Review: Key Definitions

- **Elementary Row Operations**: Given any m×n matrix A:
  - Multiplication of one row of A by a **non-zero** scalar c (**scale**)
  - Replacement of one row of A by the sum of the row and a scalar multiple of a **different** row (**replace**)
  - Interchange of two rows of A (**interchange**)
- An m×n matrix A is said to be in **echelon form** if:
  - All non-zero rows are above any all-zero rows
  - Each leading entry (***i.e. first non-zero entry***) of a row is to the right of the leading entry of the row above it
- An m×n matrix A is said to be an **RREF matrix** if:
  - A is in echelon form
  - The leading entry in each non-zero row is 1
  - Each column which contains such a leading entry (necessarily it has to be 1) has *all its other entries as 0*

Row Reduction Algorithm -1

- *The Algorithm we are about to present is commonly referred to as Gauss-Jordan elimination.*
- The input for the algorithm is an $m \times n$ matrix. The algorithm proceeds by carrying out a sequence of ***elementary row operations only*** starting with the input matrix.
- We will use the term **pivot position** to indicate a position corresponding to a leading position in an echelon form. Its column is called a **pivot column**.
- At the start, move any all-zero rows to the bottom of the matrix using interchange operations, since they will not play any further role. (*This is like a Step 0 which may be carried out before Step 1 in each repetition within the forward phase.*)

Row Reduction Algorithm - 2

- Step 1: Start with the left-most non-zero column; it will be the pivot column.
- Step 2: Use an interchange to make the top element of the pivot column non-zero (this will be the pivot position).
- Step 3: Use replacement operations to make all entries in the pivot column below the pivot position as 0's.
- Step 4: Cover the row containing the pivot position and all rows above it. Repeat Steps 1 to 4 for the uncovered sub-matrix. Iterate until all the non-zero rows have been processed.

(Steps 1 to 4 constitute the forward phase, which produces a matrix in echelon form - this portion is referred to as ***Gaussian Reduction*** or ***Gaussian Elimination*****)**

Row Reduction Algorithm - 3

- Step 5: Use scaling operations to make all the pivot elements 1. (*This step is referred to as* ***normalization*** *of the pivot elements.)*

- Step 6: Starting with the **right-most** pivot, create zeroes in the entire column above it by using replacement operations with the pivot row. Repeat this step moving **leftward** and **upward**.

(Steps 5 and 6 constitute the backward phase, which produces an RREF matrix)

Note: The Algorithm will stop after a finite number of steps. When it stops, we have obtained an RREF matrix.

Conclusion

- Definition: If A and B are m×n matrices, B is said to be **row equivalent** to A if B can be obtained from A by a **finite sequence of row operations.**
- **Proposition 1:** Gauss-Jordan Elimination row-reduces any given m×n matrix A to an RREF matrix.
- **Proposition 1 (*Alternative Statement*):** Given any m×n matrix A , there exists an RREF matrix which is row-equivalent to A.
- **Justification (Proof):** Applying the Gauss-Jordan Elimination Algorithm to A, we must terminate after a finite number of steps, and when it terminates, we have an RREF matrix row-equivalent to A.

Row Equivalence - 1

- **Proposition 2:** Row equivalence is an **equivalence relation** on the set $\mathbb{R}^{m\times n}$ of $m\times n$ matrices with entries from the set $\mathbb{R}$ of real numbers (for all $m,n \in \mathbb{Z}^+$).
- **NB**: Later on we will occasionally work with the field of complex numbers $\mathbb{C}$, i.e. we will take matrices with complex entries. Proposition 2 will continue to hold with $\mathbb{R}$ replaced by $\mathbb{C}$.
- **Proof**: You should be able to verify (prove) the above proposition (left as an exercise - *try it yourself !*)
- **Remark 1**: Recall that every equivalence relation induces a **partition** of the underlying set, the parts of the partition being the equivalence classes, i.e. the equivalence classes are pair-wise disjoint subsets whose union is the whole set. Conversely, given any partition of a set, there exists a corresponding equivalence relation. (*Again, try to justify this remark yourself !*)

Row Equivalence - 2

- **Remark 2**: In fact, the RREF matrix of any given matrix is unique, i.e. a matrix cannot be row-equivalent to two distinct RREF matrices. Alternatively, two distinct RREF matrices cannot be row-equivalent to each other.
- Remark 2 is very important, and is one of the reasons why the RREF matrix is so useful. We shall see a proof of Remark 2 later (*it is a little difficult, and requires some ideas which we haven't yet studied*).
- **Concluding Remark**: So, inside each equivalence class for this equivalence relation, there is a distinctive member, i.e. the one and only RREF matrix in it. This fact can be used to determine whether two matrices are row-equivalent to each other.