HW3: Induction

Golam Md Muktadir

May 4, 2019

$$\frac{\langle x, \sigma\rangle \Downarrow \sigma(x) \qquad \dfrac{\langle y, \sigma\rangle \Downarrow \sigma(y) \qquad \langle z, \sigma\rangle \Downarrow \sigma(z)}{\langle y+z, \sigma\rangle \Downarrow \sigma(y)+\sigma(z)}}{\langle x*(y+z), \sigma\rangle \Downarrow \sigma(x)*(\sigma(y)+\sigma(z))}$$

1 Problem 1

1.1 Question

In the WHILE language, prove or disprove the equivalence of the two commands:

$$t := x; x := y; y := t$$

and

$$t := y; y := x; x := t$$

(where x; y, and t are distinct locations).

1.2 Answer

We can disprove it with a counter example with a state s in which two sequences yield different results.

Let

$$s = [t \to 0, x \to 5, y \to 100]$$

Then,

$$\langle t := x; x := y; y := t, s\rangle \Downarrow s[t \to 5, x \to 100, y \to 5]$$

$$\langle t := y; y := x; x := t, s\rangle \Downarrow s[t \to 100, x \to 100, y \to 5]$$

So, location, t, has different values after execution of the sequences under this s.

2 Problem 2

2.1 Question

In the WHILE language, prove that if

$$\langle \text{while b do } y := y - x, s\rangle \Downarrow s'$$

then there exists an integer k such that

$$s(y) = s'(y) + k * s(x)$$

Please make it explicit if/when you reason by induction on derivations, stating your induction hypoth- esis.

2.2 Answer

We can prove it by induction on k. Let us define

$\sigma_i = s[y := y - i * x]$

So,

$\sigma_o = s$

$\sigma_k = s[y := y - k * x]$

Now, $\sigma_k = s\prime$ if we assume while terminates after k-th loop.

Base case: $i = 0$

$$\frac{\langle b, \sigma_o \rangle \Downarrow false \qquad \langle skip, \sigma_o \rangle \Downarrow \sigma_o}{\langle W, \sigma_o \rangle \Downarrow \sigma_o}$$

So, k = 0 and P(0) holds.

Assuming x $>$ 0 We can reformulate the problem as, Let,

$W =$ while b do $y := y - x$

Let, $\sigma_i = \sigma[y - (k - i) * x]$

Let, $b = y \geq (y - k * x)$

Claim: $\langle W, \sigma_i \rangle \Downarrow \sigma_o, \forall i \in N$

Base case: $i = 0$ **or** $\langle W, \sigma_o \rangle \Downarrow \sigma_o$ Observation, b evaluates to false at state σ_o

$$\frac{\dfrac{\sigma_o(y) = y - k * x \qquad \langle y = y - x, \sigma_o \rangle \Downarrow (y - k * x - x)}{y \geq (y - k * x), \sigma \Downarrow false}}{\langle W, \sigma_o \rangle \Downarrow \sigma_o}$$

So, base case holds.

Inductive case:

Notes: for x¡0, same proof can be done by setting $b = y \leq (y - k * x)$, and for x=0, the solution is trivial.

3 Problem 3

3.1 Question

In the WHILE language, prove:

$$\forall c1, c2, c3 : c1; (c2; c3) \equiv (c1; c2); c3$$

3.2 Answer