

Question Presentation

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Ramsey 4.2 Question 15

Question

Prove that the tangent to the circle

$$\|x\|^2 = 5 \quad (0.1)$$

at the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ also touches the circle

$$x^T x + (-8 \ 6) x + 20 = 0 \quad (0.2)$$

and find the coordinates of the point of contact.

Some important results

The general equation of a second degree can be expressed as:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (0.3)$$

Given the point of contact \mathbf{q} , the equation of a tangent to the conic is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \quad (0.4)$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \quad (0.5)$$

If r is radius and \mathbf{c} is the centre of the circle we have:

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (0.6)$$

$$\mathbf{c} = -\mathbf{u} \quad (0.7)$$

Solution continued...

We can rewrite (0.1) as $x^T x = 5$, And comparing (0.1) with (0.3), we get

$$u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, q = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (0.8)$$

$$f = -5 \quad (0.9)$$

using (0.6) and (0.7) we will get center of circle as $c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $r = \sqrt{5}$.

And now using (0.4), we will get tangent at the point $P \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as

$$\left(1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)^T x + \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \end{pmatrix} + -5 = 0 \quad (0.10)$$

$$\implies (1 \quad -2) x = 5 \quad (0.11)$$

Solution continued...

The equation of the tangent line is

$$(1 \quad -2) \mathbf{x} = 5 \quad (0.12)$$

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (0.13)$$

Comparing with (0.2) with (0.3)

$$\mathbf{u} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, f = 20 \quad (0.14)$$

If \mathbf{n} is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T \mathbf{x} = c \quad (0.15)$$

Solution continued...

Comparing (0.12) with (0.15)

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (0.16)$$

The point of contact \mathbf{q} , of a line with a normal vector \mathbf{n} to the conic in (0.3) is given by:

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \quad (0.17)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (0.18)$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \quad (0.19)$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \quad (0.20)$$

Solution continued...

Solving for the point of contact using the above equations we get,

$$\kappa = \pm \sqrt{\frac{(-4 \ 3) \begin{pmatrix} -4 \\ 3 \end{pmatrix} - 20}{(1 \ -2) \begin{pmatrix} 1 \\ -2 \end{pmatrix}}} \quad (0.21)$$

$$= \pm \sqrt{\frac{25 - 20}{5}} \quad (0.22)$$

$$= \pm \sqrt{1} \quad (0.23)$$

$$\mathbf{q} = - \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (0.24)$$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad (0.25)$$

Solution continued...

If the line in (0.13) touches (0.3) at exactly one point q , then

$$m^T (Vq + u) = 0 \quad (0.26)$$

It can be seen that for the tangent line,

$$m = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (0.27)$$

Solving (0.26) for given line and circle, we get

$$= (2 \ 1) \left(\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \end{pmatrix} \right) \quad (0.28)$$

$$= (2 \ 1) \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (0.29)$$

$$= 0 \quad (0.30)$$

Conclusion

And the co-ordinates of point of contact is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Hence, it is proved that the tangent to the circle $\|x\|^2 = 5$ at the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ also touches the circle $x^T x + (-8 \ 6)x + 20 = 0$ at the point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

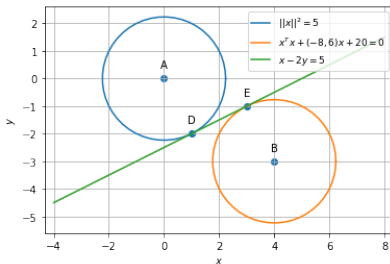


Figure: Graphical illustration

THANK YOU