Review: Definition of Vector Space

- A **vector space** is a non-empty set V of objects called **vectors** *together with an associated field* F of scalars, with two operations called addition and scalar multiplication. For **u, v** $\in$ V and c,d $\in$ F, the following properties hold:

A. **Closure under the Operations: u** + **v** $\in$ V and c**u** $\in$ V.

B. **The following properties hold for addition:**

a) associative property: (**u** + **v**) + **w** = **u** + (**v + w**)

b) identity property: there exists a "zero" vector which satisfies

$$\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} \text{ for all vectors } \mathbf{u} \in V$$

c) Every vector **u** $\in$ V has an additive inverse vector **v** $\in$ V such that **u** + **v** = **0**

d) Commutative property: **u** + **v** = **v** + **u**

Definition of Vector Space - 2

C. The following additional properties are satisfied:

a) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

b) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

c) $c(d\mathbf{u}) = (cd)\mathbf{u}$

d) $1\mathbf{u} = \mathbf{u}$ (where 1 indicates the unit element of F)

Examples of Vector Spaces - 1

1. The space $\mathbb{R}^n$ of ordered n-tuples of size n (for any $n \geq 1$). You would already have used the term vectors for these objects. The base field is $\mathbb{R}$.

If $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ are any two vectors

then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$ and $c\mathbf{u} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$ for any scalar c.

- **Note:** All the axioms can easily be verified. These are the standard examples of vector spaces, frequently referred to as Euclidean spaces.

Examples of Vector Spaces - 2

2. The space $\mathbb{R}^{m \times n}$ of m×n matrices with real entries. Again, the base field is $\mathbb{R}$.

Exercise: *Verify the vector space axioms for* $\mathbb{R}^{m \times n}$.

Remark: The fact that matrices form a vector space is of fundamental importance in both the theory and applications of linear algebra.

For example, if we consider an image, it can be regarded as a rectangular array of numbers corresponding to the light intensity at each pixel. Usually, we restrict the values to be positive integers or even just 0-1. However, while doing the computations in image processing, we treat them as real numbers. In other words, images are simply regarded as matrices in mathematics and in computer science. So these vector spaces play a major role in image processing.

Examples of Vector Spaces - 3

3. The space C[0,1] of continuous functions from the closed interval [0,1] on the real line to $\mathbb{R}$, i.e.

   $C[0,1] = \{f: f \text{ is a continuous function}, f: [0,1] \rightarrow \mathbb{R}\}$. The base field for this vector space is $\mathbb{R}$.

   Remark: This space and related spaces play a major role in signals and systems, since an analogue signal is usually thought of as a continuous function of time. In other words, a signal is nothing but a "vector" in such a vector space. Sometimes, a different interval is used, such as $[0,2\pi]$. In proving general results for such spaces, we typically use C[a,b].

   Note: The above is an example of a "function" space.

   Exercise: Verify the vector space axioms for C[0,1].

Consequences of the Vector Space Axioms

Proposition 7: Let V be a vector space over F. Then:

a) The zero vector is unique; we will use the notation $\mathbf{0}$ for the zero vector

b) The additive inverse vector of any vector $\mathbf{u} \in V$ is unique; we use the notation $-\mathbf{u}$ for the inverse vector

c) $0\mathbf{u} = \mathbf{0}$ for every vector $\mathbf{u} \in V$

d) $c\mathbf{0} = \mathbf{0}$ for every scalar $c \in F$

e) $-\mathbf{u} = (-1)\mathbf{u}$ for every vector $\mathbf{u}$

Proof: Left as an exercise. (*Hint: Observe the similarity with fields.*)

Remark: In future, when we use the term ***vector***, we simply mean: an element of a vector space. Sometimes a specific vector space, e.g. one of the above examples, is indicated. Else, it would be a general or abstract vector space V.