MTH100 A_ 20230816 _ L05 _ WED (1)

Example from <u>MON</u> left as an exercise

$$R = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ — Already an RREF matrix

x_1, x_2, x_5 basic; x_3, x_4, x_6 free.

We have to get 3 vectors on RHS

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}_3} + x_4 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}_4} + x_6 \underbrace{\begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{u_6}$$

We can check that $\bar{u}_3, \bar{u}_4, \bar{u}_6$ are solutions (see top of page (2) of these notes).

But observe, Put $\bar{w} = x_3\bar{u}_3 + x_4\bar{u}_4 + x_6\bar{u}_6$, where the x_i are arbitrary real numbers (parameters)

Then: $R\bar{w} = R(x_3\bar{u}_3 + x_4\bar{u}_4 + x_6\bar{u}_6)$

$= x_3 R\bar{u}_3 + x_4 R\bar{u}_4 + x_6 R\bar{u}_6$

$= x_3 . \bar{0} + x_4 . \bar{0} + x_6 \bar{0} = \bar{0}$,

i.e. $\bar{w}$ is again a solution.

Check: $R\bar{u}_3 = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$R\bar{u}_4 = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$Ru_6 = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓ ~~* See also page 4.~~

An example for Non-homogeneous System:

$A\bar{x} = \bar{b}$, where $\bar{b} = \begin{bmatrix} 23 \\ 16 \\ 24 \end{bmatrix}$

Work with augmented matrix $[A : \bar{b}]$

$$= \left[\begin{array}{cccc:c} 0 & 5 & 10 & 8 & 23 \\ 1 & 2 & 6 & 7 & 16 \\ 2 & 4 & 12 & 6 & 24 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc:c} 1 & 2 & 6 & 7 & 16 \\ 0 & 5 & 10 & 8 & 23 \\ 2 & 4 & 12 & 6 & 24 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1]{R_3 \to} \left[\begin{array}{cccc:c} 1 & 2 & 6 & 7 & 16 \\ 0 & 5 & 10 & 8 & 23 \\ 0 & 0 & 0 & -8 & -8 \end{array}\right] \xrightarrow[R_3 \to -1/8 R_3]{R_2 \to \frac{1}{5}R_2}$$

$$\left[\begin{array}{cccc:c} 1 & 2 & 6 & 7 & 16 \\ 0 & 1 & 2 & 8/5 & 23/5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc:c} 1 & 2 & 6 & 0 & 9 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc:c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] = [\cancel{R}\ S : \bar{b}_1]$$

→

Convert $S\bar{x} = \bar{b}_1$ to a linear system: ~~4~~ 3

$$\begin{aligned} x_1 &= 3 - 2x_3 \\ x_2 &= 3 - 2x_3 \\ x_3 &= 0 + x_3 \\ x_4 &= 1 + 0.x_3 \end{aligned} = \underbrace{\begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{\bar{u}} + x_3 \underbrace{\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}}$$

Check: $A\bar{u} = \begin{bmatrix} 0 & 5 & 10 & 8 \\ 1 & 2 & 6 & 7 \\ 2 & 4 & 12 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 23 \\ 16 \\ 24 \end{bmatrix}$ ✓

So: $\bar{u}$ is a solution of $A\bar{x} = \bar{b}$ and $\bar{v}$ is a solution of $A\bar{x} = \bar{0}$ (See notes for MON: 0814)

Also: $A(\bar{u} + x_3\bar{v}) = A\bar{u} + x_3 A\bar{v}$

$= \bar{b} + x_3\bar{0} = \bar{b}$

→ So, x_3 acts as a parameter.
Infinitely many solutions.