[redacted]

Proof of VIT:

We will proceed as follows:

(a) $\Rightarrow$ (c) $\Rightarrow$ (h) $\Rightarrow$ (a)

[(a) $\Leftrightarrow$ (d), i.e. (a) $\Rightarrow$ (d) $\Rightarrow$ (a) will be done later.]

(a) $\Rightarrow$ (c) Given: A is invertible.

To Prove: $A\bar{x} = \bar{0}$ has only the trivial solution

Suppose $\bar{y}$ is any solution of $A\bar{x} = \bar{0}$

$\therefore A\bar{y} = \bar{0}$

Multiply on left by A^{-1}!

$\therefore A^{-1}(A\bar{y}) = A^{-1}\bar{0} = 0$

LHS $= (A^{-1}A)\bar{y} = I\bar{y} = \bar{y}$.

$\therefore \bar{y} = \bar{0}$, as required.

(c) $\Rightarrow$ (h) Given: the homogeneous system $A\bar{x} = \bar{0}$ has only the trivial solution.

To prove: A is row-equivalent to I.

Now, if R is the RREF matrix of A, then $R\bar{x} = \bar{0}$ has only the trivial solution

$\Rightarrow$ R has no free variables

$\Rightarrow$ R has only basic variables

$\Rightarrow$ R has a leading 1 in each row (there are m rows)

$\Rightarrow$ R has exactly one 1 in each column (since no. of columns $= m$)

$\Rightarrow$ R is I_m

(h) $\Rightarrow$ (a)

Given: A is row-equivalent to I.

To prove: A is invertible.

Now, A is row equivalent to I

$\Rightarrow$ there are elementary row operations $e_p, e_{p-1}, \ldots, e_2, e_1$ s.t.

$$e_p(e_{p-1}(\cdots(e_2(e_1)A)\cdots)) = I \quad ①$$

If E_i is the elementary matrix corresponding to e_i, we can write ① as :-

$$E_p(E_{p-1}(\cdots\cdots(E_2(E_1A))\cdots)) = I$$

(using Prop. 5)

$$\therefore (E_p \cdots E_1)A = I \quad ②$$

Putting $B = E_p \cdots E_1$, we get from Prop. 6 and Observation 4 for Invertible Matrices, that B is invertible.

From ②, $BA = I$.

Multiplying by B^{-1} on the left,

$$B^{-1}(BA) = B^{-1}$$

$$\Rightarrow A = B^{-1}$$

Hence, A, being the inverse of an invertible matrix, is itself invertible (Observation 2).