MTH 100 - Notes on Sequences

<u>Basic Concept</u>: A sequence is an object which contains a term for each +ve integer n. The notation we will use for a sequence s is $\langle a_n \rangle$ - this indicates the sequence whose general term is a_n. ~~The~~ Another way to indicate a sequence is $s = \langle a_1, a_2, \cdots \rangle$.

At times, the general term is given by a formula. For example, $a_n = \frac{1}{n}$ or $\langle \frac{1}{n} \rangle$ - in this sequence, the n-th term is $\frac{1}{n}$.

It can also be indicated by a pattern, e.g. $s = \langle 1, 2, 3, 1, 2, 3, 1, 2, 3, \cdots \rangle$.

Here, the pattern is easily understood, but not so easy to give via a formula.

As can be seen, the term in a sequence need not be distinct.

Sequences arise in many mathematical and scientific situations and applications.

We will use the notation $\mathbb{R}^\infty$ for the set of all sequences with real terms.

Similarly, $\mathbb{C}^\infty$ would be the set of all sequences with complex terms.

We will briefly look into two aspects of sequences: algebraic and convergence.

Algebra of Sequences.

All sequences under consideration will be in $\mathbb{R}^\infty$. But the approach is similar for $\mathbb{C}^\infty$, $\mathbb{Q}^\infty$, etc.

We can define ~~two~~ the following algebraic operations on $\mathbb{R}^\infty$:- If $\langle a_n \rangle$, $\langle b_n \rangle$ are sequences and $c \in \mathbb{R}$, then:

(i) ~~the sum sequence~~ Addition:

$$\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle,$$ i.e. the sequence whose general term is $a_n + b_n$

(ii) Scalar multiplication:

$$c \langle a_n \rangle = \langle c\, a_n \rangle,$$ i.e. the sequence whose general term is $c\, a_n$.

(iii) Multiplication:

$$\langle a_n \rangle \cdot \langle b_n \rangle = \langle a_n b_n \rangle,$$ i.e. the sequence whose general term is $a_n \cdot b_n$.

Remark: ~~Thus~~ Considering (i) and (ii) above, $\mathbb{R}^\infty$ is a vector space over the field $\mathbb{R}$. This was covered in detail in the tutorials.

Convergence of Sequences

Convergence is concerned with the behaviour of a sequence as n gets very large.

A sequence is said to converge, if there exists a real number L such that the terms of the sequence lie ultimately in any interval about L, however small. L is called the limit of the sequence. It is easy to see that any convergent sequence has precisely one limit.

The phrases: $\langle a_n \rangle$ is convergent or L is the limit of $\langle a_n \rangle$ or $\langle a_n \rangle$ converges to the limit L mean exactly the same thing.

Notations: $a_n \to L$, $\langle a_n \rangle \longrightarrow L$,

$$\lim_{n \to \infty} a_n = L.$$

A sequence which is not convergent is st'd divergent or we can say that the sequence diverges.

Convergence (continued)

Examples: $\langle \frac{1}{n} \rangle \longrightarrow 0$

$\langle 1, 2, 3, 0, 0, 0, \ldots \rangle \longrightarrow 0$

Examples of Divergent sequences:

$\langle n \rangle$ or $\langle 1, 2, 3, \ldots, \ldots \rangle$

$\langle 1, 2, 1, 2, 1, 2, \ldots \rangle$

As the above examples illustrate, divergence can occur because ~~of~~ the terms in the sequence are unbounded or because of oscillatory behaviour (or a mixture of both).

The following basic results about limits are presented without proof. They follow from the basic idea that if $a_n \to L$ and ϵ is any +ve real number, however small, then the terms of $\langle a_n \rangle$ lie <u>ultimately</u> in the small interval $(L-\epsilon, L+\epsilon)$, i.e. there is some +ve integer k such that <u>all the terms</u> $a_k, a_{k+1}, a_{k+2}, \ldots$, etc must lie in the interval $(L-\epsilon, L+\epsilon)$.

Proofs can be found in standard textbooks of calculus, advanced calculus, or analysis.

Proposition: Suppose $\langle a_n \rangle \to L_1$, $\langle b_n \rangle \to L_2$, and $c \in \mathbb{R}$. Then:

(i) $\langle a_n \rangle + \langle b_n \rangle$ is convergent and

$$a_n + b_n \longrightarrow L_1 + L_2$$

(ii) $c\langle a_n \rangle$ is convergent and

$$c\, a_n \longrightarrow c L_1$$

(iii) $\langle a_n \rangle \cdot \langle b_n \rangle$ is convergent and

$$a_n b_n \longrightarrow L_1 L_2$$

(iv) If $a_n \neq 0$ for all n and $L_1 \neq 0$, then the sequence $\langle \frac{1}{a_n} \rangle$ is well-defined and $\frac{1}{a_n} \longrightarrow \frac{1}{L_1}$.

We use $\underline{c}$ (italics) for the subset of $\mathbb{R}^\infty$ consisting of all convergent sequences. Using (i) and (ii) above, we can show that $\underline{c}$ is a (vector) subspace of $\mathbb{R}^\infty$. Using Prop. 8:-

1. The zero sequence $\langle 0 \rangle = \langle 0, 0, \ldots \rangle$ is convergent, i.e. $\langle 0 \rangle \in \underline{c}$
2. Closure under addition follows from (i)
3. Closure under scalar multiplication follows from (ii).

Convergence (continued):-

In principle, it is quite difficult to show that a sequence is convergent, since we have to first identify a possible limit L and then verify that L is indeed the limit. Hence, results that show a sequence is convergent without necessarily finding the limit are very ~~I~~ useful. ~~#~~ One of the most useful math results is:

VIP :- Suppose $\langle a_n \rangle$ is a monotonically non-decreasing sequence, in $\mathbb{R}$, which is bounded above, i.e. $a_n \leq a_{n+1}$ for all n and $a_n < M$ for some fixed real number M for all n.
Then: $\langle a_n \rangle$ is convergent.
The above is called the LUB property of the real number system.

Example: $\langle a_n \rangle$ where $a_n = (1+\frac{1}{n})^n$.
It can be shown that $a_n \leq a_{n+1}$ for all n and $a_n < 3$ for all n. $\therefore$ the sequence is convergent.
Its limit is denoted by e. The real number plays a major role in mathematics.