10}x$

Group I

x	y	X = log x	Y = log y
1.2	4.2	0.0792	0.6233
1.4	6.1	0.1416	0.7853
1.6	8.5	0.2041	0.9294
1.8	11.5	0.2553	0. 0607

$\Sigma X = .68902$ $\Sigma Y = 3.3987$

$X_1 = \frac{.6802}{4}$ $Y_1 = \frac{3.3987}{4} = .8497$

Group II

X	y	X = log x	Y = logy
2.0	14.9	0.3010	1.1732 1
2.4	23.5	0.3802	1.3714
2.6	27.1	0.4150	1.4330

$\Sigma X = 1.09692$ $\Sigma Y = 3.9772$

$X_2 = \frac{1.0962}{3} = 0.3654$ $Y_2 = 1.4326$

$.8497 = A + 0.1701\ B$

$1.3257\ A + 0.3654\ B$

$A = 2\ .\ 4370$ $B = -9.3342$

$a = \text{anti}\ \log_{10}$ $A = 273.5012$

$b = B = 9.3342$

Hence the best fit is

$y - 273.\ 5012\ x^{-9.3342}$

Problem – 3:

For the following data

x	2.3	3.1	4	4.92	5.91	7.20
y	3.3	39.1	50.3	67.2	85.6	125

Fit a curve of the form y

Solution

$y = a\ e^{bx}$

$\log e^{y} = a + bx$

$Y = a + bx$

Where $Y = \log_e y$

Group I

X	y	$Y = \log_e y$
2.3	33	1.52
3.1	39.1	1.59
4	50.3	1.70

$\Sigma x = \frac{94}{3}$ $\Sigma y = \frac{4.81}{3}$

$\bar{x}_1 = 3.1333$ $\bar{y}_1 = 1.6033$

Group - II

X	y	$y = \log_x^y$
4.92	67.2	1.83
5.91	85.6	1.93
7.20	125	2.1

$\Sigma x = 18.03$ $\Sigma y = 5.86$

$\bar{x}_2 = 6.01$ $\bar{y}_2 = 1.9533$

1.6033 * a + b (3.133)

1.9533 - a + b (6.01)

Solving a = 1.2221

b = 0.1217

$y = 1.221\ e^{0.1217x}$

Problem – 4:

Using the method of averages fit a parabola $y + bx + ex^2$ to the following data

x	20	40	60	80	100	120
y	5.5	9.1	14.9	22.8	33.3	46.0

Solution

Let the point (20, 5.5) lies on the parabola

$y = a + bx + ex^2$

5.5 a a + 20b + 400c

$y - 5.5 = b(x - 20) + e\left(x^2 - 400\right)$

$\frac{y-5.5}{x-20}$ = b + c (x + 20)

Y = b + c X

Where Y = $\frac{y-5.5}{x-20}$. X – x + 20

Group I

x	y	x – 20	y – 5.5	X = x + 20	Y = $\frac{y-5.5}{x-20}$
40	9.1	20	3.6	60	0.18
60	14.9	40	9.4	80	0.235

Σx = 140 Σy = 0.415

$X_1 = \frac{140}{2}$ = 70 $Y_1 = \frac{0.415}{2}$ = .2075

Group II

x	y	x – 20	Y – 5.5	X	Y
80	22.9	60	17.3	100	0.288
100	33.3	80	27.8	120	0.348
120	46	100	40.5	140	0.405

ΣX = 360 ΣY = 1.041

$X_2 = \frac{360}{3}$ = 120 $Y_2 = \frac{1.041}{3}$ = 0.347

0.2075 = b + c (70)

.347 = b + c (l20)

Solving b = 0.0122 , c = .00279

$\frac{y-5.5}{x-20}$ = 0.0122 + 0.00279 (x + 20)

(i.e.) y = 0.00279 x^2 + 0.0122x + 4.14

Exercise

1. Fit a straight line y = a + bx to the following data by the method of group averages.

x	1	2	3	4
y	16	19	23	26

2) Fit the curve $y = ab^{x}$ to the following data by the method of group average

x	2	4	6	8	10	12
y	7.32	8.24	9.20	10.19	11.01	12.05

3) Fit a curve of the form $y = ae^{bx}$ to the following data

x	1	2	3	4	5	6
y	14	27	40	55	68	300

4) Fit a parabola for the following data

X	2	2.5	3	3.5	4	4.5	5
y	18	17.8	17.5	17	15.8	14.8	13.3

Principles of Least Squares

The values of the constants in the group average method depends upon the choice of grouping the observations. So we require some other method to get the best values of the constants. Principle of least squares provides a unique set of values to the constants.

Let (x_1, y_1) , i = 1 , 2 ..... n be the n set of observations and Let y = f(x) be the relation between x and y.

When $x = x_1$ the observed value is y1

$\therefore$ Residual $d_1 = y_1 - f(x_1)$

$$E = \sum_{i=1}^{n} di^2$$

If E = 0, then $y_1 = f(x_1)$ for all i (i.e.) all the points lie on the curve otherwise, the minimum of E results the best fitting curve to the data.

To fit a straight line y = ax + b by least squares.

$$E = \sum_{i=1}^{n} di^2 = \sum_{i=1}^{n} \left[y_i - (a x_1 + b)\right]$$

By the principle of least squares E is minimum

$\frac{\partial E}{\partial a}$ = 0 and $\frac{\partial E}{\partial b}$ = 0

$$\sum_{i=1}^{n} 2 \ (y_1 - ax_1 \text{-} b) \ (-x_1) = 0$$

(i.e.) $\sum_{i=1}^{n} 2 \ \left(y_1 - x_1 \ - ax_1^{\ 2} - bx_1\right) = 0$

and $\sum_{i=1}^{n} 2 \ \left(y_1 - ax_1^{\ 2} - b\right)$ (-1) = 0

(i.e.) $\sum_{i=1}^{n} (y_1 - ax_1 - b) = 0$

$$\sum_{i=1}^{n} y_1 \, x_1 \ = a \sum_{i=1}^{n} x_1^2 + b \sum_{i=1}^{n} x_1$$

$$\sum_{i=1}^{n} x_1 \ = \ a \sum_{i=1}^{n} x_1 + nb$$

These are known as normal equations to the curve y = ax + b solving these equation find the values of a and b.

Problem – 5:

By the method least squares find the straight line that best fits the following data

X	1	2	3	4	5
y	14	27	40	55	68

Solution

Let the straight line of best fit be y = ax + b

The normal equation are

$\Sigma xy = a \ Ex^2 + b\Sigma x$

$\Sigma y = a \ \Sigma x + 5b$

X	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\Sigma x = 15$	$\Sigma y = 204$	$x^2 = 55$	$\Sigma xy = 748$

748 = 55a + 156

204 = 15a + 5b

Solving these we have a = 13.6 and b = 0

The line of best fit in y= 13.6 x

Note: Fitting a parabola $y = ax^2 + bx + c$ the normal equations are

$\Sigma x^2 y = a \Sigma x^4 + b \Sigma x^3 + c \Sigma x^2$

$\Sigma xy = a \Sigma x^3 + b \Sigma x^2 + c \Sigma x$

$\Sigma y = \Sigma x^2 + b\Sigma x + nc$

Problem – 6:

Fit a parabola $y = ax^2 + bx + c$ by least square method to the following data.

X	10	12	15	23	20
y	14	n	23	25	21

Solution:

Let X = x - 16 and Y - y - 20 and the parabola fit to be

$y = Ax^2 + Bx + c$ whose Normal equation are

$\Sigma y = A\Sigma x^4 + B\Sigma x + 5c$

$\Sigma xy = A\Sigma x^3 + B\Sigma x^2 + c\Sigma x$

$\Sigma x^2 y - A\Sigma x^4 + B\Sigma x^3 + c\Sigma x^2$

x	Y	X	Y	x^2	x^3	x^4	xy	x^2y
10	14	-6	-6	36	-216	1296	36	-216
12	17	-4	-3	16	-64	256	12	-48
15	23	-1	3	1	1	1	-3	3
23	25	7	5	49	49	2401	35	245
20	21	4	1	16	16	256	4	16
		$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma X^5 = 128$	$\Sigma X^5 = 128$	$\Sigma X = 4120$	$\Sigma XY = 84$	$\Sigma X^2 Y = 0$

0 = 118 A + 5c

84 = 12 A + 118

0 = 4210 A+ 128 B + 118 c

on solving A = - 0.07

B = 0.79

C = 1.67

$y = 0.07\ x^2 + 0.79 \times 1.67$

(i.e.) $y = 20 = -\ 0.07\ (x-\ 16)^2 + 0.79\ (x - 16) + 1.67$

$y = -.07x^2 + 3.03\ x - 8.89$ is the curve of best fit.

Problem – 7:

Find the curve of best fit of the type $y = ae^{bx}$ to the following data by the method of least squares

x	26	36	88	110	140 |
y	2.69	3.29	9.30	14.44	26.31

Solution

Let $y = ae^{bx}$

$\mathrm{Log}_e^a = \log_e^y \ \ + bx$

Y = A + bx

Where $Y = \log_e^y$, $A + \log_e^y$

X	y	Y = log y	xY	X^2
26	2.69	0.9895	25.729	676
36	3.29	1.1909	42.872	1296
88	9.30	2.2300	196.240	7744
110	14.44	2.6700	293.700	12100
140	26.31	3.2699	457.786	19600
$\Sigma x = 400$		$\Sigma Y = 10.3503$	$\Sigma xY = 1016.327$	$\Sigma x^2 = 41416$

Normal equations are

$\Sigma y = 5A + b\Sigma x$

$\Sigma xy - A\ \Sigma x + 6\ \Sigma x^2$

10.35 = 5A + b 400

1016.326 = 400 A + 1416 b

Solving b = 0.01887

A = .5604 a = Anti log A = 1.7514

The curve of best Fit is

$y = (1.7514)\ e^{01887x}$

Exercise

1) Find the equation of the line of best fit to the following data by the principle of least squares

x	0	5	10	15	20	25
y	12	15	17	22	24	30

2) Find a line of best fit to the following data by the method of least squares

x	5	10	15	20	25
y	16	19	23	26	30

3) Fit as second degree curve $y = a + bx + cx^2$

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

4) Fit a second degree parabola to the following data.

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

5) Fit a curve $y = ab^x$ to the following data

x	2	3	4	5	6
y	144	172.8	207.4	248.8	268.5

6) Fit a curve of the form $y = ab^x$ for the data

x	1	2	3	4	5
y	176	170	180.7	240.5	300

7) Fit a exponential curve for the following data.

x	0.0	0.5	1.0	1.5	2.0	2.5
y	0.10	0.45	2.15	9.25	40.35	180.75

8) Fit a curve of the form $y = ax^b$ to the data given below

x	1	2	3	4	5
Y	7.1	27.8	62.1	110	161

Method of Moments

Let (x_1, y_1), 1 = 1.2 ….. n be n sets of observations of related data, so that the x's are equally spaced.

This method is based on the assumption that, the moments of the observed values of y are, respectively, equal to the moments of the expected values of y.

(i.e.) $h\,\Sigma y = \int_{x_1-h/2}^{x_n+h/2} y\,dx$

$h\,\Sigma y = \int_{x_1-h/2}^{x_n+h/2} xy\,dx$

$h\,\Sigma x^2 y = \int_{x_1-h/2}^{x_n+h/2} x^2 y\,dx$ and so on.

Problem – 8:

Using method of moments fit a straight line to the data.

X	1	3	5	7	9
y	1.5	2.8	4.0	4.7	6.0

Solution

x	y	xy
1	1.5	15.8
3	2.3	8.4
5	4.0	20.0
7	4.7	32.7
9	6.0	545
	Σy = 19.0	Σ xy 116.8

Let y = ax + b

$h = 2 \,.\, h\,\Sigma\, y = \int_{x_1-1}^{x_n+1} y\,dx$

$(2)\ (19) = \int_0^{10} (ax+b)\,dx$

$38 = \left(\frac{ax^2}{2} + bx\right)_0^{10}$

$38 = 50a + 10b \qquad \ldots.(1)$

$h\,\Sigma\, xy = \int_{x_1-1}^{x_n+1} xy\,dx$

$2(116.8) = \int_0^{10} x(ax+b)\,d\lambda$

$233.6 = \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^{10}$

$233.6 = \frac{1000a}{3} + 50b \qquad \ldots.(2)$

Solving (1) & (2)

$y = 0.5232\,x + 1.1838$

Exercise

1) Fit a straight line by the method of moments

X	1	2	3	4	5	6
y	4	8	10	12	16	20

2) Fit a straight line y = ax + b, Using the method of moments for the date

x	1	2	3	4	5
y	20	27	32	36	47

3) By the method of moments fit a second degree parabola to the following data

x	1	2	3	4
y	0.30	0.64	1.32	6.40

4) Fit a parabola of the form $y = ax^2 + bx + c$ to the data by the method of moments.

x	1	2	3	4
Y	1.7	1.8	2.3	3.2

S. Tamilselven
Lecturer in Mathematics
Faculty of Engineering and Technology
Annamalai University

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NUMERICAL SOLUTION TO PARTIAL DIFFERENTIAL EQUATIONS

5.1 INTRODUCTION

Numerical methods available for the solution to partial differential equations, the method of finite differences in commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed into a system of linear equation which are solved by iterative procedures.

5.2 CLASSIFICATIONS OF LINEAR SECOND ORDER EQUATIONS

The most general linear partial differential equation of second order can be written as

$$A \frac{\partial^2 u}{\partial^2 x} + B \frac{\partial^2 u}{\partial x\ \partial y} + C \frac{\partial u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + FU = G \qquad (5.1)$$

Where A. B, C, D, K, F, and G are functions of x and y.

The equation (5.1) is said to be

(a) Elliptic if $B^2 = 4\ AC < 0$

(b) Parabolic if $B^2 = 4\ AC < 0$

(c) Hyperbolic if $B^2 = 4\ AC > 0$

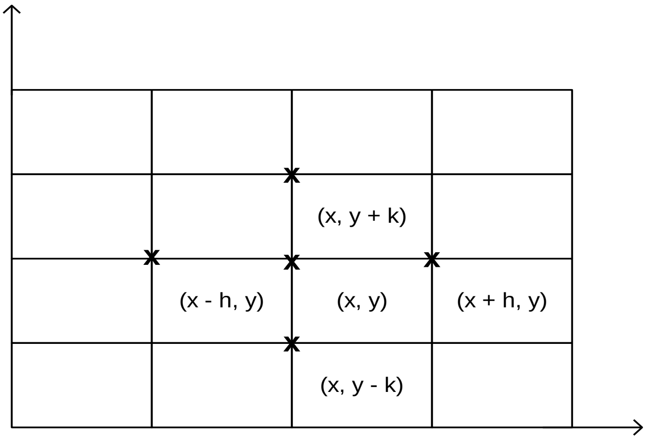
Example

Laplace equation in two dimension $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is an elliptic type. One dimensional heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ is a parabolic type one-dimensional, wave equation $x^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2}$ is a Hperbolic type

5.3 FINITE DIFFERENCE APPROXIMATIONS

The xy plane is divided into a series of rectangles whose sides are parallel to x and – y axes such that $\Delta x = h$, $\Delta y = k$. The grid points or mesh points or Lattice points are (x, y) , (x + h , y), (x + 2h, y), (x – h, y), (x – 2h, y) . (xi, yi) is any grid point.

$x_i = x_o + ih$, $y_j = y_o + jk$



Here x = ih, y = j k is denoted by (I, j)

Hence U (x, y) = u (I j, j k) = $U_{i.j}$

$$U_x = \frac{U_{1+1.j} - U_{i,j}}{h} \text{ [Forward difference]} \qquad (5.2)$$

$$U_x = \frac{U_{1.j} - U_{i,j}}{h} \text{ [Backward difference]} \qquad (5.3)$$

$$U_y = \frac{U_{1.j+1} - U_{i,j}}{h} \text{ [Backward difference]} \qquad (5.4)$$

$$U_y = \frac{U_{1.j} - U_{i,j-1}}{h} \text{ [Backward difference]} \qquad (5.5)$$

$$U_{xx} = \frac{U_{i-1.j} - 2U_{i,j} + U_{i+1.j}}{h} \qquad (5.6)$$

$$U_{yy} = \frac{U_{i.j-1} - 2U_{i,j} + U_{i.j+1}}{k^2} \qquad (5.7)$$

5.4 ELLIPTIC EQUATION

An important equation of the elliptic type is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

(i.e.) $U_{xx} + U_{yy} = 0$ This equation is called Laplace's equation. Replacing the derivatives by the corresponding difference expressions.

We get

$$\frac{U_{i-1.j} - 2U_{i.j} + U_{i+1.j}}{h^2} + \frac{U_{i.j-1} - 2U_{i.j} + U_{i.j+1}}{k^2} = 0$$

Taking a square mesh and putting h = k, we get.

$$U_{i.j} = \frac{1}{4}\left(U_{i+1.j} + U_{i-1.j} + U_{i.1+j} + U_{i.1-j}\right) \qquad (5.8)$$

(i.e.) The value of U at any interior mesh point is the arithmetic mean of its values at the four neighbouring mesh points. This is called the standard five point formula.

Instead of equation (5.8) We may use the formula

$$U_{i.j} = \frac{1}{4}\left(U_{i-1-j} + U_{i+1.j} - U_{i-1.j} + U_{i+1.j+1}\right)$$

The value of U y is the arithematic mean of its values at the four neighbouring diagonal mesh points. This is called the diagonal five point formula.

Note: The error in diagonal five point formula is four times the error in standard five point formula. Hence use the standard five point formula where ever possible.

5.5 SOLUTION TO LAPLACE'S EQUATION BY LIEBMANN'S ITERATIVE PROCESS

To solve the Laplace's equation in a bounded region R with boundary C and with the value specified every where on C. Let R be a square region. So that it can be divided into a network of small squares of side h. Let the values of u (x, y) on the boundary C be given by bi and let the interior mesh points and the boundary points be as follows:

	b_{13}	b_{12}	b_{11}	b_{10}	b_9
b_{14}	u_7	u_8	u_9		b_8
b_{15}	u_4	u_5	u_6		b_7
b_{16}	u_1	u_2	u_3		b_6
					b_5
	B_1	b_2	b_3	b_4	

Computational Procedure

1) Use the following five point formula to get the initial value of u at the centre.
$$U_5 = \frac{1}{4}\left[b_{15} + b_7 + b_3 + b_{11}\right]$$
2) Calculate the approximate values of u_1 . u_3 , u_7 , u_9 by the diagonal five point formula
$$U_1 = \frac{1}{4}\left[b_1 + b_3 + b_5 + b_{15}\right]$$
$$U_3 = \frac{1}{4}\left[b_3 + b_5 + b_5 + b_5\right]$$

$$U_7 = \frac{1}{4}\left[b_{15} + b_5 + b_{11} + b_{13}\right]$$

$$U_9 = \frac{1}{4}\left[b_5 + b_7 + b_9 + b_{11}\right]$$

3) Obtain the values of the remaining interior points by the standard five point formula.

$$U_2 = \frac{1}{4}\left[b_3 + b_3 + b_5 + b_1\right]$$

$$U_4 = \frac{1}{4}\left[u_1 + u_5 + u_7 + b_{15}\right]$$

$$U_6 = \frac{1}{4}\left[u_3 + u_7 + u_9 + b_5\right]$$

$$U_8 = \frac{1}{4}\left[u_5 + u_9 + u_{11} + b_7\right]$$

Having obtained all values $u_1, u_2 \ . \ . \ . \ u_9$. Once, their accuracy can be improved by repeated application of Gauss - Seidel iterative formula

Problem: 1

Solve $U_{xx} + U_{yy} = 0$ in $0 \le x \le 4$, $0 \le y \le 4$

given that u (o, y) = 0

u (4, y) = 8 + 2y

u (x , 0) = $\frac{x^2}{2}$

and U (x,4) = x^2

Take h = k = 1 and obtain the result correct to two decimals

Solution

Using Standard Five point formula

$U_{i.j}$ = 1/4 $\left[U_{i-1.j} + U_{i.j+1}\, U_{i+1.j} + U_{i.j-1}\right]$

Diagonal five point formula

$U_{i.j}$ = 1/4 $\left[U_{i-1.j} + U_{i-1.j+1}\, U_{i+1.j+1} + U_{i+1.j-1}\right]$

Using u (0, y) = 0 ==> x = 0 , u = 0

Using u (4 , y) = 8 + 2y

U (4, 0) = 8

U (4, 1) = 1

U(4, 2) =12

U (4, 3) = 14

U (4, 4) = 16

Using U (x , 4) = x^2

U (0,4) = 0

U (1,4) = 1

U (2, 4) = 4

U (3, 4) = 9

U (4,4) = 16

Using U (x ,0) = $\frac{x^2}{2}$

U(0,0) = 0

U(l,2) = 1/2

U (2,2) = 2

U (3,2) = 9/2

U(4.4) = 8

	↑ y – axis	u = 1	u = 4	u = 9	u = 16
u = 0 y = 4	u_1	1.9912 1.9844 2.0 2.125 u_2	4.4187 4.9121 4.9063 4.4375 u_3	8.9956 8.9961 8.9922 9.125	u = 14
u = 0 y = 3	u_4	2.0612 2.0528 2.0313 2.045 u_5	4.6875 4.6875 4.6719 45 u_6	8.0627 8.0637 8.0723 8.0625	u = 12
u = 0 x = 2	u_7	1.5669 1.5664 1.5547 1.625 u_8	3.9053 9.7063 9.1729 9.6875 u_9	6.5671 6.5675 6.5713 6.625	u = 10
u = 0 y = 1					u = 8
u = 0 x = 0 y = 0		x = 1 u = 0.5	x = 2 u = 2	x = 3 u = 4.5	→ x - axis x = 4 u = 8

Step 1

$u_5 = 1/4\ (4 + 2 + 0 + 12) = 4.5$ Using standard five point formula

$u_1 = 1/4\ (0 + 4.5 + 0 + 5) = 2.125$

$u_3 = 1/4\ (4.5 + 16 + 4 + 12) = 9.125$ Using diagonal five point formula

$u_7 = 1/4\ (0 + 4.5 + 0 + 2) = 1.625$

$u_9 = 1/4\ (4.6 + 8 + 2 + 12) = 6.625$

$u_2 = 1/4\ (2.124 + 9.125 + 4.5 + 4) = 4.9375$

$u_4 = 1/4\ (2.125 + 4.5 + 1.625 + 0) = 2.0625$

$u_6 = 1/4\ (4.5 + 12 + 9 + 125 + 6.625) = 8.0625$

$u_8 = 1/4\ (4.5 + 2 + 1.625 + 6.625) = 3.6875$

After 4 – Steps we get

$u_1 = 1.991$, $u_2 = 4.919$, $u_3 = 8.996$, $u_4 = 2.061$, $u_5 = 4.638$

$u_6 = 1.567$, $u_7 = 3.706$, $u_8 = 6.567$

Exercise

1. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Over a square region with the boundary conditions are u (o.y) - 0 , u (4,y) = 12 + y , u (x , 0) = 3x and u (x,4) = x^2,o<x<4,0<y<4 obtain the result correct to two decimal and taking mesh, length as one.

2. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown.

0	500	1000	500		0
1000	u_1	u_2	u_3		1000
2000	u_4	u_5	u_6		2000
1000	u_7	u_8	u_9		1000
0	500	1000	500		0

3. Solve $\nabla^2 u = 0$, given that

$u\ (o\ ,\ y) = 0$

$u\ (4\ .\ y) = 16y\ ,\ 0 \leq y \leq 4$

u (x ,0) = 0

$u(x,4) = x^3, \quad 0 \leq x \leq 4,\ h = 1$

4. Solve $\nabla^2 u = 0$, inside a region $0 \leq x \leq 1 \ . \ 0 \leq y \leq 1$ with $u(x, y) = 1/4\, xy (x + 1)(y + 1)$ on the boundary. Take the step length to be 1/3 in both the x and y directions.

5. With h =1, solve $\frac{\partial^2 u}{\partial xy^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ . \ 0 \leq x \leq 4 \ . \ 0 \leq y \leq 4$

   $U(x, 0) = x^2 + 2x \qquad U(x, 4) = x^2 + 2x - 24$

   $U(0, y) = -2y - y^2 \qquad U(4, y) = 24 - y^2 - 2y$

5.6 POISSON EQUATION

Problem – 1:

The partial differential equation $\frac{\partial^2 u}{\partial xy^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ is called the poission's equation. It is of elliptic type.

To solve the poisson equation numerically using

$U_{i-1.j} + U_{i+1.j} + U_{i.1-j} + 4U_{i.j} = h^2 f(x.y)$

Problem – 2:

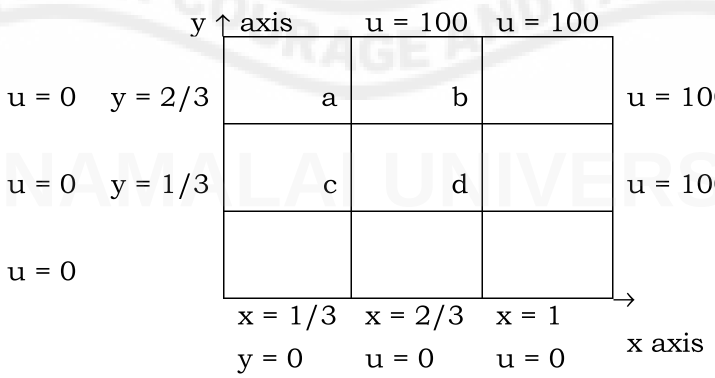
Solve the poission's equation $\nabla^2 U = 81\, xy$

$0 \leq x \leq 1,\ 0 \leq y \leq 1$ with $u(0, y) = u(x, 0) = 0$

$u(1, y) = u(x, 1) = 100$

and h= 1/3

Solution



We know that

$$U_{i-1.j} + U_{i+1.j} + U_{i.j-j} + U_{i.j+1} - 4U_{i.j} = h^2 f(ih.jk)$$

at a

$o + c + b + 100 - 4a = -81\ (1/3)\ (2/3)\ (1/9)$

$-4a + b + c = -102 \qquad \ldots (1)$

at b

$a + 100 + d + 100\ (1/9)\ (-81)\ (2/3)\ (2/3)$

$a - 4b + d = 204 \qquad \ldots (2)$

at c

$0 + a + 0 + d - 4c = 81\ (1/3)\ (1/3)\ (1/9)$

at d

$0 + b + c + 100 - 4d = 81\ (2/3)\ (1/3)\ (1/9)$

$b + c - 4d = -102 \qquad \ldots (3)$

$0 + b + c + 100 - 4d = -81\ (2/3)\ (1/3)\ (1/9)$

$b + c - 4d = -102 \qquad \ldots (4)$

From (l) and (2) a = d

1. $\Rightarrow -4a + b + c = -102$
2. $\Rightarrow 2a - 4b = -204$
3. $\Rightarrow 2a - 4c = -1$

$$\begin{array}{rcl} 16a + 4b + 4c & = & -408 \\ \underline{2a - 4b} & \underline{=} & \underline{-204} \\ -14a + 4c & = & -612 \\ \underline{2a - 4c} & \underline{=} & \underline{-1} \\ -12a & = & -613 \end{array}$$

$2\,(51.083) - 4c = -1$

$4c = 2\,(51.083) + 1 \qquad a = 51.0833$

$c = 25.7915$

$2a - 4b = -20a$

$4b = 2a + 204$

$b = 2\,(51.0833) + 204/4 \qquad b = 76.54165$

$a = 51.0833 = d$

$b = 76.5417$

$c = 25.7415$

Exercises

1) Solve $\nabla^2 u = 10\,(x^2 + y^2 + 10)$ .over the square mesh with sides x = 0, y = 0, x =3, y = 3 with u = 0 on the boundary and mesh length one unit.

2) With Step - size, h = 1/2, Slove $\nabla^2 u = -1$, $|x| \le 1$. $|y| \le 1$
$u(\pm 1, y)\ u(x, \pm 1) = 0$

3) Solve the poission equation $u_{xx} + u_{yy} = -40$, over the square bounded by the lines x = 0, x = 4, y = 0 and y = 4 with u = 0 on the boundary, taking the mesh length h = 1, Use Liebmann's iteration method.

5.7 PARABOLIC EQUATION

Bender - Schmidt method

Solution to one dimensional heat equation can be obtained Using Bender – Schmidt method.

Consider $\frac{\partial^2 u}{a h^2} = a \frac{\partial u}{\partial t}$ be the one dimensional conditions

u (0, t) = To ,u (1, t) = T_1 and this initial conditions u (x, 0) = f (x) by finite differences method.

$$\left[\frac{y_{i+1.j} - 2u_{i.j} + u_{i-1.j}}{h^2}\right] = a \left[\frac{u_{i.j+1} - u_{i.j}}{h}\right]$$

$$u_{i.j+1} - u_{i.j} = \lambda \left[u_{i+1.j} - 2_{ui.j} + u_{i-1.j}\right]$$

where $\lambda = \frac{k}{ah^2}$

$$u_{i.j+1} \; \lambda + u_{i+1.j} + (1-2\lambda)u_{i.j} + \lambda u_{i-i.j}$$

If h, k are chosen such that the coefficient of y_{ij} varishes (i.e.) 1 - 2λ = 0λ = 1/2 is called Bender - Schmidt recurrence equations.

$$u_{i.j+1} = ½ \left[u_{i+i.j} + u_{i+1.j}\right]$$

Problem 3

Find the solution of the parabolic equation

$$\frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial u}{\partial t} = 0$$

When u (0,t) = 0

u (4,t) = 0

u (x, 0) = x (4 - x)

Assume h =1, Find the values upto t = 5

Solution

$\frac{\partial^2}{\partial x^2} = 2\ \frac{\partial u}{\partial t}$ a = 2, h = 1 Bender- Schmidt's recurrence relation is

$U_{i.j+1} = 1/2\ (U_{i-1.j} + U_{i+}1.j)$ ..... (1)

u (0, t) = 0

$u_{0.j} = 0$ for j = 0 .1. 2 ....

$u_{0.0} = 0$

$u_{0.1} = 0$

$u_{0.2} = 0$

$u_{0.3} = 0$

$u_{0.4} = 0$

u(4,x) = 0 u 4.j = 0 for j = 0, 1, 2, 3, ...

$u_{4.0} = 0$

$u_{4.1} = 0$

$u_{4.2} = 0$

$u_{4.3} = 0$

$u_{4.4} = 0$

u (x, 0) = x (4 – x)

$u_{i.0}$ = ih (4 – ih)

for = 1, 2 ......

= 1(4 – 1)

$U_{1.0} = 3$

$U_{2.0} = 4$

$U_{3.0} = 4$

j \ i		x = 0	x = 1	x = 2	x = 3	x = 4
		0	1	2	3	4
t = 0	0	0	3	4	3	0
t = 1	1	0	2	3	2	0
t = 2	2	0	1.5	2	1.5	0
t = 3	3	0	1	1.5	1	0
t = 4	4	0	.75	1	.75	0
t = 5	5	0	.5	.75	.5	0

Step 1

j = 0 in (1)

$u_{i.j}$ = 1/2 ($u_{1+1.0}$ + $u_{i-1.0}$)

i = 1 $u_{i.j}$ = 1/2 (u_2 . 0 + $u_{0.0}$) = 1/2 (4 + 0) = 2

i = 2 $u_{2.1}$ = 1/2 ($u_{2.0}$ + $u_{1.0}$) = 1/2 (3 + 3) = 3

1 = 3 $u_{3.1}$ = 1/2 ($u_{4.0}$ + $u_{2.0}$) = 1/4 (0 + 4) = 2

Exercise

1) Given $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}$ = 0 u (0, t) = u (5, t) = 0, u (x , 0) = $x^2 \left(25 - x^2\right)$ find the values if u in the range h = 1 and upto 3 sec.
2) Find the solution to $u_t = u_{xx}$ subject to u (x, 0) = Sin π x , 0 ≤ x ≤ 1
3) Given $u_t = 25u_{xx}$, u (0, t) = u (10, t) = 0 u (x, 0) = 1/25 x (10 - x) Choosing h = 1 find $u_{i.j}$ for 0 ≤ i ≤ 9, 1 ≤ j ≤ 4
4) Solve $u_t = 5u_x$ with u (0, t) = 0, 'u (5, t) = 60

   u (x , 0) = $\begin{cases} 20x \; for\, 0 < x \le 3 \\ 60 \; for\, 3 < x \le 5 \end{cases}$ for 5 time steps having h = 1 by Schmidt method.

Crank – Nicholson Method

To solve parabolic equation $u_{xx} = u_t$ subject to the conditions u (0,t) = T_0 , u (1, t) = T_1 and u (x, 0) = f(x)

We know that at point $u_{i\,.\,j}$ the finite difference approximation for U_{xx} is

$u_{xx} = \frac{1}{h^2} \left[u_{i+1.j+1} - 2u_{i.j-1.j}\right]$

At point $u_{i.j+1}$ the finite difference approximation for Uxx is

$u_{xx} = \frac{1}{h^2} \left[u_{i+1.j+1} - 2u_{i.j+1} + u_{i-1.j+1}\right]$

The averages of there equations in

and $u_t = \frac{1}{k} \left[u_{i.j+1} - u_{i.j}\right]$

Then substituting these equation

$$U_{xx} = \frac{1}{2h^2} \left[u_{i+1.j+1} - 2u_{i.j+1} + u_{i-1.j+1} - u_{i+1.j} - 2u_{i.j} + u_{i-1.j} \right]$$

$$\frac{\lambda}{2} \left[u_{i+1.j+1} - u_{i.j+1} + u_{i-1.j} - u_{i.j_{=1}} + u_{i.j}\right]$$

$$= u_{i.j+1} - u_{i.j} \text{ where } \lambda = \frac{a\,k}{h^2}$$

(i.e.) $\frac{\lambda}{2} u_{i+1.j+1} - (\lambda+1) u_{i.j+1} + \frac{\lambda}{2} u_{i-1.j+1}$

$$= -\frac{1}{2} u_{i+1.j} + (\lambda-1) u_{i.j} - \frac{\lambda}{2} u_{i-1.j}$$

(i.e.) $\lambda u_{i+1.j+1} - (\lambda+1) u_{i.j+1} + \lambda u_{i-1.j+1}$

$$= -\lambda u_{i+1.j} + 2(\lambda-1) u_{i.j} - \lambda u_{i-1.j}$$

This is called Crank – Nicholson scheme

Choosing λ such that $\lambda = 1 = 0$

$\lambda = 1$

Then the Crank – Nicholson scheme becomes

$u_{i+1.j+1} - 4u_{i.j+1} + u_{i-1.j+1} = -u_{i-1.j} - u_{i+1.j}$ and $k = ah^2$

Problem 4

Solve by crank Nicholson > Scheme

$\frac{\partial u}{\partial t} = 1/16 \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1 . t > 0$

u (x. 0) = 0 , u (0, t) = 0 , u (1 , t) = 100 t

Compute u for one time step with h = 1/4.

$$\frac{\partial u}{\partial t} = 1/16 \ \frac{\partial^2 u}{\partial x^2}$$

(i.e.) $\frac{\partial^2 u}{\partial x^2} = 16 \ \frac{\partial u}{\partial t}$. a = 16 , h = ¼ take $\lambda = 1$

$k = a \lambda h^2 \qquad = 16\ (1)\ (1/16) = 1$

$x = ih = i/4 \qquad k = 1$

$t = jk = j$

$u\ (x, 0) = 0 \qquad u_{i.0} = 0 \qquad i = 0\ .1, 2, 3, 4$

$u_{1.0} = 0$

$u_{2.0} = 0$

$u_{3.0} = 0$

$u_{4.0} = 0$

$u\ (0, t) = 0 \qquad u_{0.j} = 0 \qquad j = 0\ .1$

$u_{0.0} = 0$

$u_{0.1} = 0$

$u\ (1, t) = 1001 \qquad u_{1.j} = 100j \qquad j = 1, 2\ \ldots\ldots\ldots$

	j \ i	x = 0	x = 1/4	x = 1/2	x = 3/4	x = 1
		0	1	2	3	4
t = 0	0	0	0	0	0	0
t = 1	1	0	a	b	c	100

We know that

$u_{i-1.j+1} - 4u_{i.j+1} + u_{i+j.j+1} = -u_{i+1.j} - \text{ui+1.j}$

Put j = 0

$u_{i\text{-}1.j} - 4u_{i.j} + u_{i+1.j} = -u_{i\text{-}1.0} - u_{i+j.0}$

i = 1

$u_{0.1} - 4u_{1.1} + u_{2.1} = -u_{0.0} - u_{2.0}$

0 - 4 (a) + b = – 0 -0

-4a + b = 0

4a - b =0 ....(1)

i = 2

$u_{1.1} - 4u_{1.1} + u_{2.1} = -u_{0.0} - u_{2.0}$

a – 4b + c = – 0 - 0

a – 4b + c = 0 ....(2)

i = 3

$u_{2.1} - 4u_{3.1} + u_{4.1} = -u_{2.0} - u_{4.0}$

b – 4c + 100 = 0 ....(3)

Solving (1), (2) (3)

a = 1, 79

b = 7.14

c = 26. 79

Exercise

1) Solve by crank Nicholson's method

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ 0 < x < 1, t > 0 u (x, 0) = 100 $\left(x - x^2\right)$

u (0, t) = u (1, t) = 0 compute u for one time step with h = 0.25

2) Compute u for one time step by Crank - Nicholson method if $u_t = u_{xx}$

0< x <5, t > 0 u (x, 0) = 20, u (0, t) = 0 and u (5, t) = 100

3) Obtain the Numerical solution to solve $u_t = u_{xx}$ $0 \le x \le 1$, t . 0 under th conditions that u (0, t) = u (t, t) = 0 and

u (x , 0) = $\begin{cases} 2x \text{ for } 0 \le x \le 1/2 \\ 2(1-x) \text{ for } 1/2 < x \le 1 \end{cases}$

5.8 HYPERBOLIC EQUATIONS

The one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is hyperbolic equation, solving, by method of Finite differences, subject to the conditions

u (0 , t) s 0

u (1 , t) = 0

$u_{i.j+1} - 2u_{i.j-1} = a^2 \lambda^2 \lfloor u_{i+1.j} - 2u_{i.j} + u_{i-j.j} \rfloor$

and u (x , 0) = f (x)

u_t (x, 0) = 0

Applying finite difference approximation to the wave Equation.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

(i.e.) $\dfrac{u_{i.j+1} - 2u_{i.j} + u_{i.j-1}}{k^2} = a^2 \dfrac{\lfloor u_{i+1.j} - 2u_{i.j} + u_{i-1} \rfloor}{h^2}$

Where $\lambda = \dfrac{k}{h}$

This process is converage when k < h Choose the value of h and k such that the coefficient of $u_{1.j}$ = 0.

(i.e) $1 - a^2\lambda^2 = 0$

$\lambda = 1/A$

(i.e.) $\dfrac{k}{h} = \dfrac{1}{a}$

$\therefore\ u_{i.j+1} = u_{i+1.j} + u_{1-i.j} - u_{i.j-1}$ and k = h/a

Problem 5

Evaluate the pivotal values of the following equation taking h = 1 and upto

t = 1 Sec of 16 $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial x^2}$ given that

u (0, t) = 0

u (5, t) = 0

u (x, 0) = x^2 (5 - x)

u_t (x, 0) = 0

Solution

$a^2 = 16$

a = 4 h = 1 k = $\dfrac{h}{a}$ = ¼

k < h the solution is convergent

$u(0, t) = 0$ $\qquad u_{0,j} = 0$

$u(5, t) = 0$ $\qquad u_{5.j} = 0$

$u_t (x , 0) = 0$

$u_{i.j+1} - u_{i.j} = 0$ when $t = 0$

$u_{i.j+1} - u_{i.j} = 0$ when $j = 0$

$u_{i.1} – u_{1.0} – 0$

$u_{i.1} = u_{1.0}$

$u(x, 0) = x^2 (5 – x)$

$u_{1.0} = 1_2 (5 - 1)$

$u_{1.0} = 4$

$u_{2.0} = 12$

$u_{3.0} = 15$

$u_{4.0} = 16$

Then

$u_{i.j+1} = u_{i+1.j} + u_{i-1.j} - u_{i.j-1}$

	j \ i	x = 0 0	x = 1 1	x = 2 2	x = 3 3	x = 4 4	x = 5 5
t = 0	0	0	4	12	18	16	0
t = 1/4	1	0	4	12	18	16	0
t = 2/4	2	0	8	10	10	2	0
t = 3/4	3	0	6	-3	-6	-6	0
t = 1	4	0	-2	-10	-10	-8	0

But $j = 1 \quad u_{1.2} = u_{i+1.1} + u_{i-1.1} – u_{i.0}$

$i = 1 \quad u_{1.2} = u_{2.1} + u_{0.1} – u_{1.0}$

$= 0 + 12 - 4 = 8$

$i = 2u_{2.2} = u_{3.1} + u_{i.1} = u_{2.0}$

$= 4 + 18 - 12 = 10$

$i = 3u_{3.2} = u_{4.1}\, u_{2.1} - u_{3.0}$

$= 12 + 16 - 18 = 0$

$i = 4u_{4.2} = u_{5.1} + u_{3.1} - u_{4.0}$

$= 18 + 0 - 16 = 2$

Exercise

1) Solve 4 $= \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ given the u (0, t) = 0, u (4, t) = 0 u_t (x, 0) = 0 and u (x, 0) = x (4 - x) taking h = 1
2) Solve the hyperbolic equation $u_{tt} = 25\, u_{xx}$, u (0, 1) = U (5, t) = 0

   u_t (x, 0) = 0 .u (0. t) = $\begin{cases} 2x \text{ for } 0 \le x \le 2.5 \\ 10 - 2x \text{ for } 2.5 \le x \le 5 \end{cases}$

   by taking h=l
3) Solve numerically 16 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$ upto one-half of the period of vibration, given that u (0, t) = 0 u (6, t) = 0 u (x, 0) = x^2 (6 - x) , u_t (x . 0) = 0
   (Take h = l)
4) Solve numerically 25 $U_{xx} = U_{tt}$ given U_t (x, 0) = 0, U (0, t) = U (5, t) = 0

   and U(x . 0) = $\begin{cases} 20x \text{ for } 0 \le x \le 1 \\ 5(5 - x) \text{ for } 1 \le x \le 5 \end{cases}$

   Take h = 1

S. Tamilselvan
Lecturer in Mathematics
Faculty of Engineering and Technology
Annamalai University

BSC O214 : C/500

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