# Question Presentation

Adhvik Murarisetty - AI20BTECH11015

13th September, 2021

# Ramsey 4.2 Question 15

## Question

Prove that the tangent to the circle

$$\|\mathbf{x}\|^2 = 5 \tag{0.1}$$

at the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ also touches the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -8 & 6 \end{pmatrix}\mathbf{x} + 20 = 0 \tag{0.2}$$

and find the coordinates of the point of contact.

## Some important results

The general equation of a second degree can be expressed as:

$$\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathsf{T}}\mathbf{x} + f = 0 \tag{0.3}$$

Given the point of contact $\mathbf{q}$, the equation of a tangent to the conic is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T\mathbf{q} + f = 0 \tag{0.4}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{0.5}$$

If $r$ is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{0.6}$$

$$\mathbf{c} = -\mathbf{u} \tag{0.7}$$

# Solution continued...

We can rewrite (0.1) as $\mathrm{x}^T\mathrm{x} = 5$, And comparing (0.1) with (0.3), we get

$$\mathrm{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathrm{q} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{0.8}$$

$$\mathrm{f} = -5 \tag{0.9}$$

using (0.6) and (0.7) we will get center of circle as $\mathrm{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $r = \sqrt{5}$.
And now using (0.4), we will get tangent at the point P(1,-2) as

$$\left(\mathrm{I}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)^T \mathrm{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \end{pmatrix} + -5 = 0 \tag{0.10}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \mathrm{x} = 5 \tag{0.11}$$

# Solution continued...

The equation of the tangent line is

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{0.12}$$

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{0.13}$$

Comparing with (0.2) with (0.3)

$$\mathbf{u} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, f = 20 \tag{0.14}$$

If n is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T \mathbf{x} = c \tag{0.15}$$

# Solution continued...

Comparing (0.12) with (0.15)

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{0.16}$$

The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (0.3) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{0.17}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{0.18}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{0.19}$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \tag{0.20}$$

# Solution continued...

Solving for the point of contact using the above equations we get,

$$\kappa = \pm \sqrt{\frac{\begin{pmatrix} -4 & 3 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} - 20}{\begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}}} \tag{0.21}$$

$$= \pm \sqrt{\frac{25 - 20}{5}} \tag{0.22}$$

$$= \pm \sqrt{1} \tag{0.23}$$

$$q = -\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{0.24}$$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{0.25}$$

# Solution continued...

If the line in (0.13) touches (0.3) at exactly one point q, then

$$\mathrm{m}^T(\mathrm{Vq}+\mathrm{u})=0 \tag{0.26}$$

It can be seen that for the tangent line,

$$\mathrm{m}=\begin{pmatrix}2\\1\end{pmatrix} \tag{0.27}$$

Solving (0.26) for given line and circle, we get

$$=\begin{pmatrix}2 & 1\end{pmatrix}\left(\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}-4\\3\end{pmatrix}\right) \tag{0.28}$$

$$=\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} \tag{0.29}$$

$$=0 \tag{0.30}$$

# Conclusion

And the co-ordinates of point of contact is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Hence, it is proved that the tangent to the circle $\|x\|^2 = 5$ at the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ also touches the circle $x^T x - \begin{pmatrix} -8 & 6 \end{pmatrix} x + 20 = 0$ at the point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

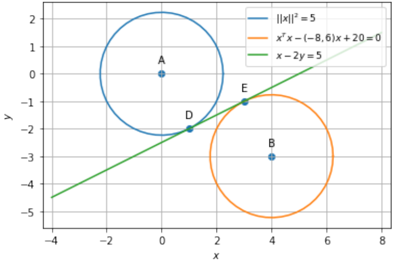


Figure: Graphical illustration

# THANK YOU