Quiz2

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https://github.com/adhvik24/EE3900/blob/main/quiz2/main.tex

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Problem 3.7 (a)

3.7 (a) The input to a casual linear time-invariant system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n] \tag{0.0.1}$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)} \tag{0.0.2}$$

(a) Determine H(z), the z-transform of the system impulse response. Be sure to specify the region of convergence.
(b) What is the region of convergence for Y(z).
(c) Determine y[n].

Solution 3.7(a)

Definition 1. *We say that a system is* ***Causal*** *if the output of a system at a given time instance is independent of the future input values, i.e the output at a particular instance only depends on the present and past input values.*

Lemma 0.1. *A system is causal if and only if the transfer function $h[n]$ satisfies $h[n] = 0, n < 0$*

Proof. Let the input signal be given by $x[n]$ and the output signal be given by $y[n]$, then, we know in an LTI system:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{0.0.3}$$

Since, $y[n]$ is causal, it should be independent of future values of n.
If $k < 0$, then $n - k > n$, which is undesirable, and thus, to keep $y[n]$ independent of future values, $h[k] = 0, k < 0$ □

Lemma 0.2. *A system is said to be causal if and only if the ROC of the impulse function lies outside the outermost pole.*

Lemma 0.3. *If $x[n] = a^n u[n]$, where*

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & otherwise \end{cases} \tag{0.0.4}$$

then $x[n] \overset{\mathcal{Z}}{\rightleftharpoons} X[z] = \frac{1}{1-az^{-1}}$ with $ROC = |z| > a$

Proof. Using the formula for the sum of an infinite GP, we get:

$$x[n] = \begin{cases} a^n & n \geq 0 \\ 0 & otherwise \end{cases} \tag{0.0.5}$$

$$\mathcal{Z}\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{0.0.6}$$

$$= \sum_{n=-\infty}^{0} 0 \times z^{-n} + \sum_{n=0}^{\infty} (az^{-1})^n \tag{0.0.7}$$

$$= \frac{1}{1-az^{-1}}, ROC = |az^{-1}| < 1 \tag{0.0.8}$$

$$= \frac{1}{1-az^{-1}}, ROC = |z| > a \tag{0.0.9}$$

□

Lemma 0.4. *If $x[n] = -a^n u[-n-1]$, then $x[n] \overset{\mathcal{Z}}{\rightleftharpoons} X[z] = \frac{1}{1-az^{-1}}$ with $ROC = |z| < a$*

Proof. Using the formula for the sum of an infinite GP, we get:

$$x[n] = \begin{cases} -a^n & n \leq -1 \\ 0 & otherwise \end{cases} \tag{0.0.10}$$

$$\begin{aligned}
\mathcal{Z}\{x[n]\} = X[z] &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1]\}\, z^{-n} \\
&= \sum_{n=-\infty}^{-1} -a^n z^{-n} + \sum_{n=-1}^{\infty} 0 \times z^{-1} \\
&= -\sum_{n=1}^{\infty} (a^{-1}z)^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \\
&= 1 - \frac{1}{1-a^{-1}z}, ROC = |a^{-1}z| < 1 \\
X[z] &= \frac{1}{1-az^{-1}}, ROC = |z| < a \qquad (0.0.11)
\end{aligned}$$

□

We are given x[n] as,

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n] \qquad (0.0.12)$$

Then the z-transform of x[n] using (0.3) and (0.4) is,

$$X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \frac{1}{2} < |z| < 1 \qquad (0.0.13)$$

We know that H(z) is,

$$H(z) = \frac{Y(z)}{X(z)} \qquad (0.0.14)$$

Using Y(z) and (0.0.13) in (0.0.14), will result in H(z) as,

$$\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= \frac{\frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}}{\frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}} \qquad (0.0.15) \\
&= \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)} \cdot \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}{-\frac{1}{2}z^{-1}} \\
H(z) &= \frac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)} \\
H(z) &= \frac{2}{\left(1+z^{-1}\right)} - 1 \qquad (0.0.16)
\end{aligned}$$

From the above decomposition, we find that the poles of $H(z)$ are $z = -1$, and the zeroes are $z = 1$, as shown in the plot-zero diagram given below. Thus, we can also say the outermost pole is $z = -1$, and thus, from (0.2), As given the system is casual, that implies **ROC is $|z| > 1$.**

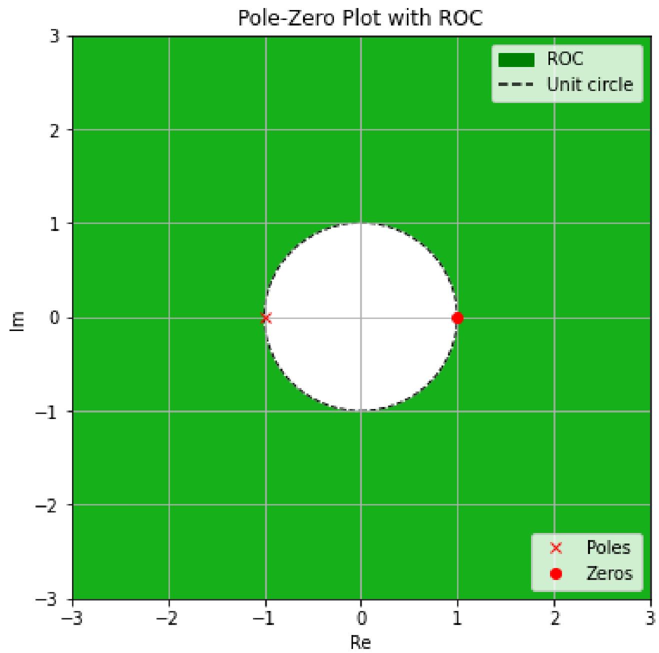

Fig. 1: Pole-Zero plot for $H(z)$

$\therefore$ Z transform of the system impulse response is $H(z) = \frac{(1-z^{-1})}{(1+z^{-1})}$ and the ROC is $|z| > 1$