Lecture 5: Recursion

Course outline

- Part 1 Introduction to Computing and Programming (first 2 weeks):
    - Problem solving: Problem statement, algorithm design, programming, testing, debugging
    - Scalar data types: integers, floating point, Boolean, others (letters, colours)
    - Arithmetic, relational, and logical operators, and expressions
    - Data representation of integers, floating point, Boolean
    - Composite data structures: string, tuple, list, dictionary, array
    - Sample operations on string, tuple, list, dictionary, array
    - Algorithms (written in pseudo code) vs. programs
    - Variables and constants (literals): association of names with data objects
    - A language to write pseudo code
    - Programming languages: compiled vs. interpreted programming languages
    - Python as a programming language
    - Computer organization: processor, volatile and non-volatile memory, I/O

Course outline (may change a bit)

- Part 2 Algorithm design and Programming in Python (balance 11 weeks):
  - Arithmetic/Logical/Boolean expressions and their evaluations in Python
  - Input/output statements (pseudo code, and in Python)
  - Assignment statement (pseudo code, and in Python)
  - Conditional statements, with sample applications
  - Iterative statements, with sample applications
  - Function sub-programs, arguments and scope of variables
  - Recursion
  - Modules
  - Specific data structures in Python (string, tuple, list, dictionary, array), with sample applications
  - Searching and sorting through arrays or lists
  - Handling exceptions
  - Classes, and object-oriented programming
  - (Time permitting) numerical methods: Newton Raphson, integration, vectors/matrices operations, continuous-time and discrete-event simulation

Recursion

- Recursion is a powerful concept, and a tool, for developing algorithms or programs
- Often certain definitions are given recursively:

Recursion

- Example 1: n factorial

  For n > 0, n! = n*(n-1)*(n-2)* … *2*1

  OR, quivalently

Called the "base case"

n! = 1, if n = 1,
 = n * (n-1)!, otherwise

Recursion

- Example 1: n factorial

  For n > 0, n! = n*(n-1)*(n-2)* … *2*1

  OR, quivalently

  Called the "base case"

  n!   = 1, if n = 1,
  = n * (n-1)!, otherwise

- Example calculation of n!
  1! = 1
  2! = 2
  3! = 6
  4! = 24
  Etc.

Recursion

- Example 2: GCD(a, b), where a > b > 1

Called the "base case"

GCD(a, b) = b, if a <u>mod</u> b = 0,
= GCD(b, a <u>mod</u> b), otherwise

Recursion

- Example 2: GCD(a, b), where a > b > 1

GCD(a, b) = b, if a <u>mod</u> b = 0,
= GCD(b, a <u>mod</u> b), otherwise

Called the "base case"

- Example calculations:

GCD (15, 6) = GCD(6, 3) = 3
GCD (42, 15) = GCD(15, 12) = GCD(12, 3) = 3
etc.

Recursion

- Example 3: Fibonacci numbers (work by an Indian mathematician in 450 BC–200 BC)

F(n) = 1, if n = 0 or n = 1,
= F(n-1) + F(n-2), otherwise (viz. n >= 2)

Called the “base case”

https://en.wikipedia.org/wiki/Fibonacci_number
Alternate definition F(0) = 0, F(1) = 1, F(2) = 1, etc.

Recursion

- Example 3: Fibonacci numbers

F(n) = 1, if n = 0 or n = 1,
= F(n-1) + F(n-2), otherwise (viz. n >= 2)

Called the "base case"

- Example calculations:

n	F(n)
0	1
1	1
2	2
3	3
4	5
Etc.	Etc.

https://en.wikipedia.org/wiki/Fibonacci_number
Alternate definition F(0) = 0, F(1) = 1, F(2) = 1, etc.

Recursion

- Example 3: Fibonacci numbers:
  (early work by an Indian mathematician in 450 BC–200 BC)

The "base case"

F(n) = 1, if n = 0 or 1,
     = F(n-1) + F(n-2), n >= 2

resulting in Fibonacci sequence

n	F(n)	F(n)/F(n-1)
0	1	
1	1	1.00000
2	2	2.00000
3	3	1.50000
4	5	1.66667
5	8	1.60000
6	13	1.62500
7	21	1.61538
8	34	1.61905
9	55	1.61765
10	89	1.61818
11	144	1.61798
12	233	1.61806
13	377	1.61803
14	610	1.61804
15	987	1.61803
16	1597	1.61803
17	2584	1.61803

Golden ratio:
solution to equation $x^2 = x + 1$

Also:
F(n) = 1.61803 * F(n-1)
when n is large

Recursion

- Example 4: Is a given string **S** a Palindrome?
- A string s is a Palindrome if **S** reads the same way forward and backwards
- Example Palindromes:
  '$'
  'anna'
  'kayak'

Recursion

- Example 4: Is a given string **S** a Palindrome?
- A string s is a Palindrome if **S** reads the same way forward and backwards

where c is a character

S is a Palindrome, if s = '', or
s = 'c', or
s[0] = s[-1], <u>and</u> s[1, -1] is a Palindrome
is NOT a Palindrome, otherwise

the "base cases"
len(s) <= 1

Recursion

- Example 4: Is a given string S a Palindrome?
- A string s is a Palindrome if S reads the same way forward and backwards

where c is a character

S is a Palindrome, if s = ‘’, or
s = ‘c’, or
s[0] = s[-1], and s[1, -1] is a Palindrome
is NOT a Palindrome, otherwise

the “base case”
len(s) <= 1

- Example Palindromes:
‘’
‘$’
‘anna’
‘kayak’
Etc.
- Example NON-Palindromes:
‘IN’
‘e-Rupee’
Etc.

The following MAY also be considered to be a palindrome:
‘Was it a cat I saw’

Recursion

- Recursion – based function program to compute n!

By definition:

n! = 1, if n = 1,
 = n * (n-1)!, otherwise

```
# compute n! using recursion
# Assumes that n > 0; returns n!
def fact(n):
    if n == 1:
        return(1)
    else:
        return(n*fact(n - 1))

for k in range(1, 6):
    print(fact(k))
```

Exercise to be done at home:
Rewrite the program without using recursion.

Recursion

	Local variables/objects	Global variables/objects
Main	f(.)	
fact(.)	n,	fact(.)

- Recursion – based function progra

By definition:

$$n! = 1, \text{ if } n = 1,$$
$$= n * (n-1)!, \text{ otherwise}$$

```
# compute n! using recursion
# Assumes that n > 0; returns n!
def fact(n):
    if n == 1:
        return(1)
    else:
        return(n*fact(n - 1))

for k in range(1, 6):
    print(fact(k))
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Recursion

- Recursion – based function program to compute n!

By definition:

n! = 1, if n = 1,
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# compute n! using recursion
# Assumes that n > 0; returns n!
def fact(n):
    if n == 1:
        return(1)
    else:
        return(n*fact(n - 1))

for k in range(1, 6):
    print(fact(k))
```

How is the "base case" handled? Could it be that this program will never terminate?

What ensures that the program will terminate?

Visualize its execution using:
https://tinyurl.com/47v4xxp6

Recursion

- Example 4: Is a given string **S** a Palindrome?
- A string s is a Palindrome if s reads the same way forward and backwards

s **is** a Palindrome, if s = ‘’, or
s = ‘c’, or
s[0] = s[-1], <u>and</u> s[1, -1] is a Palindrome
is NOT a Palindrome, otherwise

- Algorithmically:
  <u>if</u> (len[s] <= 1) <u>then</u> s is palindrome
  <u>else</u> <u>if</u> ((s[0] = s[-1]) <u>and</u> (string s[1: -1] is a palindrome)) <u>then</u> s is palindrome

The “base case”

Recursion

- Recursion – based function sub-program to test is given string s is a Palindrome
- Algorithmically:

  if (len[s] <= 1) then s is palindrome

  else if ((s[0] = s[-1]) and (string s[1: -1] is a palindrome)) then s is palindrome

```
# determine whether a given string is a palindrome
def isPal(s):
    """Assumes s is a str only of lower-case English letters.
    It returns True if s is a palindrome, False otherwise."""
    if len(s) <= 1:
        return(True)
    else:
        return((s[0] == s[-1]) and isPal(s[1:-1]))

print(isPal("wasitacatisaw"))
print(isPal("was it a cat i saw"))
```

https://tinyurl.com/3n74b3yu

Recursion

- Recursion –based algorithm/program to test whether a given string s is a Palindrome
  - This time it allows strings to be form: ‘**Was it a cat I saw’**, or ‘**dog God’**

```
# Palindrome 2
# Determine if S is a palindrome
def isPalindrome(s):
    """Assumes s is a str
    Returns True if s is a palindrome; False otherwise.
    Punctuation marks, blanks, and capitalization are ignored."‶″
#
    def toChars(s):
        s = s.lower()
        letters = '`
        for c in s:
            if c in 'abcdefghijklmnopqrstuvwxyz':
                  letters = letters + c
        return letters
#
    def isPal(s):
        print(' isPal called with', s)
        if len(s) <= 1:
            print(' About to return True from base case')
            return True
        else:
            answer = s[0] == s[-1] and isPal(s[1:-1])
            print(' About to return', answer, 'for', s)
            return answer
    return(isPal(toChars(s)))
#
Print(isPalindrome('dog..God`))
```

```
Built-in ‶method″ for
strings, as in
txt = "Hello my FRIENDS‶
x = txt.lower()
print(x)
>>>hello my friends
```

Let us visualize the execution flow using http://pythontutor.com/visualize.html

Recursion

- Example: Fibonacci numbers:

F(n) = 1, if n = 0 or 1,
= F(n-1) + F(n-2), n >= 2

resulting in Fibonacci sequence

n	F(n)	F(n)/F(n-1)
0	1	
1	1	1.00000
2	2	2.00000
3	3	1.50000
4	5	1.66667
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Golden ratio:
solution to equation $x^2 = x + 1$

Also:
F(n) = 1.61803 * F(n-1)
when n is large

https://en.wikipedia.org/wiki/Fibonacci_number
In some cases F(0) = 0, F(1) = 1, F(2) = 1, etc.

Recursion

- Recursion – based function sub-program to compute Fibonacci numbers
- By definition:

  F(n) = 1, if n = 0 or 1,

  = F(n-1) + F(n-2), n >= 2

```
# Fibonacci-1
# compute F(k), for k in [0, 1, 2, 3, 4]
def fib(x):
    """Assumes int x >= 0
       Returns F(x)"""
    if x == 0 or x == 1:
        return(1)
    else:
        return(fib(x-1) + fib(x-2))

for k in range(5):
    print('fib of', k, '=', fib(k))
```

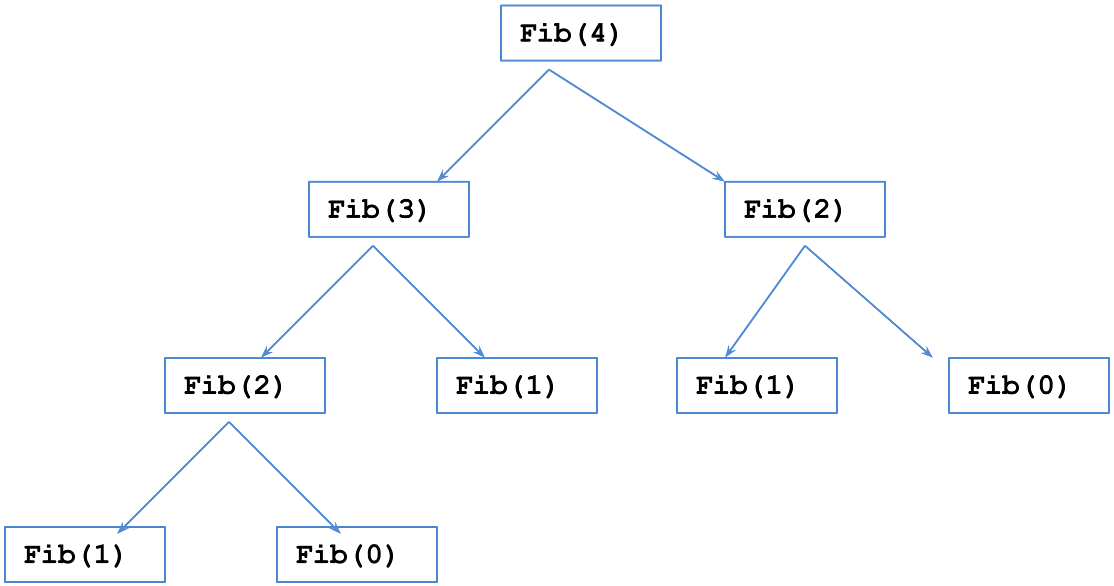
1. How is the “base case” handled?
2. Could it be that this program will never terminate?

```
Output:
fib of  0 = 1
fib of  1 = 1
fib of  2 = 2
fib of  3 = 3
fib of  4 = 5
```

http://tinyurl.com/yc78yey7

Recursion

- How many times is **`fib(n)`** called for **`n=4?`**
  - More generally, number of times fib(.) is called, M(n):
    M(n) = 1, of n <= 1
    = 1 + M(n-1) + M(n-2), otherwise



n	M(n)
0	1
1	1
2	3
3	5
4	9
5	15
6	25
7	41
8	67
9	109
10	177
11	287
12	465
13	753

Recursion

- Global variables vs. local variables
- Counting no. of calls, while computing Fibonacci numbers

```
# Fibonacci-2
# Compute F(k), k >= 0, while counting no. of calls to fib(.)
def fib(x):
global numFibCalls
     numFibCalls = numFibCalls + 1
    if x == 0 or x == 1:
        return(1)
    else:
        return(fib(x-1) + fib(x-2))

global numFibCalls
for k in range(5):
    numFibCalls = 0
    print('fib of', k, '=', fib(k))
    print("fib called", numFibCalls, "times.")
```

```
Output:
fib of  0 = 1
fib called 1 times.
fib of  1 = 1
fib called 1 times.
fib of  2 = 2
fib called 3 times.
fib of  3 = 3
fib called 5 times.
fib of  4 = 5
fib called 9 times.
```

http://tinyurl.com/czv4wd4j

Recursion

- Global variables vs. local variables
- Counting no. of calls, while computing Fibonacci numbers

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# Compute F(k), k >= 0, while counting no. of calls to fib(.)
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    else:
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global numFibCalls
for k in range(5):
    numFibCalls = 0
    print('fib of', k, '=', fib(k))
    print("fib called", numFibCalls, "times.")
```

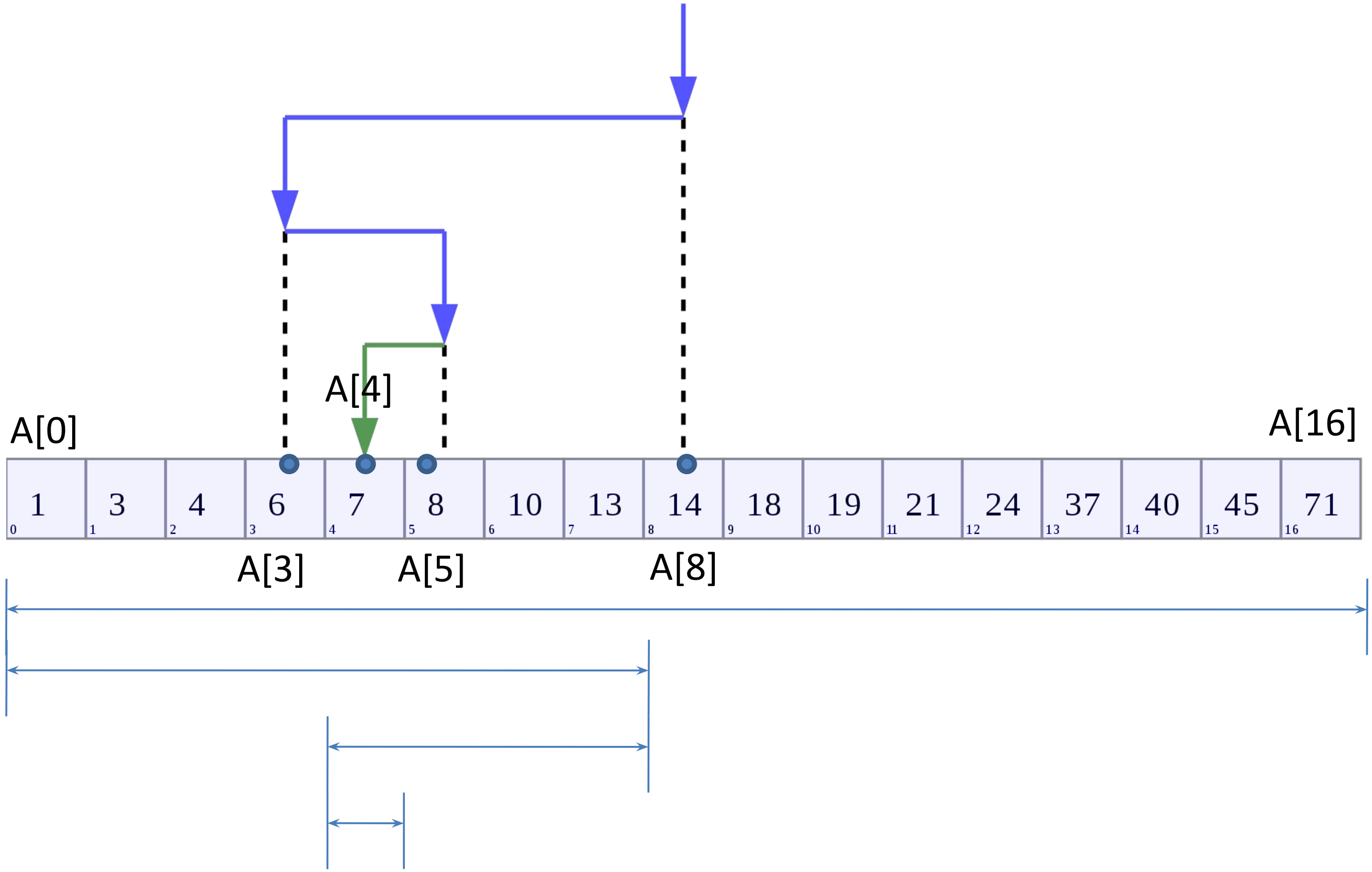
Advice: Use global variables sparingly. Since indiscriminate use of 'global' variables is likely to result in incorrect results.

```
Outpu
             1
               times.
             1
    called 1 times.
fib of  2 = 2
fib called 3 times.
fib of  3 = 3
fib called 5 times.
fib of  4 = 5
fib called 9 times.
```

Recursion

- Recursion (as a method) to solve some simple problems:
  - Search for a data object using “binary search” in an array (for the present array == vector)

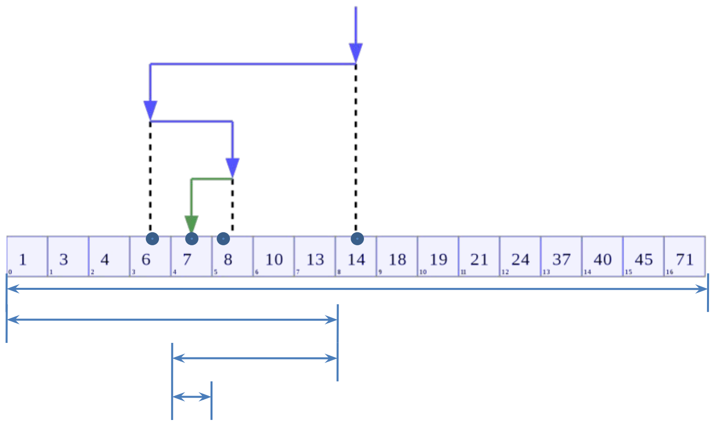
https://en.wikipedia.org/wiki/Binary_search_algorithm



Recursion

- Recursion (as a method) to solve some simple problems:
  - Search for a data object using “binary search” in an array (for the present array == vector)

```
# Binary search -2
# Use iteration to locate of data, T, in list A of size n
def binary_search(A, n, T):
    L = 0
    R = n-1
    while L <= R:
        m = int(((L + R) / 2))
        if A[m] < T:
            L = m + 1
        else:
            if A[m] > T:
                R = m-1
            else:
                return("Located at:" , m)
    return("Not found")
#
n = 17
A = [1, 3, 4, 6, 7, 8, 10, 13, 14, 18, 19, 21, 24, 37, 40, 45,
result = binary_search(A, n, 7)
print(result)
#
result = binary_search(A, n, 11)
print(result)
```

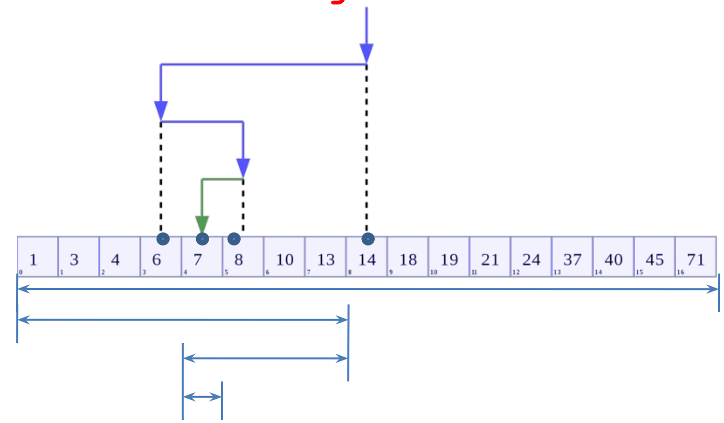


Based on while loop

Recursion

- Recursion (as a method) to solve some simple problems:
    - Search for a data object using “binary search” in an array (for the present array == vector)

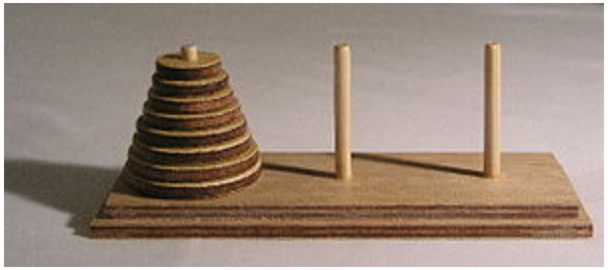
```
# Binary search -2
# Use recursion to locate of data, T, in list A indexed L through R
def binary_search(A, L, R, T):
    if L > R:
        return(-1)
    m = int(((L + R) / 2))
    if T == A[m]:
        return(m)
    if T < A[m]:
        R = m - 1
    else:
        L = m + 1
    return(binary_search(A, L, R, T))
#
A = [1, 3, 4, 6, 7, 8, 10, 13, 14, 18, 19, 21, 24, 37, 40, 45, 71]
n = len(A) # indexed A[0] through A[16]
print(binary_search(A, 0, 16, 7)) # list, low, high, data
#
print(binary_search(A, 0, 16, 9))
```



Using recursion

Recursion

- Recursion (as a method) to solve some difficult problems:
  - Tower of Hanoi problem: move N disks from tower A to tower C, possibly using tower B:
    - Only one disk can be moved at a time
    - Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod
    - No larger disk may be placed on top of a smaller disk



https://en.wikipedia.org/wiki/Tower_of_Hanoi

https://www.freecodecamp.org/news/analyzing-the-algori
hm-to-solve-the-tower-of-hanoi-problem-686685f032e3/
https://www.geeksforgeeks.org/python-program-for-towe
-of-hanoi/

For no. of disks = 1 or = 2 it is easy
For no. of disks = 3 it is not too difficult
For no. of disks = 4 or more, it is seems impossible, unless …

Here is the algorithm:

If n = 1, "move the disk to Tower C"
Else {"move the top n-1 disk from Tower A to Tower B";
"move the remaining 1 disk from Tower A to Tower C"
"move the n-1 disks from Tower B to Tower C"
}

Recursion

If n = 1, “move the disk to Tower C”
else {“move the top n-1 disk from Tower A to Tower B”;
“move the remaining 1 disk from Tower A to Tower C”
“move the n-1 disks from Tower B to Tower C”}

```
# ToH
# Solving the Tower of Hanoi puzzle
def TowerOfHanoi(n , src, dest, aux):
    print("new ToH with n = ", n, "src = ", src, "dest = ", dest)
    if n == 1:
        print("Move disk 1 from src", src,"to dest", dest)
        return
    TowerOfHanoi(n-1, src, aux, dest)
    print("Move disk",n,"from src", src,"to dest", d
    TowerOfHanoi(n-1, aux, dest, src)
#
n = 4
#Initial peg = ‘A’, Final peg = ‘B’, Via peg = ‘C’
TowerOfHanoi(n,'A','B','C')
```

Output:

```
Move disk 1 from src A to dest C
Move disk 2 from src A to dest B
Move disk 1 from src C to dest B
Move disk 3 from src A to dest C
Move disk 1 from src B to dest A
Move disk 2 from src B to dest C
Move disk 1 from src A to dest C
Move disk 4 from src A to dest B
Move disk 1 from src C to dest B
Move disk 2 from src C to dest A
Move disk 1 from src B to dest A
Move disk 3 from src C to dest B
Move disk 1 from src A to dest C
Move disk 2 from src A to dest B
Move disk 1 from src C to dest B
```

https://www.geeksforgeeks.org/python-program-for-tower-of-hanoi/

Recursion

If n = 1, "move the disk to Tower C"
else {"move the top n-1 disk from Tower A to Tower B";
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    TowerOfHanoi(n-1, src, aux, dest)
    print("Move disk",n,"from src", src,"to dest", dest)
    TowerOfHanoi(n-1, aux, dest, src)
#
n = 4
#Initial peg = 'A', Final peg = 'B', Via peg = 'C'
TowerOfHanoi(n,'A','B','C')
```

Sequence of calls to TowerOfHanoi
new ToH with n = 4 src = A dest = B
new ToH with n = 3 src = A dest = C
new ToH with n = 2 src = A dest = B
new ToH with n = 1 src = A dest = C
new ToH with n = 1 src = C dest = B
new ToH with n = 2 src = B dest = C
new ToH with n = 1 src = B dest = A
new ToH with n = 1 src = A dest = C

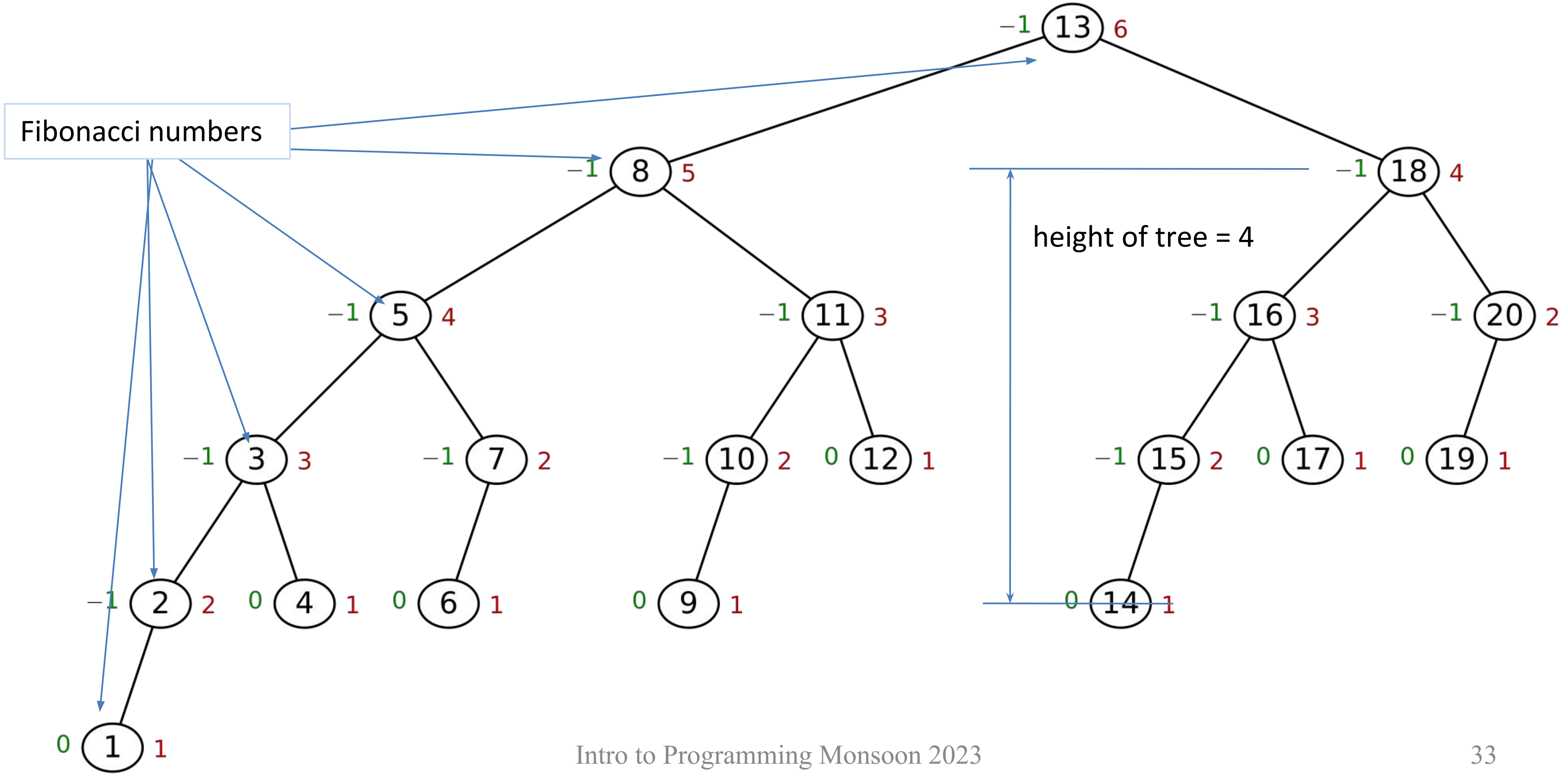
https://www.geeksforgeeks.org/python-program-for-tower-of-hanoi/

Q&A

- On recursion
- On n!
- On palindromes
- On Fibonacci(k)
- On binary search
- On Tower of Hanoi

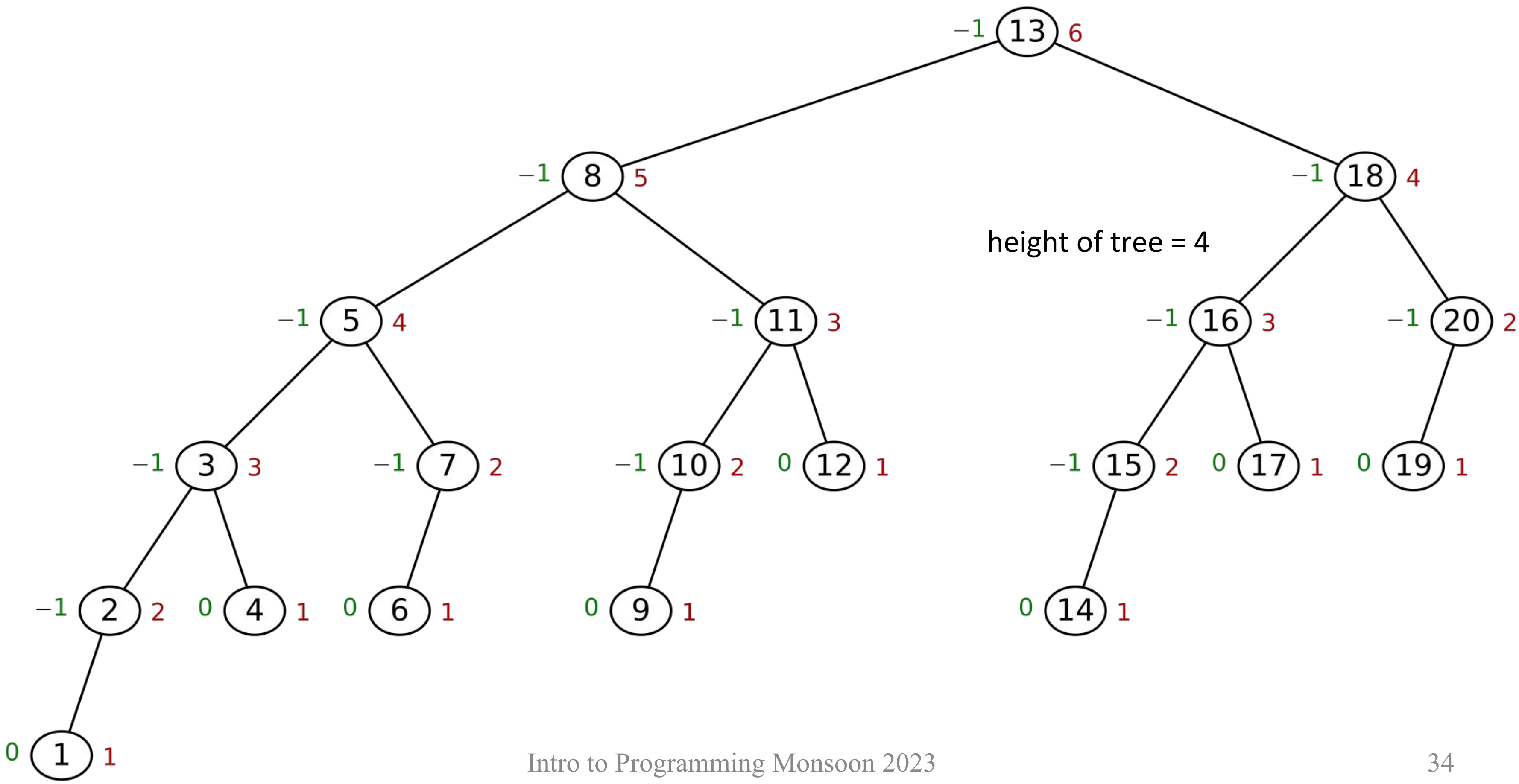
Recursion

- Applications of Fibonacci numbers (see also https://en.wikipedia.org/wiki/Fibonacci_number):
  - Fibonacci tree is a binary tree whose LEFT & RIGHT sub-trees differ in height by exactly 1
    - An AVL tree is a “binary search tree” which is also height-balanced (i.e. the difference in height of LEFT & RIGHT sub-trees is at most 1) - particularly useful in maintaining a telephone “directory” to which add/delete and search ops are efficient



Recursion

- Recursion (as a method) to solve some simple problems:
  - Search through a binary search tree for an object stored in the "node" in the tree.



Recursion

- Recursion (as a method) to solve some simple problems:
  - Convert a string consisting of digits into the corresponding integer value
    - For example:
    - **Int-value('345')**
      = Int-value('34')*10 + int('5')
      = (Int-value('3')*10 +int('4'))*10 + int('5')
      = (int('3')*10 +int('4'))*10 + int('5')

For more examples: see
https://pythonexamples.org/python-recursion/,
or simply https://pythonexamples.org/

Recursion

- Recursion (as a method) to solve some simple problems:
  - Sort an array using "merge sort" (for the present array == vector)

    https://en.wikipedia.org/wiki/Merge_sort

12	4	65	32	45	12	22	89	9	34	91	23	44	14	45	66

split

12	4	65	32	45	12	22	89

9	34	91	23	44	14	45	66

sort

sort

4	12	12	22	32	45	65	89

9	14	23	34	44	45	66	91

merge

4	9	12	12	14	22	23	32	34	44	45	45	65	66	89	91

Recursion

- Recursion (as a method) to solve some difficult problems:
    - Eight Queens problem
        - Involves recursion & back-tracking

https://en.wikipedia.org/wiki/Eight_queens_puzzle