1 Question 1

Using Hoare's rules, prove:

$$\{x = y\}x := x + 1; y := y + 1\{x = y\}$$

1.1 Answer

Here all the rules belong to Hoare's rules for While. So, ignored the $\vdash$ in the triples.

$$\dfrac{\dfrac{x = y \Rightarrow true \qquad \dfrac{\{x + 1 = y + 1\}x := x + 1\{x = y + 1\}}{\{true\}x := x + 1\{x = y + 1\}}\,(ass_p \text{ rule})}{\{x = y\}x := x + 1\{x = y + 1\}}\,(cons_p) \qquad \dfrac{\{[y + 1/y](x = y)\}y := y + 1\{x = y\}}{\{x = y + 1\}y := y + 1\{x = y\}}\,(ass_p)}{\{x = y\}x := x + 1; y := y + 1\{x = y\}}\,(\ comp_p\ )$$

Note: if we can do $x = y + 1 \Rightarrow x - 1 = y$, the solution becomes easier.

$$\dfrac{\dfrac{x = y \Rightarrow true \qquad \dfrac{\{[x + 1/x](x - 1 = y)\}x := x + 1\{x - 1 = y\}}{\{true\}x := x + 1\{x - 1 = y\}}\,(ass_p \text{ rule})}{\{x = y\}x := x + 1\{x - 1 = y\}}\,(cons_p) \qquad \dfrac{x - 1 = y\ \Rightarrow true \qquad \dfrac{\{[y + 1/y](x = y)\}y := y + 1\{x = y\}}{\{true\}y := y + 1\{x = y\}}\,(ass_p)}{\{x - 1 = y\}y := y + 1\{x = y\}}\,(\ cons_p\ )}{\{x = y\}x := x + 1; y := y + 1\{x = y\}}\,(\ comp_p\ )$$

2 Question 2

Using Hoare's rules, prove

$$\{y = z\} \text{ while b do } y := y - x\{\exists k.z = (y + k * x)\}$$

for an arbitrary boolean expression b.

2.1 Answer

Let's define W = $\text{ while b do } y := y - x$

First we need to find the lopp invariant I so that we can apply the while rule. In this case the invariant is $x = n$

Loop invariant: $\exists j \exists k.y = z - j * x \wedge k = j + 1$

$$\text{D: } \dfrac{y = z \Rightarrow \exists j \exists k.y = z - j * x \wedge k = j + 1 \qquad \dfrac{D1 : \{\exists j \exists k.y = z - j * x \wedge k = j + 1 \wedge b\}y := y - x\{\exists j \exists k.y = z - (j + 1) * x \wedge k = j + 1\}}{\{\exists j \exists k.y = z - j * x \wedge k = j + 1\}W\{Q' : \exists j \exists k.y = z - (j + 1) * x \wedge k = j + 1 \wedge \neg b\}}\,(while_p)}{\{y = z\} \text{ while b do } y := y - x\{Q : \exists k.z = (y + k * x)\}}$$

Now we show the proof of necessary parts:

$Q' : \exists j \exists k. y = z - (j+1) * x \wedge k = j+1 \wedge \neg b$
$\Rightarrow \exists j \exists k. y = z - (j+1) * x \wedge k = j+1$
$\Rightarrow \exists k. y = z - k * x$
$\Rightarrow \exists k. z = y + k * x$
$= Q$

$$\text{D1: } \dfrac{\exists j \exists k. y = z - j * x \wedge k = j+1 \wedge b \Rightarrow true \qquad \dfrac{\{[y/y-x](\exists j \exists k. y = z - (j+1) * x \wedge k = j+1)\} y := y - x \{\exists j \exists k. y = z - (j+1) * x \wedge k = j+1\}}{\{true\} y := y - x \{\exists j \exists k. y = z - (j+1) * x \wedge k = j+1\}} \, (ass_p)}{\{\exists j \exists k. y = z - j * x \wedge k = j+1 \wedge b\} y := y - x \{\exists j \exists k. y = z - (j+1) * x \wedge k = j+1\}} \, (cons_p)$$