EXAMPLES FOR LU DECOMPOSITION

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

We row-reduce ~~A~~ to echelon form w/o interchanges or scaling, and by only adding multiples of a row to a lower row at every ~~stage~~ step.

We further record all the steps:-

$$\begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 5 & +6 \end{bmatrix}$$

$$\xrightarrow[R_3 - 5R_2]{R_3 \to} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \quad \text{(echelon form)}$$

(Note: U doesn't always have 1's on diagonal)

We record the row-operations e_i and their inverses f_i :

$e_1 : R_2 \longrightarrow R_2 - R_1$ $\qquad f_1 : R_2 \longrightarrow R_2 + R_1$

$e_2 : R_3 \longrightarrow R_3 - 2R_1$ $\qquad f_2 : R_3 \longrightarrow R_3 + 2R_1$

$e_3 : R_3 \longrightarrow R_3 - 5R_2$ $\qquad f_3 : R_3 \longrightarrow R_3 + 5R_2$

Recall: the same ~~ste~~ steps which take A to U take L to I:

$\cancel{\therefore I = (E_p E_2 E_1) L \Rightarrow L = E_3}$

$$\therefore I = (E_p \cdots E_1) L \implies L = (E_p \cdots E_1)^{-1} I$$

$$\implies L = (E_1^{-1} \cdots E_p^{-1}) I.$$

(PTO)

$\therefore$ to get L we apply: $(f_1\ f_2\ f_3)\, I$

giving:
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow[R_3+5R_2]{R_3 \to} \begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}$$

$$\xrightarrow[2R_1]{R_3 \to R_3 +} \begin{bmatrix}1&0&0\\0&1&0\\2&5&1\end{bmatrix} \xrightarrow[R_2+R_1]{R_2 \to} \begin{bmatrix}1&0&0\\1&1&0\\2&5&1\end{bmatrix} = L$$

Note that L is lower-triangular with 1's on the diagonal, i.e. L is a unit lower triangular matrix.

Check:
$$LU = \begin{bmatrix}1&0&0\\1&1&0\\2&5&1\end{bmatrix}\begin{bmatrix}1&-1&-2\\0&1&1\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&-1&-2\\1&0&-1\\2&3&2\end{bmatrix} = A, \text{ as}$$

desired.

[NB: we have actually ~~done the~~ calculated L above by calculating the f's, but it can be seen that we only take $(-1) \times$ factor used in any elementary row operation to fill the initial part of every row]

Example: Solving a linear system using the LU factorization]

Recall: Solution of $A\bar{x} = \bar{b}$ is obtained by first solving $Ly = \bar{b}$, and then solving $U\bar{x} = \bar{y}$.

Find a soln. of $A\bar{x} = \bar{b}$ where $\bar{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Now $L\bar{y} = \bar{b}$ is :

$$\begin{aligned} y_1 &= 2 \\ y_1 + y_2 &= -1 \\ 2y_1 + 5y_2 + y_3 &= 1 \end{aligned}$$

We can use row-reduction but it's easier to do forward substitution

$$\boxed{\begin{aligned} y_1 &= 2 \\ y_2 &= -1 - y_1 = -1 - 2 = -3 \\ y_3 &= 1 - 2y_1 - 5y_2 = 1 - 2(2) - 5(-3) = 1 - 4 + 15 \\ &= 12 \end{aligned}}$$

Now, we solve the system $U\bar{x} = \bar{y} = \begin{bmatrix} 2 \\ -3 \\ 12 \end{bmatrix}$

Now : $U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{aligned} x_1 - x_2 - 2x_3 &= 2 \\ x_2 + x_3 &= -3 \\ x_3 &= 12. \end{aligned}$$

To solve this, we use backward substitution.

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$x_3 = 12$

$x_2 = -3 - x_3 = -15$

$x_1 = 2 + x_2 + 2x_3 = 2 + (-15) + 2(12)$
$= 11$

Check: $\begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ -15 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, as required.

If we had used ~~a~~ a general $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then

we would have got: $y_1 = b_1$
$y_2 = b_2 - b_1$
$y_3 = b_3 - 2b_1 - 5(b_2 - b_1)$.

Then, from the eqn. $U\bar{x} = \bar{y}$,

we would have got: $x_3 = y_3$

$x_2 = y_2 - x_3 = y_2 - y_3$

$x_1 = y_1 + x_2 + 2x_3 = y_1 + (y_2 - y_3) + 2y_3$.

So from the ~~e~~ entries of L and U, once they have been calculated, we can solve for any new $\bar{b}$ very easily.