EE3900 Assignment - 5

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Download latex-tikz codes from

https://github.com/adhvik24/EE3900/blob/main/Assignment_5/Assignment5.tex

Download python codes from

https://github.com/adhvik24/EE3900/blob/main/Assignment_5/codes/a_5.py

1 Problem

($QuadraticForms/Q2.16$) Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$ and verify the relationship between the zeroes and the coefficients.

2 Solution

Lemma 2.1. *A general polynomial equation $p(x, y)$ of degree 2 is given by :*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2.0.1}$$

The vector equation of $p(x, y)$ is given by :

$$\mathbf{x^T}\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}\mathbf{x} + \begin{pmatrix} D & E \end{pmatrix}\mathbf{x} + F = y \tag{2.0.2}$$

And for a quadratic polynomial we have :

$$B = 0 \tag{2.0.3}$$

$$C = 0 \tag{2.0.4}$$

$$E = 0 \tag{2.0.5}$$

If we take $A = 1$, we have :

$$\textit{Sum of zeroes} = -D \tag{2.0.6}$$

$$\textit{Product of zeroes} = F \tag{2.0.7}$$

Let $y = x^2 + 7x + 10 = 0$.
Then, y can be represented in the vector form as:

$$y = \mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} \frac{7}{2} & 0 \end{pmatrix}\mathbf{x} + 10 = 0 \tag{2.0.8}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{2.0.9}$$

Subsituting $y = 0$, we get:

$$\begin{pmatrix} x \\ 0 \end{pmatrix}^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} + 2\begin{pmatrix} \frac{7}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} + 10 = 0 \tag{2.0.10}$$

$$\implies x^2 + 7x + 10 = 0 \tag{2.0.11}$$

$$\implies \left(x + \frac{7}{2}\right)^2 = \frac{9}{4} \tag{2.0.12}$$

$$\implies \left(x + \frac{7}{2}\right) = \pm\frac{3}{2} \tag{2.0.13}$$

$$\implies x = -2, -5 \tag{2.0.14}$$

Clearly, for two values of x would we get the value of this expression to be 0, and hence, this equation has two real roots.
The roots can be verified using the python code. As we can see from the graph, $x^2 + 7x + 10$ intersect the x-axis at $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$.

Affine Transformation Consider $y = x^2 + 7x + 10$, which can be written in the vector form as:

$$\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} \frac{7}{2} & -\frac{1}{2} \end{pmatrix}\mathbf{x} + 10 = 0 \tag{2.0.15}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{7}{2} \\ \frac{-1}{2} \end{pmatrix}, f = 10 \tag{2.0.16}$$

For obtaining the affine transformation, we use:

$$\mathbf{x} = \mathbf{Py} + \mathbf{c} \tag{2.0.17}$$

The corresponding eigenvalues of $\mathbf{V}$ are:

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.18}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.19}$$

The corresponding eigenvectors are:

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.21}$$

Since $|\mathbf{V}| = 0$,

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{2.0.22}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{2.0.23}$$

$$\implies \eta = \frac{-1}{2} \tag{2.0.24}$$

$$\implies \begin{pmatrix} \frac{7}{2} & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ \frac{-7}{2} \\ 0 \end{pmatrix} \tag{2.0.25}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-7}{2} \\ \frac{-9}{4} \end{pmatrix} \tag{2.0.26}$$

Verifying the relationship between the zeroes and coefficients. By comparing (2.0.8) with (2.0.2),

$$\implies \text{sum of the zeroes} = -7 = -D \tag{2.0.27}$$

$$\text{product of zeroes} = 10 = F. \tag{2.0.28}$$

$\therefore$ The zeroes of equation $x^2 + 7x + 10$ are -2, -5.

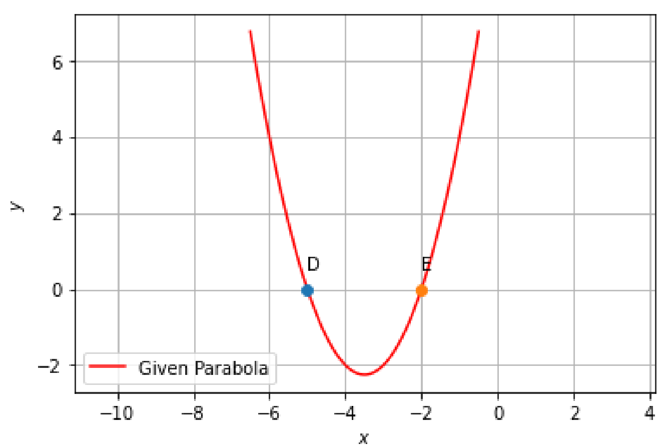

Fig. 1: Quadratic polynomial $x^2 + 7x + 10$