Challenging problem1

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Download all latex-tikz codes from

https://github.com/adhvik24/AI1103−PROBABILITY−AND−RANDOM−VARIABLES/blob/main/Challenging_1/main.tex

1 Question

Let X be a random variable such that $E(X) = E(X^2) = 1$. Then $E(X^{100}) =$?

1) 0
2) 1
3) 2^{100}
4) $2^{100} + 1$

Solution:
Given $E(X) = E(X^2) = 1$,Let f(x) be the PDF of random variable X.
Important property: Cauchy-Schwartz Inequality

$$\left(\int_a^b g(x)h(x)dx\right)^2 \leq \int_a^b g^2(x)dx \int_a^b h^2(x)dx \tag{1.0.1}$$

Where equality occurs when g(x)=k(h(x)).(Where k is a constant)
Let us assume $g(x) = x\sqrt{f(x)}$, $h(x) = \sqrt{f(x)}$ and $(a, b) = (-\infty, \infty)$ to use in (1.0.1).

$$\left(\int_{-\infty}^{\infty} xf(x)dx\right)^2 \leq \int_{-\infty}^{\infty} x^2 f(x)dx \int_{-\infty}^{\infty} f(x)dx$$

$$(E(X))^2 \leq E(X^2)(1) \tag{1.0.2}$$

But given $E(X) = E(X^2)$.Using equality condition,

$$g(x) = k(h(x)) \tag{1.0.3}$$

$$x\sqrt{f(x)} = k(\sqrt{f(x)}) \tag{1.0.4}$$

$$x = k \tag{1.0.5}$$

$$\therefore x = constant \tag{1.0.6}$$

$$\implies E(X) = X = 1. \tag{1.0.7}$$

Therefore random variable X is a constant and equals to 1.
Hence,

$$E(X^{100}) = 1 \tag{1.0.8}$$

ANSWER: Option (2).