EE3900 Gate Assignment - 2

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https://github.com/adhvik24/EE3900/blob/main/
    Gate_A2/main.tex
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1 EC 2007/Q.48

A Hilbert transformer is a

1) non-linear system
2) non-casual system
3) time-varying system
4) low-pass system

2 SOLUTION

Definition 1. *The Hilbert transform $\mathcal{H}(x(t))$ of a signal x(t) is defined as*

$$\hat{x}(t) = \mathcal{H}(x(t)) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k)}{t-k}dk \tag{2.0.1}$$

Definition 2. *We say that a system is* ***linear*** *if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.*

Definition 3. *A system is said to be* ***casual*** *if and only if its output is independent of the future value of the input. So, A casual system output depends only on the past and present values of th system. That implies,* ***For a non casual system the impulse response h(t) should be non zero i.e.,*** $h(t) \neq 0$ ***for*** $t < 0$.

Definition 4. *A system is said to be* ***time invariant*** *if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.*

Definition 5. *A system is said to be* ***low-pass*** *if it filter the signals with a frequency lower than a selected cutoff frequency and attenuates signals with frequencies higher than the cutoff frequency.*

Lemma 2.1. *A Hilbert transformer is a non-causal, linear, time-invariant and non low-pass filter.*

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k)}{t-k}dk \tag{2.0.2}$$

Proof. 1) **Linearity**

From (2), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$\hat{x}_1(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_1(k)}{t-k}dk \tag{2.0.3}$$

$$\hat{x}_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_2(k)}{t-k}dk \tag{2.0.4}$$

$$\hat{x}_1(t) + \hat{x}_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_1(k)}{t-k}dk + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_1(k)}{t-k}dk$$

$$\hat{x}_1(t) + \hat{x}_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_1(k)+x_2(k)}{t-k}dk \tag{2.0.5}$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $\hat{x}(t)$:

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_1(k)+x_2(k)}{t-k}dk \tag{2.0.6}$$

Clearly, from (2.0.5) and (2.0.6):

$$\hat{x}(t) = \hat{x}_1(t) + \hat{x}_2(t) \tag{2.0.7}$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $cx(t)$, where k is any constant. Let the corresponding output be given by $\hat{x}(t)$, then:

$$\hat{x}_1(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{cx(k)}{t-k}dk \tag{2.0.8}$$

$$= c\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k)}{t-k}dk \tag{2.0.9}$$

$$= c\hat{x}(t) \tag{2.0.10}$$

Clearly, from (2.0.10),

$$\hat{x}_1(t) = c\hat{x}(t) \tag{2.0.11}$$

Thus, the Law of Homogeneity holds.
Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

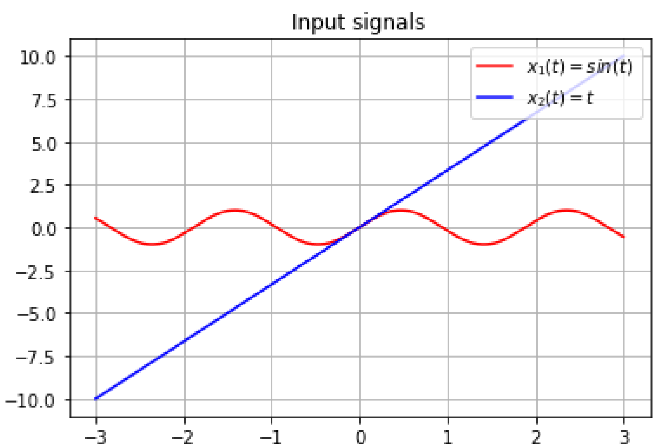

Fig. 1: $x_1(t)$ and $x_2(t)$

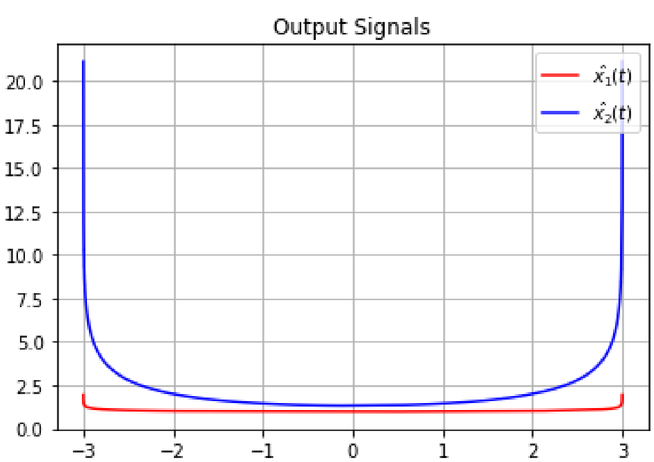

Fig. 2: $\hat{x}_1(t)$ and $\hat{x}_2(t)$

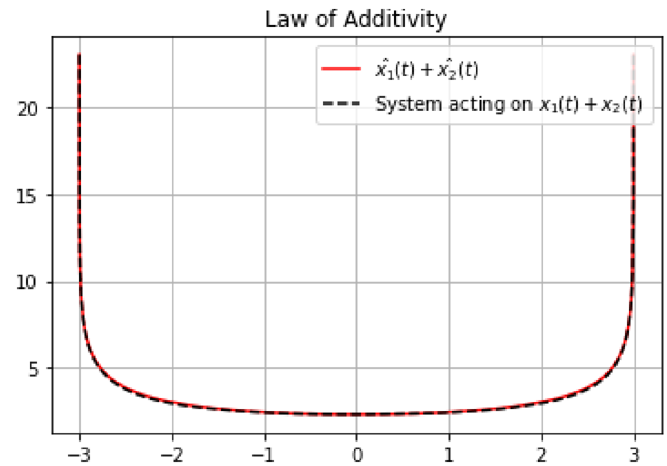

Fig. 3: Law of Additivity

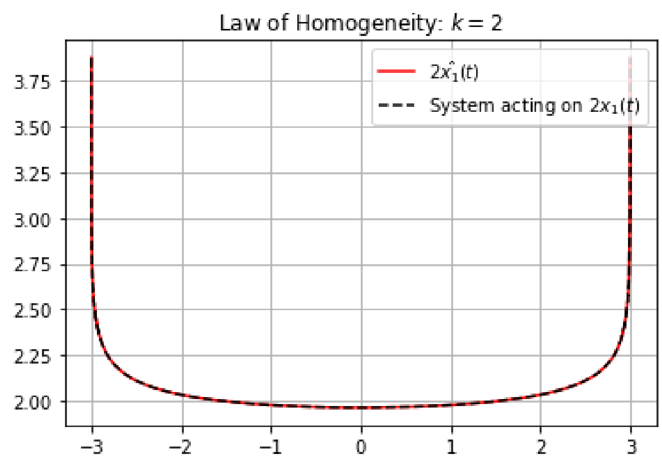

Fig. 4: Law of Homogeneity

2) **Time invariance**
From (4) , to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.
Now, we consider an input signal with a delay of t_0, given by $x(t-t_0)$, and let the corresponding output signal be given by $\hat{x}(t-t_0)$, then:

$$\hat{x}(t-t_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k)}{t-t_0-k}dk \tag{2.0.12}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(u-t_0)}{t-t_0-(u-t_0)}du \tag{2.0.13}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(u-t_0)}{t-u}du \tag{2.0.14}$$

We can rewrite $\hat{x}(t)$,

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k)}{t-k}dk$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k-t_0)}{t-k}dk \tag{2.0.15}$$

Using (2.0.15) in (2.0.14),

$$\hat{x}(t-t_0) = \hat{x}(t) \tag{2.0.16}$$

Thus, the system is **time-invariant**.
$\therefore$ The hilbert transformer is a **linear time invariant system(LTI)**.

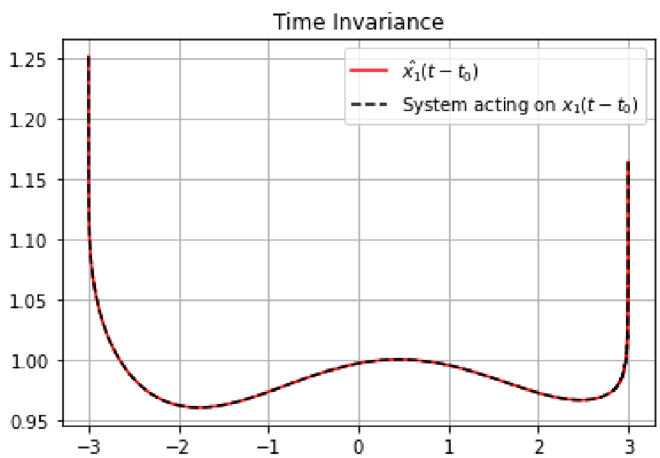

Fig. 5: Time Invariant

3) **Impulse response and non casual system**
Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\delta(k)}{t-k}dk \tag{2.0.17}$$

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases}\infty & t = 0\\ 0 & otherwise\end{cases} \quad and \int_{-\infty}^{\infty}\delta(t)dt = 1$$

$$\int_{-a}^{a} f(x)\delta(x-a)dx = f(a) \tag{2.0.18}$$

By using (2.0.18) in (2.0.17),

$$h(t) = \frac{1}{\pi t} \tag{2.0.19}$$

Here $h(t)$ is non zero for $t < 0$.
Using (3), That implies Hilbert transform is non casual system.

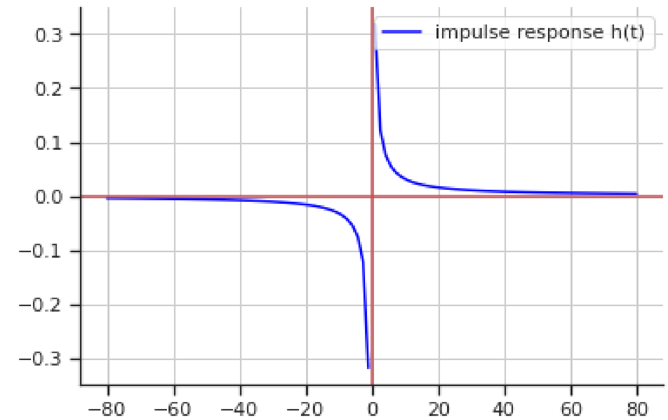

Fig. 6: Impulse response $h(t)$

$\therefore$ Hilbert transformer is a non casual system.

4) **Low-pass system**
Using (5) and by observing the Fig.8, We can observe that the output signal consists of signal component corresponds to all frequencies i.e., there is no such cutoff frequency observed.

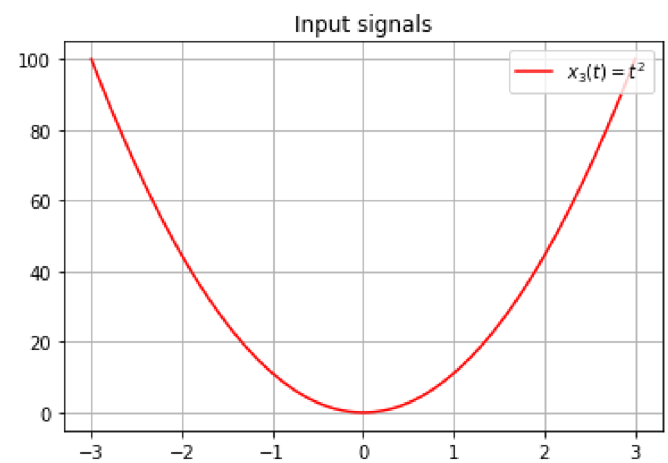

Fig. 7: Input signal $x_3(t) = t^2$

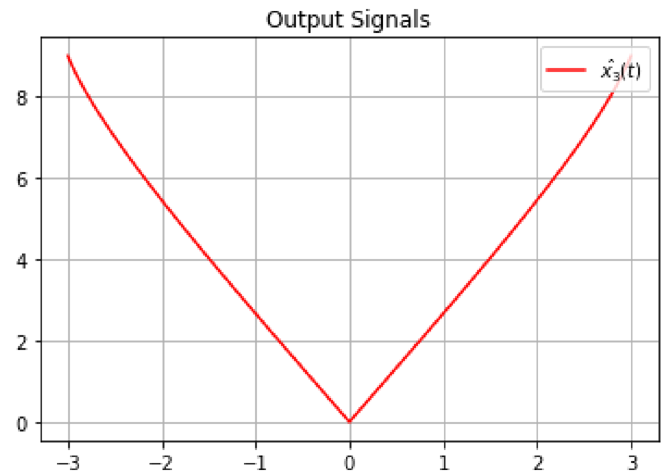

Fig. 8: Output signal $\hat{x}_3(t)$

$\therefore$ We can say that the hilbert transform is **not a low-pass system**.

$\therefore$ Correct option is **2) Non casual system**. $\square$