

EE3900 Assignment - 3

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Download latex-tikz codes from

<https://github.com/adhvik24/EE3900/blob/main/Assignment3/Assignment3.tex>

Download python codes from

<https://github.com/adhvik24/EE3900/blob/main/Assignment3/Assignment3.py>

1 RAMSEY 4.2 QN 15

Prove that the tangent to the circle

$$\|\mathbf{x}\|^2 = 5 \quad (1.0.1)$$

at the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ also touches the circle

$$\mathbf{x}^T \mathbf{x} + (-8 \ 6) \mathbf{x} + 20 = 0 \quad (1.0.2)$$

and find the coordinates of the point of contact.

2 SOLUTION

The general equation of a circle can be expressed as:

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

If r is radius and \mathbf{c} is the centre of the circle we have:

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (2.0.2)$$

$$\mathbf{c} = -\mathbf{u} \quad (2.0.3)$$

Given the point of contact \mathbf{q} , the equation of a tangent to the circle is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \quad (2.0.4)$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \quad (2.0.5)$$

We can rewrite (1.0.1) as $\mathbf{x}^T \mathbf{x} = 5$, And comparing (1.0.1) with (2.0.1), we get

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (2.0.6)$$

$$\mathbf{f} = -5 \quad (2.0.7)$$

using (2.0.2) and (2.0.3) we will get center of circle as $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $r = \sqrt{5}$.

And now using (2.0.4), we will get tangent at the point $P \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as

$$\left(\mathbf{I} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)^T \mathbf{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \end{pmatrix} + -5 = 0 \quad (2.0.8)$$

$$\implies (1 \ -2) \mathbf{x} = 5 \quad (2.0.9)$$

The equation of the tangent line is

$$(1 \ -2) \mathbf{x} = 5 \quad (2.0.10)$$

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (2.0.11)$$

Comparing with (1.0.2) with (2.0.1)

$$\mathbf{u} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, f = 20 \quad (2.0.12)$$

If \mathbf{n} is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T \mathbf{x} = c \quad (2.0.13)$$

Comparing (2.0.10) with (2.0.13)

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (2.0.14)$$

The point of contact \mathbf{q} , of a line with a normal vector \mathbf{n} to the conic in (2.0.1) is given by:

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \quad (2.0.15)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (2.0.16)$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \quad (2.0.17)$$

$$\mathbf{I}\mathbf{x} = \mathbf{x} \quad (2.0.18)$$

Solving for the point of contact using the above

equations we get,

$$\kappa = \pm \sqrt{\frac{(-4 \ 3) \begin{pmatrix} -4 \\ 3 \end{pmatrix} - 20}{(1 \ -2) \begin{pmatrix} 1 \\ -2 \end{pmatrix}}} \quad (2.0.19)$$

$$= \pm \sqrt{\frac{25 - 20}{5}} \quad (2.0.20)$$

$$= \pm \sqrt{1} \quad (2.0.21)$$

$$q = -\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (2.0.22)$$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad (2.0.23)$$

If the line in (2.0.11) touches (2.0.1) at exactly one point \mathbf{q} , then

$$\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \quad (2.0.24)$$

It can be seen that for the given line,

$$\mathbf{m} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.0.25)$$

Solving (2.0.24) for given line and circle, we get

$$= (2 \ 1) \left(\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \end{pmatrix} \right) \quad (2.0.26)$$

$$= (2 \ 1) \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.0.27)$$

$$= 0 \quad (2.0.28)$$

And the co-ordinates of point of contact is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Hence, it is proved that the tangent to the circle $\|\mathbf{x}\|^2 = 5$ at the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ also touches the circle

$\mathbf{x}^T \mathbf{x} - (-8 \ 6) \mathbf{x} + 20 = 0$ at the point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

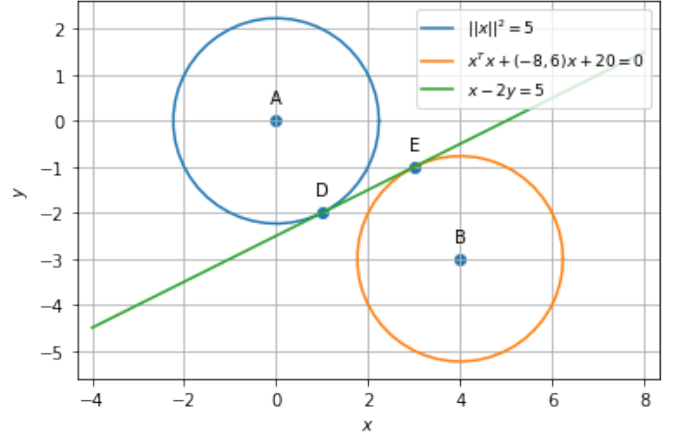


Fig. 1: Graphical illustration