MA 102 (Mathematics II)
IIT Guwahati

Tutorial Sheet No. 1 | Linear Algebra | March 20, 2023

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1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Prove or disprove the following statements.

   (a) The equality $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2)$ holds.

   (b) The equality $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$ holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   (c) There exist $\mathbf{u}$ and $\mathbf{v}$ such that $\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 2$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 3$.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$. What does this say about parallelogram in $\mathbb{R}^2$? Further, show that $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\|\,\|\mathbf{v}\|$ if and only if $\mathbf{u} = \alpha\mathbf{v}$ for some scalar α. Furthermore, show that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal $\iff$ $\|\mathbf{u}\| = \|\mathbf{v}\|$.

3. Express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, where

   (a) $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$;

   (b) $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

4. Check whether the following statements are True or False? Give proper justifications.

   (a) If $\hat{A}$ is the matrix obtained from A by replacing the ith column $\mathbf{a}_i$ of A by $2\mathbf{a}_i$ then the systems $\hat{A}\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ are equivalent.

   (b) If the rref of a 5×5 matrix A has the third column as $[1, 2, 0, 0, 0]^\top$ then $[-1, -2, 1, 0, 0]^T$ is a solution of $A\mathbf{x} = \mathbf{0}$.

   (c) For an $n \times n$ matrix A, the systems $A\mathbf{x} = \mathbf{0}$ and $A^\top\mathbf{x} = \mathbf{0}$ are equivalent.

5. The *trace* of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of its diagonal entries and is denoted by $\text{tr}(A)$, i.e. $\text{tr}(A) = a_{11} + \cdots + a_{nn}$.
   Prove the following: if A and B are $n \times n$ matrices and α is scalar, then

   1. $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$;
   2. $\text{tr}(\alpha A) = \alpha\text{tr}(A)$;
   3. $\text{tr}(AB) = \text{tr}(BA)$.

6. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two distinct solutions of the system $A\mathbf{x} = \mathbf{b}$. Prove that there are infinitely many solutions to this system. Interpret your findings geometrically.

7. Decide whether the following pairs are row-equivalent:

   (a) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 4 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \end{bmatrix}$

8. Find all the solutions of the linear system with the augmented matrix $[A \mid \mathbf{b}]$ as given below:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 2 \\ 5 & 6 & 7 & 8 & 5 \\ 9 & 10 & 11 & 12 & 8 \end{array}\right]$$

(a) Find $\hat{\mathbf{b}}$ such that $A\mathbf{x} = \hat{\mathbf{b}}$ does not have a solution.

(b) By changing exactly one entry of A, find an $\hat{A}$ such that $\hat{A}\mathbf{x} = \mathbf{b}$ will be consistent for all $\mathbf{b} \in \mathbb{R}^3$.

9. Determine the reduced row echelon form and the rank of the following matrices

$$(a)\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (b)\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad (c)\begin{bmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{bmatrix}$$

10. Find the coefficients a, b, c, d so that the graph of $y = ax^3 + bx^2 + cx + d$ passes through $(1, 2), (-1, 6)$, $(2, 3)$ and $(0, 1)$.

11. Show that matrices $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ are row equivalent.

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