Non-Homogeneous Systems

In this case, we work with the augmented matrix and reduce it to an RREF matrix, say R:

- **Proposition 4** (Existence and Nature of Solutions): The system is consistent if and only if the rightmost column of R **is not a pivot column**, i.e. if there is no row of the form

  [0 ……… 0 b] with b non-zero.

If the system is consistent, then it has either (i) a unique solution if there are no free variables or (ii) infinitely many solutions when there is at least one free variable.

Remark: The main idea behind the above proposition will be explained with the help of examples (one was done on Wednesday and one one today). *You may try the proof as an exercise or refer to the textbook.*

Observation 5: The non-homogeneous system A$\mathbf{x}$ = $\mathbf{b}$ can be inconsistent in either of the two cases of the associated homogenous system A$\mathbf{x}$ = $\mathbf{0}$ having unique solution or infinitely many solutions.

Vector Interpretation of Solutions

- **Remark**: Using Gauss-Jordan Elimination, the general solution of a linear system is obtained compactly in vector form, which is also standard as it always produces the same answer.
- Let $A\mathbf{x} = \mathbf{b}$ be a non-homogeneous system, with associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Assume that the system $A\mathbf{x} = \mathbf{b}$ is consistent so that it has at least one solution $\mathbf{u}$. By necessity, $\mathbf{u} \neq \mathbf{0}$.
- **Observation 6**: A vector $\mathbf{w}$ is a solution of the system $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{w}$ is of the form $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{v}$ is a solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.
- In case the homogeneous system has only the trivial solution, then $\mathbf{v} = \mathbf{0}$, and there is a unique solution $\mathbf{u}$. Otherwise we have infinitely many solutions. For the infinite case, this was illustrated by Wednesday’s example.

Geometric Interpretation of Solutions

- We can have a geometrical interpretation in case we are working with 2-tuples or 3-tuples. In this case, each vector corresponds to a point in either 2-space or 3-space.
- **Observation 7:** Then, the solution of a homogeneous system is either the origin only or all the points on a line or a plane through the origin.
- **Observation 8:** If a non-homogenous system has even a single solution (point), then its entire solution set consists of either only that point or the line or plane through that point which is parallel to the solution of the associated homogeneous system.

Example

- We solve the system: $x_1 + x_2 + x_3 = 1$
  $2x_1 - x_2 + x_3 = 2$

in the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix} = \mathbf{u} + t\mathbf{v}$ (t scalar)

where **u** is a solution of the system and **v** is a solution of the associated homogeneous system

- Reminder from *Coordinate Geometry*:

Equation for the Line through $P_0(x_0,y_0,z_0)$ parallel to a given vector $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ (i.e. the line segment from (0,0,0) to (A,B,C) is **:**

$$x = x_0 + tA,\ y = y_0 + tB,\ z = z_0 + tC,\ -\infty < t < \infty$$

- The solution we have obtained corresponds to the geometrical equation of the line through (1,0,0) which is parallel to the vector determined by (−2/3, −1/3,1).

Linear Systems - Summary

A. **Homogeneous Systems:** System is always consistent. If the system has a unique solution, then it is the trivial solution of all zeroes – in this case the RREF is either the n×n identity matrix I_n itself or has I_n as its upper portion with only zero rows below. Else, the system contains free variables and has infinitely many solutions; this happens when number of non-zero rows in the RREF is less than the number of variables.

- **Non-Homogeneous Systems:** The system is consistent if and only if the **rightmost column** of the RREF matrix of the augmented matrix is **not** a pivot column. If the system is consistent, then it has either (i) a unique solution if there are no free variables or (ii) infinitely many solutions when there is at least one free variable.