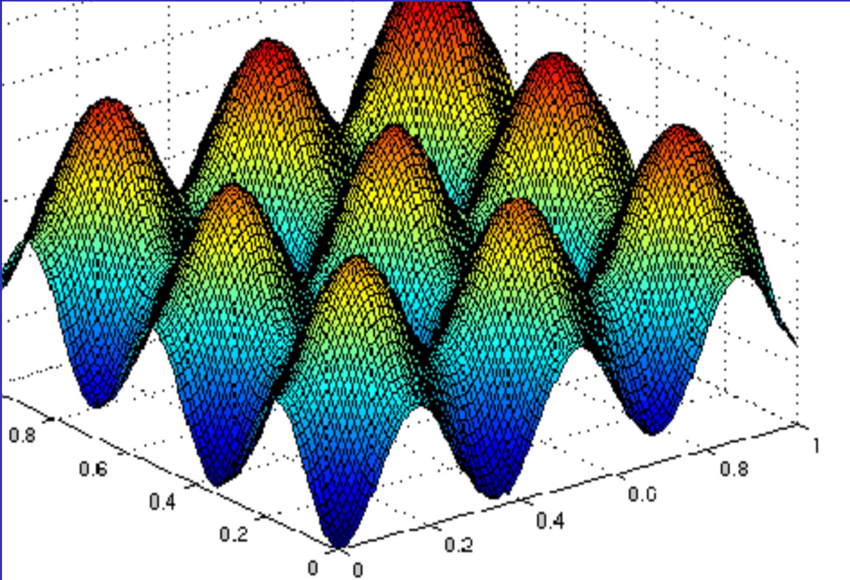
Greedy Algorithms



Greedy Algorithms

- Greedy algorithms are mainly designed to solve combinatorial optimization problems:

  Given an input, we want to compute an optimal solution according to some objective function.
- The solutions are formed by a sequence of elements.
- For example: Given a graph $G = (V, E)$ and two vertices $u, v \in V$, we want to find a path from u to v having the minimum number of edges.

  The solution is a sequence of vertices or edges.

Greedy Algorithms

A greedy algorithm obtains an optimal solution to a combinatorial optimization problem by making a sequence of choices (without backtracking).

- Greedy algorithms make locally optimal myopic choices to construct incrementally a global solution.
- In some cases this will lead to a globally optimal solution.
- Often easy greedy algorithms are used to obtain quickly solutions to optimization problems, even though they do not always yield optimal solutions.
- For many problems the greedy technique yields good heuristics, or even good approximation algorithms.

Greedy Algorithms

- At each step we choose the best (myopic) choice at the moment for the corresponding component of the solution, and then solve the subproblem that arise by taking this decision.
- The choice may depend on previous choices, but not on future choices.
- At each choice, the algorithm reduces the problem into a smaller one, and obtains one component of the solution.
- A greedy algorithm never backtracks.

Greedy Algorithms

For the greedy strategy to work correctly, it is necessary that the problem under consideration has two characteristics:

- Greedy choice property: We can arrive to the global optimum by selecting a local optimums.
- Optimal substructure: After making some local decision, it must be the case that there is an optimal solution to the problem that contains the partial solution constructed so far.

In many cases, the local criteria for selecting a part of the solution allow us to define a global order that directs the greedy algorithm.

The FRACTIONAL KNAPSACK problem

FRACTIONAL KNAPSACK: Given as input a set of n items, where item i has weight w_i and value v_i, together with a maximum total weight W permissible. We want to select a set of items or fractions of item, to maximize the profit, within allowed weight W.

Observe that from each item we can select any arbitrary fraction of its weight.

Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Fractional knapsack: Greedy schema

GreedyFKnapsack (n, v, w, W)
$O = \{1, \dots, n\}$; $S = \emptyset$; $Val = 0$; $i = 0$;
while $W > 0$ **do**
  Let $i \in O$ be the item with property P
  if $w[i] \leq W$ **then**
    $S = S \cup \{(i, 1)\}$; $W = W - w[i]$; $Val = Val + v[i]$;
  else
    $S = S \cup \{(i, W/w[i])\}$; $W = 0$;
    $Val = Val + v[i] * W/w[i]$;
  end if
  Remove i from O.
end while
return S

GreedyFKnapsack: most valuable object

Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
Selected	0	0	1	0.5	1

Total selected weight 100 and total value 146

Selecting the most valuable object is a correct greedy rule?

GreedyFKnapsack: the lighter object

Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
Selected	1	1	1	1	0

Total selected weight 100 and total value 156

Selecting the most valuable object does not provide a correct solution.

Selecting the lighter object is a correct greedy rule?

GreedyFKnapsack: the highest ratio value/weight

Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
ratio	2.0	1.5	2.2	1.0	1.2
Selected	1	1	1	0	0.8

Total selected weight 100 and total value 164

Selecting the lighter object does not provide a correct solution.

Highest ratio value/weight is a correct greedy rule?

GreedyFKnapsack: highest ratio value/weight

Theorem

The GreedyFKnapsack selecting the item with the best ratio value/weight always finds an optimal solution to the FRACTIONAL KNAPSACK *problem*

Proof.

Assume that the n items are sorted so that

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \cdots \geq \frac{v_n}{w_n}$$

GreedyFKnapsack: highest ratio value/weight

Let $X = (x_1, \dots, x_n)$, $x_i \in [0, 1]$, be the portions of items selected by the algorithm.

- If $x_i = 1$, for all i, the computed solution is optimal. We take all!
- Otherwise, let j be the smallest value for which $x_j < 1$.
- According with the algorithm,
  $x_i = 1$, for $i < j$, and
  $x_i = 0$, for $i > j$.
- Furthermore, $\sum_{i=1}^{n} x_i w_i = W$

GreedyFKnapsack: highest ratio value/weight

Let $Y = (y_1, \ldots, y_n)$, $y_i \in [0, 1]$, be the portions of items selected in a feasible solution, i.e.,

$$\sum_{i=1}^{n} y_i w_i \leq W$$

- We have, $\sum_{i=1}^{n} y_i w_i \leq W = \sum_{i=1}^{n} x_i w_i$
- So, $0 \leq \sum_{i=1}^{n} x_i w_i - \sum_{i=1}^{n} y_i w_i = \sum_{i=1}^{n} (x_i - y_i) w_i$
- Then, the value difference can be expressed as

$$v(X) - v(Y) = \sum_{i=1}^{n} x_i v_i - \sum_{i=1}^{n} y_i v_i = \sum_{i=1}^{n} (x_i - y_i) v_i$$

$$= \sum_{i=1}^{n} (x_i - y_i) w_i \frac{v_i}{w_i}$$

GreedyFKnapsack: highest ratio value/weight

We want to bound $v(x) - v(y) = \sum_{i=1}^{n}(x_i - y_i)w_i\frac{v_i}{w_i}$.

- If $i < j$, $x_i = 1$, so $x_i - y_i \geq 0$ but, as $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$,

$$(x_i - y_i)\frac{v_i}{w_i} \geq (x_i - y_i)\frac{v_j}{w_j}$$

- If $i > j$, $x_i = 0$, so $x_i - y_i \leq 0$ but, as $\frac{v_i}{w_i} \leq \frac{v_j}{w_j}$,

$$(x_i - y_i)\frac{v_i}{w_i} \geq (x_i - y_i)\frac{v_j}{w_j}$$

- The same inequality in both cases.

GreedyFKnapsack: highest ratio value/weight

- Using the derived inequalities, we have

$$v(x) - v(y) = \sum_{i=1}^{n} (x_i - y_i) w_i \frac{v_i}{w_i}$$

$$\geq \sum_{i=1}^{n} (x_i - y_i) w_i \frac{v_j}{w_j} \geq \frac{v_j}{w_j} \sum_{i=1}^{n} (x_i - y_i) w_i \geq 0$$

- So, $v(X) - v(Y) \geq 0$, and x is an optimal solution.

End Proof

GreedyFKnapsack: highest ratio value/weight

GreedyFKnapsack (n, v, w, W)
$O = \{1, \dots, n\}$; $S = \emptyset$; $Val = 0$; $i = 0$;
while $W > 0$ **do**
  Let $i \in O$ be an item with highest value/weight
  if $w[i] < W$ **then**
    $S = S \cup \{(i, 1)\}$; $W = W - w[i]$; $Val = Val + v[i]$;
  else
    $S = S \cup \{(i, W/w[i])\}$; $W = 0$;
    $V = Val + v[i] * W/w[i]$;
  end if
  Remove i from O
end while
return S

Cost? $O(n^2)$ a better implementation?

Fractional knapsack

GreedyFKnapsack (n, v, w, W)
Sort the items in decreasing value of v_i/w_i
$S = \emptyset$; $Val = 0$; $i = 0$;
while $W > 0$ and $i < n$ **do**
  if $w[i] < W$ **then**
    $S = S \cup \{(i, 1)\}$; $W = W - w[i]$; $Val = Val + v[i]$;
  else
    $S = S \cup \{(i, W/w[i])\}$; $W = 0$;
    $Val = Val + v[i] * W/w[i]$;
  end if
  $++i$;
end while
return S

This algorithm has cost of $T(n) = O(n \log n)$.

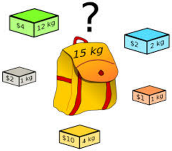
Fractional knapsack

Theorem

The Fractional Knapsack *problem can be solved in time* $O(n \log n)$.

0-1 Knapsack

0-1 Knapsack Given as input a set of n items, where item i has weight w_i and value v_i, together with a maximum total weight W permissible. We want to select a set of items to maximize the profit, within allowed weight W.

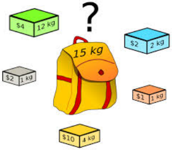


Items cannot be fractioned, you have to take all or nothing.

The greedy algorithm for the fractional version does not work for 0-1 KNAPSACK

Example: $n = 3$ and $W = 50$

Item	1	2	3
w	10	20	30
v	60	100	120
v/w	6	5	4



The algorithm will select item 1, with value 60. This is not an optimal solution, as 2 and 3 form a better solution, with value 220.

But, 0-1 KNAPSACK is known to be NP-hard.

Tasks or Activities Scheduling problems

General Setting:

- Given: A set of n tasks (with different characteristics) to be processed by a single/multiple processor system (according to different constrains).
- Provide a schedule, (when and where a (each) task must be executed), so as to optimize some objective criteria.

Some mono processor scheduling problems

1. INTERVAL SCHEDULING problem: Tasks have start and finish times. The objective is to make an executable selection with maximum size.
2. WEIGHTED INTERVAL SCHEDULING problem: Tasks have start and finish times and its execution produce profits. The objective is to make an executable selection giving maximum profit.
3. JOB SCHEDULING problem (Lateness minimization): Tasks have processing time (could start at any time) and a deadline, define the lateness of a task as the time from its deadline to its starting time. Find an executable schedule, including all the tasks, that minimizes the total lateness.

The INTERVAL SCHEDULING problem

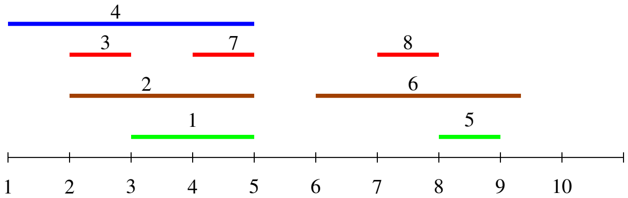
The INTERVAL SCHEDULING (aka ACTIVITY SELECTION problem)

- Given a set of n tasks where, for $i \in [n]$, task i has a start time s_i and a finish time f_i, with $s_i < f_i$.
- The processor is a single machine, that can process only one task at a time.
- A task must be processed completely from its starting time to its finish time.
- We want to find a set of mutually compatible tasks , where activities i and j are compatible if $[s_i f_i) \cap (s_j f_j] = \emptyset$, with maximum size.

A solution is a set of mutually compatible activities, and the objective function to maximize is the cardinality of the solution set.

Example: one input

Task :	1	2	3	4	5	6	7	8
Start (s):	3	2	2	1	8	6	4	7
Finish (f):	5	5	3	5	9	9	5	8



Designing a greedy algorithm

To apply the greedy technique to a problem, we must take into consideration the following,

- A local criteria to allow the selection,
- having in mind a property ensuring that a partial solution can be completed to an optimal solution.

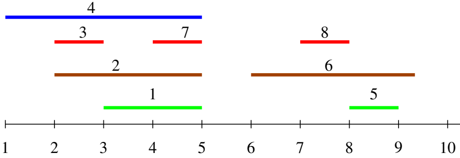
As for the FRACTIONALKNAPSACK problem, the selection criteria might lead to a sorting criteria. In such a case, greedy processes the input in this particular order.

The Interval Scheduling problem: Earlier finish time

IntervalScheduling(A)
$S = \emptyset$; $T = \{1, \ldots, n\}$;
while $T \neq \emptyset$ **do**
  Let i be the task that finishes earlier among those in T
  $S = S \cup \{i\}$;
  Remove from T, i and all tasks $j \in T$ with $s_j \leq t_i$
end while
return S.

task :	3	4	2	7	8	5	6
s :	3	1	2	4	8	5	6
f :	3	5	5	5	8	9	9

SOL: 3 1 8 5



IntervalScheduling: correctness

Theorem

The **IntervalScheduling** *algorithm produces an optimal solution to the* INTERVAL SCHEDULING*problem.*

Proof.

We want to prove that:

There is an optimal solution that includes the task with the earlier finishing time.

We will assume that this is not the case and reach contradiction.

IntervalScheduling: correctness

- Let i be a task that finishes at the earliest finish time.
- Let S be an optimal solution with $i \notin S$.
  Let $k \in S$ be the task with the earlier finish time among those in S.
- Any task in S finishes after time $A[k].f$, so they start also after $A[k].f$. As $A[i].f \leq A[k].f$, $S' = (S - \{k\}) \cup \{i\}$ is a set of mutually compatible tasks.
- As $|S'| = |S|$, S' is an optimal solution that includes i.

IntervalScheduling: correctness

Optimal substructure

After each greedy choice, we are left with an optimization subproblem, of the same form as the original. In the subproblem we removed the selected task and all tasks that overlap with the selected one.

An optimal solution to the original problem is formed by the selected task (one that finishes earliest possible) and an optimal solution to the corresponding subproblem.

End Proof

Interval Scheduling: cost

IntervalScheduling(A)
$S = \emptyset$; $T = [n]$; O(n)
while $T \neq \emptyset$ **do**
  Let i be the task that finishes earlier among those in T O(n)
  $S = S \cup \{i\}$;
  Remove i and all tasks overlapping i from T O(n)
end while
return S.

It takes $O(n^2)$ Too slow, a better implementation?

We have to find a fastest way to select i and discard i and the overlapping tasks.

The Interval Scheduling problem: algorithm 2

IntervalScheduling2(A)
Sort A in increasing order of $A.f$
$S = \{0\}$
$j = 0$ {pointer to last task in solution}
for $i = 1$ **to** $n - 1$ **do**
  if $A[i].s \geq A[j].f$ **then**
    $S = S \cup \{i\}$; $j = i$;
  end if
end for
return S.

IntervalScheduling2: correctness

Theorem

The **IntervalScheduling2** *algorithm produces an optimal solution to the* INTERVAL SCHEDULING *problem in time* $O(n \log n)$

Proof.

- A tasks that does not verify $A[i].s \geq A[j].f$ overlaps with task $j \in S$. It starts before j and finishes after j finishes. Therefore, it cannot be part of a solution together with j.
- As the tasks are sorted by finish time at each step, we select, among those tasks that start later than j, the one that finishes earlier.

IntervalScheduling2: correctness

- **IntervalScheduling2** makes the same greedy choice as **IntervalScheduling**, therefore it computes an optimal solution.
- The most costly step in **IntervalScheduling2** is the sorting, which can be done in $O(n \log n)$ time using Merge sort.

End Proof

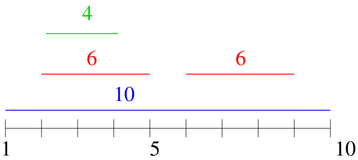
IntervalScheduling2: particular case

If we know that the tasks start and finish time are given in seconds within a day (24 hours),

IntervalScheduling2 can be implemented with cost $O(n)$

Adding weights: greedy choice does not always work.

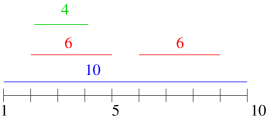
WEIGHTED ACTIVITY SELECTION problem:
Given a set of n activities to be processed by a single machine, where each activity i has a start time s_i and a finish time f_i, with $s_i < f_i$, and a weight w_i.
We want to find a set S of mutually compatible activities so that $\sum_{i \in S} w_i$ is maximum among all such sets.



IntervalScheduling2 selects the green and the second red activity with weight 10 which is not an optimal solution.

What about maximizing locally the selected weight?

WeightedAS-max-weight (A)
$S = \emptyset$; $T = [n]$;
while $T \neq \emptyset$ **do**
  Let i be the task with highest weight among those in T.
  $S = S \cup \{i\}$
  Remove i and all tasks overlapping i from T
end while
return S



The algorithm chooses the blue task with weight 10, and the optimal solution is formed by the two red intervals with total weight of 12.

Greedy approach

- Easy to come up with one or more greedy algorithms
- Easy to analyze the running time.
- Hard to establish correctness.
- Most greedy algorithms we came up are not correct on all inputs.

A Job Scheduling problem

LATENESS MINIMIZATION problem.

- We have a single processor and n tasks (or jobs) to be processed.
- Once a task starts to be processed it continues using the processor until its completion.
- Processing task i takes time t_i. Furthermore, task i has a deadline d_i.
- The goal is to schedule all the tasks, i.e., determine the time at which to start processing each tasks.
- We want to minimize, over all the tasks, the maximum amount of time that the finish time of a tasks exceeds its deadline.

Minimize Lateness: a more formal formulation

- We have a single processor
- We have n jobs such that job i:
  - requires $t_i > 0$ units of processing time,
  - it has to be finished by time d_i,
  - A schedule will determine a finish time f_i
- Under this schedule lateness of i is:

$$L_i = \begin{cases} 0 & \text{if } f_i \leq d_i, \\ f_i - d_i & \text{otherwise.} \end{cases}$$

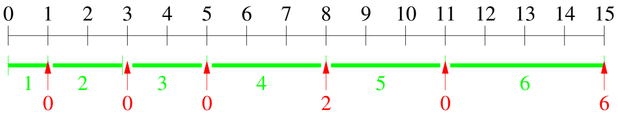
- The lateness of a valid schedule is $\max_i L_i$.

Goal: find a schedule with minimum lateness

Minimize Lateness: an example

We must assign starting time s_i to each i, making sure that the processor only processes a job at a time, in such a way that $\max_i L_i$ is minimum.

6 tasks: t: 1 2 2 3 3 4 d: 9 8 15 6 14 9



Minimize Lateness

We can try different task selection criteria to schedule the jobs following a generic greedy algorithm.

LatenessXX (A)
Sort A according to XX
$S[0] = 0;\ t = A[0].t;\ L = \max(0, t - A[0].d);$
for $i = 1$ **to** $n - 1$ **do**
  $S[i] = t$
  $t = t + A[i].t$
  $L = \max(L, \max(0, t - A[i].d))$
end for
return (S, L)

Minimize Lateness: selection criteria

Process jobs with short time first

i	t_i	d_i
1	1	6
2	5	5

1 at time 0 and 2 at time 1 lateness 1, but 2 at time 0 and 1 at time 5 has lateness 0. It does not work.

Process first jobs with smaller $d_i - t_i$ time

i	t_i	d_i	$d_1 - t_i$
1	1	2	1
2	10	10	0

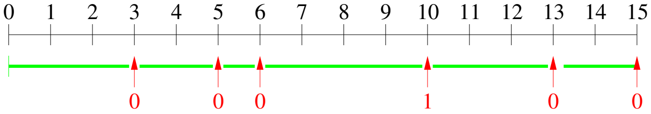
2 should start at time 0, that does not minimize lateness.

Process urgent jobs first

Sort in increasing order of d_i.

LatenessUrgent (A)
Sort A by increasing order of $A.d$
$S[0] = 0;\ t = A[0].t;$
$L = \max(0, t - A[0].d);$
for $i = 1$ **to** $n - 1$ **do**
  $S[i] = t$
  $t = t + A[i].t$
  $L = \max(L, \max(0, t - A[i].d))$
end for
return (S, L)

i	t_i	d_i	sorted i
1	1	9	3
2	2	8	2
3	2	15	6
4	3	6	1
5	3	14	5
6	4	9	4



Process urgent jobs first: Complexity

LatenessUrgent (A)
Sort A by increasing order of $A.d$
$S[0] = 0$; $t = A[0].t$; $L = \max(0, t - A[0].d)$;
for $i = 1$ **to** $n - 1$ **do**
    $S[i] = t$
    $t = t + A[i].t$
    $L = \max(L, \max(0, t - A[i].d))$
end for
return (S, L)

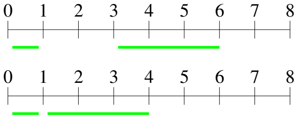
Time complexity
Running-time of the algorithm without sorting $O(n)$
Total running-time: $O(n \lg n)$

Process urgent jobs first: Correctness

Lemma

There is an optimal schedule minimizing lateness that does not have idle steps.

From a schedule with idle steps, we always can eliminate gaps to obtain another schedule with the same or better lateness:



LatenessUrgent has no idle steps.

Inversions and exchange argument

A schedule S has an inversion if $S(i) < S(j)$ and $d_j < d_i$.

Lemma

Exchanging two adjacent inverted jobs reduces the number of inversions by 1 and does not increase the max lateness.

Proof.

Assume that in schedule S, i is scheduled just before j and that they form an inversion.
Let S' be the schedule obtained from S interchanging i with j.

- $S[k] = S'[k]$ for $k \neq j$ and $k \neq j$.
- Thus, only i and j can change lateness.

Inversions and exchange argument: Cont.

- Let L_i, L_j and L'_i, L'_j be the lateness of jobs i and j in S and S', respectively.
- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S', respectively.
- We have $f_i < f_j$, $f'_j < f'_i$, $f'_i = f_j$, and $f'_j < f_j$. Also $d_j < d_i$,
- If $f_j < d_j$, $L_i = L_j = L'_i = L'_j = 0$

  Both schedules have the same latency.

Inversions and exchange argument: Cont.

- Let L_i, L_j and L'_i, L'_j be the lateness of jobs i and j in S and S', respectively.
- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S', respectively.
- We have $f_i < f_j$, $f'_j < f'_i$, $f'_i = f_j$, and $f'_j < f_j$. Also $d_j < d_i$,
- If $d_i < f_i$,

$$L'_i = f'_i - d_i = f_j - d_i < f_j - d_j = L_i$$
$$L'_j = f'_j - d_j < f_j - d_j = L_j$$

S' has the same or better latency than S.

Inversions and exchange argument: Cont.

- Let L_i, L_j and L'_i, L'_j be the lateness of jobs i and j in S and S', respectively.
- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S', respectively.
- We have $f_i < f_j$, $f'_j < f'_i$, $f'_i = f_j$, and $f'_j < f_j$. Also $d_j < d_i$,
- if $f_i \le d_i < d_j \le f_j$, $f'_j \le d_i < d_j \le f'_i = f_j$

$$L'_i = 0 \le L_j$$
$$L'_j = f'_j - d_j < f_j - d_j = L_j$$

S' has the same or better latency than S.

Inversions and exchange argument: Cont.

Therefore, in all the three cases, the swapping does not increase the maximum lateness of the schedule.

End Proof

Correctness of **LatenessUrgent**

Theorem

Algorithm **LatenessUrgent** *solves correctly the* LATENESS MINIMIZATION *problem. in* $O(n \log n)$ *time*

Proof.

According to the design, the schedule S produced by **LatenessUrgent** has no inversions and no idle steps.

Assume $\hat{S}$ is an optimal schedule. We can assume that it has no idle steps.

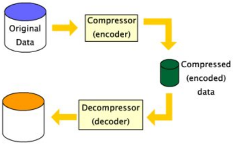
Correctness of **LatenessUrgent**

- If $\hat{S}$ has 0 inversions, S sorts jobs by deadlines and $\hat{S} = S$.
- Otherwise, $\hat{S}$ has an inversion on two adjacent jobs.
  Let i, j be an adjacent inversion.
  As we have seen, exchanging i and j does not increase lateness but it decreases the number of inversions.
  As $\hat{S}$ is optimal, the new schedule is also optimal but has one inversion less.
- Repeating, if needed the interchange of adjacent inversions, we will reach an optimal schedule with no inversions. Therefore, S is optimal.

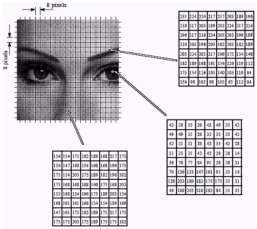
End Proof

Data Compression

Given as input a text $\mathcal{T}$ over a finite alphabet Σ. We want to represent $\mathcal{T}$ with as few bits as possible.

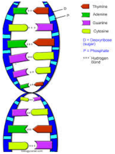


The goal of data compression is to reduce the time to transmit large files, and to reduce the space to store them.
If we are using variable-length encoding we need a system easy to encode and decode.



Example.

$$\underbrace{AAACAGTTGCAT \cdots GGTCCCTAGG}_{130.000.000}$$



- *Fixed-length encoding*: $A = 00$,$C = 01$, $G = 10$ and $T = 11$. Needs 260Mbites to store.
- *Variable-length encoding*: If A appears 7×10^8 times, C appears 3×10^6 times, G 2×10^8 and T 37×10^7, better to assign a shorter string to A and longer to C

Prefix codes

Given a set of symbols Σ, a prefix code, is $\phi : \Sigma \to \{0,1\}^+$ (symbols to chain of bits) where for distinct $x, y \in \Sigma$, $\phi(x)$ is not a prefix of $\phi(y)$.

- $\phi(A) = 1$ and $\phi(C) = 101$ then ϕ is not a prefix code.
- $\phi(A) = 1, \phi(T) = 01, \phi(G) = 000, \phi(C) = 001$ is a prefix code.
- Prefix codes easy to decode (left-to-right):

$$\underbrace{000}_{G}\underbrace{1}_{A}\underbrace{01}_{T}\underbrace{1}_{A}\underbrace{001}_{C}\underbrace{1}_{A}\underbrace{01}_{T}\underbrace{000}_{G}\underbrace{001}_{C}\underbrace{01}_{T}$$

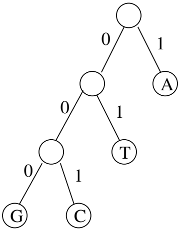
Prefix tree

We can identify an encoding with prefix property with a labeled binary tree.

A prefix tree T is a binary tree with the following properties:

- One leaf for symbol,
- Left edge labeled 0 and right edge labeled 1,
- Labels on the path from the root to a leaf specify the code for the symbol in that leaf.

Σ	code
A	1
T	01
G	000
C	001



The encoding length

- Given a text S on Σ, with $|S| = n$, and a prefix code ϕ, $B(S)$ is the length of the encoded text.
- For $x \in \Sigma$, define the frequency of x as

$$f(x) = \frac{\text{number occurrencies of } x \in S}{n}$$

Note: $\sum_{x \in \Sigma} f(x) = 1$.

- We get the formula,

$$B(S) = \sum_{x \in \Sigma} n\, f(x)|\phi(x)| = n \sum_{x \in \Sigma} f(x)|\phi(x)|.$$

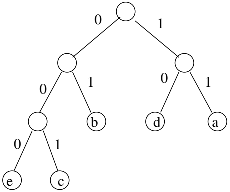
- $\alpha(S) = \sum_{x \in \Sigma} f(x)|\phi(x)|$ is the average number of bits per symbol or compression factor.

The encoding length

- In terms of the prefix tree of ϕ, the length of a codeword $|\phi(x)|$ is the depth of the leaf labeled x in T ($d_T(x)$).
- Thus, $\alpha(T) = \sum_{x \in \Sigma} f(x) d_T(x)$.

Fixed versus variable length codes: Example.

- Let $\Sigma = \{a, b, c, d, e\}$ and let S be a text over Σ with frequencies:
  $f(a) = .32, f(b) = .25, f(c) = .20, f(d) = .18, f(e) = .05$
- If we use a fixed length ϕ code, we need $\lceil \lg 5 \rceil = 3$ bits, we get compression 3.
- Consider the prefix-code ϕ_1:

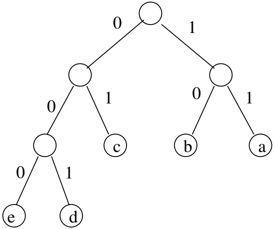


$\alpha = .32 \cdot 2 + .25 \cdot 2 + .20 \cdot 3 + .18 \cdot 2 + .05 \cdot 3 = 2.25$

- In average, ϕ_1 reduces the bits per symbol over the fixed-length code from 3 to 2.25, about 25%

Fixed versus variable length codes: Example.

Is 2.25 the maximum compression? Consider the prefix-code ϕ_2:



$$\alpha = .32 \cdot 2 + .25 \cdot 2 + .20 \cdot 2 + .18 \cdot 3 + .05 \cdot 3 = 2.23$$

is that the best? (the maximum compression using a prefix code)

Optimal prefix code.

Given a text, an optimal prefix code is a prefix code that minimizes the total number of bits needed to encode the text, i.e., α.

Intuitively, in the prefix tree of an optimal prefix code, symbols with high frequencies should have small depth ans symbols with low frequency should have large depth.

Before describing the algorithm we analyze some properties of optimal prefix trees.

A property of optimal prefix trees.

A binary tree T is full if every interior node has two sons.

Lemma

The prefix tree describing an optimal prefix code is full.

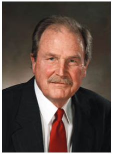
Proof.

- Let T be the prefix tree of an optimal code, and suppose it contains a u with a unique son v.
- If u is the root, construct T' by deleting u and using v as root. Otherwise, let w be the father of u. Construct T' by deleting u and connecting directly v to w.
- In both cases T' is a prefix tree and all the leaves in the subtree rooted at v reduce its height by 1 in T'.
- T' yields a code with less bits, so T is not optimal.

□

Greedy approach: Huffman code

Greedy approach due to David Huffman (1925-99) in 1952, while he was a PhD student at MIT



Wish to produce a labeled binary full tree, in which the leaves are as close to the root as possible. Moreover symbols with low frequency will be placed deeper than the symbol with high frequency.

Greedy approach: Huffman code

- Given the frequencies $f(x)$ for every $x \in \Sigma$
- The algorithm keeps a dynamic sorted list in a priority queue Q.
- Construct a tree in bottom-up fashion
  - Insert symbols as *leaves* with key f.
  - Extract the two first elements of Q and join them by a new *virtual node* with key the sum of the f's of its children. Insert the new node in Q.
- When Q has size 1, the resulting tree will be the prefix tree of an optimal prefix code.

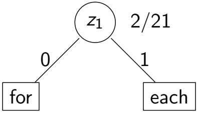
Huffman Coding: Construction of the tree.

Huffman Σ, S
Given Σ and S {compute the frequencies $\{f\}$}
Construct priority queue Q of leaves for Σ, ordered by increasing f
while $Q.size() > 1$ **do**
  create a new node z
  $x =$Extract-Min (Q)
  $y =$Extract-Min (Q)
  make x, y the sons of z
  $f(z) = f(x) + f(y)$
  Insert $(Q, z, f(z))$
end while
$\phi =$Extract-Min (Q)

If Q is implemented by a Heap, the algorithm takes time $O(n \lg n)$.

Example

Consider the text: *for each rose, a rose is a rose, the rose*
with $\Sigma = \{\text{for}/ \text{ each}/ \text{ rose}/ \text{ a}/ \text{ is}/ \text{ the}/ \text{ ,}/ \ \flat \ \}$
Frequencies:
$f(\text{for}) = 1/21$, $f(\text{rose}) = 4/21$, $f(\text{is}) = 1/21$,
$f(\text{a}) = 2/21$, $f(\text{each}) = 1/21$, $f(,) = 2/21$,
$f(\text{the}) = 1/21$, $f(\flat) = 9/21$.
Priority Queue:
Q=((for:1/21), (each:1/21), (is:1/21), (a:2/21), (,:2/21), (the:2/21), (rose:4/21), ($\flat$: 9/21))



Then, Q=((is:1/21), (a:2/21), (,:2/21), (the:2/21),(z_1:2/21), (rose:4/21), ($\flat$:9/21))

Example.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

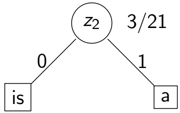
data compression

prefix codes

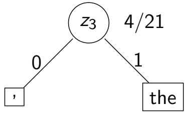
Huffman code

Approximation algorithms

Q=((is:1/21), (a:2/21), (,:2/21), (the:2/21),(z_1:2/21), (rose:4/21), ($\flat$:9/21))



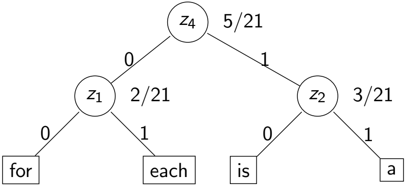
Then, Q=((,:2/21), (the:2/21),(z_1:3/21), (z_2:3/21),(rose:4/21), ($\flat$:9/21))



Then, Q=((z_1:2/21), (z_2:3/21),(rose:4/21), (z_3:4/21), ($\flat$:9/21))

Example.

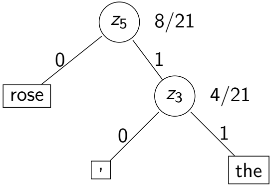
Q=((z_1:2/21), (z_2:3/21),(rose:4/21), (z_3:4/21), ($\flat$:9/21))



Then, Q=((rose:4/21), (z_3:4/21), (z_4:5/21), ($\flat$:9/21))

Example.

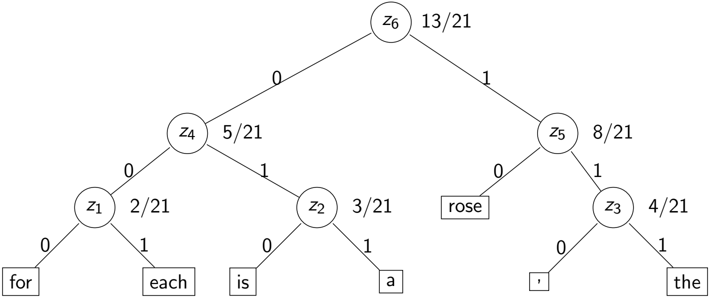
Q=((rose:4/21), (z_3:4/21), (z_4:5/21), ($\flat$:9/21))



Then, Q=((z_4:5/21), (z_5:8/21), ($\flat$:9/21))

Example.

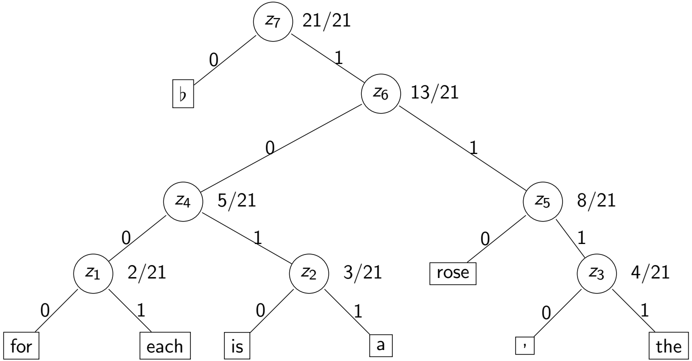
Q=((z_4:5/21), (z_5:8/21), ($\flat$:9/21))



Then, Q=(($\flat$:9/21),(z_6:13/21))

Example.

Q=(($\flat$:9/21),(z_6:13/21))



Then, Q=((z_7:21/21))

Example

- Therefore *for each rose, a rose is a rose, the rose* is Huffman coded as 10000100101101110010110110010100101101101110011110110
- The solution is not unique!
- The encoded length is 53, and compression is $53/21 = 2.523\ldots$.
- With a fixed size code, we need 4 bits per symbol, length 84 bits instead of 53.
- Why does the Huffman's algorithm produce an optimal prefix code?

Correctness

Theorem (Greedy property)

Let Σ be an alphabet, and let x, y be two symbols with the lowest frequency. There is an optimal prefix code ϕ in which $|\phi(x)| = |\phi(y)|$ and both codes differ only in the last bit.

Proof.

Assume that T is optimal but that x and y have not the same code length. In T there must be two symbols a and b siblings at max. depth. Assume $f(a) \leq f(b)$ and $f(x) \leq f(y)$, otherwise sort them accordingly.
We construct T' by exchanging x with a and y with b. As $f(x) \leq f(a)$ and $f(y) \leq f(b)$ then $B(T') \leq B(T)$. So T' is optimal and verifies the property. □

Correctness

Theorem (Optimal substructure)

Assume T' is an optimal prefix tree for $(\Sigma - \{x, y\}) \cup \{z\}$ where x, y are two symbols with the lowest frequencies, and z has frequency $f(x) + f(y)$. The T obtained from T' by making x and y children of z is an optimal prefix tree for Σ.

Proof.

Let T_0 be any prefix tree for Σ. We must show $B(T) \leq B(T_0)$.

By the previous result, we only need to consider T_0 where x and y are siblings, their parent has frequency $f(x) + f(y)$.

Correctness

- Let T_0' be obtained by removing x, y from T_0. As T_0' is a prefix tree for $(\Sigma - \{x, y\}) \cup \{z\}$, then $B(T_0') \geq B(T')$.
- Comparing T_0 with T_0' we get,

$$B(T_0) = B(T_0') + f(x) + f(y),$$
$$B(T) = B(T') + f(x) + f(y) = B(T).$$

- Putting together the three identities, we get $B(T) \leq B(T_0)$.

End Proof

More on Huffman codes

Huffman is optimal under assumptions:

- The compression is lossless, i.e. *uncompressing the compressed file yield the original file*.
- We must know the alphabet beforehand (characters, words, etc.),
- We must pre-compute the frequencies of symbols, i.e. read the data twice, which make it very slow for many real applications.
- A good source for extensions of Huffman encoding compression is the Wikipedia article on it: `https://en.wikipedia.org/wiki/Huffman_coding`.

Approximation algorithms

- Many times the Greedy strategy yields a feasible solution with value which is near to the optimum solution.
- In many practical cases, when finding the global optimum is hard, the greedy may yield a *good enough* feasible solution: An approximation to the optimal solution.
- An approximation algorithm for the problem always computes a close valid output. Heuristics also could yield good solutions, but they do not have a theoretical guarantee of closeness.
- Greedy is one of the algorithmic techniques used to design approximations algorithms.

Greedy and approximation algorithms

- For any optimization problem, let $c(*)$ be the value of the optimization function, let $\mathcal{A}px$ be an algorithm, that for each input x produces a valid solution $\mathcal{A}px(x)$ to x. Let $\text{opt}(x)$ be the cost of an optimal solution to x.
- We want to design a fast algorithm that produce solutions close to the optimal.
- For a NP-hard problem, we don't know if it has polynomial time algorithms, we want to design algorithms that are fast (polynomial) and that outputs good solutions always.

Approximation algorithm: Formal definition

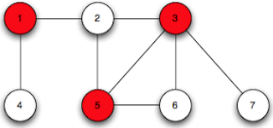
- For a given optimization problem, let $\mathcal{A}px$ be an algorithm, that for each input x produces a valid solution with cost $\mathcal{A}px(x)$ to x. Let $\mathrm{opt}(x)$ be the cost of an optimal solution to x.
- For $r > 1$, $\mathcal{A}px$ is an r-approximation algorithm if, for any input x:

$$\frac{1}{r} \leq \frac{\mathcal{A}px(x)}{\mathrm{opt}(x)} \leq r.$$

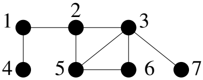
- r is called the approximation ratio.
- Given an optimization problem, for any input x, we require
  - in a MAX problem, $\mathcal{A}px(x) \leq \mathrm{opt}(x) \leq r\mathcal{A}px(x)$.
  - in a MIN problem, $\mathrm{opt}(x) \leq \mathcal{A}px(x) \leq r\mathrm{opt}(x)$.

An easy example: VERTEX COVER problem

Recall the problem of Vertex cover: Given a graph $G = (V, E)$ with $|V| = n, |E| = m$ find the minimum set of vertices $S \subseteq V$ such that it covers every edge of G.

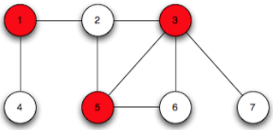


GreedyVC for I: $G = (V, E)$
$E' = E$, $S = \emptyset$,
while $E' \neq \emptyset$ **do**
  Pick $e \in E'$, say $e = (u, v)$
  $S = S \cup \{u, v\}$,
  $E' = E' - \{(u, v) \cup \{\text{edges incident to } u, v\}\}$
end while
return S.

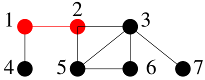


An easy example: VERTEX COVER problem

Given a graph $G = (V, E)$ with $|V| = n, |E| = m$ find the minimum set of vertices $S \subseteq V$ such that it covers every edge of G.

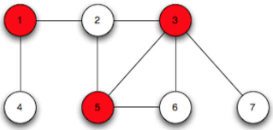


GreedyVC $G = (V, E)$
$E' = E$, $S = \emptyset$,
while $E' \neq \emptyset$ **do**
  Pick $e \in E'$, say $e = (u, v)$
  $S = S \cup \{u, v\}$,
  $E' = E' - \{(u, v) \cup \{\text{edges incident to } u, v\}\}$
end while
return S.

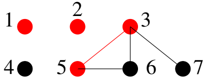


An easy example: Vertex cover problem

Given a graph $G = (V, E)$ with $|V| = n, |E| = m$ find the minimum set of vertices $S \subseteq V$ such that it covers every edge of G.

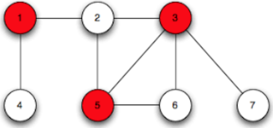


GreedyVC $G = (V, E)$
$E' = E$, $S = \emptyset$,
while $E' \neq \emptyset$ **do**
  Pick $e \in E'$, say $e = (u, v)$
  $S = S \cup \{u, v\}$,
  $E' = E' - \{(u, v) \cup \{\text{edges incident to } u, v\}\}$
end while
return S.

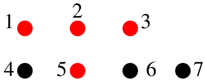


An easy example: VERTEX COVER problem

Given a graph $G = (V, E)$ with $|V| = n, |E| = m$ find the minimum set of vertices $S \subseteq V$ such that it covers every edge of G



GreedyVC $G = (V, E)$
$E' = E$, $S = \emptyset$,
while $E' \neq \emptyset$ **do**
  Pick $e \in E'$, say $e = (u, v)$
  $S = S \cup \{u, v\}$,
  $E' = E' - \{(u, v) \cup \{\text{edges incident to } u, v\}\}$
end while
return S.



An easy example: Vertex cover

Theorem

GreedyVC *runs in* $O(m+n)$ *steps. Moreover, if* S *is solution computed on input* G, $|S| \le 2opt(G)$.

Proof.

- The edges selected among by **GreedyVC** do not share any vertex.
- Therefore, an optimal solution must have at least one of the two endpoints of each edge while **GreedyVC** takes both.
- So, $|S| \le 2\text{opt}(G)$.

$\square$

An easy example: Vertex cover

The decision problem for Vertex Cover: given G and k, does G have a vertex cover with k or less vertices?, is NP-complete.

Moreover, unless P=NP, vertex cover can not be approximated within a factor $r \leq 1.36$