Research Paper Presentation

Adhvik Murarisetty - AI20BTECH11015

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On the Product of Two Correlated Complex Gaussian Random Variables

Abstract

1. We will derive the exact joint PDF of the amplitude and phase of the product of two correlated non-zero mean complex Gaussian random variables with arbitrary variances.
2. We determine the joint pdf in terms of an infinite summation of modified Bessel functions of the first and second kinds, which generalizes the existing results and observes some special cases.
3. We will also study truncation error when a truncated sum is employed.
4. Finally, we evaluate the derived expressions through numerical experiments.

Introduction

Complex Gaussian RVs are used more extensively in signal processing. We consider the following complex random variable,

$$Z = XY^* \tag{1}$$

where X and Y are two correlated complex Gaussian RVs with non-zero means and arbitrary variances. This RV is useful in many applications:

1. In a single channel M-ary phase-shift-keying (MPSK) communication system, the linear combiner output can be characterized by the product of two complex random RVs as (1).
2. In time reversal detection, the aggregate random channel can be modeled by the RV (1).
3. For radar applications, in the case of multipath scattering, for over-the-horizon (OTH) radar for example, the RV is useful to characterize the overall reflection coefficients.

NOTATIONS

SYMBOL	DENOTES
μ, m	Mean
σ	Variance
ρ	Correlation factor
E	Expectation
$(.)^*$	Conjugate
$(.)^T$	Transpose
$(.)^H$	Conjugate transpose
j	Imaginary unit $\left(\implies j^2 = -1\right)$
$R(.)$	Real part of a complex number
$J(.)$	Imaginary part of a complex number
$I_\mu(.)$	Modified Bessel function of first kind with order μ
$K_\mu(.)$	Modified Bessel function of second kind with order μ

Some important results

Theorem 1

Let X and Y be two bivariate normal random variables,Then there exist independent standard normal random variables Z_1 and Z_2 such that,

$$X = \sigma_X Z_1 + \mu_X \tag{2}$$

$$Y = \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y \tag{3}$$

Proof

To prove the theorem, define

$$Z_1 = \frac{X - \mu_X}{\sigma_Y} \tag{4}$$

$$Z_2 = -\frac{\rho}{\sqrt{1-\rho^2}} \frac{X - \mu_X}{\sigma_X} + \frac{1}{\sqrt{1-\rho^2}} \frac{Y - \mu_Y}{\sigma_Y} \tag{5}$$

Rearranging (4) will result in (2) and Using (4) in (5) will result in (3)

Some important results Contd..

Theorem 2

Suppose X *and* Y are jointly normal random variables with parameters μ_X, σ_X, μ_Y, σ_Y and ρ. Then, given X, $Y = y$ is normally distributed with,

$$E[X|Y = y] = \mu_X + \rho\sigma_X \frac{y - \mu_Y}{\sigma_Y} \tag{6}$$

$$\sigma_{X|Y=y} = \left(1 - \rho^2\right)\sigma_X^2 \tag{7}$$

Proof

Using theorem 1, Let us define

$$Y = \sigma_Y Z_1 + \mu_Y \tag{8}$$

$$X = \sigma_X \left(\rho Z_1 + \sqrt{1 - \rho^2} Z_2\right) + \mu_X \tag{9}$$

Some important results Contd..

Proof Contd..

Thus given $Y = y$, we have

$$Z_1 = \frac{y - \mu_Y}{\sigma_Y} \tag{10}$$

and

$$X = \sigma_X \left(\rho \frac{y - \mu_Y}{\sigma_Y} + \sqrt{1 - \rho^2} Z_2 \right) + \mu_X \tag{11}$$

$$= \sigma_X \rho \frac{y - \mu_Y}{\sigma_Y} + \sigma_X \sqrt{1 - \rho^2} Z_2 + \mu_X \tag{12}$$

Since Z_1 *and* Z_2 are independent. We have shown that given $Y = y$, X is a linear function of Z_2, thus it is normal. So,

Some important results Contd..

Proof Contd..

$$E[X|Y=y] = \sigma_X \rho \frac{y-\mu_Y}{\sigma_Y} + \sigma_X \sqrt{1-\rho^2} E[Z_2] + \mu_X \tag{13}$$

$$= \mu_X + \rho \sigma_X \frac{y-\mu_Y}{\sigma_Y} \tag{14}$$

and

$$\sigma_{X|Y=y} = \left(1-\rho^2\right) \sigma_X^2 \sigma_{Z_2}^2 \tag{15}$$

$$= \left(1-\rho^2\right) \sigma_X^2 \tag{16}$$

Since Z_2 is a standard normal variable $\implies Z_2 \sim N(0,1)$

Some important results Contd..

PDF of a complex normal random variable

Let X be a complex normal random variable, Then its PDF is

$$f_X(x) = \frac{1}{\sigma_X^2 \pi} e^{-\frac{|x-\mu_X|^2}{\sigma_X^2}} \tag{17}$$

Important Integrals

$$I_\mu(u) = \left(\frac{u}{2}\right)^\mu \sum_{m=0}^{\infty} \frac{\frac{u}{2}^{2m}}{m!(m+\mu)!} \tag{18}$$

$$\int_0^\infty x^{v-1} e^{(-\beta x^p - \gamma x^{-p})} dx = \frac{2}{p} \left(\frac{\gamma}{\beta}\right)^{\frac{v}{2p}} K_{\frac{v}{p}} \left(2\sqrt{\beta\gamma}\right) \tag{19}$$

DERIVATION OF JOINT PDF

The following theorem provides a general expression for the product of two correlated complex Gaussian random variables.
Let $v = [X, Y]^T \sim CN(m, \Sigma)$ be a 2×1 complex Gaussian vector, where

$$m = E[v] = [m_x, m_y]^T \tag{20}$$

and

$$\Sigma = E\left[(v - m)(v - m)^H\right] \tag{21}$$

$$= \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho^*\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \tag{22}$$

DERIVATION Contd..

Theorem

Let

$$Z = XY^* = Z_I + jZ_Q = Re^{j\Theta} \tag{23}$$

The joint pdf of (R, Θ) is given by

$$f_{R,\Theta}(r,\theta) = \frac{2re^{-g}}{c\pi\sigma_y^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\beta_1^n \beta_2^p}{n!p!\beta_3^{\frac{n+p}{2}}} \left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} \times K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right) I_{n+p}\left(2\sqrt{\beta_3}\right) \tag{24}$$

DERIVATION Contd..

Where,

$$\alpha = \frac{|b|^2}{c} + \frac{1}{\sigma_y^2} \tag{25}$$

$$g = \frac{|a|^2 - 2R\left\{re^{-j\theta}b\right\}}{c} + \frac{|m_y|^2}{\sigma_y^2} \tag{26}$$

$$\beta_1 = \frac{r^2|a|^2}{c^2}, \quad \beta_2 = | - \frac{b^*a}{c} + \frac{m_y}{\sigma_y^2}|^2 \tag{27}$$

$$\beta_3 = 2R\left\{\frac{re^{-j\theta}a}{c}\left(-\frac{ba^*}{c} + \frac{m_y^*}{\sigma_y^2}\right)\right\} \tag{28}$$

in which,

$$a = m_x - \frac{\rho\sigma_x}{\sigma_y}m_y, b = \frac{\rho\sigma_x}{\sigma_y} \; and \; c = \sigma_x^2(1 - |\rho|^2) \tag{29}$$

DERIVATION Contd..

The joint pdf for the amplitude R and phase Θ can be obtained by

$$f_{R,\Theta}\left(r,\theta\right) = rf_Z\left(r\ cos\theta, r\ sin\theta\right) \tag{30}$$

$$f_Z\left(z_I, z_Q\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Z|Y}\left(z_I, z_Q\right) f_Y\left(y_I, y_Q\right)\ dy_I\ dy_Q \tag{31}$$

By using (6) and (7),

$$X|Y \sim\ CN\left(m_X + \rho\sigma_X \frac{Y - m_Y}{\sigma_Y}, \left(1-\rho^2\right)|\sigma_X|^2\right) \tag{32}$$

$$\implies\ Z|Y = XY^*|Y \sim CN\left(aY^* + b|Y|^2, c|Y|^2\right) \tag{33}$$

$$and\ Y \sim\ CN\left(m_y, \sigma_y^2\right) \tag{34}$$

DERIVATION Contd..

Using (17),(33) and (34) in (31),

$$f_Z\left(z_I, z_Q\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi c|y|^2} e^{-\frac{\left|z-\left(ay^*+b|y|^2\right)\right|^2}{c|y|^2}} \frac{1}{\sigma_Y^2\pi} e^{-\frac{|y-m_y|^2}{\sigma_y^2}}\, dy_I\, dy_Q \quad (35)$$

where $y = y_I + jy_Q$ and $z = z_I + jz_Q$

$$\begin{aligned}\left|z - \left(ay^* + b|y|^2\right)\right|^2 = {} & |z|^2+|a|^2|y|^2+|b|^2|y|^4-2R\left\{z^* a\right\} y_I-2J\left\{z^* a\right\} y_Q \\ & - 2R\left\{z^* b\right\}|y|^2 + 2R\left\{b^* a\right\}|y|^2 y_I + 2J\left\{b^* a\right\}|y|^2 y_Q \quad (36)\end{aligned}$$

$$|y-\mu_Y|^2 = |y|^2 + |m_y|^2 - 2R\left\{m_y\right\} y_I - 2J\left\{m_y\right\} y_Q \quad (37)$$

Using (36) and (37) and letting $y_I = tcos\psi$ and $y_Q = tsin\psi$, we will get

DERIVATION Contd..

$$f_Z(z_I, z_Q) = \frac{e^{-g}}{c\pi^2\sigma_y^2} \int\limits_{-\infty}^{\infty} \frac{1}{t} e^{-\alpha t^2 - \frac{|z|^2}{ct^2}} \int\limits_{0}^{2\pi} e^{\lambda_1(t,z)cos\psi + \lambda_2(t,z)sin\psi} \; d\psi \; dt \quad (38)$$

$$= \frac{2e^{-g}}{c\pi\sigma_y^2} \int\limits_{-\infty}^{\infty} \frac{1}{t} e^{-\alpha t^2 - \frac{|z|^2}{ct^2}} I_0(\lambda(t,z)) \; dt \quad (39)$$

where,

$$\lambda_1(t,z) = \frac{2R\{z^* a\}}{ct} - \frac{2R\{b^* a\} t}{c} + \frac{2R\{m_y\} t}{\sigma_y^2} \quad (40)$$

$$\lambda_2(t,z) = \frac{2J\{z^* a\}}{ct} - \frac{2J\{b^* a\} t}{c} + \frac{2J\{m_y\} t}{\sigma_y^2} \quad (41)$$

$$\lambda(t,z) = \sqrt{\lambda_1(t,z)^2 + \lambda_2(t,z)^2} = 2\sqrt{\beta_1 t^{-2} + \beta_2 t^2 + \beta_3} \quad (42)$$

DERIVATION Contd..

Using (18) to get (44) and use it in (39),

$$I_0(\lambda(t,z)) = \sum_{m=0}^{\infty} \frac{(\beta_1 t^{-2} + \beta_2 t^2 + \beta_3)^m}{(m!)^2} \tag{43}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \sum_{p=0}^{n} \frac{(\beta_1 t^{-2})^n (\beta_2 t^2)^p (\beta_3)^{m-n-p}}{m!n!p!(m-n-p)!} \tag{44}$$

$$\implies f_Z(z_I, z_Q) = \frac{2e^{-g}}{c\pi\sigma_y^2} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \sum_{p=0}^{n} \frac{(\beta_1)^n (\beta_2)^p (\beta_3)^{m-n-p}}{m!n!p!(m-n-p)!} \int_0^{\infty} \frac{t^{(2p-2n)}}{t} e^{-\alpha t^2 - \frac{|z|^2}{ct^2}} \, dt \tag{45}$$

DERIVATION Contd..

Rewiting integral in (45) using (19)

$$\int_{0}^{\infty} \frac{t^{(2p-2n)}}{t} e^{-\alpha t^2 - \frac{|z|^2}{ct^2}}\, dt = \left(\frac{|z|^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2|z|\sqrt{\frac{\alpha}{c}}\right) \tag{46}$$

$$\implies f_Z\left(z_I, z_Q\right) = \frac{2e^{-g}}{c\pi\sigma_y^2} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \sum_{p=0}^{n} \frac{(\beta_1)^n(\beta_2)^p(\beta_3)^{m-n-p}}{m!n!p!(m-n-p)!} \left(\frac{|z|^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2|z|\sqrt{\frac{\alpha}{c}}\right) \tag{47}$$

Reordering the summation gives us,

DERIVATION Contd..

$$f_Z(z_I, z_Q) = \frac{2e^{-g}}{c\pi\sigma_y^2} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\beta_1)^n(\beta_2)^p(\beta_3)^{-n-p}}{n!p!} \left(\frac{|z|^2}{c\alpha}\right)^{\frac{p-n}{2}} \times K_{p-n}\left(2|z|\sqrt{\frac{\alpha}{c}}\right) \sum_{m=n+p}^{\infty} \frac{\beta_3^m}{m!(m-n-p)!} \quad (48)$$

$$\sum_{m=n+p}^{\infty} \frac{\beta_3^m}{m!(m-n-p)!} = (\beta_3)^{\frac{n+p}{2}} I_{n+p}\left(2\sqrt{\beta_3}\right) \quad (\because using(18)) \quad (49)$$

$$\implies f_{R,\Theta}(r,\theta) = \frac{2re^{-g}}{c\pi\sigma_y^2} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\beta_1^n\beta_2^p}{n!p!\beta_3^{\frac{n+p}{2}}} \left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right) I_{n+p}\left(2\sqrt{\beta_3}\right) \quad (50)$$

Special Cases

Independent RVs

In this case $\rho = 0$, using this in (24) gives us

$$f_{R,\Theta}(r,\theta) = \frac{2re^{-\left(k_x^2+k_y^2\right)}}{\pi\sigma_x^2\sigma_y^2} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{n!p!}\left(\frac{k_x}{k_y}\right)^{n-p}\left(\frac{|\phi(r,\theta)|}{2\sqrt{R\{\phi(r,\theta)\}}}\right) \times K_{p-n}\left(\frac{2r}{\sigma_x\sigma_y}\right) I_{n+p}\left(2\sqrt{R\{\phi(r,\theta)\}}\right) \quad (51)$$

where, $k_x = \frac{|m_x|}{\sigma_x}$, $k_y = \frac{|m_y|}{\sigma_y}$ and $\phi(z) = 2\frac{R\left\{re^{-j\theta m_x m_y^*}\right\}}{\sigma_x^2\sigma_y^2}$.
This result is agrees with result in **N. O'Donoughue and J. M. Moura, "On the product of independent complexGaussians," IEEE Trans. Signal Process., vol. 60, no. 3, pp. 1050– 1063, Mar. 2012.**

Special cases contd..

Zero means

As $m_x, m_y \to 0$ then the terms of (24) become,

$$\lim_{m_x,m_y \to 0} g = \frac{-2R\left\{re^{-j\theta}b\right\}}{c} = g_0 \ and \lim_{m_x,m_y \to 0} \beta_i = 0, \ i = 1,2,3 \quad (52)$$

$$\implies \lim_{\beta_3 \to 0} \frac{I_{n+p}\left(2\sqrt{\beta_3}\right)}{\beta_3^{\frac{n+p}{2}}} = \lim_{\beta_3 \to 0} \sum_{m=0}^{\infty} \frac{\beta_3^m}{m!(m+n+p)!} = \frac{1}{(n+p)!} \quad (53)$$

$$\implies \lim_{m_x,m_y \to 0} f_{R,\Theta}(r,\theta) = \lim_{\beta_1,\beta_2,\beta_3 \to 0, g \to g_0} f_{R,\Theta}(r,\theta)$$

$$= \lim_{\beta_1,\beta_2 \to 0} \frac{2re^{-g_0}}{c\pi\sigma_y^2} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\beta_1^n \beta_2^p}{n!p!} \left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right) \frac{1}{(n+p)!} \quad (54)$$

Special cases contd..

In the (54), the limit of general summation term is non-zero only when n=0,p=0.

$$f_{R,\Theta}(r,\theta) = \frac{2re^{-g_0}}{c\pi\sigma_y^2} K_0\left(2r\sqrt{\frac{\alpha}{c}}\right)$$
$$= \frac{2r}{\pi(1-|\rho|^2)\sigma_x^2\sigma_y^2} e^{\frac{2R\left\{\rho r e^{-j\theta}\right\}}{(1-|\rho|^2)\sigma_x\sigma_y}} K_0\left(\frac{2r}{(1-|\rho|^2)\sigma_x\sigma_y}\right) \quad (55)$$

This result is consistent with result in **Z. Zheng, L.Wei, J. Hamalainen, and O. Tirkkonen, "A blind time-reversal detector in the presence of channel correlation," IEEE Signal Process. Lett., vol. 20, no. 5, pp. 459–462, May 2013.**

Truncated joint PDF

Truncated joint PDF

Since (24) involves a doubly-infinite summation, we define a truncated joint pdf approximation of (24) as

$$\bar{f}_{R,\Theta}\left(r,\theta,M\right)=\frac{2re^{-g}}{c\pi\sigma_y^2}\sum_{n=0}^{M}\sum_{p=0}^{M}\frac{\beta_1^n\beta_2^p}{n!p!\beta_3^{\frac{n+p}{2}}}\left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} \times K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right)I_{n+p}\left(2\sqrt{\beta_3}\right) \tag{56}$$

Thus the truncation error is

$$\epsilon_M=f_{R,\Theta}\left(r,\theta\right)-\bar{f}_{R,\Theta}\left(r,\theta,M\right) \tag{57}$$

Truncated joint PDF Contd..

Substituting (24) and (56) in (57), using inequalities and approximations, it can be proved that for sufficiently large M, the truncation error (57) is bounded by

$$|\epsilon_m| \leq \frac{4re^{-g}}{c\pi\sigma_y^2}\left[\sum_{n=0}^{M+1}\psi_{n,M+1} + \sum_{p=0}^{M+1}\psi_{M+1,p}\right] = B_M \tag{58}$$

Where,

$$\psi_{n,p} = \frac{\beta_1^n \beta_2^p}{n!p!|\beta_3|^{\frac{n+p}{2}}}\left(\frac{r^2}{c\alpha}\right)^{\frac{p-n}{2}} K_{p-n}\left(2r\sqrt{\frac{\alpha}{c}}\right) I_{n+p}\left(2\sqrt{\beta_3}\right) \tag{59}$$

and for any arbitrary $C > 1$,

$$\lim_{M\to\infty}\frac{B_M}{C^{-M}} = 0 \tag{60}$$

$\implies$ the decay of truncation error is faster than exponential.

Numerical Results

Verifying the accuracy of truncated series in (56) through numerical results by setting parameters as

$$m_x = 1, m_y = 0.5 - j0.5, \sigma_x^2 = 1, \sigma_y^2 = 0.8 \ and \ \rho = -\frac{1}{2}e^{j\frac{\pi}{4}}$$

The below graphs are the cut of truncated PDF along $\theta = 0$ and $r = 1$ for M=3,4,5,6 and 20.we see that the cuts of the truncated pdf for M $=$ 6 and for M $=$ 20 are almost the same, which implies that for M$>$6, the truncation error can be very small in the tested example.
Other r or θ cuts of the truncated pdf are also tested and the results are similar.

Numerical Results Contd..

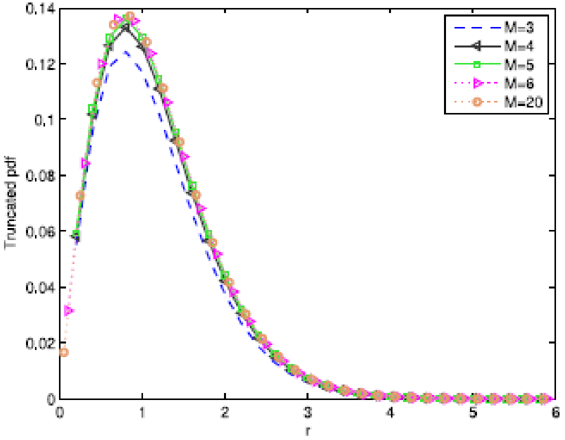

Fig. 1. $\theta = 0$ cut of the truncated pdf for different M.

Numerical Results Contd..

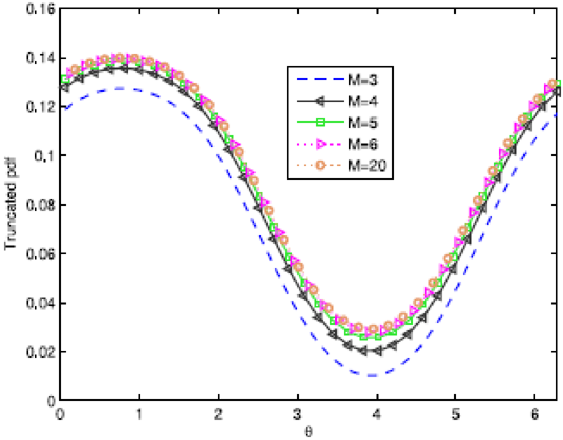

Fig. 2. $r = 1$ cut of the truncated pdf for different M.

Numerical Results Contd..

Compare the numerical result obtained from Monte- Carlo method with the computation result from (56).
In Fig. 3(a), we plot the numerical result by using 10^7 independent samples sampled from the RV in (24). In Fig. 3(b), the computation result from (56) with M $=$ 20 is plotted.
To evaluate the difference between Fig. 3(a) and Fig. 3(b), define

$$\epsilon(r,\theta) = \left| f(r,\theta) - \bar{f}_{R,\Theta}(r,\theta,20) \right| \tag{61}$$

Numerical Results Contd..

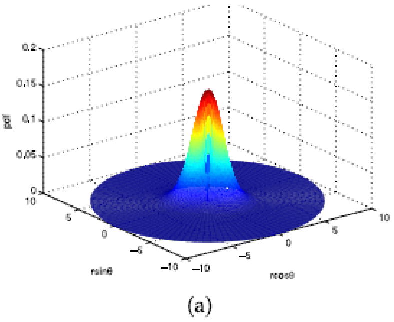

Fig. 3. Plot of the joint pdf. (a) Numerical result using Monte Carlo method. (b) Computation result from (41) with $M = 20$.

Numerical Results Contd..

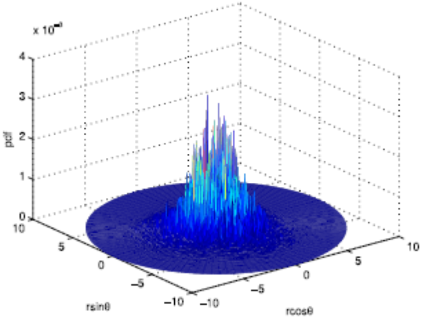

Fig. 4. Difference between numerical result and computation result.

We see that the peak value of $\epsilon(r, \theta)$ is no larger than 4×10^{-3}, which is very small compared with the peak values in Fig. 3(a) and Fig. 3(b). This implies that the derived pdf matches numerical result.

THANK YOU