Existence + Uniqueness.

① **Proposition 1:** Given any ~~many~~ $m \times n$ matrix A, there exists an RREF matrix which is row equivalent to A

Proof → Each matrix obtained at a step of Gauss Jordan reduction process is row equivalent and similarly, the RREF is also row equivalent to A.

② **Proposition 2:** Row equivalence is an equivalence relation on the set of $R^{m \times n}$ of $m \times n$ matrices with entries from the field (set) R of real numbers.

Let R be the relation.

Proof → Reflexive

Let $A \in R^{m \times n}$.

For every row operation, there exists a reverse operation

i.e. $e_1(e(A)) = A$ and $(A, A) \in R$

∴ Row equivalence is ~~an equivalence~~ reflexive relation. ie ARA

Symmetric

Let $A, B \in R^{m \times n}$

∴ $e(A) = B$

For every row operation, there exists a reverse operation

$e_1(B) = A$ ie $ARB \Leftrightarrow BRA$

$\therefore (A,B) \in R$ and $(B,A) \in R$
$\therefore$ Row equivalence is a symmetric relation.

Transitive:
Let $(A,B) \in R$ and $(B,C) \in R$
where $\{A,B,C\} \in R^{m\times n}$

$e_1(A) = B$ and $e_2(B) = C$

$\therefore e_2(e_1(A)) = C \Rightarrow e_2 e_1(A) = C$

$\therefore (A,C) \in R \rightarrow$ Transitive and Equivalent

Remark: RREF is unique i.e. a matrix cannot be row equivalent to 2 distinct RREF's.
Alternatively, 2 distinct RREF matrices cannot be row equivalent to each other.

i.e. 2 matrices are row equivalent to each other if they have similar type of RREF.

★ Application to Determinants →
A = n×n matrix → Echelon
B = n×n matrix obtained from A without scaling

$$\boxed{\begin{aligned} \det(A) &= (-1)^k \det(B) \\ &= (-1)^k\, b_{11} b_{22} \cdots b_{nn} \end{aligned}}$$

Q If A is a square matrix, then A is row equivalent to identity matrix if and only $Ax = 0$ has only trivial solution.

Important

Proof → Identity matrix =
$$\begin{bmatrix} 1 & & & \vdots \\ & 1 & & \vdots \\ & & 1 & \vdots \\ \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

Due to the presence of no free variables, it must have a solution, and the augmented matrix gives
$x_1 = 0$
$x_2 = 0$
$\vdots$
$x_n = 0$

Q The system is consistent if and only the right most column is not a pivot column, i.e of the form $[0\ 0\ 0\ 0 \ldots b]$

Proof →

Given → RREF augmented matrix with rightmost column as pivot column and row of the form [0, 0, 0, ... 1]

RTP → The system is inconsistent

The given RREF has a row corresponding to [0 0 0 0 .... 1] which equals.

$$0x_1 + 0x_2 + \dots 0x_n = 1$$

Thus, this case is not possible as $0 \neq 1$
Hence, it has no solution

Q. A vector is a solution of the system $Ax = b$ if and only if of the form $u + v$ where v is the solution of associated homogeneous system.

Proof:
Given: A vector $\vec{w}$ is the solution of $A\vec{x} = \vec{b}$

$\vec{w}$ is of the form $\vec{u} + \vec{v_1}$, where is solution of $Ax = 0$

$\therefore \vec{v_1} = \vec{w} - \vec{u}$

If $V=0 \rightarrow$ there exists a unique solⁿ of $\vec{u}$.
Otherwise there are ∞ solutions

$A(\vec{w}-\vec{u})=0$
$A\vec{w}=A\vec{u}=\vec{b}$

- Given: $w=\vec{u}+\vec{v}$ where $\vec{v}$ is a solⁿ of $Ax=0$

RTP: $\vec{w}$ is solⁿ of $A\vec{x}=\vec{b}$

$A\vec{w}=A(\vec{u}+\vec{v})$
$=A\vec{u}+A\vec{v}$
$=\vec{b}+0=\vec{b}$

Hence, Proved.

★ If $\underset{\text{Coeff}}{A}$ is an $m\times n$ matrix with $m<n$, then the homogeneous system $Ax=0$, has a non-trivial solution

$\Rightarrow$ Due to the presence of atleast 1 free variable, there have to be infinite solutions

★ If $\underset{\text{Coeff.}}{A}$ is a Square matrix, then there is only 1 solution, i.e. trivial solution

$\Rightarrow$ Due to absence of any free variable.

(3.) **Proposition 3:** If A is a square matrix, then A is row equivalent to identity matrix if and only if homogeneous system $Ax = 0$ has only trivial solution.

Proof:
Given:- A is square matrix
RTP – A is row equivalent to identity matrix

⇒ Using proposition 1, we know that every matrix has a unique RREF which is row equivalent to it.

For a square matrix, the RREF is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & 1 & & & & \\ \vdots & & & 1 & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ & & & & & & 1 \end{bmatrix}_{n\times n}$$

due to absence of free variables and gives an identity matrix.

∴ The given system corresponds to

$$x_1 = 0$$
$$x_2 = 0$$
$$\vdots$$
$$x_n = 0$$

Hence, it has only a trivial solution.

If the system (homogeneous) is consistent and has a unique solⁿ, then it has the trivial solution; in this case it's either a $I_{n \times n}$ matrix or has I_n as its upper portion with only zero rows below it.

④ **Proposition 4 :** The system is consistent if and only if the rightmost column of R is not a pivot column

i.e. the augmented matrix has no row of the form $[0\,0\,0 \ldots\ldots b]$, $b \neq 0$

A system is consistent if it has either (i) a unique solution if no free variables or infinitely many solutions due to free variables.

★ The solution of homogeneous solution is a line (2-tuple) or a plane (3-tuple) passing through the origin.

★ If a non homogeneous system has even a single solution (point), then ~~there exists a~~ its entire solution set consists of only that point or a line or plane through that point which is parallel to the solution of associated homogeneous solution.

★ Invertible Matrices → Non Singular
Non-Invertible → Singular

★ Inverse of A is unique.
If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$

★ If A and B are invertible, so is AB and $(AB)^{-1} = B^{-1}A^{-1}$

If $C = A_1 A_2 A_3 \dots A_n$
$C^{-1} = (A_1 A_2 A_3 \dots A_n)^{-1} = A_n^{-1} \dots A_3^{-1} A_2^{-1} A_1^{-1}$

★ Elementary matrix is matrix obtained from the (square) identity matrix by an elementary row operation.

(5.) **Proposition 5** : If e is an elementary row operation and E is the $m \times n$ elementary matrix $e(I_m)$, then for every $m \times n$ matrix A, $e(A) = EA$

i.e. elementary row operation ~~to~~ is equivalent to left multiplication by

Proof:

(a.) Scaling

E is obtained by multiplying an I_m row by c.

$$E = \begin{bmatrix} e_1 \\ ce_i \\ \vdots \\ e_m \end{bmatrix} \Rightarrow EA = \begin{bmatrix} e_1 A \\ ce_i A \\ \vdots \\ e_m A \end{bmatrix} = \begin{bmatrix} a_1 \\ ca_i \\ \vdots \\ a_m \end{bmatrix}$$

where e_i denotes i^{th} row whose i^{th} position is 1 and other entries are zero. Thus EA is same as multiplying the i^{th} row of A by c

(b.) Interchange

$$E = \begin{bmatrix} e_1 \\ \vdots \\ e_j \\ e_i \\ \vdots \\ e_m \end{bmatrix} \Rightarrow EA = \begin{bmatrix} e_1 A \\ e_j A \\ e_i A \\ \vdots \\ e_m A \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ a_j \\ \vdots \\ a_m \end{bmatrix}$$

Thus, EA is same is same as exchanging the same as exchanging the i and j^{th} row of A. The converse is also true

(c) Row Replacement

$$E = \begin{bmatrix} \vdots \\ e_i \\ \vdots \\ ce_i + e_j \end{bmatrix} \Rightarrow EA = \begin{bmatrix} e_i A \\ ce_i A + e_j A \end{bmatrix} = \begin{bmatrix} \vdots \\ a_i \\ \vdots \\ ca_i + a_j \end{bmatrix}$$

Thus EA is same as adding c times of i^{th} row to j^{th} row of A

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6. **Proposition 6** → "Every elementary Matrix is Invertible"

Proof → Let E be an elementary matrix ~~As~~ and e be the corresponding E.R.O

Similarly there must exist another row operation which reverses e operation. Let it be f.

Let F be corresponding elementary matrix of f

So,

$$EF = (EF)I = E(FI) = e(f(I))$$
$$= I$$

Similarly $FE = (FE)I = F(EI) = f(e(I))$
$$= I$$

Remark: Inverse of elementary matrix is also elementary matrix (of same type).

★ **Very Important Theorem** ⇒

Theorem 1: The following are equivalent for an $m \times m$ square matrix A

(a.) A is invertible
(b.) A is row equivalent to identity matrix
(c.) Homogeneous system $Ax = 0$ has only trivial sol'n
(d.) The system $Ax = b$ has atleast 1 sol^n for every b in R^m

Proof : We will proceed as (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a)
(d) will done later

(a $\Rightarrow$ c)

Given $\rightarrow$ A = invertible
RTP $\rightarrow$ $A\vec{x} = 0$ has only trivial sol^n

Suppose $\vec{y}$ is sol^n of $A\vec{x} = \vec{0}$

Then $A\vec{y} = \vec{0}$

$A^{-1}A\vec{y} = A^{-1}\vec{0}$

$(A^{-1}A)\vec{y} = A^{-1}0$

$\boxed{\vec{y} = 0}$

(c $\Rightarrow$ b)

Given $\rightarrow$ $A\vec{x} = 0$ has only the trivial sol^n
RTP $\rightarrow$ A is row equivalent to I

Now if R is RREF of A,
$R\vec{x} = 0$ has only the trivial solⁿ.

$\Rightarrow$ R has no free variables and only basic variables
$\Rightarrow$ every column has leading 1
$\Rightarrow$ R is $m \times m$, so R has a 1 in every row
$\Rightarrow$ R is I_m.

(b) $\Rightarrow$ (a)

Given $\Rightarrow$ A is row equivalent to I
RTP $\Rightarrow$ A is invertible.

Now, A is row equivalent to I

Thus using reverse row operations, i.e $e_p, e_{p-1}, \ldots e_1$

$$e_p(e_{p-1}(\ldots\ldots(((e_1(A))))) = I$$ which correspond to F

We can write as

$$E_p(E_{p-1}\ldots\ldots E_2(E_1 A))\ldots) = I$$

Using Prop 5,
$\therefore (E_p \ldots E_1)A = I$

$$FA = I$$
$$(F^{-1}F)A = F^{-1} I$$
$$A = F^{-1}$$

Hence the inverse exists and its invertible.

(a) $\Rightarrow$ (d)

<u>Given</u> $\Rightarrow$ A is invertible
<u>RTP</u> $\rightarrow$ $A\vec{x} = \vec{u}$ has atleast 1 sol^n for every $\vec{u}$

Let $\vec{v}$ be any arbitary vector and since A is invertible

$$\vec{v} = A^{-1}\vec{u}.$$

Then $A\vec{v} = (AA^{-1})\vec{u} = I\vec{u} = \vec{u}$

In other words v is a sol^n of $A\vec{x} = \vec{u}$ for every $\vec{u}$

(d) $\Rightarrow$ (a)

<u>Given</u> $\rightarrow$ $A\vec{x} = \vec{u}$ has a solution for every $\vec{u}$
<u>RTP</u> $\rightarrow$ A is invertible

Corollary 1.4
we consider n vectors

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \ldots\ldots \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let $\vec{v}_i$ be solⁿ of $Ax = e_i$

$\therefore A\vec{v}_i = \vec{e}_i$

Let B be a matrix with $B = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_m]$

$$AB = [Av_1 \ Av_2 \ \dots \ A\vec{v}_m]$$
$$= [\vec{e}_1 \ \vec{e}_2 \ \dots \ \vec{e}_m] = I_m$$

$\therefore$ By Corollary 1.2, A has right inverse and is invertible

In order to calculate inverse of matrix, we use

<u>Corollary 1.1</u> → An invertible matrix is product of elementary matrix. Any sequence of ROs that reduces A to I also transforms I to A^{-1}

<u>Proof</u> → Using VIT (c)

$$I = e_p(e_{p-1} \dots ((e_1 A) \dots))$$

$$= (e_p e_{p-1} \dots e_1)A$$

$$I = (E_p E_{p-1} \dots E_1)A$$

$$A = (E_p \dots E_1)^{-1} I$$

$$A = E_1^{-1} \dots E_p^{-1}$$

$$\boxed{A^{-1} = E_p \dots E_1}$$

Hence, proved

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Corollary 1.2 → If A has a left inverse or a right inverse, then it has an inverse.

invertible Eg. $BA = I$
↳ Left Inverse

Proof →
Case I Suppose A has left inverse, i.e
$BA = I$
⟨Use KIT (c)⟩

Considering homogeneous solution,
$A\vec{x} = 0$ with $\vec{y}$ as solution

$A\vec{y} = 0$
$BA\vec{y} = B0$
$I.\vec{y} = 0 \Rightarrow \vec{y} = 0$
∴ A is invertible

$BA = I$
~~$(B^{-1}B)A = B^{-1}I$~~ $BAA^{-1} = IA^{-1}$
~~$A = B^{-1}$~~ $\boxed{B = A^{-1}}$
∴ Left inverse ~~of other~~ has to be inverse of other & vice versa.

Case 2 $AB = I$ → Right inverse
$A = B^{-1}$

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Corollary 1.3 : Suppose square matrix A is factored by as a product of square matrices i.e $A = A_1 A_2 \dots A_n$ $(n \geq 2)$. Then A is invertible if and only if each A_i is invertible

Proof by induction →

Step 1 $n = 2$

$A = A_1 A_2$

Let us first see if A_2 is invertible

$A_2 \vec{x} = 0$ → has a solution

$A_2 \vec{u} = 0$

$A_1 A_2 \vec{u} = A_1 0$

$A \vec{u} = 0$

∴ A is invertible with only trivial solution

⇒ A_2 is invertible

$A = A_1 A_2$

$A A_2^{-1} = A_1 A_2 A_2^{-1}$

$A A_2^{-1} = A_1$

∴ A is invertible as it's represented as product of 2 invertible matrices

Step 3 If invertible matrix can be expressed as (n-1) factors, then each factor is invertible

$$A = \underbrace{A_1 A_2 \dots A_{n-1}}, A_n$$

$$= B A_n$$

By base case, B and A_n are both invertible

So, $A_1, A_2, A_3 \dots A_n$ are all invertible.

★ Formula for inverse of 2X2 matrix

$A(\text{adj} A) = |A|$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $ad - bc = 0 \Rightarrow$ Non invertible.

- A vector space is a non empty set V of objects on which addition and multiplication are subject to following axioms.

1 → For $\vec{u}$ and $\vec{v}$ in V, u+v is in V
2 → $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
8 → $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
3 → $(\vec{u}+\vec{v})+w=\vec{u}+(\vec{v}+\vec{w})$
4 → There is 0 vector in V
5 → For $\vec{u}$, there is $-\vec{u}$ such that $\vec{u}+(-\vec{u})=0$
6 → $c\vec{u}$ is in V
7 → $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
9 → $d(c\vec{u})=(dc)\vec{u}$
10 → $1\vec{u}=\vec{u}$

Subspaces → It's a subset of V and follows the 3 axioms →

→ It includes 0 vector
→ $(\vec{u}+\vec{v})\in V$ → Closed under addition
→ $c\vec{u}\in V$ → Closed under multiplication

★ Every subspace is a vector space. Conversely, every vector space is a subspace of itself or other larger spaces.

★ Set consisting of only 0 vector is also always a subspace written as $V\{0\}$

Subspace spanned by a set

if v_1 and v_2 are in a vector space V, and $H = \text{span}(v_1, v_2)$; then H is subspace of V.

✓ as H contains 0 for $0v_1 + 0v_2$

✓ H gives $(\vec{u}) = s_1 v_1 + s_2 v_2 + (\vec{v}) = t_1 v_1 + t_2 v_2 \in H$

✓ H gives $c(\vec{u}) \in H$

★ span$\{v_1, v_2, v_3, \ldots v_p\}$ is always subspace of V containing $v_1, v_2, v_3 \ldots v_p$

called as subspace spanned

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Proposition 7 → Let V be a vector space. Then:

(a.) 0 vector is unique

(b.) Additive vector of any vector $\vec{u}$ is unique : i.e $-\vec{u}$

(c.) $0\vec{u} = 0$ for every vector $\vec{u}$

(d.) $c\vec{0} = \vec{0}$ for every scalar.

(e.) $-\vec{u} = (-1)\vec{u}$ for every vector $\vec{u}$

(f.) Cancellation law $\vec{u} + \vec{v} = \vec{u} + \vec{w}$, then $\vec{v} = \vec{w}$

Proof →

(a.) Suppose $\vec{0}$ and $\vec{0}'$ are both zero vectors in V.
Then $\vec{x} + \vec{0} = \vec{x}$ and $\vec{x} + \vec{0}' = \vec{x}$, for all $\vec{x}$ in V

Therefore, $0' = 0' + 0$ as 0 is zero vector
$= 0 + 0'$, by commutativity
$= 0$, as $0'$ is a zero vector

Hence, $0 = 0'$, showing that zero vector is unique

(b.) Suppose $\vec{x} + \vec{x}' = 0$ and $\vec{x} + \vec{x}'' = 0$
Then, $\vec{x}'' = \vec{x}'' + 0$
$= \vec{x}'' + (\vec{x} + \vec{x}')$ as x' is $-ve$ of x
$= (\vec{x}'' + \vec{x}) + \vec{x}'$ ~~as distributive law~~ associative law
$= 0 + \vec{x}'$ as x'' is $-ve$ of x
$= \vec{x}'$ as x'' is $-ve$ of x

$\therefore \vec{x}' = \vec{x}''$, there is only 1 additive vector.

Also, $0 = 0.\vec{x} = (1-1).\vec{x} = \vec{x} + (-\vec{x})$

$\Rightarrow -\vec{x}$ is additive inverse of $\vec{x}$ and unique.

(c.) $0\vec{u} = 0 + 0\vec{u}$

$= -\vec{u} + \vec{u} + 0\vec{u}$ (Additive ~~FA~~ vector)

$= -\vec{u} + (1+0)\vec{u}$

$= -\vec{u} + (1)\vec{u}$

$= -\vec{u} + \vec{u}$ (Additive vector)

$= 0$

Hence, proved

(d.) $c\vec{0} = c(0_1, 0_2, \ldots 0_n)$ Def of vector

$= (c.0_1, c.0_2, \ldots c0_n)$ Def of scalar multiplication

$= (0_1, 0_2, 0_3 \ldots 0_n)$

$= \vec{0}$

(e.) $0 = 0\vec{u}$

$= (1-1)\vec{u}$

$= 1\vec{u} + (-1)\vec{u}$

$= \vec{u} + (-1)\vec{u} \Rightarrow -\vec{u} = (-1)\vec{u}$

law

$\therefore (-1)\vec{u} = -\vec{u}$

(f) $\vec{u}+\vec{v} = \vec{u}+\vec{w}$

$\Rightarrow (\vec{u}+\vec{v})+(-\vec{u}) = (\vec{u}+\vec{w})+(-\vec{u})$ [Adding $-\vec{u}$ to both sides]

$\Rightarrow \cancel{\vec{u}+}(\vec{v}+(\vec{u}+(-\vec{u}))) = (\vec{w}+(\vec{u}+(-\vec{u}))$ [Associativity]

$\Rightarrow \vec{v}+\vec{0} = \vec{w}+\vec{0}$ (Additive Inverse)

$\Rightarrow \therefore \boxed{\vec{v}=\vec{w}}$

Hence, proved.

<u>Proposition 8</u> → A subset W of V is a subspace if and only if it satisfies the following 3 properties →

(i) The zero vector is in W

(ii) W is closed under addition i.e. $\vec{u}+\vec{w} \in W$ for all $\vec{u}$ and $\vec{v} \in$

(iii) W is closed under scalar multiplication i.e. for each $\vec{u}$ in W, and each scalar, $c\vec{u}$ is in W.

Proof

For a vector space, by definition it has to follow (i), (ii) and (iii) i.e 10 conditions in all.

⇒ Given W is subspace, it follows (i) (2 axioms), (ii) (4 axioms) and (iii) 4 axioms, Hence satisfied.

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$\Leftarrow$ Given W satisfying (i), (ii), (iii)
RTP → W is subspace
→ (i) proves that it follows axiom 1 and 7 (Closure under addition + multiplication)
→ (ii) (a) Associativity
(b.) Zero vector
(c.) Additive Inverse
sides] (d) Commutative

Proposition 9 → A non empty subset W of V is a subspace if and only if for each $\vec{u}$ and $\vec{v}$ in W and each scalar c, the sum $c\vec{u} + \vec{v} \in W$

Proof $\Rightarrow$

Proposition 10 → Let V be a vector space and let $v_1, v_2, \ldots, v_p$ be vectors in V. Then,
and $\vec{v} \in W$
span S = span $\{v_1, v_2, \ldots, v_p\}$ is a subspace of V

Proof → Let us 1st use prop 8 to test for subspace

→ Zero vector : $0 = 0v_1 + 0v_2 + \ldots - 0v_p$ is a linear combination of the v's

cons) → $w_1 = c_1v_1 + c_2v_2 + \ldots c_pv_p$ and $w_2 = d_1v_1 + d_2v_2 + \ldots d_pv_p$

$\Rightarrow w_1 + w_2 = (c_1+d_1)\vec{v_1} + (c_2+d_2)\vec{v_2} + \ldots (c_p+d_p)\vec{v_p}$
$\in$ ~~W~~ span S

→ If c is any scalar and w_1 is a linear combination,
Then, $cw_1 = c(c_1v_1 + c_2v_2 + \dots c_pv_p)$
$= cc_1v_1 + cc_2v_2 + \dots cc_pv_p$

Corollary 10.1

Let V be vector space,

(a.) If U and W are 2 subspaces of V, then $U \cap W$ is also subspace of V

(b.) If $S = \{v_1, v_2, \dots v_p\}$ is a set of vectors, then span S is smallest subspace which contain S i.e. $S \subseteq W$, then span $S \subseteq W$

⇒ In terms of this, span S is sometimes described as intersection of all subspaces of V containing S.

Proof →

(a.) Since U and W are subspaces $\{0\} \in U$ and $\{0\} \in$

For any arbitrary element 'a', if $\{a\} \in U \cap W$, then $a \in U$ and $a \in W$

So, for $a, b \in U$ and $a, b \in W$, $a + b \in U \cap W$ as $a + b \in U$ and $a + b \in W$.

Similarly ca ~~and cb~~ $\in U$ and W, ∴ $ca \in U \cap W$

∴ $U \cap W$ is also a subspace of V

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(b.) Let $A_1, A_2, A_3 \ldots\ldots A_\infty$ be all subspaces of V containing $\{v_1, v_2, \ldots\ldots v_p\}$.

Then, by 10.1 (A), $A_1 \cap A_2 \cap A_3 \ldots \cap A_\infty$ is also a subspace and must contain $\{v_1, v_2, \ldots v_p\}$.

By intersection we get the smallest case and it's represented by all linear combination of $v_1, v_2 \ldots v_p$.

$\therefore$ spans is smallest subspace which contains V.

<u>Proposition 11</u> → $B = \{v_1, v_2 \ldots\ldots v_n\}$ is a basis of vector space V <u>if and only if</u> every vector $v \in V$ is uniquely expressible as a linear combination of elements of B.

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Remarks → (1) Any list that contains $\vec{0}$ is linearly dependent

(2) A single non zero vector is L I

(3) 2 vectors are L.D if they are multiple of each other

(4) A list of vectors is L.D if and only if at least 1 of the vector is a linear combination of other.

(5.) Any list which contains a repeated vector is linearly dependent

(6.) Any list which contains a LD list has to be LD

(7.) Any subset of LI set is LI

(1) **Proof** → Assume V is a vector space.

Consider $\{\vec{0}, \vec{v_1}, v_2 \dots \vec{v_p}\}$, $\vec{v_i} \in V$

Considering

$$c_0 0 + c_1 \vec{v_1} + c_2 \vec{v_2} + \dots c_p \vec{v_p} = 0$$

If $c = 0$ and $c_1, c_2, \dots c_p \neq 0$, $\Rightarrow 0$

Assuming $c \neq 0$ and $c_1, c_2 \dots c_p = 0$

We get $\underbrace{c.0}_{0} + \underbrace{c_1 \vec{v_1} + c_2 \vec{v_2} + \dots c_p \vec{v_p}}_{0} = 0$

⇒ Linearly dependent.

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(2) Proof →

(3.) Proof → For 2 vectors $\vec{V_1}$ and $\vec{V_2}$

Considering,

$$c_1\vec{V_1} + c_2\vec{V_2} = 0$$

Given → They are L.D where $c_1 \neq 0$ or $c_2 \neq 0$

⇒ ⇒ If $c_1 \neq 0$,

$$c_1\vec{V_1} + c_2\vec{V_2} = 0 \Rightarrow \vec{V_1} = \frac{-c_2}{c_1}\vec{V_2}$$

⇒ A multiple of $\vec{V_1}$

⇒ If $c_2 \neq 0$

$$c_1\vec{V_1} + c_2\vec{V_2} = 0 \Rightarrow V_2 = \frac{-c_1}{c_2}\vec{V_1}$$

⇒ A multiple of $\vec{V_2}$

⇐ Given → They are multiples of each other

i.e. $V_1 = c V_2$

⇒ $1.V_1 - c V_2 = 0$

→ Linearly dependent

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4. **Proof** → S is L.D $\Leftrightarrow$ One of vector is Linear comb. of oth

$\Leftarrow$ If v_i is linear combination of others,

$$\underbrace{V_i}_{\text{~~Not zero~~}} = c_1 V_1 + c_2 V_2 + \dots c_{i-1} V_{i-1}, c_{i+1} V_{i+1} \dots c_k V_k$$

$$0 = c_1 v_1 + \dots (-1) v_i + \dots c_k v_k$$

$\neq 0 \Rightarrow \therefore$ L.D

$\Rightarrow$ S is L.D., $\therefore$ $0 = c_1 v_1 + c_2 v_2 + \dots c_k v_k$ (not all 0's)

$\Rightarrow -c_j v_j = c_1 v_1 + c_2 v_2 + \dots c_k v_k$.

non zero $\Rightarrow$ $\boxed{v_j = \frac{c_1 v_1}{-c_j} + \frac{c_2 v_2}{-c_j} + \dots \frac{c_k v_k}{-v_j}}$

5. **Proof** →

6. **Proof** →

Let $\{v_1, v_2, v_3 \dots v_k\}$ be linearly dependent

Consider the set $\{v_1, v_2 \dots v_k, v_{k+1}, \dots v_n\}$

We know that

$$c_1 v_1 + c_2 v_2 + \dots c_k V_k = 0$$

where all $c_1, c_2, \dots c_k$ are not zero

$\therefore \underbrace{c_1 v_1 + c_2 v_2 + \dots \dots c_k v_k}_{\text{not all } 0} + 0 v_{k+1} + \dots \dots 0 v_n = 0$

Hence it's linearly dependent.

7. Proof →

Let U_1 be a L.I set and let U_2 be subset of U_1

$\therefore \quad U_2 \subset U_1$

If U_2 is L.D then U_1 is L.D by remark 6.
Therefore by contradiction, U_2 Has to be linearly independent

---

• Proposition 12 →

(Steinitz Exchange Lemma)

Suppose $\vec{v}_1, \vec{v}_2, \dots v_n$ are L.I vectors in vector space V and suppose $V = \text{span}\{\vec{w}_1, \vec{w}_2 \dots \vec{w}_m\}$

Then:

(a.) $n \leq m$

(b.) $\{v_1, v_2, \dots v_n, w_{n+1} \dots, w_m\}$ span V, after reordering the w's if necessary

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Proof →

Since w's span V, we must have

$$\vec{v_1} = c_1\vec{w_1} + \dots + c_m \vec{w_m} \quad \text{— (1.)}$$

Now atleast one of the c is not 0, ∵ if all are 0, then $v_1 = 0$

Let us assume $c_1 \neq 0$

Rewrite (1) as,

$$c_1\vec{w_1} = \vec{v_1} - c_2\vec{w_2} - \dots - c_m\vec{w_m}$$

Multiply by c_1^{-1}, we get,

$$\vec{w_1} = d_1\vec{v_1} + d_2 w_2 + \dots d_m \vec{w_m} \quad \text{— (2.)}$$

∴ $\boxed{\text{span of } \vec{w_1}, \vec{w_2} \dots \vec{w_m} = \text{span } \vec{v_1}, \vec{w_2} \dots \vec{w_m}}$

To explain (*), suppose some

$$\vec{v} = \alpha_1\vec{w_1} + \dots \alpha_m\vec{w_m}$$

Substitute $\vec{w_1}$ from (2.)

$$\vec{v} = \alpha_1(d\vec{v_1} + d_2\vec{w_2} + \dots + d_m\vec{w_m}) + \alpha_2\vec{w_2} + \dots$$

*

Collecting coefficients, we get R.H.S terms of

$\vec{v_1}, \vec{w_2}, \vec{w_3} \ldots\ldots \vec{w_m}$

i.e. we have been able to replace $\vec{w_1}$ by $\vec{v_1}$

Doing same for $\vec{v_2}$

$$v_2 = e_1 \vec{v_1} + e_2 \vec{w_2} + \ldots\ldots e_m \vec{w_m}$$

$$\vec{w_2} = f_2 v_2 + f_1 v_1 + \ldots\ldots + f_m \vec{w_m}$$

$$\boxed{\text{Span of } \vec{w_1}, \vec{w_2} \ldots\ldots \vec{w_m} = \text{span of } \vec{v_1}, \vec{v_2}, \ldots\ldots \vec{w_m}}$$

The process can be repeated further till it stops.

The result obtained is summed as →

Case I $n \leq m$

$$\text{span}(\vec{w_1}, \vec{w_2}, \ldots\ldots \vec{w_m}) = \text{span}(\vec{v_1}, \vec{v_2} \ldots \vec{v_m} \ldots \vec{w_m})$$

Case II $n > m$

i.e. $\vec{v_1}, \vec{v_2}, \ldots\ldots \vec{v_m}, \vec{v_{m+1}} \ldots\ldots \vec{v_n}$

$$\begin{array}{ccccc} \vec{v_1} & \vec{v_2} & \ldots & \vec{v_m} \\ \downarrow & \downarrow & & \downarrow \\ \vec{w_1} & \vec{w_2} & & \vec{w_m} \end{array}$$

$\alpha_m \vec{w_m}$ (3.)

This implies that $\vec{v_{m+1}}, \ldots\ldots \vec{v_n}$ must be L.D and this is a contradiction

★ The 0 subspace has a dimension 0 but has no basis.

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Proposition 13 → If V is a finite dimensional vector space, then any 2 bases of V have the same no. of elements.
(Dimension = No. of elements in basis for V)

Proof →
Let B_1 and B_2 be 2 bases of V with k_1 and k_2 vecto

Using 12(a.) we get $k_1 \leq k_2$
because B_1 is L.I set and B_2 spans V.

Similarly,
$$k_2 \leq k_1$$
because B_2 is L I set and B_1 spans V.

$\therefore k_1 = k_2$
Hence, proved.

Proposition 14 → Suppose $S = \{v_1, v_2, \dots v_n\}$ is a linearly independent set in a vector space V. Suppose $\vec{v}$ is a vector not in span S, then set obtained by adjoining $\vec{v}$ to S is L I.

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Proof → Suppose $\vec{v}$ is not in span S, and consider any expression

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_n\vec{v}_n + c\vec{v} = 0$$

If $c \neq 0$,

$$c\vec{v} = -(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_n\vec{v}_n)$$

$$c^{-1}c\vec{v} = -c^{-1}(c_1\vec{v}_1 + \dots c_n\vec{v}_n)$$

$$\underbrace{\vec{v} = -c^{-1}c_1\vec{v}_1 + \dots -c_nc^{-1}\vec{v}_n}$$

This is a contradiction as c is not included in span S.

Hence, $c = 0$ and $S \cup \{v\} = L.I$

Proposition 15 → Any L.I. set S in a finite dimensional vector space can be expanded to the basis.

Let $B = \{v_1, v_2, \dots v_k\}$ be L.I set

If $k = n$

By Prop 17, it's a basis.

If $k < n$ Then B is not a basis as it's not the the ~~###~~maximal L.I set

Therefore, we can add some vector v_{k+1} to B such that it remains L.I by Steinitz exchange lemma

The process must terminate before we reach maximal set. Therefore any L I set can be extended

~~If $k > n$~~

This is not possible by Prop 17.

Proposition 16 → Any finite spanning set S in a non zero vector space can be contracted to a basis.

Proof:

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Proposition 17 → Let V be a non zero-finite dimensional vector space with dimension n. Then:

* Any linearly independent set of vectors have $\leq n$ vectors. If a L I set has n vectors, it's a basis.
* Any spanning set must have $\geq n$ vectors. If a spanning set has n vectors → It's a basis.

Proof → (a.) Given: $\dim(V) = n$

∴ Any basis has exactly n elements.

We know that the elements of a basis span V and by Steinitz exchange Lemma, any L I set has $\leq n$ elements.

• The $\dim(V) = n$ and thereby if B ~~that~~ is a L I set with n elements, it's a maximal L. I set or a minimal spanning set and thus a basis.

(b.) Given: $\dim(V) = n$

∴ Any basis has exactly n elements

We know that elements of basis are L. I and by Steinitz exchange lemma, any spanning set has $\geq n$ elements..

• The $\dim(V) = n$ and thereby if B is a spanning set with n elements, it's a ~~minimal~~ maximal L. I set or a minimal spanning set and thus a basis

Proposition 18 → If W is a proper subspace of a finite dimensional space V, then W is also finite dimensional and $0 < \dim W < \dim V$.

Proof →

Since W is a proper subspace, it contains a vector $w_1 \neq 0$. If w_1 spans W, then W is finite dimensional. If not there is a vector w_2 in W outside Span w_1 and by adjoining w_2 to w_1, we still have a L I (Prop 14) set.
Continuing this, we get a basis with atmost dim V elements (Prop 12 a)
Hence, W is finite dimensional and since W is a proper subspace, there is a vector v outside W. Adjoining v to any basis of W, we still have a L I set.
$\therefore \boxed{\dim W < \dim V}$

★ Sum of Subspaces →

Let U and W be subspaces of vector space V.
The Sum of U and W, $U+W = \{u+w : u \in U, w \in W\}$

We can see that $U+W$ is a subspace of V.
⇒ Infact $U+W$ is smallest subspace containing U and W

Proof →

Proposition 19 → If U and W are subspace of vector space V, then,

$$\dim(U+V) = \dim U + \dim W - \dim(U \cap W)$$

Proof →
For convenience, let us put $K = U \cap W$ and $Z = U+W$, so that K and Z are both subspaces of V.
If either U or $W = \{0\}$, the result is obvious

We will use Prop 15 to construct a basis for Z.

Let $B = \langle \vec{k_1}, \vec{k_2} \dots \vec{k_m} \rangle$ be a basis for K.
[It is finite dimensional by Prop 18]

Since $K \subseteq U$, we expand basis of B to a basis B_1 of U by adjoining the vectors $\vec{u_1}, \dots \vec{u_n}$

i.e $B_1 = \langle \vec{k_1}, \vec{k_2} \dots \vec{k_m}, \vec{u_1} \dots \vec{u_n} \rangle$

Similarly we expand B to get basis for W

i.e. $B_2 = \langle \vec{k_1}, \vec{k_2} \dots \vec{k_m}, \vec{w_1} \dots \vec{w_p} \rangle$

Put $C = B \cup B_1 \cup B_2 = B_1 \cup B_2 = \langle \vec{k_1} \dots \vec{k_m}, \vec{u_1} \dots \vec{u_n}, \vec{w_1} \dots \vec{w_p} \rangle$

We claim C is a basis of Z. To justify it, we need to prove that.

(i) Span $C = Z$
(ii) C is linearly independent

(i) Let $\vec{v} = \vec{u} + \vec{w}$

$\vec{w} \in W$

$\vec{u} \in U$

$\vec{u} = c_1 \vec{k_1} + \dots c_m \vec{k_m} + d_1 \vec{u_1} + \dots d_n \vec{u_n}$

$\vec{w} = f_1 \vec{k_1} + \dots f_m \vec{k_m} + g_1 \vec{w_1} + \dots g_p \vec{w_p}$

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so that $\vec{w} = (c_1 + f_1)\vec{k_1} + \dots (c_m + f_m)\vec{k_m} + d_1\vec{u_1} + \dots d_n\vec{u_n} + g_1\vec{w_1} + \dots g_p\vec{w_p}$
i.e. a linear independence of the elements of C

(ii) Suppose $c_1\vec{k_1} + \dots c_m\vec{k_m} + d_1\vec{u_1} + \dots d_n\vec{u_n} + g_1\vec{w_1} + \dots + \dots + g_p\vec{w_p} = 0$ — (1.)

i.e. $c_1\vec{k_1} + \dots c_m\vec{k_m} + d_1\vec{u_1} + \dots d_n\vec{u_n} = g_1\vec{w_1} + \dots - g_p\vec{w_p}$ — (2.)

Now L.H.S of (2) is a vector in U and R.H.S is a vector in W; hence a vector in $U \cap W = K$ i.e. we can write it as

$c_1\vec{k_1} + \dots c_m\vec{k_m} + d_1\vec{u_1} + \dots d_n\vec{u_n} = f_1\vec{k_1} + \dots f_m\vec{k_m}$ (3)

or

$h_1\vec{k_1} + \dots h_m\vec{k_m} + d_1\vec{u_1} + \dots d_n\vec{u_n} = \vec{0}$ — (4.)

But now since B_1 is a basis for U, hence, we get
$d_1 = d_2 = \dots = d_n = 0$ — (5)
$\therefore$ (1) becomes:
$c_1\vec{k_1} + \dots c_m\vec{k_m} + g_1\vec{w_1} + \dots g_p\vec{w_p} = \vec{0}$ — (6)

But then again, since B_2 is a basis for W and hence l.i, we get $c_1 = c_2 = \dots c_m = g_1 = \dots g_p = 0$

So, we have proved (i) and (ii) and so C is indeed a basis for $Z = U+W$.

$$\dim(U+W) = \dim(Z) = m+n+p \quad \text{--- (7)}$$

OTOH, $\dim U + \dim W - \dim(U \cap W)$
$= (m+n) + (m+p) - m$
$= m+n+p.$

$$\therefore \dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

Proposition 20 → If U and W are subspaces of vector space V, then $V = U \oplus W$ if and only if $V = U+W$ and $U \cap W = \{0\}$. → direct sum

★ Direct Sum:
V is said to be the direct sum of the subspaces U and W if every vector $v \in V$ is uniquely expressible in the form,
$$v = \vec{u} + \vec{w}$$

Proof:
iff (if and only if) direct sum $\Leftrightarrow$ $U \cap W = \{0\}$

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$\Rightarrow$ Given:

$V = U \oplus W$

To prove:

$U \cap W = \{0\}$

Proof →

Assume $v \in U \cap W$

$\vec{v} = \vec{v} + \vec{0} = \vec{0} + \vec{v}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$v \in U \quad 0 \in W \quad 0 \in U \quad v \in W$

Since it's a ~~two~~ direct sum, it can only be expressed ~~as~~ in a unique way.

Sum

$\Rightarrow \therefore$ since it's a unique, $v = 0$

$\therefore U \cap W = \{0\}$

$\Leftarrow$ Given:

$U \cap W = \{0\}$

To prove:

$U + W$ ~~is~~ a direct sum

Proof →

Taking $V = U + W$

Let $\vec{v} = \vec{u}_1 + \vec{w}_1 = \vec{u}_2 + \vec{w}_2$

To show that it's a direct sum, $u_1 = u_2$ and $w_1 = w_2$

$\Rightarrow u_1 - u_2 = w_2 - w_1$

$\Rightarrow \therefore \vec{u}_1 - \vec{u}_2 \in U$ and $\vec{w}_2 - \vec{w}_1 \in W$

Since $U \cap W = \{0\}$

and $\vec{u}_1 - \vec{u}_2 = \vec{w}_2 - \vec{w}_1$

$\therefore \vec{u}_1 - \vec{u}_2 = \vec{w}_2 - \vec{w}_1 = 0 \Rightarrow u_1 = u_2$

$w_1 = w_2$

$\therefore U + W$ is a direct sum.

Hence, proved

★ Null Space → It is set of all solutions to homogeneous system $Ax = 0$ where A is a $m \times n$ matrix.

⇒ Remark ⇒ Nul A is subset of $\mathbb{R}^n$

i.e $\text{Nul } A = \{x : x \in \mathbb{R}^n \text{ and } A\vec{x} = 0\}$

Proposition 21 → The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Or equivalently, set of all solutions of a homogeneous system of m equations in n variables is a subspace of $\mathbb{R}^n$

★ The solⁿ set of non homogeneous system is not a subspace.

Proof :

(i) $\{0\} \in \text{Nul } A$ as the trivial solution exist for all homogeneous systems.

(ii) If $\vec{u}$ and $\vec{v} \in \text{Nul } A$, then $A\vec{u} = 0$ and $A\vec{v} = 0$, hence, $A(\vec{u} + \vec{v}) = 0 + 0 = 0$

(iii) If $\vec{u} \in \text{Nul } A$, $A\vec{u} = 0$ and $A(c\vec{u}) = c(A\vec{u})$
$= c(0) = 0$

$\Rightarrow$ Either Nul A is the zero subspace, or the dimensions of Nul A is equal to number of free variables in the solution.

★ **Column Space** → The column space of an $m \times n$ matrix A, written col A, is the set of all linear combinations of columns of A i.e. span of the column vectors obtained from A.

If $A = [a_1\ a_2\ a_3 \ldots a_n]$, then $\text{col } A = \text{span}\{a_1, a_2 \ldots a_n\}$.

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Proposition 22 → Col A is a subspace of R^m

Proof : Since A is an $m \times n$ matrix, its columns are vectors in R^m.

(i) Since col A is a span of vectors, it must contain $\{0\}$

(ii) For $\vec{u} \in$ col A and $\vec{v} \in$ col A,

where $\vec{u} = c_1\vec{a_1} + \dots c_n\vec{a_n}$

$\vec{v} = d_1\vec{a_1} + \dots d_n\vec{a_n}$

$\vec{u}+\vec{v} = (c_1+d_1)\vec{a_1} + \dots (c_n+d_n)\vec{a_n} \in$ col A

(iii) For $\vec{u} \in$ col A, $c\vec{u} \in$ col A

The column space is a subspace of R^m, since each column has m elements and hence we get a m-tuple.

Proposition 23 → The pivot columns of a matrix A form a basis for col A.

Justification (Concise Proof):

Any linear dependence relationship between columns of A can be expressed as $Ax = 0$. when A is reduced to R, the colums change, but the solution set of $Rx = 0$ remains same. In other words, row reduction does not change dependence relationship btw. columns. The pivot columns are always l.i. and also non-pivot columns are a l.c of ~~the~~ preceding pivot columns.

Note: We must take columns of A for basis.

Nul A	Col A
• Subspace of $\mathbb{R}^n$	• Subspace of $\mathbb{R}^m$
• Defined implicitly	• Defined explicitly
• $\text{Nul } A = \{0\}$, iff $Ax = 0$ has only trivial solⁿ	• $\text{Col } A = \mathbb{R}^m$ iff $Ax = b$ has a solution for every b in $\mathbb{R}^m$

★ **Row Space** → The row space of an $m \times n$ matrix A, is the set of all linear combinations of rows of A, i.e the span of the (row) vectors obtained from A.

Each row is considered as n tuple and its a vector in $\mathbb{R}^n$

If $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$, then row A = span$\{r_1, r_2, \ldots r_m\}$.

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Proposition 24: Row A is a subspace of R^n

Proof: Since A is a matrix of m×n order, it's rows are vectors in R^n.

(i) Since row A = span of vectors, it must contain $\vec{0}$

(ii) For $\vec{u} \in$ row A and $\vec{v} \in$ row A

$$\vec{u} = c_1 \vec{r_1} + c_2 \vec{r_2} + \dots c_m \vec{r_m}$$
$$\vec{v} = d_1 \vec{r_1} + d_2 \vec{r_2} + \dots d_m \vec{r_m}$$
$$(\vec{u}+\vec{v}) = (c_1+d_1)\vec{r_1} + \dots (c_m+d_m)\vec{r_m}$$

(iii) For $\vec{u} \in$ row A
$c\vec{u} \in$ row A

The row A is a subspace of R^n; since each row is a n-tuple.

Proposition 25: Row equivalent matrices have the same row space.

~~Elemen~~ Proof:

elementary row operations replace rows of original matrix by rows which are the same or linearly dependent on them. Hence, the row space does not get enlarged by row operations. If B is got from A by an elementary row operation, then Row B ⊆ Row A. But since elementary row operations are reversible Row A ⊆ Row B. Therefore Row B = Row A.

Hence Proved.

⇒ Finding a basis for row A → • The non zero rows of A for its RREF form a basis for row R and row A.

• Alternate Method: Take A^T and find basis for col A, i.e all pivot columns.

★ Observation →

$$\boxed{\text{dimension of Col } A = \text{dimension of Row } A}$$

⇒ If A is an $m \times n$ matrix, then the column rank of A is defined to be dim(col A). Similarly, the row rank of A = dim (Row A). The nullity of A = dim(Nul A)

(2.) Rank Theorem

(a.) The row rank or column rank of matrix A are equal. This is called rank of A

(b.) Rank = No. of pivot columns

(c) $\boxed{\text{rank}(A) + \text{nullity}(n) = \text{no. of columns of } A = n}$

Proof: (a.) and (b.) correspond to the fact that no. of basis vectors correspond to no. of pivot elements.

Corollary 2.1 → A square matrix of $m \times m$ is invertible if and only if rank(A) = m, i.e. it forms a basis of $\mathbb{R}^m$.

Linear Transformation ⇒

★ A map or a function $T: V \rightarrow W$, from a vector space V to a vector space W is said to be a linear transformation if:

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $u, v \in V$
(ii) $T(c\vec{u}) = c\,T(u)$ for $\vec{u} \in V$ and all $c \in F$

⇒ The space W (the co-domain) may be the space V or a subspace of V or may be an entirely different space (but over same field F)
⇒ Homomorphism is same as linear Transformation

Eg. of L T

(i) O Transformation → $O : V \to W$ such that $O(u) = 0$
(ii) Identity Transformation → $I : V \to W$ such that $I(u) = u$

Some remarks →
(i) if T is linear, then $T(0) = 0$ and $T(-v) = -T(v)$

Proof (i) Let $T : V \to W$
$u \in V$ and $T(\vec{u}) = \vec{w}$

$$T(\vec{0}) = T(0.\vec{u}) = 0.(T(\vec{u})) = 0$$

Hence proved.

Similarly, $T(v) = T(v)$ and $T(-\vec{v}) = -T(\vec{v})$
Hence Proved.

(ii) If T is linear, then T "preserves" linear combinations, i.e. $T(c_1\vec{v}_1 + \dots c_K\vec{v}_K) = c_1 T(v_1) + c_2 T(v_2) + \dots c_K T(v_K)$

By basic definition of linear transformation,
$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

So, $T(c_1\vec{v}_1 + \dots c_K\vec{v}_K) = T(c_1\vec{v}_1) + \dots T(c_K\vec{v}_K)$ — (1)

Also $T(c\vec{u}) = c(T(\vec{u}))$

So $T(c_1\vec{v}_1) + \dots T(c_K\vec{v}_K) = c_1 T_1(\vec{v}_1) + \dots c_K T_K(\vec{v}_K)$

Proposition 26 (a.) → A linear transformation $T: V \to W$ is completely determined by its action on a basis of V.

V assumed to be finite dimensional

Proposition 26 (b.) → Conversely, given a basis $B = \{v_1, \dots, v_n\}$ of V, and a list of n vectors $w_1, \dots, w_n$ (not necessarily distinct) in the codomain space W, there is a unique linear transformation T such that $T(\vec{v}_1) = \vec{w}_1, \dots T(\vec{v}_n) = \vec{w}_n$

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Other remarks:

2 important subspaces are associated with any LT

(i) $\ker T = \{\vec{v} \in V : T v = \vec{0} \in U\}$ is a subspace of V.

Proof → • $\{0\} \in \ker T$ as $T(0) = 0$ always

• for $\vec{u} \in \ker T$ and $\vec{v} \in \ker T$ $T(u) = T(v) = 0$

$\therefore T(u+v) = T(u) + T(v) = 0$

• for $\vec{u} \in \ker T$

$T(c\vec{u}) = 0 = cT(\vec{u}) = 0$

(ii) The range of T, Range $T = \{\vec{w} \in W : \vec{w} = T\vec{v}$ for some $\vec{v} \in W\}$ is a subspace of W.

• $\{0\} \in$ Range T as $T(0) = 0$

• For $\vec{w}_1, \vec{w}_2 \in$ Range T $\Rightarrow \vec{w}_1 + \vec{w}_2 = Tv_1 + Tv_2 = T(v_1 + v_2)$

• For $\vec{w} \in$ Range T $\Rightarrow c\vec{w} \in$ Range T as $c\vec{w}_1 = cT(\vec{v}_1)$

☆ It's easy to see that T is injective if and only if $\ker T = \{0\}$

Proof → For proving injection, we see $f(x_1) = f(x_2)$

$\Downarrow$

$x_1 = x_2$

If it's not an injection, then we have $Tv = 0$ and $Tw = 0$ where $v \neq w$.

$\therefore$ Ker T has more than one element

Ker T will have $\{0\}$ always and hence T is injective if and only if Ker $T = \{0\}$.

★ Rank of a linear transformation:

Let $T: V \to W$ be a linear transformation.

The rank of T is the dimension of range of T

⇒ we can see that

$$\boxed{\dim(\text{Range}(T)) \leq \dim V} \Bigg\} \text{ derived from 2o (a) and (b)}$$

Also, dimension of $\text{Ker}(T) = $ Nullity of T

Theorem 2 (Rank Theorem for linear transformation):

Suppose $T: V \to W$,

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim V}$$

(If V is finite dimensional, then so is range(T))

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Proof of rank theorem →

Suppose that $\dim V = n$ and nullity$(T) = K$.
Then let $v_1, v_2 \dots v_K$ be a basis for Ker T and let us expand the basis as $\boxed{K \le n}$

New Basis = span $\{v_1, v_2 \dots v_K, v_{K+1}, \dots v_n\}$
We claim that vectors $T(v_{K+1}), \dots, T(v_n)$ forms a basis for Range(T)

All vectors $T(\vec{v_1}), T(\vec{v_2}) \dots T(\vec{v_n})$ span Range(T), and since $T(\vec{v_1}) = T(\vec{v_2}) = \dots T(v_K) = 0$),
actually $T(v_{K+1}), \dots, T(v_n)$ span Range T

Suppose $c_{K+1} T(\vec{V}_{K+1}) + c_{K+2} T(\vec{V}_{K+2}) + \dots + c_n T(\vec{V}_n) = \vec{0}$

Then $T(c_{K+1} \vec{V_{K+1}} + \dots + c_n \vec{V_n}) = 0$

Hence; $c_{K+1} \vec{v_{K+1}}, \dots, c_n \vec{V_n}$ belong to Ker T
$\therefore c_{K+1} \vec{V_{K+1}} + \dots c_n \vec{V_n} = b_1 \vec{v_1} + b_2 \vec{v_2} + \dots b_K \vec{v_K}$

$$\Rightarrow \underbrace{b_1 \vec{v_1} + \dots b_K \vec{v_K} + \dots c_{K+1} \vec{V_{K+1}} + \dots c_n \vec{V_n}}_{\text{Basis for } V} = 0$$

$\therefore c_{K+1} = c_{K+2} = \dots c_n = 0$
This proves the claim, rank(T) = dim Range(T) = $(n-K)$

★ **Isomorphism** : A linear transformation $T: V \to W$ is said to be isomorphism if its injective and surjective.

Proposition 27 → Let V and W be finite dimensional spaces.

(a.) An isomorphism $T: V \to W$ takes any arbitrary basis of V to a basis of W

(b.) Conversely, if a linear transformation $T: V \to W$ takes some basis of V to a basis of W, then its an isomorphism.

Proof :

Proposition 28 : 2 finite dimensional vector Spaces V and W (over the same field) are isomorphic if and only if $\dim V = \dim W$

Proof :

def Multiplication by a matrix is also a linear transformation. $T_A : R^n \rightarrow R^m$ defined by $T_A(\vec{x}) = A\vec{x}$

Coordinate Systems

Given a basis for a finite-dimensional vector space V, we recall that a vector can be expressed in one and only one way as a linear combination of basis vectors.

An ordered basis is a basis taken in a specified fixed order.

Given an ordered basis $B = \{u_1, u_2 \ldots u_n\}$

$$\vec{u} = \underbrace{x_1 \vec{u}_1 + x_2 \vec{u}_2 + \ldots x_n \vec{u}_n}$$

Coordinates

The coordinate vector $\vec{u}$ (relative to basis B) is written as $[\vec{u}]_B$

$(u \rightarrow [u]_B) \Rightarrow$ Coordinate Mapping

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Matrix of a Linear Transformation :

Suppose V and W are finite-dimensional vector spaces over field F, and $T: V \to W$

Suppose $\dim V = n$ and $\dim W = m$
we take fixed ordered basis $= \{v_1, v_2, \dots v_n\} = B$
$\{w_1, w_2, \dots w_m\} = C$

Since Tv_i belongs to W,
$T\vec{v_i} = A_1\vec{w_1} + \dots A_m\vec{w_m}$

Matrix of T with respect to B and C is ~~are~~ denoted by $[T]_{B,C}$

★ A matrix is formed with these coefficients as columns

eg. $T: R^3 \to R^2$
$T(x,y,z) = (x+y+z, x+2y+3z)$

$T(1,0,0) = (1,1) = 1e_1 + 1e_2$
$T(0,1,0) = (1,2) = 1e_1 + 2e_2$
$T(0,0,1) = (1,3) = 1e_1 + 3e_2$

$\Rightarrow$ matrix of T is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

★ For any vector $\vec{v}$ in V, we can find coordinate of $T\vec{v}$ in W by left ~~mult~~ multiplying the coordinate vector of $\vec{v}$ by matrix $A = [T$

$$[T(v)]_C = [T]_{B \to C} [v_B]$$

★ In case of a linear operator, i.e. linear transformation from V to itself, the eqⁿ becomes

$$[T(\vec{v})]_B = [T]_B [\vec{v}]_B$$

Change of Basis ⇒

It's the change in matrix when the basis is changed.
we will restrict our attention to case when T is a linear operator from V to V.

Proposition 29 : let $B = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_n\}$ and $C = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ be 2 ordered bases of vector space V. Then there is an invertible matrix of $n \times n$ order such that

$$[x]_C = P[x]_B$$

Not Imp

Proof: Let $\vec{x} \in V$; since B is a basis for V, we can write $\vec{x} = b_1\vec{u_1} + b_2\vec{u_2} + \dots + b_n\vec{u_n}$ — (1.)

so that $[\vec{x}]_B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ — (2.)

Since C is also a basis for V, we can write:

$$\left.\begin{aligned} \vec{u_1} &= A_{11}\vec{V_1} + A_{21}\vec{V_2} + \dots + A_{n1}\vec{V_n} \\ u_2 &= A_{12}\vec{V_1} + A_{22}\vec{V_2} + \dots + A_{n2}V_n \\ &\vdots \\ \vec{u_n} &= A_{1n}\vec{V_1} + \dots + A_{nn}\vec{V_n} \end{aligned}\right\} \text{(3.)}$$

From (3.), $[u_i]_C = \begin{bmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{bmatrix}$ for $i = 1, 2 \dots n$ — (4.)

Substituting from (3) in (1), we get:

$$\vec{x} = b_1(A_{11}\vec{V_1} + \dots A_{n1}\vec{V_n}) + \dots b_n(A_{1n}\vec{V_1} + \dots A_{nn}\vec{V_n})$$ — (5.)

Rearranging and collecting the coefficients of $\vec{V_1}, \vec{V_2}, \dots, \vec{V_n}$, we get:

$$\vec{x} = (A_{11}b_1 + A_{12}b_2 + \dots A_{1n}b_n)\vec{V_1} + \dots + (A_{n1}b_1 + \dots + A_{nn}b_n)\vec{V_n}$$ — (6.)

Hence $[\vec{x}]_C = \begin{bmatrix} A_{11} b_1 + A_{12} b_2 + \dots A_{1n} b_n \\ \vdots \\ A_{n1} b_1 + \dots \dots + A_{nn} b_n \end{bmatrix}$

$$= \begin{bmatrix} A_{ij} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = P[\vec{x}]_B$$

As already noted from (4), the columns $\vec{p}_1, \dots, \vec{p}_n$ of the matrix P are nothing but coordinate vectors of old basis, i.e.

Finally, we note that P must be invertible for the foll. reason: it's columns $\vec{p}_1, \dots, \vec{p}_n$ are images of vectors $\vec{u}_1, \vec{u}_2, \dots \vec{u}_n$ under coordinate mapping relative to the basis B.

Since the coordinate mapping $V \to F^n$ is an isomorphism by Prop 27(a), vectors $\vec{p}_1, \dots, \vec{p}_n$ form a basis of F^n. Thus by VIT, P is invertible.

$\Rightarrow$ To change coordinates between 2 bases, we need the coordinate vectors of the old basis relative to new basis. These become the columns of the change of coordinate matrix P.

$P = Q^{-1}$, where Q has its columns the coordinate vectors of new basis C relative to old basis B

★ 2 matrices are said to be similar if there exist an invertible matrix P such that

$$\boxed{B = PAP^{-1}}$$

Proposition 30: Similarity of matrices is an equivalence relation on $F^{n\times n}$, i.e. the set of $n\times n$ matrices with entries taken from a field F.

Proof: Let R be the relation on $F^{n\times n}$ of similarity of matrix.

(i) Reflexive $\rightarrow$ ~~ARA~~ $A = I^{-1}AI$

$\therefore \boxed{ARA}$

(ii) Symmetric: Let $ARB \Rightarrow B = P^{-1}AP$

By left and right multiplication,

$P^{-1}BP = A$ $\therefore$ BRA

Hence symmetric.

(iii) Transitive: Let ARB and BRC

$\Rightarrow B = PAP^{-1}$ and $C = QBQ^{-1}$

$\Rightarrow C = Q(PAP^{-1})Q^{-1}$
$= (QP)A(P^{-1}Q^{-1})$
$= (QP)A(QP)^{-1}$
$= SAS^{-1}$
$\therefore ARC$

Hence transitive.

So, by definition, matrix similarity is an equivalence relation.

Proposition 31: Suppose A and B are the matrices of the linear operator T relative to ordered bases α and β respectively. Then A and B are similar matrices, in fact $B = PAP^{-1}$ where $P = P_{\alpha\to\beta}$ is the change of basis matrix.

Proof: We use the fact that if P is the change of basis matrix from α to β, then P^{-1} is the change of basis matrix from β to α.

Let $A = [T]_\alpha$

- Therefore, for any $\vec{v} \in V$:

$$(PAP^{-1})[\vec{v}]_\beta = (PA)P^{-1}[\vec{v}]_\beta$$
$$= (PA)[\vec{v}]_\alpha$$
$$= P(A[\vec{v}]_\alpha)$$
$$= P([T]_\alpha[\vec{v}]_\alpha)$$
$$= P[T\vec{v}]_\alpha$$
$$= [T\vec{v}]_\beta = [T]_\beta[\vec{v}]_\beta$$

Since the above holds for all vectors $\vec{v} \in V$, it follows that $PAP^{-1} = [T]_\beta = B$

Let V and W be vectors spaces over a field F

Proposition 32: (a.) The set W^V of all functions from V to W is a vector space ~~of all functions from V to W~~ is a ~~[illegible]~~ vector space over F

(b.) The set of all linear transformations from V to W is a subspace of W^V

Proof →

(a.) For any functions f and g from V to W and any scalar c, we define the functions $(f+g)$ and (cf) by:

$*$ $(f+g)(\vec{u}) = f(\vec{u}) + g(\vec{u})$ for all $\vec{u}$ in V

$*$ $(cf)(\vec{u}) = cf(\vec{u})$ for all $\vec{u}$ in V.

⇒ Because W is a vector space over F, W^V becomes a vector space over F

(b.) To show it's a subspace →

(i) 0 function is a LT, hence belongs to $L(V,W)$

(ii) $T+U(\vec{u}+\vec{v}) = T(\vec{u}+\vec{v}) + U(\vec{u}+\vec{v})$

$$= T(\vec{u}) + T(\vec{v}) + U(\vec{u}) + U(\vec{v})$$

$$= (T+U)(\vec{u}) + (T+U)(\vec{v})$$

Similarly, $(T+U)(c\vec{u}) = c(T(\vec{u}) + U(\vec{u}))$

$$= c(T+U)(\vec{u})$$

(iii) $(cT)(\vec{u}+\vec{v}) = cT(\vec{u}+\vec{v}) = c(T(\vec{u}) + T(\vec{v}))$

$$= (cT)(\vec{u}) + (cT)(\vec{v})$$

Proposition 33 : Let V, W and Z be vector spaces over a field F. Let T be a linear transformation from V into W. Then the ~~de~~ composed function UT from V into Z (U be a linear transformation from W into Z) defined by $UT(\vec{v}) = U(T(\vec{v}))$ for all $\vec{v}$ in V is a linear transformation from V to Z.

Proof →

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☆ Linear Operators →

Any linear transformation of the form $L(V, V)$ is known as linear operator.

Properties →

(a.) $IU = UI = U$ for linear operators U

(b.) $(T_1 T_2)T_3 = T_1(T_2 T_3)$

(c.) $U(T_1 + T_2) = UT_1 + UT_2$

(d.) $(T_1 + T_2)U = UT_1 + UT_2$

(e.) $c(UT_1) = (cU)T_1 = U(cT_1)$

(f.) However, this multiplication is not commutative

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Proposition 34 : Let V be a n dimension vector space over F, and let W be an m-dimensional vector space over F. Then there is an isomorphism between $L(V, W)$ and $F^{m \times n}$

Proof →

Let us take an ordered matrix $\alpha = \{v_1, v_2, \ldots, v_n\}$ for V and $\beta = \{w_1, w_2, \ldots, w_m\}$ for W

Let T be any linear transformation in $L(V, W)$. Then we can find the matrix of T w.r.t to bases α and β and let us call it $[T]_{\alpha \to \beta}$

The mapping $\phi : L(V, W) \to F^{m \times n}$ which take a linear transformation T to its matrix $[T]_{\alpha \to \beta}$ is an isomorphism!

Note: ϕ depends on choice of α and β

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Proposition 35: If $\dim(V) = n$ and $\dim W = m$, then $\dim L(V, W) = mn$

Proof:

method 1 — We use the fundamental isomorphism of proposition 34. Since $L(V, W)$ is isomorphic to $F^{m \times n}$, and $\dim(F^{m \times n}) = mn$, the result follows from Proposition 28.

method 2 — We take a fixed ordered basis $\alpha = \{v_1, \dots v_n\}$ and $\beta = \{w_1, w_2 \dots, w_m\}$ for W.

We define LT $E_{ij} : V \to W$ by $E_{ij}(\vec{v_j}) = w_i$ and $E_{ij}(\vec{v_k}) = 0$ (for $k \neq j$).

It is evident that $S = \{E_{ij} : i = 1 \text{ to } m, j = 1 \text{ to } n\}$ forms a basis for $L(V, W)$. Since $|S| = mn$, result follows.

★ $[T(\vec{v})]_\beta = [T]_\beta [\vec{v}]_\beta$

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Proposition 36: Suppose T and U are linear operators on a finite dimensional vector space V and β is a fixed ordered basis for V. Then, $[UT]_\beta = [U]_\beta [T]_\beta$

[Proposition 36a] → The mapping $\phi : L(V, V) \to F^{n \times n}$ given by $\phi(T) = [T]_\beta$ is a vector space isomorphism which also preserves products, i.e. $\phi(UT) = \phi(U)\phi(T)$

[Proposition 36 b] → Suppose dim V = n, dim W = m, dim Z = k. $UT : V \to Z$ would be a linear transformation from a space of dimension n to a space of dimension k, i.e. its matrix would be an $k \times n$ matrix.

$T : V \to W$, $U \to W \to Z$

Let α, β, γ be bases of V, W, Z

Then $\underset{k \times n}{[UT]_{\alpha \to \gamma}} = \underset{k \times m}{[U]_{\beta \to \gamma}} \underset{m \times n}{[T]_{\alpha \to \beta}}$

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☆ Invertibility of Linear Transformations

Any function f from V to W is said to be invertible if there exists a function g from W into V such that gf is identity function on V and fg is identity function on W.

⇒ In case f is invertible, then the function g is unique and denoted as f^{-1}

⇒ Any function f is invertible if and only if f is bijective

Proposition 37: If $T: V \to W$ is an invertible linear transformation, its function $T^{-1}: W \to V$ is also a linear transformation.

Proof: Consider the function $T^{-1}: W \to V$

(i) consider $\vec{w_1}, \vec{w_2} \in W$
We have to show $T^{-1}(\vec{w_1} + \vec{w_2}) = T^{-1}(\vec{w_1}) + T^{-1}(\vec{w_2})$

Let us apply T to both sides of (1).

L.H.S $T(T^{-1}(\vec{w_1} + \vec{w_2})) = (TT^{-1})(\vec{w_1} + \vec{w_2})$
$= I(\vec{w_1} + \vec{w_2}) = \vec{w_1} + \vec{w_2}$ — (2)

R.H.S $T(T^{-1}(\vec{w_1}) + T^{-1}(\vec{w_2})) = T(T^{-1}(\vec{w_1})) + T(T^{-1}(\vec{w_2}))$ — (3)
$= \vec{w_1} + \vec{w_2}$

Since T is injective from (2.) and (3.), we get
L.H.S of (1) = R.H.S of (1.)

(ii) We also need to show $T^{-1}(c\vec{w_1}) = cT^{-1}(\vec{w_1})$ — (4)

L.H.S $T(T^{-1}(c\vec{w_1})) = (TT^{-1})(c\vec{w_1})$
$= c\vec{w_1}$

R.H.S $T(cT^{-1}(\vec{w_1})) = c(TT^{-1})(\vec{w_1})$
$= c\vec{w_1}$

Since T is injective, from (5.) and (6.) we get that

L.H.S of (4) = R.H.S of 4 as reqd.

(1.) ↳ Corollary 37.1: Isomorphism is an equivalence relation on the set of all vector spaces over a given field F

Proof Outline → Reflexive → Obvious
Symmetric → Prop 37
Transitive → Prop 33

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Proof :

☆ Singular and Non-Singular Linear Transformations

⇒ A linear transformation T from $V \to W$ is said to be **non singular** if null space of T is $\{0\}$, i.e. $T\vec{v} = 0$ ~~and~~ implies $\vec{v} = 0$
i.e. T is injective

Proposition 38 → Let T be a linear transformation from $V \to W$. Then T is non-singular if and only if T carries every linear independent subset of V into a linearly independent subset of W.

Proof →

Proposition 39 → Let V and W be a finite dimensional spaces with dim V = dim W. Let T be a linear transformation from V → W. Then the following are equivalent:

(a.) T is invertible
(b.) T is non singular i.e. range of T is W.
(c.) T is surjective
(d.) T carries every basis of V into a basis of W ★

Proof →

dim W

★ Remark : Note for finite dimensional spaces with equal dimension, if the LT is non-singular (i.e. injective), then it must be surjective, and if surjective, it must be injective.

For ∞ dimensional spaces, this might not be true

Theorem 4 (Basis Theorem or Fundamental Theorem of Linear Algebra)

Every vector space V has a basis, more precisely, if $\vec{v} \in V$ is a non-zero vector, then there exists a basis a basis B of V such that $\vec{v} \in B$.

☆ Eigenvectors and Eigen Values

- An eigenvector of an ~~m×n~~ n×n matrix A is a non-zero vector $\vec{x}$ such that $\boxed{A\vec{x} = \lambda\vec{x}}$ for some scalar λ. A scalar $\boxed{\lambda}$ is called eigen value if there is a non trivial solution to $A\vec{x} = \lambda\vec{x}$ ($\vec{0}$ is never a eigen vector)

- If v is an eigen vector corresponding to λ_1, then it cannot be an eigenvector corresponding to λ_1, then it cannot correspond to different value λ_2 (~~$\vec{0}$ is not~~

☆ Eigen Values → Characterisic Values / Latent Roots

Eigen Vectors → Characteristic vectors.

- 0 is allowed to be eigen value, but not an eigen vector.

☆ An n×n matrix A is invertible if and only if 0 is not an eigenvalue of A.

Because for $A\vec{x} = 0\vec{x}$, it must have a non trivial soln

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⇒ An eigen vector is not unique, since all scalar multiples of it are also eigen vector.

~~Actually, the set of all n×n matrix A together with zero~~

Actually, the set of all eigenvectors corresponding to a fixed eigenvalue λ of the $n \times n$ matrix A together with $\vec{0}$ forms a subspace of

$$V = F^n$$

$$X = \{ \vec{v} \in V : \vec{v} \text{ is an eigenvector for } \lambda \} \cup \{ 0 \}$$
$$= \{ \vec{v} : A\vec{v} = \lambda \vec{v} \}$$

X is a subspace of V corresponding to eigenvalue λ and called the eigen space for λ.

Proof of Proposition 40 : If $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_p$ of matrix A, then the set $\{ \vec{v}_1, \vec{v}_2 \ldots \vec{v}_p \}$ is linearly independent.

Corollary 40.1 → An $n \times n$ matrix A can have atmost n distinct eigen~~vector~~values.

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Proof :

Proof By Contradiction →

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are linearly dependent. Let m be the smallest no. such that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are L.I and $\vec{v}_{m+1}$ is a linear combination of preceding vectors.

$$\vec{v}_{m+1} = c_1\vec{v}_1 + \ldots c_m \vec{v}_m \quad — (1)$$

Left multiplying by A

$\rightarrow A\vec{v}_{m+1} = c_1 A_1 \vec{v}_1 + c_2 A_2 \vec{v}_2 + \ldots + c_m A \vec{v}_m$

$\rightarrow \lambda_{m+1} \vec{v}_{m+1} = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \ldots + c_m \lambda_m \vec{v}_m$

From (1) multiplied by λ_{m+1}, we get.

$$c_1(\lambda_1 - \lambda_{m+1})\vec{v}_1 + \ldots + c_m(\lambda_m - \lambda_{m+1})\vec{v}_m = 0 \quad — (2)$$

However, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are lin independent;

so,

$c_1(\lambda_1 - \lambda_{m+1}) = 0$ & $(\lambda_2 - \lambda_{m+1}) = 0$ implies

$c_1 = c_2 = \ldots c_m = 0$

From (1), we get $\vec{v}_{m+1} = 0$

But this is not possible, since all $\vec{v}$'s are eigen vectors.

$\therefore$ Due to no contradiction, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is LI

To determine eigen vectors and eigen values, we solve the <u>characteristic equation</u> i.e $\det(A - \lambda I) = 0$.

→ It's possible for a matrix with real entries to have no real eigen values.

<u>Proposition 41</u>: A scalar λ is an eigen value of an $n \times n$ matrix A if and only if $\det(A - \lambda I) = 0$.

<u>Proof</u>:

- Consider λ is an eigen value of A

$\Leftrightarrow$ There is a non zero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$

$\Leftrightarrow$ The system $(A - \lambda I)\vec{x} = 0$ has a non trivial soln

$\Leftrightarrow$ Matrix $(A - \lambda I)$ is not invertible by (VIT)

$\Leftrightarrow$ $\det(A - \lambda I) = 0$ (again by VIT)

$\Leftrightarrow$ λ is a root of characteristic equation

Proposition 42 → If the $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigen values with same multiplicities.

(However, eigen vectors won't be same always)

Proof → Let us consider

$$\det(B-\lambda I) = \det(PAP^{-1} - \lambda I)$$
$$= \det(PAP^{-1} - P(\lambda I)P^{-1})$$
$$= \det(P(A-\lambda I)P^{-1})$$
$$= (\det P)(\det A-\lambda I)(\det(P^{-1}))$$
$$= \det A - \lambda I$$

✯ Polynomials applied to Matrices

Matrix Powers → If A is an $n \times n$ square matrix, with coefficients as either real or complex, $A^m = AAA \ldots A$ (m times)

and $A^{-m} = (A^{-1})^m$

Also $A^i A^j = A^{(i+j)}$

$\boxed{(A^i)^j = A^{ij}}$

i and j must be non-ve for A when noninvertible

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For 2 polynomials, P and q, $Pq(A) = P(A)q(A)$
$= q(A)P(A) = qp(A)$

★ **Minimal Polynomial of a Matrix →**

Given an $n \times n$ square matrix A, the minimal polynomial of A is the (nonzero) monic polynomial of minimal degree such that $p(A) = 0$, i.e. the 0 matrix. The monic condition is inserted so that we make the minimal polynomial unique.

NOTE: A monic polynomial is a polynomial in which coefficient of highest power of variable is 1.

⇒ Every square matrix must have a minimal polynomial.

Proof → Suppose A is an $n \times n$ matrix in F, then set $\{I, A, A^2, \dots, A^k\}$, where $k = n^2$, cannot be LI because dim of $F^{n \times n}$ is n^2 and there are n^2+1 matrices in set.

Let m be smallest possible +ve integer such that $\{I, A, A^2, \dots, A^m\}$ is linearly dependent. Hence A^m is a linear combination of the preceding matrices such that $A^m + a_{m-1}A^{m-1} + \dots + a_1 A + a_0 I = 0$

Theorem 5 (Carey Hamilton Th.)

Let q denote the characteristic polynomial of the $n \times n$ square matrix A.

Then $q(A) = 0$

Corollary 5.1: The degree of the ~~most~~ minimal polynomial of any $n \times n$ square matrix is at most n.

Remark → It's easy to see, using remainder theorem for polynomial division, that if $p(x)$ is the minimal polynomial of A and if $q(x)$ is any other polynomial satisfied by A, then $p(x)$ divides $q(x)$

(5.) Lemma 5.5 → Suppose $q \in F[t]$. Then $q(A) = \vec{0}$ if and only if the minimal polynomial of A divides q

Proof → Suppose $q(A) = 0$

Let $p(t)$ be minimal polynomial of A, then $p \neq 0$, so using the remainder theorem (lemma 5.3) we can write,

$q(t) = p(t)s(t) + r(t)$ —— (I)

* F = Field of real or Complex

(1) Lemma 5.1 → Suppose $p \in F[t]$ is a polynomial of degree $m \geq 1$. Then λ is a root of p if and only if there exists a polynomial $q \in F[t]$ with degree $m-1$ such that $p(t) = (t-\lambda) q(t)$

(2) Lemma 5.2 → Suppose $p \in F[t]$ is a polynomial of degree $m \geq 0$, then p has at most m distinct roots in F

(3) Lemma 5.3 → (Division Algorith or Remainder Algorithm)
: Suppose $p, q \in F[t]$ with $p \neq 0$, then there exists polynomials $r, s \in F[t]$ with $q(t) = p(t)s(t) + r(t)$ and either $r = 0$ or $\deg(r) < \deg p$

(4.) Lemma 5.4 → Suppose $p \in \mathbb{C}[t]$ is a polynomial of degree $m \geq 1$, then p has a root. Furthermore, p has a factorisation of the form $p(t) = c(t-\lambda_1)(t-\lambda_2) \ldots (t-\lambda_m)$

5.3

$q(A) = p(A)s(A) + r(A) = 0$

But this is only possible if $r(A) = 0$ and $p(A) = 0$

$\therefore q(t) = p(t)s(t)$ and so, p divides q

Diagnolization of Matrices

If A is diagnol matrix, then its diagnol elements are its eigen values and the standard basis vectors are its eigenvectors.

A is said to be similar to a diagnol matrix D such that $A = PDP^{-1}$

if A is diagonalizable, $A^K = PD^K P^{-1}$
if A is diagonalizable, eigen values are same.

Method → (1) Charact Eq^n
(2) Get eigen values
(3) Make P
(4) Get D

Theorem 6 (Diagonalization Theorem)

(a) An $n \times n$ matrix A is diagonalizable if and only if A has n LI eigen vectors.

(b) $A = PDP^{-1}$ where columns of P are n LI eigenvectors of A, and the diagnol entries of D are corresponding eigen values.

Case I: An $n \times n$ matrix A has n distinct (real) eigenvalues. Then we get:

Proposition 43: An $n \times n$ matrix A with n distinct eigen values is diagonalizable

Proof: By prop. 40, eigen vectors corresponding to distinct eigen values are LI.

$\therefore$ A has n LI independent eigen vectors, and thus by Th 6, A is diagonalizable.

ALGEBRAIC MULTIPLICITY of λ_1 → Power of $(\lambda - \lambda_1)$ in ch poly-nomial

λ_1: eigen value

GEOMETRIC MULTIPLICITY of λ_1 → Dimension of eigenspace corresponding to λ_1

Case 2: An $n \times n$ matrix has $p < n$ distinct eigenvalues, but counting the (algebraic) multiplicities, there are n real eigen values (not distinct), we get:

Proposition 44: Let A be an $n \times n$ matrix with n (real) eigen values of which only $\lambda_1, \lambda_2 \ldots \lambda_p$ are distinct ($p<n$), Then:

(a.) For $1 \le k \le p$, the geometric multiplicity is less than or equal to algebraic multiplicity of λ_K

(b.) A is diagonalizable if sum of ~~different~~ dimensions of diff. eigenspaces is n, and this happens if and only if AM = GM for each λ_K

(c.) If A is diagonalizable, and B_K is a basis for the eigenspace corresponding to λ_K for each k, then ~~total~~ collection of vectors in $B_1, B_2 \ldots B_K$ forms an eigenvector basis for R^n.

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Case 3: An $n \times n$ matrix A has $p < n$ distinct eigen values, but even after counting AM's, it is $< n$ real eigen values (p could be 0), then A is not diagonalizable

→ In such a case, we need to consider complex eigen values and eigen vectors.

→ Even after we consider complex no.'s, there is no guarantee that it would be diagonalizable always.

Proposition 45: Basic Result for Complex Eigen Values:

Suppose A is a real 2×2 matrix with complex eigen value $\lambda = a - bi$, $b \neq 0$ and associated eigen vector $\vec{v}$ in $\mathbb{C}^2$

Then $A = PBP^{-1}$, $P = [\mathrm{Re}\,\vec{v} \;\; \mathrm{Im}\,\vec{v}]$

$$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$r = |\lambda| = \sqrt{a^2 + b^2}$ = Modulus of λ

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~~eg~~: Suppose $A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$

$\det(A - \lambda I) = 8 - 4\lambda + \lambda^2 \Rightarrow \lambda = 2 \pm 2i$

Take $\lambda = 2+2i$ (i.e $= a = 2$, $b = -2$)

$$A - \lambda i = \begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix}$$

$$(-2-2i)x + y = 0$$
$$-8x + (2-2i)y = 0$$

Since $A - \lambda I$ has non trivial solution, it's two rows are LD.

In 1st eq., putting $x = 1$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2+2i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$P = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

$PBP^{-1} = \Lambda$

Also, $B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \sqrt{8}\begin{bmatrix} 2/\sqrt{8} & 2/\sqrt{8} \\ -2/\sqrt{8} & 2/\sqrt{8} \end{bmatrix}$

i.e rotation through $7\pi/4$ followed by scaling through $\sqrt{8}$

★ Determinant of a Linear Operator

Let $T: V \to V$ be a linear operator for V a finite dimension vector space of dim 'n'.
Let α be any basis for V, and let A be matrix of T wrt ordered α.

Then $\boxed{\det T = \det A}$

★ Eigenvector of Linear Operators

An eigen vector of a linear operator $T: V \to V$, is a non zero vector such that $T\vec{v} = \lambda\vec{v}$

$\lambda \to$ eigen value

By Prop 42, eigen values of T coincide with eigenvalues for matrix of T with respect to suitable basis of V

Inner Product

It's equivalent to dot product and represented by $\langle\ \rangle$

1. $\langle u, v\rangle = \langle v, u\rangle$
2. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
3. $\langle cu, v\rangle = c\langle u, v\rangle$
4. $\langle u, u\rangle \geq 0$ and if $\langle u, u\rangle = 0 \rightarrow u = 0$

A vector space with an inner product is called an Inner product space.

For complex inner products, axiom 1 is modified as $\langle u, v\rangle = \overline{\langle v, u\rangle}$

eg. The space $R_n[t]$ of all polynomials of degree less than or equal to n.
Let $t_0, t_1, \ldots, t_n$ be distinct real no.s

Then for p, q in $R_n[t]$

$$\langle p, q\rangle = p(t_0) \cdot q(t_0) + p(t_1)q(t_1) + \ldots \ldots p(t_n)q(t_n)$$

eg. $\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt$

→ $C[a,b]$ of all continuous functions in $[a,b]$ is a space

★ **Length / Norm of any vector** →

$$\|u\| = \sqrt{\langle u, u\rangle}$$
$$= \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

For a unit vector, we can write $\dfrac{\vec{u}}{\|u\|}$] Called as normalization

★ **Orthogonality** →
2 vectors are said to be orthogonal if $\langle u, v\rangle = 0$
or we can say $u \perp v$

A set of vectors $\{u_1, u_2, \dots, u_p\}$ is said to be an orthogonal set if any 2 distinct vectors in set are orthogonal to each other, i.e $\langle u_i, u_j\rangle = 0$ for $i \neq j$

★ 0 is orthogonal to every vector.

Proposition 46 → An orthogonal set of nonzero vectors in V is linearly independent

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Proof → Suppose $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors and suppose

$c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots c_p\vec{u}_p = 0$

Taking inner product with $\vec{u}_1$

$c_1\langle\vec{u}_1 . \vec{u}_1\rangle + c_2\langle\vec{u}_1 . \vec{u}_2\rangle + \ldots + c_p\langle\vec{u}_1 . \vec{u}_p\rangle = 0$

Since $\vec{u}_1 . \vec{u}_1 > 0 \therefore c_1 = 0$

Similarly $c_1 = c_2 = \ldots = c_p = 0$

Hence, S is linearly independent

★ If W is subspace of V, then a vector $\vec{v}$ is said to be orthogonal to W ~~is called~~ orthogonal ~~complement of W~~, to every vector in W. The set of all vectors orthogonal to W is called the orthogonal complement of W, $W^{\perp}$

$W^{\perp} = \{v \in V : v \perp w \text{ for any } \vec{w} \in W\}$

Proposition 47 → (a) $\vec{v}$ belongs to $W^{\perp}$ if and only if $\vec{v}$ is orthogonal to every vector in a spanning set for W

(b) $W^{\perp}$ is a subspace of V and $W \cap W^{\perp} = \{0\}$

- Actually, if S is any subset of V, then $S^\perp = \{\vec{v} \in V : \vec{v} \perp \vec{u}$ for every $u \in S\}$ is a subspace of V.

Proof:

(a) $\Rightarrow$ it is obvious, since $\vec{v} \in W^\perp$, it is actually orthogonal to every vector $\vec{w} \in W$

$\Leftarrow$ Suppose $\vec{v}$ is orthogonal to every vector in spanning set K for W.

Let $\vec{w} \in W$.

Then $\vec{w} = c_1\vec{w_1} + \dots + c_p\vec{w_p}$.

$$\langle \vec{w}, \vec{v} \rangle = c_1\langle \vec{w_1}, \vec{v} \rangle + \dots + c_p\langle \vec{w_p}, \vec{v} \rangle$$
$$= 0$$

$\therefore \vec{v} \in W^\perp$ as required.

(b) (i) $\vec{0} \in W^\perp$ as 0 is orthogonal to every vector

(ii) Suppose $\vec{v_1}, \vec{v_2} \in W^\perp$ and let $\vec{w} \in W$

Then, $\langle \vec{v_1} + \vec{v_2}, \vec{w} \rangle = \langle \vec{v_1}, \vec{w} \rangle + \langle \vec{v_2}, \vec{w} \rangle = 0 + 0 = 0$

$\therefore \vec{v_1} + \vec{v_2} \in W^\perp$

(iii) $c \in \mathbb{R}$, $\vec{v_1} \in W^\perp$ and $w \in W$

Then $\langle c\vec{v_1}, \vec{w} \rangle = c\langle \vec{v_1}, \vec{w} \rangle = c \cdot 0 = 0$

Hence proved

★ An orthogonal basis for a subspace W is a basis which is also an orthogonal set.

Proposition 48: Let $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_p}\}$ be an orthogonal basis for a subspace W. Then if $\vec{y} = c_1\vec{u_1} + c_2\vec{u_2} + \ldots c_p\vec{u_p}$ is any vector in W,

$$c_j = \frac{\langle y, \vec{u_j}\rangle}{\langle u_j, u_j\rangle} \quad \text{for } j = 1, \ldots, p$$

Proof: Since $\vec{y} \in W$, $\vec{y}$ is uniquely expressible as a L.C of $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_p}$

$$\vec{y} = c_1\vec{u_1} + c_2\vec{u_2} + \ldots\ldots + c_p\vec{u_p}$$

We need to determine coefficients $c_1, c_2, \ldots c_p$

Let us take inner product with u_j

$$\langle \vec{y}, \vec{u_j}\rangle = c_1\langle \vec{u_1}, \vec{u_j}\rangle + \ldots c_j\langle u_j, u_j\rangle + \ldots c_p\langle \vec{u_p}, \vec{u_j}\rangle$$

$$= c_j\langle \vec{u_j}, \vec{u_j}\rangle \text{ since all terms are 0}$$

$$\therefore c_j = \frac{\langle \vec{y}, \vec{u_j}\rangle}{\langle \vec{u_j}, \vec{u_j}\rangle}$$

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Theorem 7 (Orthogonal Decomposition Theorem)

Let W be any finite dimensional subspace of V. Then each vector $\vec{y}$ in V can be written as uniquely in form of $\vec{y} = \vec{y}\,\hat{} + \vec{z}$, where $\vec{y}\,\hat{}$ is in W and z is in $W^{\perp}$.

In fact if $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_p\}$ is any orthogonal basis of W, then $\hat{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_p\vec{u}_p$

with $c_j = \dfrac{\langle y, u_j \rangle}{\langle u_j, u_j \rangle}$

and $z = \vec{y} - \hat{y}$

Th 7 Alternate Statement: Given any finite-dimensional subspace ~~of~~ W of V, then we can express $V = W + W^{\perp}$, with $W \cap W^{\perp} = \{0\}$ i.e $V = W \oplus W^{\perp}$

Proof: We assume that ~~the original res~~ any finite-dimensional subspace W of an inner product space has an orthogonal basis.

Proving Uniqueness

$$\left.\begin{aligned} \vec{y} &= \hat{y} + \vec{z} \\ \vec{y} &= \hat{y}_1 + \vec{z}_1 \end{aligned}\right\} \quad \begin{aligned} &\hat{y}, \hat{y}_1 \in W \\ &\vec{z}, \vec{z}_1 \in W^{\perp} \end{aligned}$$

$\vec{0} = (\hat{y} - \hat{y}_1) + (\vec{z}_0 - \vec{z}_1)$

~~L.H.S~~ $(\hat{y} - \hat{y}_1) = -(\vec{z} - \vec{z}_1)$

L.H.S $\in W$, R.H.S $\in W^\perp$ i.e they both are in $W \cap W^\perp = \{0\}$

$\therefore$ It's a unique representation

Proving decomposition exists

i.e. $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y} \in W$ and $\vec{z} \in W^\perp$

Let's put $\hat{y} = c_1\vec{u}_1 + \dots + c_p\vec{u}_p$ where $c_j =$

$$c_j = \frac{\langle \vec{y}, u_j \rangle}{\langle u_j, u_j \rangle} \quad \text{— (1)}$$

So, clearly $\hat{y} \in W$

Now put $\vec{z} = \vec{y} - \hat{y}$ so $\Rightarrow \vec{y} = \hat{y} + \vec{z}$

This shows that $\vec{z} \in W^\perp$.

We use proposition 47(a) for this, so it is suffic to show that $\langle \vec{z}, u_j \rangle = 0$

But $\langle z, u_j \rangle = \langle y - \hat{y}, u_j \rangle$

$= \langle y, u_j \rangle - \langle \hat{y}, u_j \rangle$

$= \langle y, u_j \rangle - c_j \langle \vec{u}_j, u_j \rangle$ from (1)

$= 0$ as desired.

Hence proved

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Note 1: The vector $\hat{y}$ is called the orthogonal projection of $\vec{y}$ onto W, written $proj_W \vec{y}$.

In case $W = span\{u\}$ is 1 dimensional subspace,

$$\underbrace{\hat{y} = \left(\frac{\langle y, u\rangle}{\langle u, u\rangle}\right) u}_{\text{Projection of } y \text{ onto } u}$$

Note 2: In case $\vec{y} \in W$, its orthogonal projection onto W is itself, i.e. $\hat{y} = \vec{y}$ for $\vec{y} \in W$ (This follows from Prop 48 + ODT (Th 7))

Theorem 8 (Gram-Schmidt Process

Given a basis $\{\vec{x}_1, \vec{x}_2 \ldots, \vec{x}_p\}$ for subspace W of V, we can generate orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ for W such that $span\{\vec{v}_1, \vec{v}_2, \ldots v_k\} = span\{x_1, x_2, \ldots, x_k\}$ for $k = 1, 2 \ldots p$

$$\vec{v}_1 = x_1$$

$$\vec{v}_2 = x_2 - \frac{\langle x_2 . v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$\vec{v}_3 = x_3 - \frac{\langle x_3 . v_1\rangle}{\langle v_1, v_1\rangle} v_1 \quad \frac{\langle x_3 , v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p$$

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Proposition 49 (Pythagoras Th): $\vec{u}$ and $\vec{v}$ are orthogonal to each other if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Proof: $\Rightarrow$ $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,v\rangle + 2\langle u,v\rangle$ —(1)

Since $u \perp v$, $\langle u,v\rangle = 0$

and $\therefore \|u+v\|^2 = \|u\|^2 + \|v\|^2$

$\Leftarrow$ $2\langle u,v\rangle = 0 \Rightarrow \vec{u} \perp \vec{v}$

Proposition 50 (Best Approximation Theorem): Let W be any finite dimensional subspace of V, $\vec{y}$ any vector in V and $\hat{y}$ be orthogonal projection of $\vec{y}$ onto W. Then $\|y-\hat{y}\| < \|y-v\|$ for all $\vec{v}$ in W distinct from $\hat{y}$.

In other words, $\hat{y}$ is closest vector in W to $\vec{y}$

Corollary 50.1 → If $\vec{y}$ is any vector, and W is a finite subspace then

$$\|\text{proj}_W \vec{y}\| \leq \|y\|$$

Proof: Let $\vec{v} \in W$

Then $\|\vec{y}-\vec{v}\|^2 = \langle \vec{y}-\vec{v}, \vec{y}-\vec{v}\rangle$

$$= \langle (\vec{y}-\hat{y}) + (\hat{y}-\vec{v}), (\vec{y}-\hat{y}) + (\hat{y}-\vec{v}) \rangle$$
$$= \langle \vec{y}-\hat{y}, \vec{y}-\hat{y} \rangle + \langle \hat{y}-\vec{v}, \hat{y}-\vec{v} \rangle + 2\langle \vec{y}-\hat{y}, \hat{y}-\vec{v} \rangle \quad \text{— (1)}$$

Now, $\vec{y}-\hat{y} \in W^{\perp}$ whereas $\hat{y}-\vec{v} \in W$

Hence 3rd term on R.H.S of (1) $= 0$

$$\therefore \ ||\vec{y}-\vec{v}||^2 = ||\vec{y}-\hat{y}||^2 + ||\hat{y}-\vec{v}||^2 \quad \text{— (2)}$$

Now, if $\vec{y} = \vec{v}$, then $\vec{y} \in W \Rightarrow \vec{y} = \hat{y} = \vec{v}$, Not allowed

Hence $||\hat{y}-\vec{v}||^2 > 0$

$$||\vec{y}-\vec{v}|| > ||\vec{y}-\hat{y}||$$

<u>Proof of 50.1</u> → We have that $\vec{v} = \text{Proj}_W \vec{v} + \vec{z}$, $\vec{z} \in W^{\perp}$

∴ Applying pythagoras theorem,

$$||\vec{v}||^2 = ||\text{Proj}_W \vec{v} + z||^2$$
$$= ||\text{Proj}\,\vec{v}||^2 + \underbrace{||z||^2}_{\geq 0}$$

Hence, result follows.

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Proposition 51 (Cauchy-Schwarz Inequality):

For all $\vec{u}, \vec{v}$ in V, $|\langle u, v\rangle| \leq \|u\| \|v\|$

Proof: Clearly result holds if either $\vec{u}$ or $\vec{v} = 0$

So let both $\vec{u}$ and $\vec{v}$ be non-zero, and applying corollary 50.1,

$$\|Proj_w \vec{u}\| \leq \|\vec{u}\| \quad -(1)$$

$$\|Proj_w \vec{u}\| = \left\| \frac{\langle \vec{u}, \vec{v}\rangle \vec{v}}{\langle \vec{v}, \vec{v}\rangle} \right\|$$

$$= \frac{|\langle \vec{u}, \vec{v}\rangle| \|\vec{v}\|}{\|\vec{v}\|^2} \quad -(2)$$

From (1) and (2) we can prove the Prop.

Proposition 52 (Triangle Inequality):

For all $\vec{u}, \vec{v}$ in V, $\|\vec{u} + \vec{v}\| \leq \|u\| + \|v\|$

Proof → $\|u+v\|^2 = \langle u+v, u+v\rangle$

$$= \|u\|^2 + \|v\|^2 + 2\langle \vec{u}, \vec{v}\rangle \leq \|u\|^2 + \|v\|^2 + 2\|\vec{u}\| \|\vec{v}\|$$

Hence, Result follows.

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Proposition 53 : If A is symmetric, then any 2 eigen vectors from different eigenspaces are orthogonal

Proof : Let $\vec{u}_1$ and $\vec{u}_2$ be eigen vectors corresponding to different eigen values λ_1 and λ_2.

Then : $\lambda_1(u_1 . u_2) = (\lambda_1 u_1 . u_2)$

$= (\lambda_1 u_1)^T . u_2$

$= (Au_1)^T u_2$

$= u_1^T A^T u_2$

$= u_1^T A u_2$

$= u_1^T (\lambda u_2)$

$= u_1 . \lambda u_2$

$= \lambda_2 (u_1 . u_2)$

Since $\lambda_1 \neq \lambda_2$, $u_1 . u_2 = 0$

★ **Orthogonal Matrices** → A square matrix P is said to be orthogonal if its columns are orthonormal.

Proposition 54 → An orthogonal matrix is necessarily invertible and $P^{-1} = P^T$

Proof : Suppose $P = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$, Then :

$$P^T P = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} [v_1 \; v_2 \ldots v_n] = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \ldots & v_1^T v_n \\ v_2^T v_1 & \ldots & \ldots & v_2^T v_n \\ \vdots & & & \\ v_n^T v_1 & \ldots & \ldots & v_n^T v_n \end{bmatrix}$$

We get that $P^T P = I$ as $v_i^T v_j = v_i \cdot v_j = \delta_{ij}$

Hence, the result

★ A square matrix A is said to be orthogonally diagonalizable if there is an orthogonal matrix P and a diagonal matrix D such that

$$A = PDP^{-1} = PDP^T$$

Proposition 55: If A is an $n \times n$ matrix which is orthogonally diagonalizable, then A is symmetric.

Proof: Suppose $A = PDP^{-1} = PDP^T$

$$A^T = (P^T)^T (D)^T (P)^T = PDP^T$$

$\therefore A = A^T$ = Symmetric

★ The set of eigen values of matrix A is called the Spectrum of A.

Theorem 9 (Spectral Theorem for symmetric matrices)

An $n \times n$ symmetric matrix A has following properties →

(a.) Eigen spaces are mutually orthogonal
(b.) A has n real eigen values, counting AM's
(c.) A is orthogonally diagonalizable
(d.) AM = GM for each λ

Corollary 9.1 → A is orthogonally ~~diag~~ diagonalizable if and only if A is symmetric

★ Quadratic Forms

A quadratic form on R^n is a function Q from R^n to R

$$Q(\vec{x}) = x^T A x$$ → Symmetric $n \times n$ matrix

⇒ A change of variable in R^n is an eq^n of the form
$\vec{x} = P\vec{y}$ or $\vec{y} = P^{-1}\vec{x}$

$$\vec{x}^T A \vec{x} = (Py)^T A (Py) = \vec{y}^T (P^T A P) \vec{y}$$

Since A is ~~det~~ symmetric, P can be selected as orthogonal matrix such that $P^T A P = P^{-1} A P = D$ where D is diagonal matrix

Thus $x^T A x = y^T D y$

Theorem 10 (Principal Axis Theorem)

Let A be a symmetric $n \times n$ square matrix. Then there is ~~no~~ an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms quadratic form $x^T A x$ into $y^T D y$ which has no cross product terms (D being diagonal matrix)

The columns of P in theorem are called principal axes of quadratic form $x^T A x$.
The vector $\vec{y}$ is the coordinate vector of any vector x relative to a normal basis of R^n given by the axes.

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Classification →

(i) Positive definite if $Q(\vec{x}) > 0$ for all $x \neq 0$
(ii) -ve definite if $Q(\vec{x}) < 0$ for all $x \neq 0$
(iii) Indefinite if $Q(\vec{x})$ can be +ve or -ve
(iv) +ve semidefinite if $Q(\vec{x}) \geq 0$ for all x
(v) -ve semidefinite if $Q(\vec{x}) \leq 0$ for all x

Proposition 56 → Let A be a symmetric $n \times n$ matrix. $Q(\vec{x}) = x^T A x$ is:

(i) +ve definite if and only if eigen values of A are all +ve
(ii) -ve " " " " " " " " " " -ve
(iii) Indefinite if both values are possible.

★ Constrained Optimization of quadratic forms is finding min and max value for $Q(x)$ when x is unit vector. → m → M

Proposition 57 → Let A be a symmetric matrix and let m and M be as above. Then M is greatest eigen value of A and m is least eigen value of A and x are the corresponding eigen vector