EE3900 Gate Assignment - 2

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https://github.com/adhvik24/EE3900/blob/main/Gate_A2/main.tex

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1 EC 2007/Q.48

A Hilbert transformer is a

1) non-linear system
2) non-casual system
3) time-varying system
4) low-pass system

2 SOLUTION

Definition 1. *The Hilbert transform $\mathcal{H}(x(t))$ of a signal x(t) is defined as*

$$\hat{x}(t) = \mathcal{H}(x(t)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(k)}{t-k} dk \tag{2.0.1}$$

Definition 2. *We say that a system is* ***linear*** *if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.*

Definition 3. *A system is said to be* ***casual*** *if and only if its output is independent of the future value of the input. So, A casual system output depends only on the past and present values of th system. That implies,* ***For a non casual system the impulse response h(t) should be non zero i.e.,*** $h(t) \neq 0$ ***for*** $t < 0$.

Definition 4. *A system is said to be* ***time invariant*** *if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.*

Lemma 2.1. *A Hilbert transformer is a non-causal linear time-invariant filter.*

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(k)}{t-k} dk \tag{2.0.2}$$

Proof. 1) **Linearity**

From (2), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$\hat{x}_1(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_1(k)}{t-k} dk \tag{2.0.3}$$

$$\hat{x}_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_2(k)}{t-k} dk \tag{2.0.4}$$

$$\hat{x}_1(t) + \hat{x}_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_1(k)}{t-k} dk + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_1(k)}{t-k} dk$$

$$\hat{x}_1(t) + \hat{x}_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_1(k)+x_2(k)}{t-k} dk \tag{2.0.5}$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $\hat{x}(t)$:

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_1(k)+x_2(k)}{t-k} dk \tag{2.0.6}$$

Clearly, from (2.0.5) and (2.0.6):

$$\hat{x}(t) = \hat{x}_1(t) + \hat{x}_2(t) \tag{2.0.7}$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $cx(t)$, where k is any constant. Let the corresponding output be given by $\hat{x}(t)$, then:

$$\hat{x}_1(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{cx(k)}{t-k}dk \tag{2.0.8}$$

$$= c\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k)}{t-k}dk \tag{2.0.9}$$

$$= c\hat{x}(t) \tag{2.0.10}$$

Clearly, from (2.0.10),

$$\hat{x}_1(t) = c\hat{x}(t) \tag{2.0.11}$$

Thus, the Law of Homogeneity holds.
Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

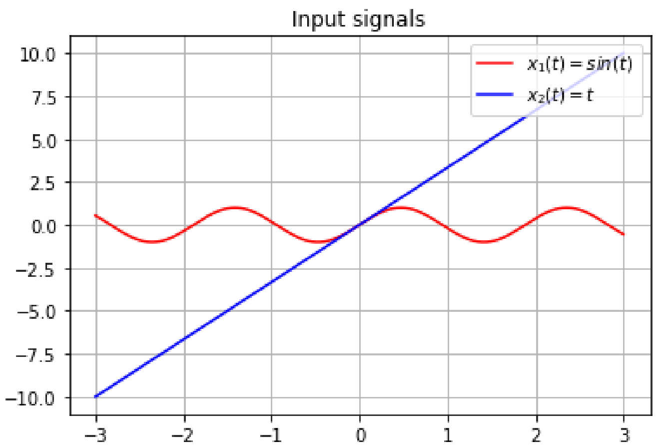

Fig. 1: $x_1(t)$ and $x_2(t)$

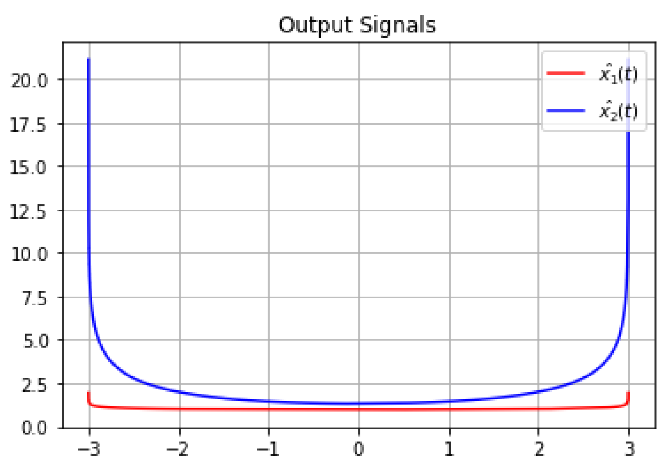

Fig. 2: $\hat{x}_1(t)$ and $\hat{x}_2(t)$

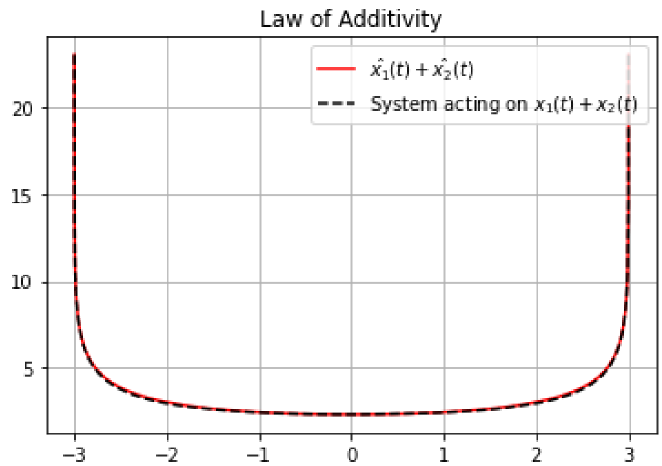

Fig. 3: Law of Additivity

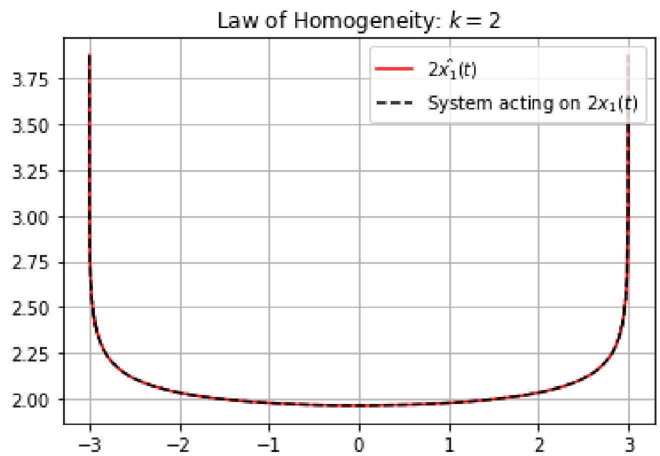

Fig. 4: Law of Homogeneity

2) <u>**Time invariance**</u>
From (4) , to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.
Now, we consider an input signal with a delay of t_0, given by $x(t-t_0)$, and let the corresponding output signal be given by $\hat{x}(t-t_0)$, then:

$$\hat{x}(t-t_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(k)}{t-t_0-k}dk \tag{2.0.12}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(u-t_0)}{t-t_0-(u-t_0)}du \tag{2.0.13}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(u-t_0)}{t-u}du \tag{2.0.14}$$

We can rewrite $\hat{x}(t)$,

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(k)}{t-k}dk$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(k-t_0)}{t-k}dk \tag{2.0.15}$$

Using (2.0.15) in (2.0.14),

$$\hat{x}(t-t_0) = \hat{x}(t) \tag{2.0.16}$$

Thus, the system is **time-invariant**.
$\therefore$ The hilbert transformer is a **linear time invariant system(LTI)**.

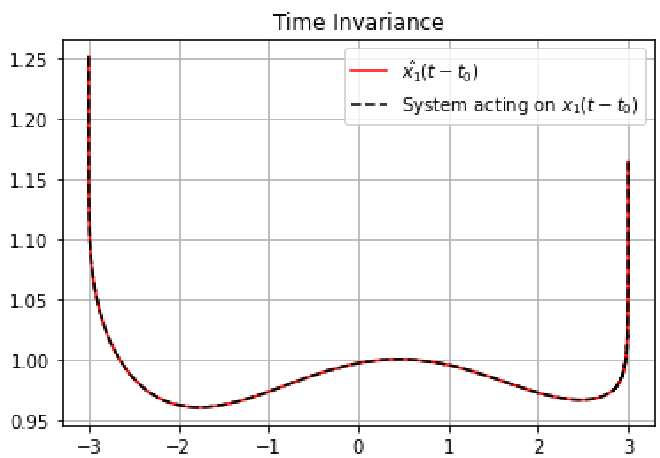

Fig. 5: Time Invariant

3) **Impulse response and non casual system**
Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\delta(k)}{t-k}dk \tag{2.0.17}$$

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} \quad and \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-a}^{a} f(x)\delta(x-a)dx = f(a) \tag{2.0.18}$$

By using (2.0.18) in (2.0.17),

$$h(t) = \frac{1}{\pi t} \tag{2.0.19}$$

Here $h(t)$ is non zero for $t < 0$.
Using (3), That implies Hilbert transform is non casual system.

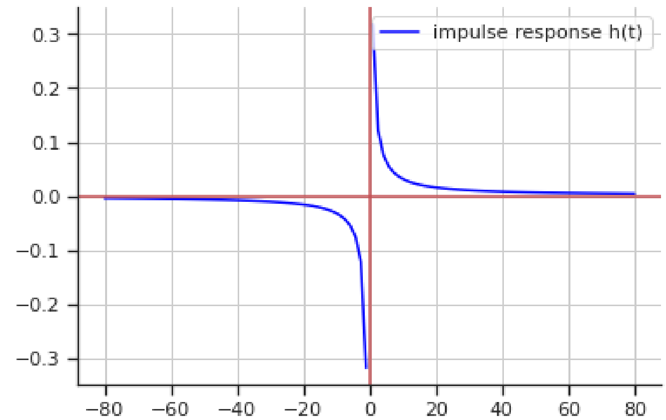

Fig. 6: Impulse response $h(t)$

$\therefore$ Hilbert transformer is a non casual system.
$\therefore$ Correct option is **2) Non casual system**. $\square$