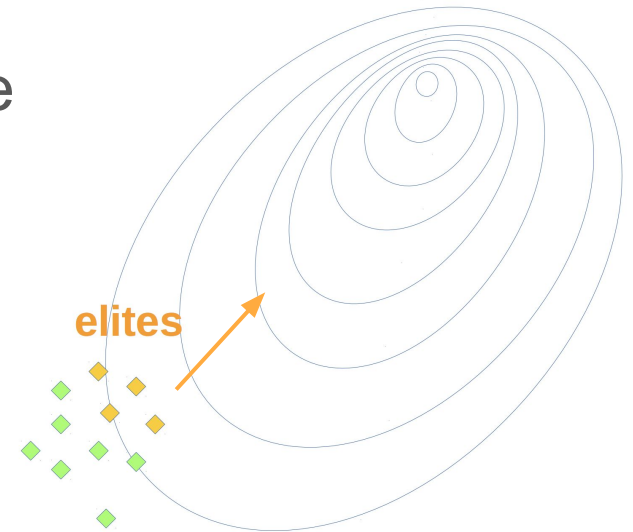


Practical RL – Week 2

Shvechikov Pavel

Previously in the course

- The MDP formalism
 - State, Action, Reward, next State
- Cross-Entropy Method (CEM)
 - easy to implement, good results
 - rich theoretical background
 - black box
 - no knowledge of environment
 - no knowledge of intermediate rewards



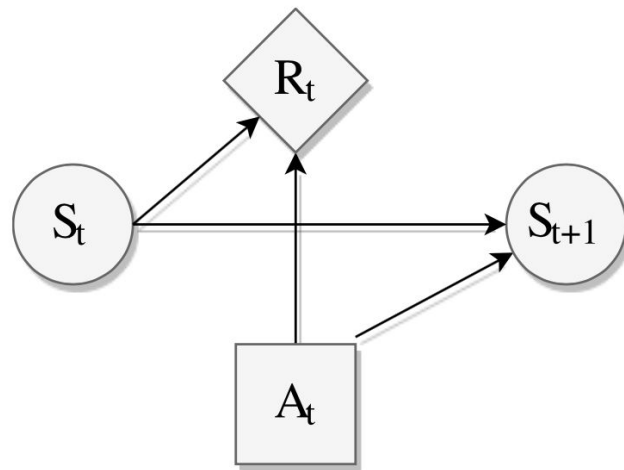
Improve on the CEM → dive into the black box

Given dynamics, how to find an optimal policy?

Definition of Markov Decision Process

MDP is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R} \rangle$, where

- ① \mathcal{S} – set of states of the world
- ② \mathcal{A} – set of actions
- ③ $\mathcal{P} : \mathcal{S} \times \mathcal{A} \mapsto \Delta(\mathcal{S})$ – state-transition function, giving us $p(s_{t+1} | s_t, a_t)$
- ④ $\mathcal{R} : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$ – reward function, giving us $\mathbb{E}_R [R(s_t, a_t) | s_t, a_t]$.



Markov property

$$p(r_t, s_{t+1} | s_0, a_0, r_0, \dots, s_t, a_t) = p(r_t, s_{t+1} | s_t, a_t)$$

(next state, expected reward) depend on (previous state, action)

Goal: solve an MDP by finding **an** optimal policy

1. What is the objective?
 - a. Reward: discounting and design
 - b. Expected objective: state- and action-value function
2. How to evaluate the objective?
 - a. Bellman **expectation** equations
3. How to improve the objective?
 - a. Bellman **optimality** equations
4. Combine evaluation and improvement:
 - a. Generalized Policy Iteration

Explaining goals to agent through reward

Reward hypothesis (R.Sutton)

Goals and purposes can be thought of as the maximization of the expected value of the cumulative sum of a received scalar signal

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Cumulative reward is called a return:

$$G_t \triangleq R_t + R_{t+1} + R_{t+2} + \dots + R_T$$

E.g.: reward in **chess** – value of taken opponent's piece

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Diagram illustrating the components of the return equation:

- A blue arrow points to G_t (blue box).
- A green arrow points to R_t (green box), labeled "immediate reward".
- A red arrow points to R_T (red box), labeled "end of an episode".

E.g.: reward in **chess** – value of taken opponent's piece

E.g.: data center non-stop cooling system

- **States** – temperature measurements
- **Actions** – different fans speed
- **R = 0** for exceeding temperature thresholds
- **R = +1** for each second system is cool

What could go wrong with such a design?

E.g.: data center non-stop cooling system

- **States** – temperature measurements
- **Actions** – different fans speed
- **R = 0** for exceeding temperature thresholds
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What could go wrong with such a design?

Infinite return for **non optimal** behaviour!

$$G_t = 1 + 1 + 0 + 1 + 1 + 0 + \dots = \sum_{t=1}^{\infty} R_t = \infty$$

E.g.: moving to destination



- **State** – position, velocities of joints
- **Actions** – actuator forces to joints
- **$R = \max(0, d(x, B) - d(x', B))$**

What could go wrong with such a design?

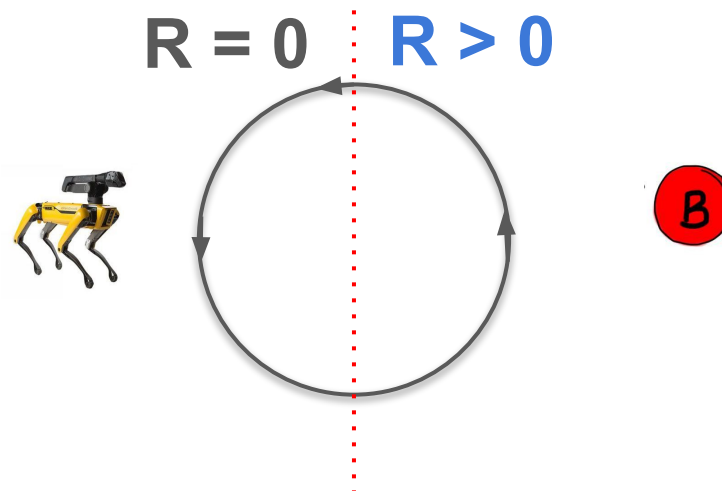
E.g.: moving to destination



- **State** – position, velocities of joints
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- **$R = \max(0, d(x, B) - d(x', B))$**

What could go wrong with such a design?

**Positive
feedback
loop!**



Reward discounting

Reward discounting

Get rid of infinite sum by **discounting** $0 \leq \gamma < 1$

$$G_t \triangleq R_t + \underset{\substack{\text{discount factor} \quad \nearrow}}{\gamma} R_{t+1} + \gamma^2 R_{t+2} + \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

The same cake compared to today's one worth

- γ times less tomorrow
- γ^2 times less the day after tomorrow



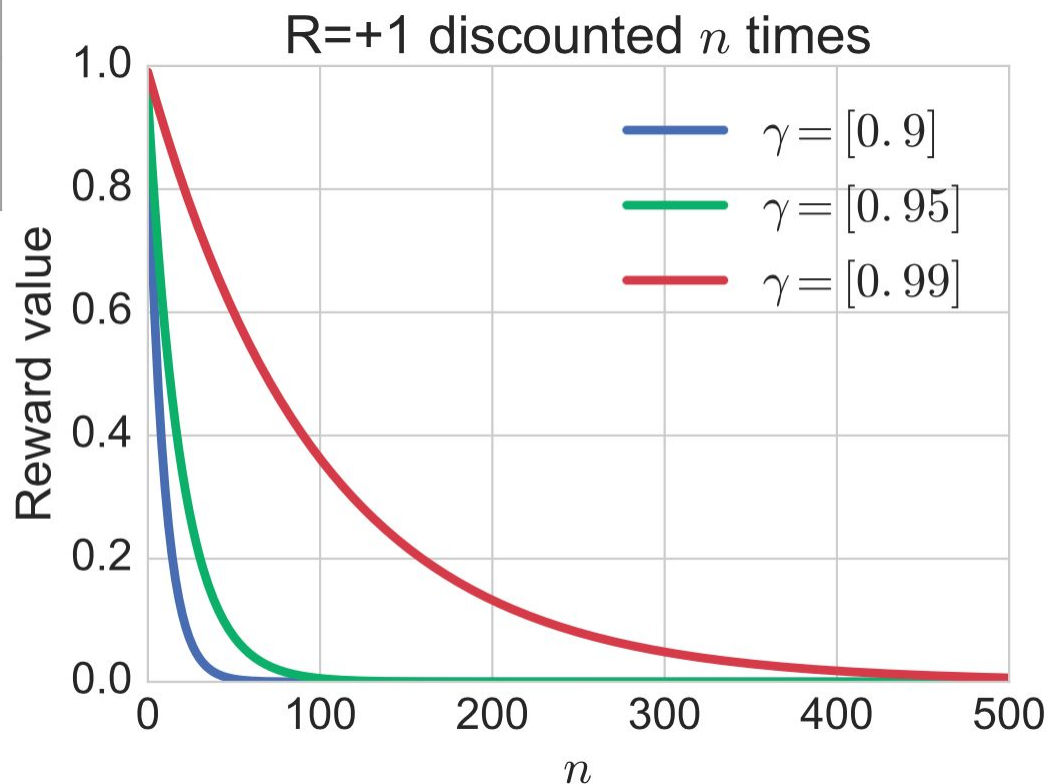
γ will eat it day by day

Discounting makes sums finite

Maximal return for **R = +1**

$$G_0 = \sum_{k=0}^{\infty} \gamma^k = \frac{1}{1 - \gamma}$$

γ	0.9	0.95	0.99
$\frac{1}{1-\gamma}$	10	20	100



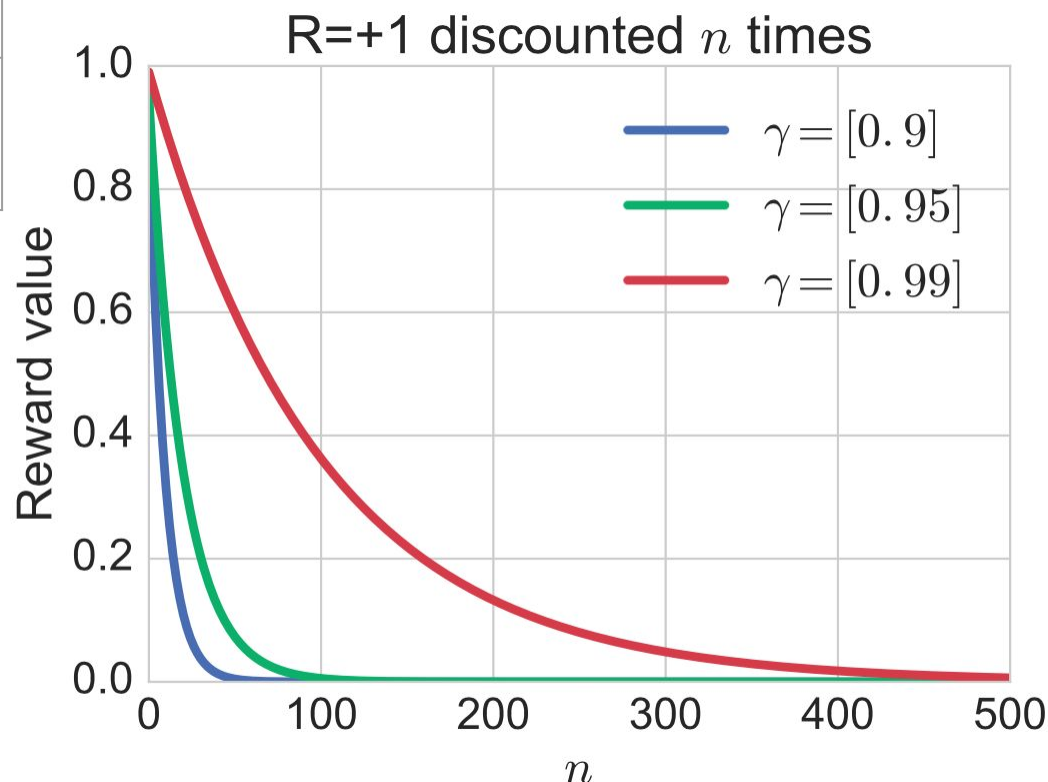
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γ	0.9	0.95	0.99
$\frac{1}{1-\gamma}$	10	20	100

Any **discounting**
changes optimisation
task and its solution!



Discounting is inherent to humans

- Quasi-hyperbolic $f(t) = \beta\gamma^t$
- Hyperbolic discounting $f(t) = \frac{1}{1 + \beta t}$

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Mathematical convenience

$$\begin{aligned} G_t &= R_t + \gamma(R_{t+1} + \gamma R_{t+2} + \dots) \\ &= \boxed{R_t + \gamma G_{t+1}} \end{aligned}$$

Remember this one!
We will need it later

Discounting is a stationary end-of-effect model

Any action affects (1) immediate reward (2) next state

Discounting is a stationary end-of-effect model

Any action affects (1) immediate reward (2) next state

Action indirectly affects future rewards 

But how long does this effect lasts?

$$\begin{aligned} G_0 &= R_0 + \gamma R_1 + \gamma^2 R_2 + \dots + \gamma^T R_T \\ &= (1 - \gamma) R_0 \\ &\quad + (1 - \gamma) \gamma (R_0 + R_1) \\ &\quad + (1 - \gamma) \gamma^2 (R_0 + R_1 + R_2) \\ &\quad \dots \\ &\quad + \gamma^T \cdot \sum_{t=0}^T R_t \end{aligned}$$

G is expected return under stationary end-of-effect model

Discounting is a stationary end-of-effect model

Any action affects (1) immediate reward (2) next state

Action indirectly affects future rewards _____↑

But how long does this effect lasts?

$$\begin{aligned} G_0 &= R_0 + \gamma R_1 + \gamma^2 R_2 + \dots + \gamma^T R_T \\ &= \boxed{(1 - \gamma)} R_0 + \boxed{(1 - \gamma)} \gamma (R_0 + R_1) \\ &\quad + \boxed{(1 - \gamma)} \gamma^2 (R_0 + R_1 + R_2) \\ &\quad \dots \\ &\quad + \gamma^T \cdot \sum_{t=0}^T R_t \end{aligned}$$

“End of effect” probability →

“Effect continuation” probability →

G is expected return under stationary end-of-effect model

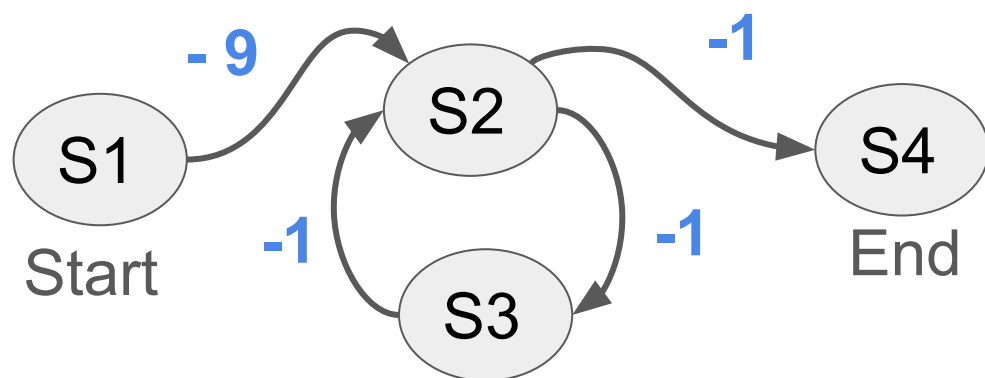
Reward design – don't shift, reward for WHAT

- E.g.: chess – value of taken opponent's piece
 - Problem: agent will not have a desire to win!
- E.g.: moving to destination
 - Problem: agent will not bother about the goal!

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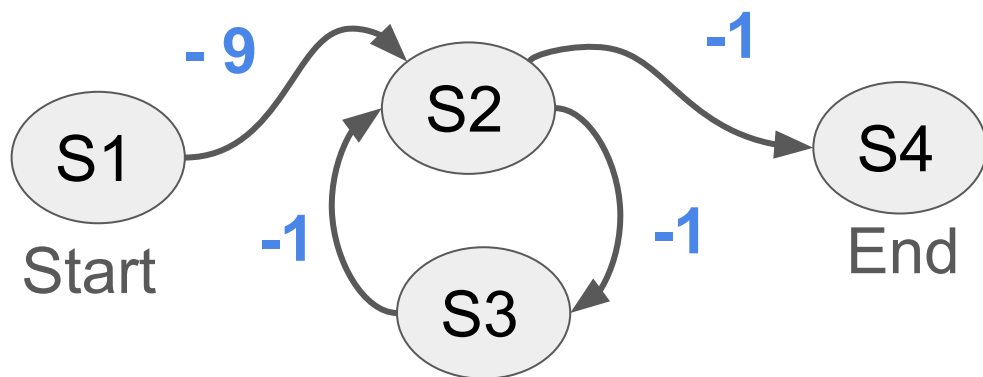
Take away: reward only for **WHAT**, but never for **HOW**



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 - **Problem:** agent will not have a desire to win!
- **E.g.:** moving to destination
 - **Problem:** agent will not bother about the goal!

Take away: reward only for **WHAT**, but never for **HOW**



Take away: do not **subtract** mean from rewards

Faulty reward functions

- Reward for ball possession in soccer
 - Vibrating near the ball
- Cyclic behaviours



Reward design – scaling, shaping

What transformations do not change optimal policy?

- Reward **scaling** – division by positive constant
 - May be useful in practise for approximate methods

Reward design – scaling, shaping

What transformations do not change optimal policy?

- Reward **scaling** – division by positive constant
 - May be useful in practise for approximate methods
- Reward **shaping** – add a **potential-based shaping function** $F(s, a, s')$:

$$R'(s, a, s') = R(s, a, s') + F(s, a, s')$$

$$F(s, a, s') = \gamma \Phi(s') - \Phi(s)$$

Intuition: when no discounting F adds as much as it subtracts from the total return

Expected objective

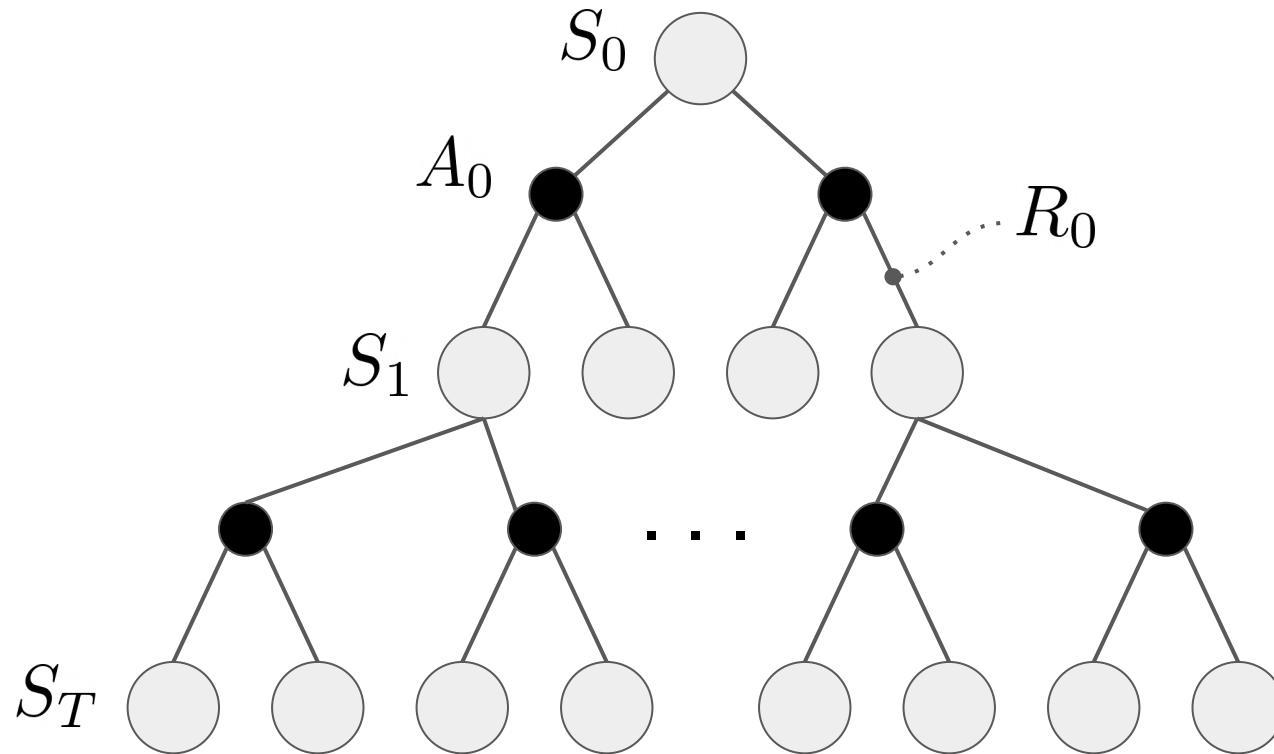
Optimal policy maximizes **expected** return

$$\begin{aligned}\mathbb{E}[G_0] &= \mathbb{E}[R_0 + \gamma R_1 + \dots + \gamma^T R_T] \\&= \mathbb{E}_{E, \pi_\theta}[G_0] \\&= \mathbb{E}_{\pi_\theta}[G_0] \\&= \mathbb{E}[G_0 \mid \pi_\theta] \\&= \mathbb{E}_{\substack{s_{0:T} \\ a_{0:T}}}[G_0] \\&= \mathbb{E}_{s_0} \left[\mathbb{E}_{a_0|s_0} \left[R_0 + \mathbb{E}_{s_1|s_0, a_0} \left[\mathbb{E}_{a_1|s_1} [\gamma R_1 + \dots] \right] \right] \right] \\&= \sum_{t=0}^T \mathbb{E}_{(s_t, a_t) \sim p_\theta} [\gamma^t R_t] \\&= \mathbb{E}_{\tau \sim p_\theta(\tau)} [G(\tau)]\end{aligned}$$

$\tau \triangleq (s_0, a_0, s_1, \dots, a_{T-1}, s_T)$

$p_\theta(\tau) = p(s_0) \prod_{t=0}^{T-1} \pi_\theta(a_t|s_t) p(s_{t+1}|s_t, a_t)$

Backup Tree: how to find an optimal policy?



State- and **Action-value** functions

State-value function $v(s)$

$v(s)$ is expected **return** conditional on state:

$$\begin{aligned} v_{\pi}(s) &\triangleq \mathbb{E}_{\pi} [G_t \mid S_t = s] \\ &= \mathbb{E}_{\pi} [R_t + \gamma G_{t+1} \mid S_t = s] \\ &= \sum_a \pi(a \mid s) \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \mathbb{E}_{\pi} [G_{t+1} \mid S_{t+1} = s'] \right] \\ &= \sum_a \pi(a \mid s) \sum_{r, s'} p(r, s' \mid s, a) [r + \gamma v_{\pi}(s')] \end{aligned}$$

Intuition: value of following policy π from state s

State-value function $v(s)$

$v(s)$ is expected **return** conditional on state:

— stochasticity in policy & environment

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$v(s)$ is expected **return** conditional on state:

stochasticity in policy & environment

$$v_{\pi}(s) \triangleq \mathbb{E}_{\pi} [G_t \mid S_t = s]$$

Environment
stochasticity

$$= \mathbb{E}_{\pi} [R_t + \gamma G_{t+1} \mid S_t = s]$$

$$= \sum_a \pi(a \mid s) \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \mathbb{E}_{\pi} [G_{t+1} \mid S_{t+1} = s'] \right]$$

Policy
stochasticity

$$= \sum_a \pi(a \mid s) \sum_{r, s'} p(r, s' \mid s, a) [r + \gamma v_{\pi}(s')]$$

Intuition: value of following policy π from state s

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Diagram annotations:

- stochasticity in policy & environment** (red line pointing to \mathbb{E}_{π})
- Environment stochasticity** (blue line pointing to $p(r, s' \mid s, a)$)
- Policy stochasticity** (blue line pointing to $\pi(a \mid s)$)
- By definition** (red line pointing to $v_{\pi}(s')$)

Intuition: value of following policy π from state s

Action-value function $q(s, a)$

Is expected **return** conditional on state and action:

Intuition: value of following policy π after committing action **a** in state **s**

$$\begin{aligned} q_{\pi}(s, a) &= \mathbb{E}_{\pi} [G_t \mid S_t = s, A_t = a] \\ &= \mathbb{E}_{\pi} [R_t + \gamma G_{t+1} \mid S_t = s, A_t = a] \\ &= \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \mathbb{E}_{\pi} [G_{t+1} \mid S_{t+1} = s'] \right] \\ &= \sum_{r, s'} p(r, s' \mid s, a) [r + \gamma v_{\pi}(s')] \end{aligned}$$

Action-value function $q(s, a)$

Is expected **return** conditional on state and action:

Intuition: value of following policy π after committing action **a** in state **s**

No policy
stochasticity at
first step

$$\begin{aligned} q_{\pi}(s, a) &= \mathbb{E}_{\pi} [G_t \mid S_t = s, A_t = a] \\ &= \mathbb{E}_{\pi} [R_t + \gamma G_{t+1} \mid S_t = s, A_t = a] \\ &= \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \mathbb{E}_{\pi} [G_{t+1} \mid S_{t+1} = s'] \right] \\ &= \sum_{r, s'} p(r, s' \mid s, a) [r + \gamma v_{\pi}(s')] \end{aligned}$$

Relations between $v(s)$ and $q(s,a)$

We already know how to write $q(s,a)$ in terms of $v(s)$

$$q_{\pi}(s, a) = \sum_{r, s'} p(r, s' | s, a) [r + \gamma v_{\pi}(s')]$$

What about $v(s)$ in terms of $q(s,a)$?

Relations between $v(s)$ and $q(s,a)$

We already know how to write $q(s,a)$ in terms of $v(s)$

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What about $v(s)$ in terms of $q(s,a)$?

$$\begin{aligned} v_{\pi}(s) &= \sum_a \pi(a | s) \sum_{r, s'} p(r, s' | s, a) [r + \gamma v_{\pi}(s')] \\ &= \sum_a \pi(a | s) q_{\pi}(s, a) \end{aligned}$$

So, we could now write $q(s, a)$ in terms of $q(s,a)$!

$$q_{\pi}(s, a) = \sum_{r, s'} p(r, s' | s, a) \left[r + \gamma \sum_{a'} \pi(a' | s') q_{\pi}(s', a') \right]$$

Bellman **expectation** equations

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For $v(s)$:

$$\begin{aligned}v_{\pi}(s) &= \sum_a \pi(a | s) \sum_{r, s'} p(r, s' | s, a) [r + \gamma v_{\pi}(s')] \\&= \mathbb{E}_{\pi} [R_t + \gamma v_{\pi}(S_{t+1}) | S_t = s]\end{aligned}$$

For $q(s, a)$:

$$\begin{aligned}q_{\pi}(s, a) &= \sum_{r, s'} p(r, s' | s, a) [r + \gamma v_{\pi}(s')] \\&= \sum_{r, s'} p(r, s' | s, a) \left[r + \gamma \sum_{a'} \pi(a' | s') q_{\pi}(s', a') \right]\end{aligned}$$

What do we gonna do with value functions?

Already know

- Return, value- and action-value functions
- Bellman equations – assess policy performance

Optimal policy makes

- best actions in each possible state

But how to know which policy **is better**?

How to compare them?

Bellman optimality equations

Optimal policy is the one with the largest $v(s)$

We could compare policies on the basis of $v(s)$

$$\pi \geq \pi' \iff v_\pi(s) \geq v_{\pi'}(s) \quad \forall s$$

Best policy π_* is better or equal to any other policy

Use optimal policy from s



$$v_*(s) = \max_{\pi} v_\pi(s)$$

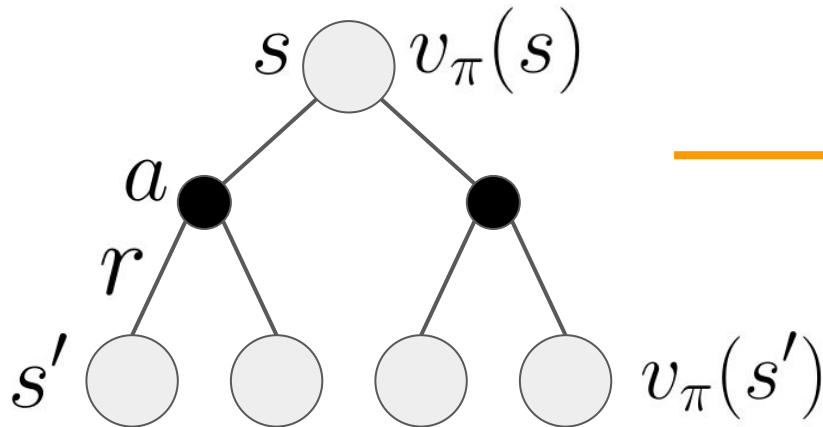
$$q_*(s, a) = \max_{\pi} q_\pi(s, a)$$



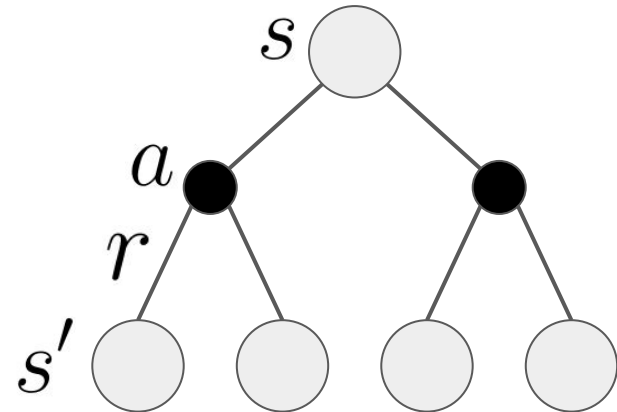
Commit action a , and **afterwards** use optimal policy

In any finite MDP there is
always **at least one**
deterministic optimal policy

Bellman **optimality** equation for $v(s)$

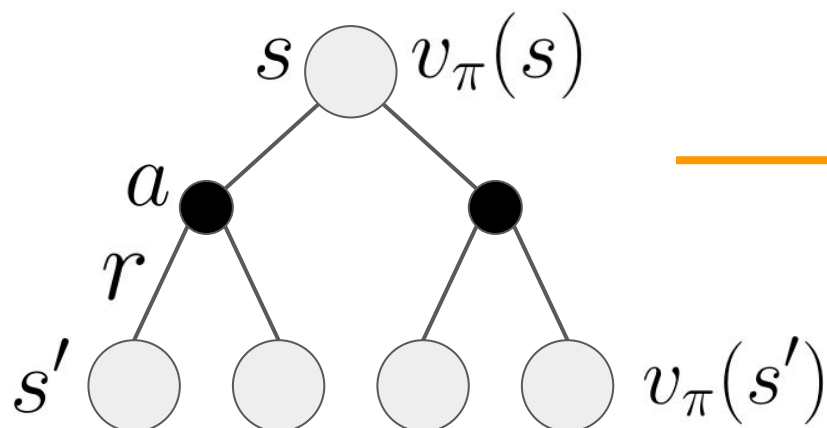


Bellman **expectation**
equation for $v(s)$

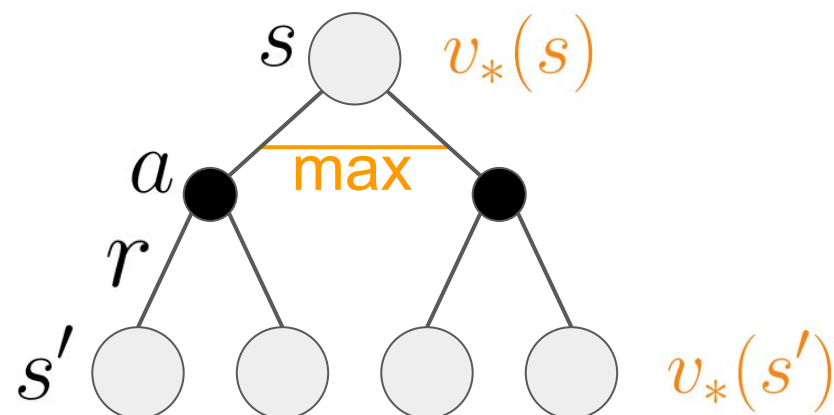


Bellman **optimality**
equation for $v_*(s)$

Bellman **optimality** equation for $v(s)$

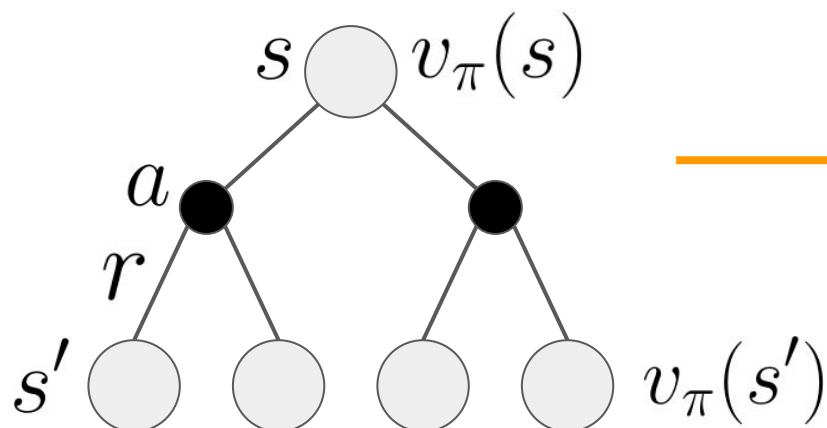


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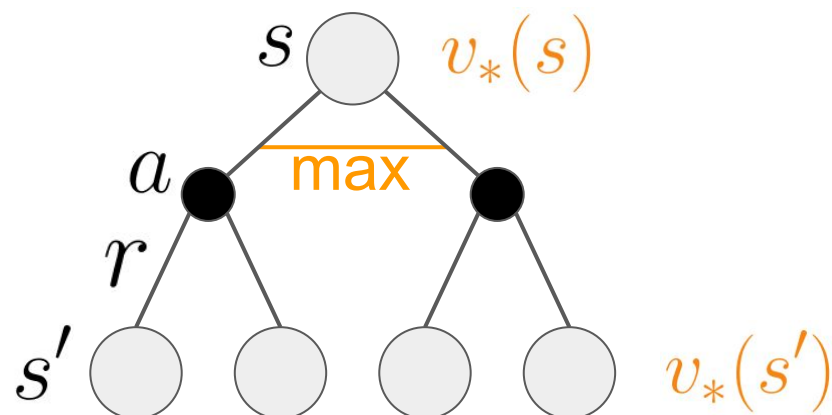


Bellman **optimality**
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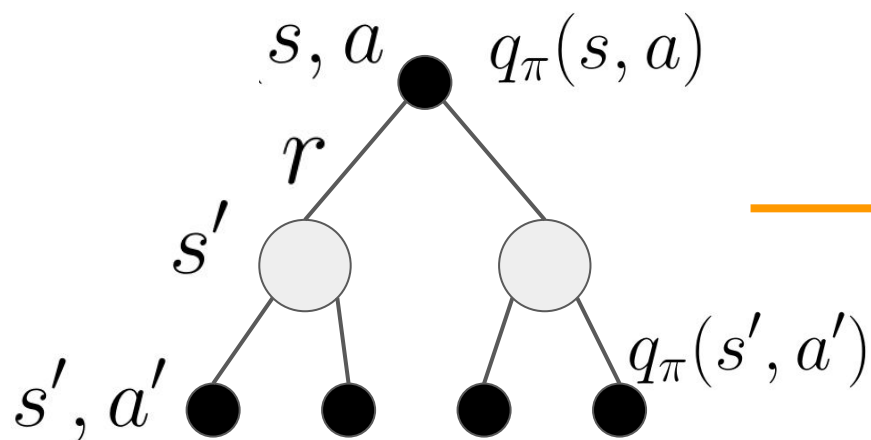
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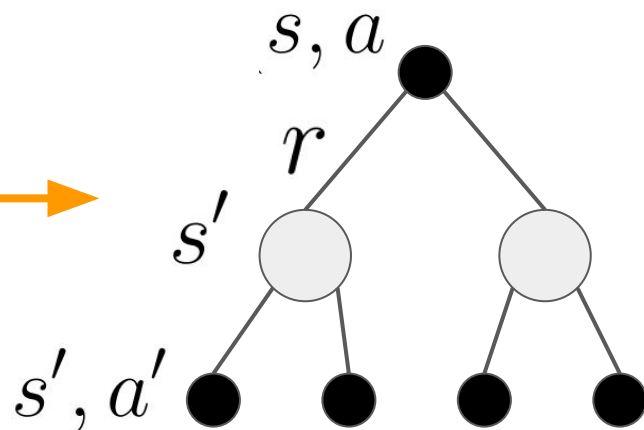
Bellman **optimality**
equation for $v_*(s)$

$$\begin{aligned} v_*(s) &= \max_a \sum_{r, s'} p(r, s' | s, a) [r + \gamma v_*(s')] \\ &= \max_a \mathbb{E} [R_t + \gamma v_*(S_{t+1}) | S_t = s, A_t = a] \end{aligned}$$

Bellman **optimality** equation for $q(s,a)$

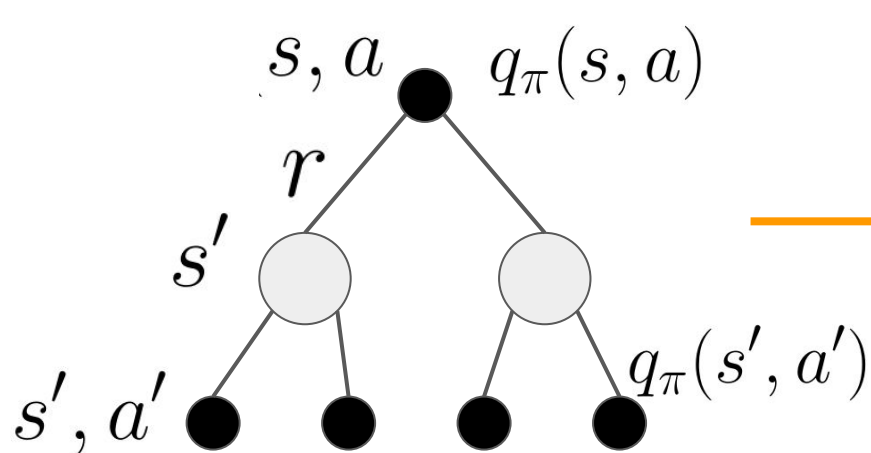


Bellman **expectation**
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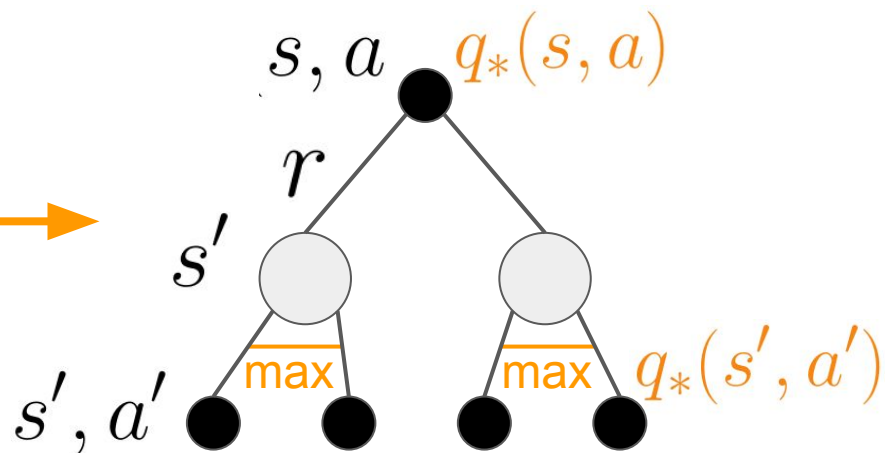


Bellman **optimality**
equation for $q_*(s, a)$

Bellman **optimality** equation for $q(s,a)$

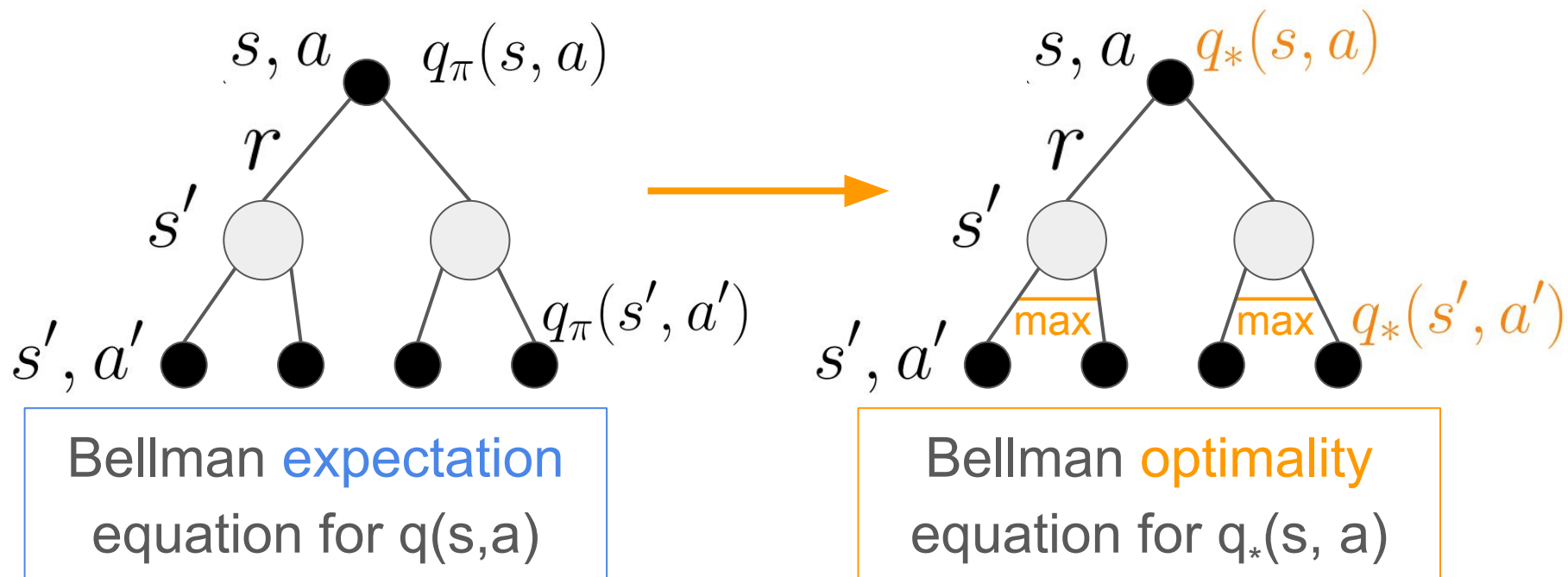


Bellman **expectation**
equation for $q(s,a)$



Bellman **optimality**
equation for $q_*(s, a)$

Bellman **optimality** equation for $q(s,a)$



$$\begin{aligned} q_*(s, a) &= \mathbb{E} \left[R_t + \gamma \max_{a'} q_*(S_{t+1}, a') \mid S_t = s, A_t = a \right] \\ &= \sum_{r, s'} p(r, s' \mid s, a) \left[r + \gamma \max_{a'} q_*(s', a') \right] \end{aligned}$$

Bellman equations: operator view

Bellman equations: operator view

$$[\mathcal{T}^\pi V](s) = \mathbb{E}_{r,s'|s,a=\pi(s)} [r + \gamma V(s')]]$$

$$[\mathcal{T}^\pi Q](s, a) = \mathbb{E}_{r,s'|s,a} [r + \gamma \mathbb{E}_{a' \sim \pi(s')} [Q(s', a')]]]$$

$$[\mathcal{T}V](s) = \max_a \mathbb{E}_{r,s'|s,a} [r + \gamma V(s')]]$$

$$[\mathcal{T}Q](s, a) = \mathbb{E}_{r,s'|s,a} \left[r + \gamma \max_{a'} Q(s', a') \right]$$

Bellman equations: operator view

Bellman **expectation** equation for $\mathbf{v}(\mathbf{s})$

$$[\mathcal{T}^\pi V](s) = \mathbb{E}_{r,s'|s,a=\pi(s)} [r + \gamma V(s')]]$$

Bellman **expectation** equation for $\mathbf{q}(\mathbf{s},\mathbf{a})$

$$[\mathcal{T}^\pi Q](s, a) = \mathbb{E}_{r,s'|s,a} [r + \gamma \mathbb{E}_{a' \sim \pi(s')} [Q(s', a')]]]$$

Bellman **optimality** equation for $\mathbf{v}_*(\mathbf{s})$

$$[\mathcal{T}V](s) = \max_a \mathbb{E}_{r,s'|s,a} [r + \gamma V(s')]]$$

Bellman **optimality** equation for $\mathbf{q}_*(\mathbf{s},\mathbf{a})$

$$[\mathcal{T}Q](s, a) = \mathbb{E}_{r,s'|s,a} \left[r + \gamma \max_{a'} Q(s', a') \right]$$

Bellman equations are **contractions**

Contraction: $d(\mathcal{T}(v), \mathcal{T}(u)) \leq \gamma d(v, u), \quad 0 \leq \gamma < 1$

Operator: $[\mathcal{T}v](s) = \max_a \mathbb{E}_{r, s'|s, a} [r + \gamma v(s')]$

Denote $a^* = \arg \max_a \mathbb{E}_{r, s'|s, a} [r + \gamma v(s')]$, then, for any s:

$$\begin{aligned} [\mathcal{T}v](s) - [\mathcal{T}u](s) &\leq r(s, a^*) + \gamma \mathbb{E}_{s'|s, a^*} [v(s')] - r(s, a^*) - \gamma \mathbb{E}_{s'|s, a^*} [u(s')] \\ &= \gamma \mathbb{E}_{s'|s, a^*} [v(s') - u(s')] \quad \text{🗨️} \\ &\leq \gamma \mathbb{E}_{s'|s, a^*} [|v(s') - u(s')|] \\ &\leq \gamma \max_{s'} |v(s') - u(s')| \\ &= \gamma \|v - u\|_\infty \quad \text{🗨️} \end{aligned}$$

Hence, taking max over s:

$$\|\mathcal{T}v - \mathcal{T}u\|_\infty \leq \gamma \|v - u\|_\infty$$

Generalized Policy Iteration:

1. Policy Evaluation
2. Policy Improvement

Policy evaluation

Policy evaluation: motivation

Policy evaluation is also called **prediction problem**:

- predict value function for a particular policy.

Bellman **expectation** equation

$$\begin{aligned} v_{\pi}(s) &= \sum_a \pi(a | s) \sum_{r, s'} p(r, s' | s, a) [r + \gamma v_{\pi}(s')] \\ &= \mathbb{E}_{\pi} [R_t + \gamma v_{\pi}(S_{t+1}) | S_t = s] \end{aligned}$$

is basically a system of linear equations where

- # of unknowns = # of equations = # of states

Policy evaluation: algorithm

Input π , the policy to be evaluated

Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$

Repeat

$\Delta \leftarrow 0$

For each $s \in \mathcal{S}$:

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

Bellman **expectation**
equation for $v(s)$

until $\Delta < \theta$ (a small positive number)

Output $V \approx v_\pi$

Policy improvement

Policy improvement: an idea

Once we know what is $v(s)$ for a particular policy

We could improve it by acting greedily w.r.t. $q(s, a)$!

$$\pi'(s) \leftarrow \underset{a}{\operatorname{arg\,max}} \sum_{r, s'} \overbrace{p(r, s' \mid s, a) [r + \gamma v_{\pi}(s')]}^{q_{\pi}(s, a)}$$

This procedure is guaranteed to produce a better policy!

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This procedure is guaranteed to produce a better policy!

if $q_{\pi}(s, \pi'(s)) \geq v_{\pi}(s)$ for all states

then $v_{\pi'}(s) \geq v_{\pi}(s)$

meaning that $\pi' \geq \pi$

Policy improvement: convergence

If new policy after improvement

$$\pi'(s) \leftarrow \underset{a}{\operatorname{arg\,max}} \overbrace{\sum_{r, s'} p(r, s' \mid s, a) [r + \gamma v_{\pi}(s')]}^{q_{\pi}(s, a)}$$

is the same as the old one

$$\pi' = \pi \quad \rightarrow \quad v_{\pi'} = v_{\pi}$$

then it is optimal, since it satisfies:

$$v_{\pi'}(s) = \underset{a}{\operatorname{max}} \sum_{r, s'} p(r, s' \mid s, a) [r + \gamma v_{\pi}(s')]$$

Policy improvement: convergence

If new policy after improvement

$$\pi'(s) \leftarrow \arg \max_a \overbrace{\sum_{r, s'} p(r, s' | s, a) [r + \gamma v_\pi(s')]}^{q_\pi(s, a)}$$

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$$v_{\pi'}(s) = \max_a \sum_{r, s'} p(r, s' | s, a) [r + \gamma v_\pi(s')]$$

Bellman
optimality
equation

Determining optimal policy from $v_*(s)$, $q_*(s,a)$

If q^* is known – how to recover the optimal policy?

$$\pi_*(s) \leftarrow \arg \max_a q_*(s, a)$$

If v^* is known – how to recover the optimal policy?

Determining optimal policy from $v_*(s)$, $q_*(s,a)$

If q^* is known – how to recover the optimal policy?

$$\pi_*(s) \leftarrow \arg \max_a q_*(s, a)$$

If v^* is known – how to recover the optimal policy?

$$\pi_*(s) \leftarrow \arg \max_a \overbrace{\sum_{r, s'} p(r, s' | s, a) [r + \gamma v_*(s')]}^{q_*(s, a)}$$

Unknown model dynamics → unable to recover optimal policy from v^*

Precise evaluation is excessive











Value function

0 iteration

	0.000	0.000	0.000
0.000	0.000	0.000	0.000
0.000	0.000	0.000	

Greedy policy

0 iteration

Value function

0 iteration

	0.000	0.000	0.000
0.000	0.000	0.000	0.000
0.000	0.000	0.000	

5 iteration











	-7.598	-4.986	-3.127
-7.816	-5.834	-2.963	0.543
-6.115	-4.186	0.332	

9999 iteration

	-13.827	-13.289	-11.318
-14.768	-14.193	-10.722	-5.346
-16.111	-13.454	-6.059	

Greedy policy

0 iteration

5 iteration

9999 iteration

Roadmap

Now we know what is

- Policy evaluation (based on Bellman **expectation** eq)
- Policy improvement (based on Bellman **optimality** eq)

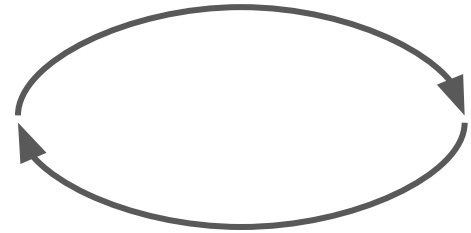
The finishing touches:

how to combine them to obtain optimal policy?

Generalized Policy Iteration

The idea of policy and value iterations

Policy evaluation



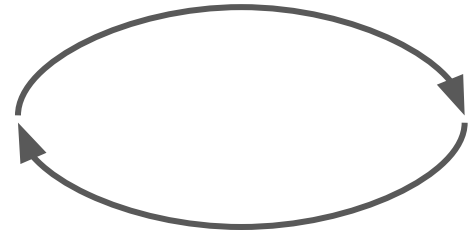
Policy improvement

The idea of policy and value iterations

Generalized policy iteration

1. Evaluate given policy
2. Improve policy by acting greedily w.r.t. to its value function

Policy evaluation



Policy improvement

The idea of policy and value iterations

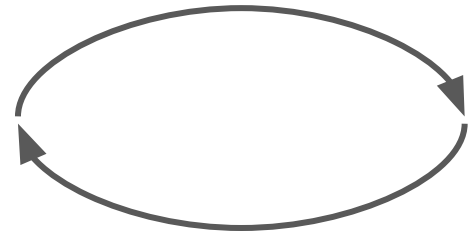
Generalized policy iteration

1. Evaluate given policy
2. Improve policy by acting greedily w.r.t. to its value function

Robustness:

- No dependence on initialization
- No need in complete policy evaluation (states / converg.)
- No need in exhaustive update (states)
 - Example of update robustness:
 - Update only one state at a time
 - in a random direction
 - that is correct only in a expectation

Policy evaluation



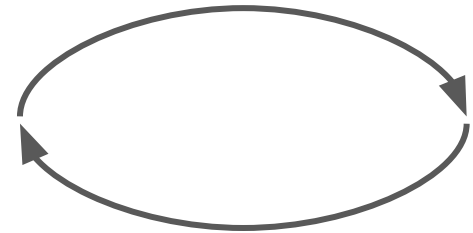
Policy improvement

The idea of policy and value iterations

Generalized policy iteration

1. Evaluate given policy
2. Improve policy by acting greedily w.r.t. to its value function

Policy evaluation



Policy improvement

Policy iteration

1. Evaluate policy until convergence (with some tolerance)
2. Improve policy

Value iteration

1. Evaluate policy only with single iteration
2. Improve policy

Policy iteration

Policy iteration: scheme

1. Initialization

$V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation

Repeat

$\Delta \leftarrow 0$

For each $s \in \mathcal{S}$:

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_{s',r} p(s', r | s, \pi(s)) [r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until $\Delta < \theta$ (a small positive number)

Bellman expectation
equation for $v(s)$

3. Policy Improvement

policy-stable \leftarrow true

For each $s \in \mathcal{S}$:

old-action $\leftarrow \pi(s)$

$\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r | s, a) [r + \gamma V(s')]$

If *old-action* $\neq \pi(s)$, then *policy-stable* \leftarrow false

If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

$q(s,a)$

Value iteration

Value iteration

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat

$$\Delta \leftarrow 0$$

For each $s \in \mathcal{S}$:

$$v \leftarrow V(s)$$

$$V(s) \leftarrow \max_a \sum_{s',r} p(s', r | s, a) [r + \gamma V(s')]$$

$$\Delta \leftarrow \max(\Delta, |v - V(s)|)$$

until $\Delta < \theta$ (a small positive number)

Bellman **optimality**
equation for $v(s)$

Output a deterministic policy, $\pi \approx \pi_*$, such that

$$\pi(s) = \operatorname{argmax}_a \sum_{s',r} p(s', r | s, a) [r + \gamma V(s')]$$

Value iteration (VI) vs. Policy iteration (PI)

- VI is **faster** per iteration – $O(|A||S|^2)$
- VI requires **many** iterations
- PI is **slower** per iteration – $O(|A||S|^2 + |S|^3)$
- PI requires **few** iterations

No silver bullet → experiment with # of steps spent in policy evaluation phase to find the best