Very Important Theorem – Ver 1.0

- **Theorem 1**: The following are equivalent for an m×m square matrix A:

  a. A is invertible

  b. A is row equivalent to the identity matrix

  c. The homogeneous system A**x** = **0** has only the trivial solution

  d. The system of equations A**x** = **b** has at least one solution for every **b** in R^m.

Calculation of the Inverse Matrix - I

In order to calculate the inverse of a matrix, we use the following result:

- **Corollary 1.1:** An invertible matrix A is a product of elementary matrices. Any sequence of row operations that reduces A to I also transforms I into A^{-1}.
- **NB**: We are implicitly using Theorem 1(b) here.

Proof of Corollary 1.1

- **Proof:** If A is invertible, then by VIT, A is row equivalent to the identity, i.e. $I = (e_p\, e_{p-1} \ldots\ldots e_1)A$ for some sequence of elementary row operations. If E_1 to E_p are the corresponding elementary matrices, then $I = (E_p \ldots\ldots E_1)A$. Each E_i being invertible, we can write $A = (E_p \ldots\ldots E_1)^{-1} I = E_1^{-1} \ldots\ E_p^{-1}$

Hence A is a product of elementary matrices.

Furthermore, $A^{-1} = (E_1^{-1} \ldots\ E_p^{-1})^{-1}$

$$= (E_p \ldots\ldots E_1) = (E_p \ldots\ldots E_1)I = (e_p\, e_{p-1} \ldots\ldots e_1)I$$

In other words, the same sequence of row operations that reduces A to I also reduces I to A^{-1}.

Calculation of the Inverse Matrix - II

- **Method:** Form the augmented matrix [A : I] (this is sometimes known as the *enlarged matrix* of A) and carry out elementary row operations till the A part becomes I. The final result has the form $[I : A^{-1}]$.
- **Example**: A numerical example was presented in the lecture (see Notes).

Invertible Matrices – cont'd

- **Corollary 1.2**: If A has a left inverse or a right inverse, then it has an inverse.
- **Corollary 1.3**: Suppose a square matrix A is factored as a product of square matrices, i.e. $A = A_1 A_2 \ldots\ldots A_n$ (***all square matrices***) with $n \geq 2$. Then A is invertible if and only if each A_i is invertible.
  - **Note:** The above Corollary 1.3 applies only if the matrices A_i are square. We had earlier seen that if each A_i is invertible, then so is A. So we only need to show that if A is invertible, then so is each A_i.

Proof of Corollary 1.2 – Case 1

- **Corollary 1.2**: If A has a left inverse or a right inverse, then it has an inverse.
- **Proof**: ***Case 1***: Suppose A has a left inverse.

Then there exists a matrix C such that CA = I.

Let $\mathbf{v}$ be any solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$, so $A\mathbf{v} = \mathbf{0}$.

Multiplying on the left by C, we get:

$(CA)\,\mathbf{v} = C\mathbf{0}$

$\Rightarrow I\mathbf{v} = \mathbf{0} \;\Rightarrow \mathbf{v} = \mathbf{0}$

In short, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Hence, by VIT, A is invertible.

Furthermore, $I = CA = A^{-1}A$.

Multiplying on the right by A^{-1}, we get $C = A^{-1}$

- **Remark:** *We have shown a little more: A is invertible, and its inverse is equal to its left inverse.*

Proof of Corollary 1.2 – Case 2

- **Corollary 1.2**: If A has a left inverse or a right inverse, then it has an inverse.
- **Proof**: ***Case 2***: Suppose A has a right inverse.

  Then there exists a matrix D such that $AD = I$.

  In other words, D has a left inverse.

  So D is invertible by ***Case 1***.

  Hence, $(AD)D^{-1} = ID^{-1}$ or $A = D^{-1}$

  Thus, A, being the inverse of an invertible matrix, is itself invertible, and $A^{-1} = D$.
- **Remark:** *Again, we have shown a little more: A is invertible, and its inverse is equal to its right inverse.*