EE3900 Assignment - 3

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Download latex-tikz codes from

https://github.com/adhvik24/EE3900/blob/main/Assignment3/Assignment3.tex

Download python codes from

https://github.com/adhvik24/EE3900/blob/main/Assignment3/Assignment3.py

1 Ramsey 4.2 qn 15

Prove that the tangent to the circle

$$\|\mathbf{x}\|^2 = 5 \tag{1.0.1}$$

at the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ also touches the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -8 & 6 \end{pmatrix}\mathbf{x} + 20 = 0 \tag{1.0.2}$$

and find the coordinates of the point of contact.

2 Solution

The general equation of a circle can be expressed as:

$$\mathbf{x^T x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.1}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.2}$$

$$\mathbf{c} = -\mathbf{u} \tag{2.0.3}$$

If $\mathbf{P}$ be a point on the line and $\mathbf{n}$ is the normal vector, the equation of the line can be expressed as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{2.0.4}$$

$$\implies \mathbf{n}^T\mathbf{x} = c \tag{2.0.5}$$

Where,

$$c = \mathbf{n}^T\mathbf{P} \tag{2.0.6}$$

We can rewrite (1.0.1) as $\mathbf{x}^T\mathbf{x} = 5$, And comparing (1.0.1) with (2.0.1), we get

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{f} = -5 \tag{2.0.8}$$

using (2.0.2) and (2.0.3) we will get center of circle as $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $r = \sqrt{5}$.

And now using (2.0.4), we will get tangent at the point P(1,-2) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.9}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = 0 \tag{2.0.10}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix}\mathbf{x} = 5 \tag{2.0.11}$$

The equation of the tangent line is

$$\begin{pmatrix} 1 & -2 \end{pmatrix}\mathbf{x} = 5 \tag{2.0.12}$$

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.13}$$

Comparing with (1.0.2) with (2.0.1)

$$\mathbf{u} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, f = 20 \tag{2.0.14}$$

If $\mathbf{n}$ is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.15}$$

Comparing (2.0.12) with (2.0.15)

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.16}$$

The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (2.0.1) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{2.0.17}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.18}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.19}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{2.0.20}$$
$$\mathbf{I}\mathbf{X} = \mathbf{X} \tag{2.0.21}$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix}-4 & 3\end{pmatrix}\begin{pmatrix}-4\\3\end{pmatrix} - 20}{\begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}}} \tag{2.0.22}$$
$$= \pm\sqrt{\frac{25-20}{5}} \tag{2.0.23}$$
$$= \pm\sqrt{1} \tag{2.0.24}$$
$$q = -\begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}-4\\3\end{pmatrix} \tag{2.0.25}$$
$$= \begin{pmatrix}3\\-1\end{pmatrix} \tag{2.0.26}$$

If the line in (2.0.13) touches (2.0.1) at exactly one point $\mathbf{q}$, then

$$\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{2.0.27}$$

It can be seen that for the given line,

$$\mathbf{m} = \begin{pmatrix}2\\1\end{pmatrix} \tag{2.0.28}$$

Solving (2.0.27) for given line and circle, we get

$$= \begin{pmatrix}2 & 1\end{pmatrix}\left(\begin{pmatrix}3\\-1\end{pmatrix} + \begin{pmatrix}-4\\3\end{pmatrix}\right) \tag{2.0.29}$$
$$= \begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} \tag{2.0.30}$$
$$= 0 \tag{2.0.31}$$

And the co-ordinates of point of contact is $\begin{pmatrix}3\\-1\end{pmatrix}$. Hence, it is proved that the tangent to the circle $\|\mathbf{x}\|^2 = 5$ at the point $\begin{pmatrix}1\\-2\end{pmatrix}$ also touches the circle $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}-8 & 6\end{pmatrix}\mathbf{x} + 20 = 0$ at the point $\begin{pmatrix}3\\-1\end{pmatrix}$.

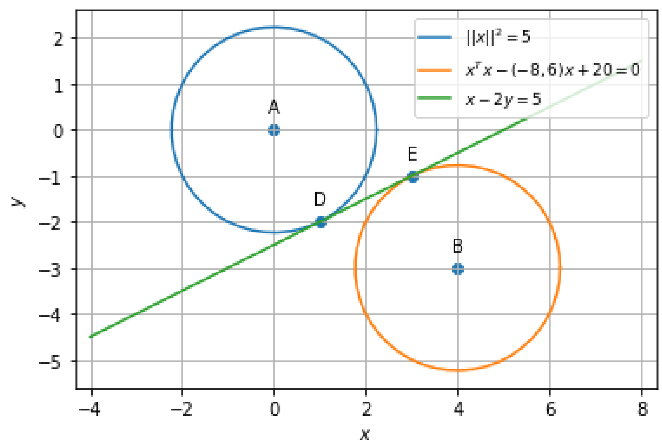

Fig. 1: Graphical illustration