Assignment 3

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Download all python codes from

https://github.com/adhvik24/AI1103−PROBABILITY−AND−RANDOM−VARIABLES/tree/main/ASSIGNMENT_3/codes

and latex-tikz codes from

https://github.com/adhvik24/AI1103−PROBABILITY−AND−RANDOM−VARIABLES/blob/main/ASSIGNMENT_3/AI1103_Assignment3.tex

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Three fair dies are rolled simultaneously. The probability of getting a sum of 5 is
(A) $\frac{1}{108}$ (B) $\frac{1}{72}$ (C) $\frac{1}{54}$ (D) $\frac{1}{36}$

2 Solution

Let $X_i \in \{1, 2, 3, 4, 5, 6\}$, i = 1,2,3, be the random variables representing the outcome for each die. As the dies are fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{2.0.1}$$

Let X be a random variable denotes the desired outcome,

$$X = X_1 + X_2 + X_3 \tag{2.0.2}$$
$$\implies X \in \{3, 4, \cdots, 18\} \tag{2.0.3}$$

We have to find $P_X(n) = \Pr(X_1 + X_2 + X_3 = n)$

$$\begin{aligned} p_X(n) &= \Pr(X_1 + X_2 + X_3 = n) \\ &= \Pr(X_1 + X_2 = n - X_3) \\ &= \Sigma_k \Pr(X_1 + X_2 = n - k | X_3 = k)\, p_{X_3}(k) \end{aligned} \tag{2.0.4}$$

As X_1, X_2, X_3 are independent, After unconditioning

$$\Pr(X_1 + X_2 = n - k | X_3 = k) = \Pr(X_1 + X_2 = n - k) \tag{2.0.5}$$

Using (2.0.5) in (2.0.4)

$$\begin{aligned} p_X(n) &= \Sigma_k \Pr(X_1 + X_2 = n - k | X_3 = k)\, p_{X_3}(k) \\ &= \Sigma_k \Pr(X_1 + X_2 = n - k)\, p_{X_3}(k) \\ &= \Sigma_k (\Sigma_a \Pr(X_1 = n - k - a | X_2 = a) \Pr(X_2 = a))\, p_{X_3}(k) \\ &= \Sigma_k (\Sigma_a \Pr(X_1 = n - k - a) \Pr(X_2 = a))\, p_{X_3}(k) \\ &= \Sigma_k (\Sigma_a p_{X_1}(n - k - a)\, p_{X_2}(a))\, p_{X_3}(k) \end{aligned} \tag{2.0.6}$$

Equation (2.0.6) can be written as follows using convolution operation,

$$\begin{aligned} p_X(n) &= \Sigma_k (\Sigma_a p_{X_1}(n - k - a)\, p_{X_2}(a))\, p_{X_3}(k) \\ &= p_{X_1}(n) * p_{X_2}(n) * p_{X_3}(n) \end{aligned} \tag{2.0.7}$$

The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.0.8}$$

From (2.0.1) and (2.0.8),

$$P_{X_1}(z) = P_{X_2}(z) = P_{X_3}(z)$$
$$= \frac{1}{6} \sum_{n=1}^{6} z^{-n} \tag{2.0.9}$$
$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{2.0.10}$$

upon summing up the geometric progression. From (2.0.7),

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n) * p_{X-3}(n), \tag{2.0.11}$$
$$P_X(z) = P_{X_1}(z) P_{X_2}(z) P_{X_3}(z) \tag{2.0.12}$$

The above property follows from Fourier analysis and is fundamental to signal processing.
From (2.0.10) and (2.0.12),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^3 \tag{2.0.13}$$

$$= \frac{1}{216} \frac{z^{-3}\left(1 - 3z^{-6} + 3z^{-12} - z^{-18}\right)}{\left(1 - z^{-1}\right)^3} \tag{2.0.14}$$

Using the fact that,

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} ZP_X(z)z^{-k}, \tag{2.0.15}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}}{(1-z^{-1})^2} \tag{2.0.16}$$

$$n^2u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}\left(1+z^{-1}\right)}{(1-z^{-1})^3} \tag{2.0.17}$$

$$(n^2+n)u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{2z^{-1}}{(1-z^{-1})^2} \tag{2.0.18}$$

after some algebra, it can be shown that,

$$\begin{aligned}
&\frac{1}{216\times 2}\Big[\left((n-2)^2+n-2\right)u(n-2) \\
&\quad -3\left((n-8)^2+n-8\right)u(n-8) \\
&\quad +3\left((n-14)^2+n-14\right)u(n-14) \\
&\quad -\left((n-20)^2+n-20\right)u(n-20)\Big] \\
&\overset{\mathcal{H}}{\longleftrightarrow} Z\frac{1}{216}\frac{z^{-3}\left(1-3z^{-6}+3z^{-12}-z^{-18}\right)}{(1-z^{-1})^3}
\end{aligned} \tag{2.0.19}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{2.0.20}$$

From (2.0.8),(2.0.14) and (2.0.19),

$$\begin{aligned}
p_X(n) = &\frac{1}{216\times 2}\Big[\left((n-2)^2+n-2\right)u(n-2) \\
&-3\left((n-8)^2+n-8\right)u(n-8) \\
&+3\left((n-14)^2+n-14\right)u(n-14) \\
&-\left((n-20)^2+n-20\right)u(n-20)\Big]
\end{aligned} \tag{2.0.21}$$

From (2.0.20) and (2.0.21),

$$p_X(n) = \begin{cases} 0 & n < 3 \\ \frac{n^2-3n+2}{432} & 3 \leq n \leq 8 \\ \frac{42n-2n^2-166}{432} & 8 < n \leq 14 \\ \frac{n^2-39n+380}{432} & 14 < n \leq 18 \\ 0 & n > 18 \end{cases} \tag{2.0.22}$$

We need probability of getting sum of 5,
$\implies$ n=5

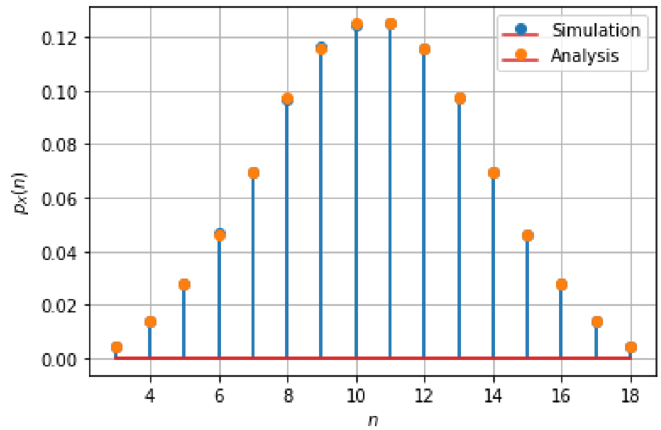

Fig. 1: Probability mass function of X (simulations are close to analysis)

from (2.0.22) and using n=5,

$$p_X(5) = \frac{5^2-3(5)+2}{432} \tag{2.0.23}$$

$$p_X(5) = \frac{12}{432} \tag{2.0.24}$$

$$p_X(5) = \frac{1}{36} \tag{2.0.25}$$

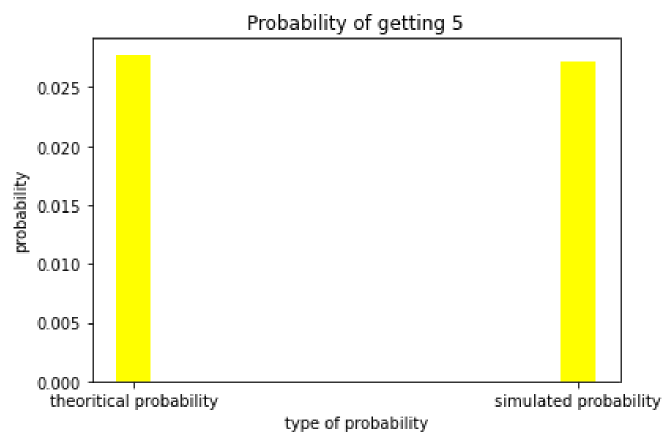

Fig. 2: Probability of getting sum of 5

Therefore the probability of getting a sum of 5 when three fair dies are rolled is $\frac{1}{36}$.
Ans: Option (D)