MTH100A **Math 1 (Linear Algebra)** **Monsoon Semester – 2023-24**

Tutorial Exercise for Tuesday 20230919

1. Given the following vectors in $\mathbb{R}^3$: **u** = (1,3,5), **v** = (1,4,6), **w** = (2, –1, 3) and **b** = (6,5,17).
   a) Does **b** $\in$ W = span {**u**, **v**, **w**}
   b) If the answer to b) is yes, express **b** as a linear combination of **u**,**v**,**w**.

2. Let U and W be two subspaces of the vector space V. Show that U $\cap$ W is also a subspace of V.

3. Obtain an LU decomposition of the symmetric matrix A given below. Find <u>**three**</u> conditions on a, b, c, d to ensure that an LU decomposition exists, i.e. A can be reduced to an echelon form matrix <u>**without row interchanges**</u>.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 1 & b & c & d \end{bmatrix}$$

4. In the following is W a subspace of V? Base field is $\mathbb{R}$ in all. Justify your answer.

   a. V = $\mathbb{R}_n[t]$ = vector space of polynomials of degree $\le$ n, W = {p(t) $\in$ V: deg p(t) = n} $\cup$ {**0**(t)}. Here **0**(t) indicates the zero polynomial.

   b. V = $\mathbb{R}^3$, W = {(x,y,z): x, y, z $\in \mathbb{Q}$}.

   c. V = $\mathbb{R}^3$, W = {(x,y,z): xy = 0}.

   d. V = $\mathbb{R}^3$, W = {(x,y,z): $x^2 + y^4 + z^6 = 0$}

5. Consider the space V of all 2×2 matrices over $\mathbb{R}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.
   - All symmetric matrices (**Definition**: For any m×n matrix A = $[a_{ij}]$, its **transpose** is the n×m matrix B = $[b_{ij}]$, given by $b_{ij} = a_{ji}$. The standard notation for the transpose of A is A^T. A matrix is symmetric if $A = A^T$.)
   - All A such that AB = BA where B is some fixed matrix in V
   - All A such that BA = 0 where B is some fixed matrix in V
   - Would the above results hold for all n×n matrices where n is a general positive integer

6. Consider the space V of all n×n matrices over $\mathbb{R}$. and let W be the subset consisting of all upper triangular matrices.
   a) Show that W is a subspace of V.
   b) Show further that W satisfies closure with regard to products and multiplicative inverses, i.e. if A, B $\in$ W, then AB $\in$W, and if A $\in$ W happens to be invertible, then $A^{-1} \in$ W.

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1. Given the following vectors in R^3: $\mathbf{u} = (1,3,5)$, $\mathbf{v} = (1,4,6)$, $\mathbf{w} = (2, -1, 3)$ and $\mathbf{b} = (6,5,17)$.
   a) Does $\mathbf{b} \in W = \text{span}\ \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$
   b) If the answer to b) is yes, express **b** as a linear combination of **u,v,w**.

2. Let U and W be two subspaces of the vector space V. Show that $U \cap W$ is also a subspace of V.

3. Obtain an LU decomposition of the symmetric matrix A given below. Find four conditions on a, b, c, d to ensure that an LU decomposition exists, i.e. A can be reduced to an echelon form matrix **without row interchanges**.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 1 & b & c & d \end{bmatrix}$$

4. In the following is W a subspace of V? Base field is $\mathbb{R}$ in all. Justify your answer.
   a. $V = R_n[t]$ = vector space of polynomials of degree $\leq n$, $W = \{p(t) \in V: \deg p(t) = n\} \cup \{\mathbf{0}(t)\}$. Here $\mathbf{0}(t)$ indicates the zero polynomial.
   b. $V = R^3$, $W = \{(x,y,z): x, y, z \in \mathbb{Q}\}$.
   c. $V = R^3$, $W = \{(x,y,z): xy = 0\}$.
   d. $V = R^3$, $W = \{(x,y,z): x^2 + y^4 + z^6 = 0\}$

5. Consider the space V of all 2×2 matrices over $\mathbb{R}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.
   - All symmetric matrices (**Definition**: For any $m\times n$ matrix $A = [a_{ij}]$, its **transpose** is the $n\times m$ matrix $B = [b_{ij}]$, given by $b_{ij} = a_{ji}$. The standard notation for the transpose of A is A^T. A matrix is symmetric if $A = A^T$.)
   - All A such that AB = BA where B is some fixed matrix in V
   - All A such that BA = 0 where B is some fixed matrix in V
   - Would the above results hold for all $n\times n$ matrices where n is a general positive integer

6. Consider the space V of all $n\times n$ matrices over $\mathbb{R}$. and let W be the subset consisting of all upper triangular matrices.
   a) Show that W is a subspace of V.
   b) Show further that W satisfies closure with regard to products and multiplicative inverses, i.e. if $A, B \in W$, then $AB \in W$, and if $A \in W$ happens to be invertible, then $A^{-1} \in W$.

7. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$ (YES/NO) ? Justify your answer briefly.

8. Let $V = \{x \in \mathbb{R} : x > 0\}$. Define addition for V by $x \oplus y = xy$, and scalar multiplication by any $\alpha \in \mathbb{R}$ by $\alpha * x = x^\alpha$.

(a) (7 marks) Verify the closure axioms, the commutative, zero and inverse properties for addition, and the property $1 * x = x$ for all $x \in V$.
(***Remark***: *V is in fact a vector space over the field* $\mathbb{R}$. *However, you need not verify the other properties of a vector space.)*
(b) (3 marks) Is V a subspace of $\mathbb{R}$ regarded as a vector space over itself (YES/NO) ? Justify your answer clearly.

(*This question was given as an exam problem for a previous batch.*)

9. Prove Remark 6 related to linear dependence/independence : Any list which contains a linearly dependent list is linearly dependent.

10. Prove Remark 7 related to linear dependence/independence : Any subset of a linearly independent set is linearly independent .

11. Determine whether the given matrices in the vector space $\mathbb{R}^{2\times 2}$ are linearly dependent or linearly independent.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

12. In the vector space $V = C[0, 2\pi]$, determine whether the given vectors (i.e. functions) are linearly dependent or linearly independent :

$f_1(x) = 1$, $f_2(x) = \sin(x)$, $f_3(x) = \sin(2x)$.

(You must justify your answer.)

13. Let $F = \mathbb{Z}_2$, and consider the vector space $V = F^4$, the space of all ordered 4-tuples with entries from F.

a) Suppose $\mathbf{v} \in V$, $\mathbf{v} \neq \mathbf{0}$. What can you say about the additive inverse of $\mathbf{v}$?
b) Consider the vectors $\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0, 0)$ and $\mathbf{v}_3 = (0. 0. 1. 1)$. Determine Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ and Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
c) Construct subspaces U and W of V which have 3 and 5 vectors, respectively.
d) Apply what you have learned from a), b), and c) to state and prove a result about the possible orders of subspaces of V. ***Note***: For any finite set X, the **order** of X is the number of elements in X, notation $|X|$.
e) Generalize your result in d) to subspaces of F^n for any arbitrary positive integer n. Can you prove your result ? ***Hint:*** *Attempt a bounded stepwise ascent type of proof (you may recall that the proof of* ***Corollary 1.3*** *was a bounded stepwise descent).*

[Solutions follow, may not be in same order.]

Q5. $V = \mathbb{R}^{2\times 2}$ = vector space of all 2×2 matrices with real entries. (1)

(a) Put $U = \{A \in V : A^T = A\}$.

$\boxed{U \text{ is}}$ often notated as $Sym_2(\mathbb{R})$, or $Sym_n(\mathbb{R})$, in the general case.

We apply Prop. 8:

(i) Clearly, $[0]^T = [0]$, where $[0]$ is the matrix with all zeroes. So $[0] \in U$.

(ii) Let $A = [a_{ij}]$ and $B = [b_{ij}] \in U$. (1)

Then: $(A+B)^T = ([a_{ij}] + [b_{ij}])^T$

$= [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}]$

$= [a_{ij} + b_{ij}]$, since $a_{ji} = a_{ij}$, $b_{ji} = b_{ij}$ because of (1)

$= A + B$

$\therefore A+B \in U$

(iii) If $c \in \mathbb{R}$, then $(cA)^T = (c[a_{ij}])^T$

$= [ca_{ij}]^T = [ca_{ji}] = [ca_{ij}]$, by (1)

$= cA$. $\therefore cA \in U$.

$\rightarrow U$ is a subspace.

(2)

b) Put $W = \{A \in V : AB = BA$, where B is a fixed matrix in $V\}$.

(i) Clearly, $[0]B = B[0] = [0]$, ~~so~~ $[0] \in W$

(ii) Let $A, C \in W$.
Then

$(A+C)B = AB + CB = BA + BC = B(A+C)$

$\Rightarrow A + C \in W$

iii) Let $A \in W$ and $c \in \mathbb{R}$

$\Rightarrow AB = BA$

Now $(cA)B = c(AB) = c(BA) = (cB)A = B(cA)$

$\Rightarrow cA \in W$

Hence, W is a subspace of V.

3

c) $W = \{A \in V \mid BA = 0\}$

i) Clearly, $0 \in W$ as $B0 = 0$

ii) Let $A, C \in W \Rightarrow BA = 0,\ BC = 0$

$\Rightarrow B(A+C) = BA + BC = 0 + 0 = 0 \Rightarrow A+C \in W$

(iii) Let $A \in W$ and $c \in \mathbb{R}$

$\Rightarrow BA = 0$

Now, $B(cA) = (Bc)A = c(BA) = c \cdot 0 = 0$

$\Rightarrow cA \in W$

Hence, W is a subspace of V.

d) Yes, all the above results will hold for $n \times n$ matrices

Q6. Let $V = \mathbb{R}^{n\times n}$ = space of all $n\times n$ matrices with real entries, and let W be the subset of all upper triangular matrices.

(a) Show that U is a subspace.

Answer: Note that a matrix ~~is upper~~ $A = [a_{ij}]$ is upper triangular if and only if $a_{ij} = 0$ whenever $j < i$. (1)

We will apply Proposition 8.

(i) Clearly $[0]$ satisfies (1), so $[0] \in W$

(ii) Let $A, B \in W$, and put $A+B = C = [c_{ij}]$. When $j < i$, $c_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0$. $\therefore$ $C \in W$

(iii) If $c \in \mathbb{R}$, then $cA = c[a_{ij}] = [ca_{ij}]$. When $j < i$, $ca_{ij} = c.0 = 0$.

$\therefore$ $cA \in W$.

Hence, W is a subspace of V.

Q6 (h) To show W satisfies closure under products and inverses.

Ans: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be in W. Put $AB = C = [c_{ij}]$. To show that $C \in W$, we need to consider c_{ij} for $j < i$. So suppose $j < i$, $i, j = 1, \dots, n$.

Then $c_{ij} = (a_{i1}b_{1j} + \dots + a_{i\,i-1}b_{i-1\,j})$ I

$+$

$(a_{ii}b_{ij} + a_{i\,i+1}b_{i+1\,j} + \dots + a_{in}b_{nj})$ II

Now, terms in parenthesis I are all 0,

~~since $a_{i1}, \dots, a_{i\,i-1} = 0$ as $A \in W$~~

since $a_{i1}, a_{i2}, \dots, a_{i\,i-1}$ are all 0 as $A \in W$.

Similarly, terms in parenthesis II are all 0, since b_{ij}, $b_{i+1\,j}, \dots, b_{nj}$ are all 0 since $B \in W$.

$\therefore c_{ij} = 0$ as required.

Q6 (h) <u>cont'd</u>

Finally, suppose $A \in W$ is invertible.

Now, $A = \begin{bmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{bmatrix}$ in form.

Since A is invertible, A is row-equivalent to I_n by VIT. ∴ there must be a pivot element in each row, i.e. $a_{ii} \neq 0$ for all i. In order to row-reduce A to I_n, we only need to normalize the pivot elements, and perform the backward phase of the Gauss-Jordan Elimination Algorithm. Every elementary row operation required is either a scale or a ~~rows~~ replacement of the form $R_i \to R_i + cR_k$, where $k > i$. ∴ each such row operation corresponds to an elementary matrix which is upper triangular.

$$\therefore \text{ if } I = (e_p e_{p-1} \cdots e_1)A = (E_p \cdots E_1)A = EA.$$

Now, E being a product of upper triangular matrices, is also upper triangular, by the first part of Q6(h). But $E = A^{-1}$.

∴ A^{-1} is upper triangular, as reqd.

Tutorial [redacted] 05

Q(1) a) Let $b \in W = \text{span}\{u, v, w\}$

$\Rightarrow b = x_1 u + x_2 v + x_3 w, \quad x_i \in \mathbb{R}, \; i = 1, 2, 3$

$\Rightarrow b = A\mathbf{x},$

where $A = [u \; v \; w]_{3\times 3}$

Consider
$$[A : b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 3 & 4 & -1 & 5 \\ 5 & 6 & 3 & 17 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 1 & -7 & -13 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2$$

So, the system $A\mathbf{x} = b$ reduces to

$x_1 + 9x_3 = 19 \Rightarrow x_1 = 19 - 9x_3$

$x_2 - 7x_3 = -13 \Rightarrow x_2 = -13 + 7x_3$

$$\bar{\mathbf{x}} = \begin{bmatrix} 19 \\ -13 \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 7 \\ 1 \end{bmatrix} x_3, \quad x_3 \in \mathbb{R} \quad \text{— (1)}$$

Hence, $b = \bar{x}_1 u + \bar{x}_2 v + \bar{x}_3 w$, for all $\bar{x}$ given by (1),

i.e $b \in W$.

(b) $b = 19u - 13v + 0w$ $\left(\text{for } \bar{\mathbf{x}} = \begin{bmatrix} 19 \\ -13 \\ 0 \end{bmatrix}\right)$

Q(2) i) As U & W are subspaces therefore $0 \in U \cap W$.

ii) Let $u, v \in U \cap W$

$\Rightarrow u, v \in U$ and $u, v \in W$

$\Rightarrow u + v \in U$ and $u + v \in W$ (as U & W are subspaces)

$\Rightarrow u + v \in U \cap W$

iii) Let $u \in U \cap W$ and $c \in F$

$\Rightarrow cu \in U$ and $cu \in W$ (U & W are subspaces)

$\Rightarrow cu \in U \cap W$

Hence, $U \cap W$ is a subspace.

Q 4 a) $V = \mathbb{R}_n[t]$

$W = \{p(t) \in V \mid \deg p(t) = n\} \cup O(t)$

Let $p(t) = x^{n-1} + x^n \in W$ and $q(t) = -x^n \in W$

$\Rightarrow p(t) + q(t) = x^{n-1} \notin W$ ($\deg = n-1$)

$\Rightarrow W$ is not a subspace of V.

(b) Let $(1,1,1) \in W$ and $\sqrt{2} \in \mathbb{R}$

then $\sqrt{2} \cdot (1,1,1) = (\sqrt{2}, \sqrt{2}, \sqrt{2}) \notin W$

$\Rightarrow W$ is not a subspace of $\mathbb{R}^3(\mathbb{R})$

(c) Let $(1,0,0)$ and $(0,1,0) \in W$. Then,

$(1,0,0) + (0,1,0) = (1,1,0) \notin W$ (as $1 \cdot 1 \neq 0$)

(d) Let $(x,y,z) \in W \Rightarrow x^2 + y^4 + z^6 = 0 \Rightarrow x = y = z = 0$

$\Rightarrow W = \{(0,0,0)\}$ is a subspace of V. (as $x^2, y^4, z^6 \geq 0$ and their sum $= 0$

$W = \{A \in V \mid A \text{ is upper triangular matrix}\}$

Q (4) (a) Yes. i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an upper triangular matrix.

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$$

ii) $$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}} \underset{}{+} \underset{B}{\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{bmatrix} \in W$$

iii) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \in W$ and $c \in \mathbb{R}$

$\Rightarrow cA = \begin{bmatrix} ca_{11} & ca_{12} \\ 0 & ca_{22} \end{bmatrix} \in W$, and hence W is a subspace of V.

b) $W = \{A \in V \mid AB = BA,\ B \text{ is fixed}\}$

i) Clearly, $O_{2\times2} \in W$ as $O_{2\times2}B = BO_{2\times2}$.

ii) Let $A, C \in W \Rightarrow AB = BA$ and $CB = BC$

3

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - R_1} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U$$

The operation are

$e_1 : R_2 \to R_2 - R_1$

$e_2 : R_3 \to R_3 - R_1$

$e_3 : R_4 \to R_4 - R_1$

$e_4 : R_3 \to R_3 - R_2$

$e_5 : R_4 \to R_4 - R_2$

$e_6 : R_4 \to R_4 - R_3$

The inverse operations are

$f_1 : R_2 \to R_2 + R_1$

$f_2 : R_3 \to R_3 + R_1$

$f_3 : R_4 \to R_4 + R_1$

$f_4 : R_3 \to R_3 + R_2$

$f_5 : R_4 \to R_4 + R_2$

$f_6 : R_4 \to R_4 + R_3$

So, $L = f_1 f_2 \cdots f_6 I$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{R_4 \to R_4 + R_3} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix} \xrightarrow{R_4 \to R_4 + R_2} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&1&1\end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&1&1\end{bmatrix} \xrightarrow{R_4 \to R_4 + R_1} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\1&1&1&1\end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix} = L$$

So, $$\boxed{\begin{bmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{bmatrix} = \begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&0&d-c\end{bmatrix}}$$

For four pivots we need $a \neq 0,\ b-a \neq 0,\ c-b \neq 0,\ d-c \neq 0$

i.e. $\boxed{a \neq 0,\ b \neq a,\ c \neq b \text{ and } d \neq c}$

Clarify ~~...~~ No condition $d \neq c$

TUT05 - SOLUTION

Q7. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

ANSWER: $\mathbb{R}^2$ **is not a subspace of** $\mathbb{R}^3$! **This is because** $\mathbb{R}^2$ **is not even a subset of** $\mathbb{R}^3$. **However,** the **set**

$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x,y \in \mathbb{R} \right\}$$ **is a subspace of** $\mathbb{R}^3$

which behaves very much like $\mathbb{R}^2$, **but is logically distinct from** $\mathbb{R}^2$**. We will shortly find a suitable description and terminology for this type of situation.**

Q8 [redacted] Let 5

$$V = \{x \in \mathbb{R} \,:\, x > 0\},$$

and define addition for V by

$$x \oplus y := xy$$

and scalar multiplication by any $\alpha \in \mathbb{R}$ by

$$\alpha * x = x^{\alpha}.$$

(a) (7 marks) Verify the closure axioms, the commutative, zero and inverse properties for addition, and the property $1 * x = x$ for all $x \in V$.

(Remark: V is in fact a vector space over the field $\mathbb{R}$. You need not verify the other properties of a vector space.)

(b) (3 marks) Is V a subspace of $\mathbb{R}$ regarded as a vector space over itself (YES/NO) ? Justify your answer clearly.

Solution.

(a)
- First closure axiom:
  Let $x, y \in V$. Then $xy \in \mathbb{R}$ and $x > 0, y > 0 \implies xy > 0$. Therefore $xy \in V \implies x \oplus y \in V$.
- Second closure axiom:
  Let $x \in V$ and $\alpha \in \mathbb{R}$. As $x > 0$ and $x \in \mathbb{R}$, it follows that $x^{\alpha} \in \mathbb{R}$ and $x^{\alpha} > 0$ (A proof of this fact is outlined in the first chapter of the text "Principles of Mathematical Analysis" by W. Rudin - p. 22, 3rd edition). Hence $x^{\alpha} \in V \implies \alpha * x \in V$.
- Commutativity of addition:
  Let $x, y \in V$. Then
  $$x \oplus y = xy = yx = y \oplus x.$$
- Existence of additive identity:
  For every $x \in V$,
  $$x \oplus 1 = x \times 1 = x$$
  Hence 1 is a zero vector in V.
- Existence of additive inverse:
  Let $x \in V$. Since $\frac{1}{x} > 0$ it follows that $\frac{1}{x} \in V$. Further
  $$x \oplus \frac{1}{x} = 1.$$
  Hence $\frac{1}{x}$ is an additive inverse of x in V.
- Lastly, let $x \in V$. Then $1 * x = x^1 = x$.

Q8 (cont'd) 5a

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- 1 mark for each correctly verified axiom.
- Add 1/2 mark, if at least three axioms are verified correctly.
- Add 1/2 mark, if all axioms are verified correctly.

(b) When we consider the question whether V is a subspace of $\mathbb{R}$ (with the usual vector space structure on $\mathbb{R}$), we strip V of the vector space structure defined in part (a) and view it simply as a subset of $\mathbb{R}$. In order for V to be a subspace of $\mathbb{R}$ it must be a vector space under the **usual operations of vector addition and scalar multiplication which are defined on** $\mathbb{R}$.

Thus the answer to our question is the following.

No, V is not a subspace of $\mathbb{R}$, when $\mathbb{R}$ is regarded as a vector space over itself.

One reason is that V is not closed under scalar multiplication. Any of the following counterexamples can be cited.

1. Let $x \in V$. Then $0 \times x = 0$, but $0 \notin V$.
2. Let $x \in V$. Then $-1 \times x = -x$, but $-x \notin V$.
3. Let $x \in V$. Let $c \in \mathbb{R}$ be chosen such that $c \leq 0$. Then

$$x > 0 \implies cx \leq 0.$$

   Hence $cx \notin V$.
4. $13 \in V, -13 \in \mathbb{R}$, but $-169 \notin V$.

Alternatively, the fact that $0 \notin V$ can be used as a standalone justification. In this case, V is not a subspace of $\mathbb{R}$, by the definition listed on p. 220 of the course textbook (3rd edition).

Rubric:

- 1 mark for answering NO. Remaining 2 marks distributed as:
  - 1 mark for stating that V is not closed under scalar multiplication.
  - 1 mark for giving any of the above counterexamples. Please note that students may have chosen numbers other than 13 and -13. There are infinitely many correct answers.

OR

- 1 mark for answering NO. 2 marks for the reason that $0 \notin V$.

(6+7) (3)

Q9

Ques: Prove Remark 8: Any list which contains a linearly dependent list is linearly dependent

Proof: Let $\overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots, \overline{u}_k, \overline{u}_{k+1}, \ldots \overline{u}_r$ be the list which contains a l.d. list $\overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots, \overline{u}_k$, $k \geq 2$ and $r \geq k$.

Since $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k$ is l.d. list, $\exists$ constants $c_1, c_2, \ldots, c_k$ such that

$c_1\overline{u}_1 + c_2\overline{u}_2 + \cdots + c_k\overline{u}_k = 0$ where not all the c_i are zero. — (1)

Now consider the relation:

$c_1\overline{u}_1 + c_2\overline{u}_2 + \cdots + c_k\overline{u}_k + 0\,\overline{u}_{k+1} + 0\,\overline{u}_{k+2} + \cdots + 0\,\overline{u}_r = 0$ — (2)

In (2) there is at least one non-zero c_i from (1)

So, the list $\overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots, \overline{u}_k, \ldots, \overline{u}_r$ is l.d.

Q10.

Ques: Prove Remark 7: Any subset of linearly independent set is linearly independent

Proof: Suppose $\{\overline{u}_1, \ldots, \overline{u}_n\}$ is linearly independent.

Suppose the set $\{\overline{u}_{i_1}, \overline{u}_{i_2}, \ldots, \overline{u}_{i_k}\}$ is linearly dependent. Then $\exists$ scalars $c_{i_1}, \ldots c_{i_k}$ not all zero such that $c_{i_1}u_{i_1} + c_{i_2}u_{i_2} + \cdots + c_{i_k}u_{i_k} = 0$

For the indices $i \notin \{i_1, i_2, \ldots, i_k\}$, put $c_i = 0$.

Then we have $c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0$, where for some index i_j, $c_{i_j} \neq 0$.

which is contradiction to $u_1, u_2, \ldots, u_n$ are l.i.

Scanned by CamScanner

[redacted]

Que. 11. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Suppose $\alpha A + \beta B + \gamma C = \bar{0}$ —— (1)

$\bar{0}$ = zero matrix

$$\alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha+\beta+\gamma & \alpha+\gamma \\ \alpha & \alpha+\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \alpha+\beta+\gamma = 0$; $\alpha = 0$; $\alpha+\beta = 0$

$\Rightarrow \beta = 0$

$\alpha + \gamma = 0$

$\Rightarrow \gamma = 0$

$\therefore \alpha = \beta = \gamma = 0$

$\therefore$ The matrices A, B and C are linearly independent in $\mathbb{R}^{2\times2}$.

Que: [redacted]
12.

Suppose $\alpha f_1(x) + \beta f_2(x) + \gamma f_3(x) = \bar{0}(x)$ —— ①

where $f_1(x) = 1$, $f_2(x) = \sin x$

$f_3(x) = \sin 2x$; $\bar{0}(x)$ is zero function $\in C[0, 2\pi]$

equation ① holds for all $x \in [0, 2\pi]$

put $x = 0$ in ① gives $\alpha = 0$

put $x = \pi/2$ gives $\alpha + \beta = 0$

$\Rightarrow \cancel{\alpha} \beta = 0$

put $x = \pi/4$ gives $\alpha + \frac{\beta}{\sqrt{2}} + \gamma = 0$

$\Rightarrow 0 + 0 + \gamma = 0$

$\Rightarrow \gamma = 0$

$\therefore \alpha = \beta = \gamma = 0$

$\Rightarrow f_1(x)$, $f_2(x)$ and $f_3(x)$ are linearly independent

Solution 13:- Let $V = F^4$ be the space of all ordered 4-tuples with entries from $F = \mathbb{Z}_2$. V is a vector space over F with the following operation:

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$, $\mathbf{v} = (v_1, v_2, v_3, v_4) \in F^4$, $u_i, v_i \in \mathbb{Z}_2$, $1 \le i \le 4$

$$\mathbf{u}+\mathbf{v} = ((u_1+v_1) \bmod 2, (u_2+v_2) \bmod 2, (u_3+v_3) \bmod 2, (u_4+v_4) \bmod 2)$$

$$\Rightarrow \mathbf{u}+\mathbf{v} \in F^4$$

Let c be a scalar from field $\mathbb{Z}_2$. Then $c=0$ or $c=1$. Since F^4 is non-empty. Let $\mathbf{u} \in F^4$.

$c=0 \Rightarrow 0 \cdot \mathbf{u} = \mathbf{0} \in F^4$, if $c=1$ $\quad 1 \cdot \mathbf{u} = \mathbf{u} \in F^4$

Thus F^4 is closed under addition and scalar multiplication.

a) Let $\mathbf{v} \in V$ be a non-zero element.

$\mathbf{v} = (v_1, v_2, v_3, v_4)$ where $v_i \neq 0$ for some i, $1 \le i \le 4$.

Since $v_i \in \mathbb{Z}_2$. In $\mathbb{Z}_2$, each element is its own additive inverse. We see here in F^4, that each tuple is its own inverse

$$\mathbf{v}+\mathbf{v} = (v_1+v_1, v_2+v_2, v_3+v_3, v_4+v_4) = (0,0,0,0)$$

$(\because v_i + v_i = 0 \ \forall\ 1 \le i \le 4)$

b) Let $\mathbf{v_1} = (1,0,1,0)$, $\mathbf{v_2} = (1,1,0,0)$, $\mathbf{v_3} = (0,0,1,1)$

$$\text{Span}\{\mathbf{v_1}, \mathbf{v_2}\} = \{a\mathbf{v_1} + b\mathbf{v_2} : a, b \in \mathbb{Z}_2\} = \{(a+b, b, a, 0) : a, b, c \in \mathbb{Z}_2\}$$

There are 4 choices for a and b

$a=0, b=0 \rightarrow (0,0,0,0)$

$a=0, b=1 \rightarrow (1,1,0,0)$

$a=1, b=0 \rightarrow (1,0,1,0)$

$a=1, b=1 \rightarrow (0,1,1,0)$

$$\text{Span}\{\mathbf{v_1}, \mathbf{v_2}\} = \{(0,0,0,0), (1,1,0,0), (1,0,1,0), (0,1,1,0)\}$$

$\text{Span}\{v_1, v_2, v_3\} = \{av_1 + bv_2 + cv_3 : a, b, c \in \mathbb{Z}_2\}$
$= \{(a+b, b, a+c, c) : a, b, c \in \mathbb{Z}_2\}$

There are 8 choices for a, b, c.

$a=0, b=0, c=0 \rightarrow (0,0,0,0)$
$a=0, b=0, c=1 \rightarrow (0,0,1,1)$
$a=0, b=1, c=0 \rightarrow (1,1,0,0)$
$a=0, b=1, c=1 \rightarrow (1,1,1,1)$
$a=1, b=0, c=0 \rightarrow (1,0,1,0)$
$a=1, b=0, c=1 \rightarrow (1,0,0,1)$
$a=1, b=1, c=0 \rightarrow (0,1,1,0)$
$a=1, b=1, c=1 \rightarrow (0,1,0,1)$

$\text{Span}\{v_1, v_2, v_3\} = \{(0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1), (1,0,1,0),$
$(1,0,0,1), (0,1,1,0), (0,1,0,1)\}$

c) Consider a subspace W of V.

Case 1: $W = \{(0,0,0,0)\}$

If W is the zero subspace then W has one element.

Case 2:- Let W contain a non-zero element, say $\mathbf{u}$

Then, $W = \{\mathbf{0}, \mathbf{u}\}$ is a subspace of F^4. Thus W has two elements.

Case 3:- Let W contains a non-zero element other than $\mathbf{u}$, say $\mathbf{v}$ such that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

Then, W has 4 elements

$W = \{\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}\}$

Case 4:- Let W contains another non-zero element $\mathbf{z}$, such that $\mathbf{z} \neq \mathbf{u}$, $\mathbf{z} \neq \mathbf{v}$, $\mathbf{z} \neq \mathbf{u}+\mathbf{v}$.

Then $W = \{0, u, v, z, u+v, v+z, u+z, u+v+z\}$
Thus W has 8 elements.

Case 5 :- we see, whenever we add a non-zero element to the subspace which is not in the above case, the no. of elements increases by 2 times.
Thus, W can have 16 elements. In this Case,
$W = F^4$ ($\because$ F^4 itself has 16 elements).

Hence, any subspace of F^4 can have either 1, 2, 4, 8 or 16 elements.

(c) There does not exist subspaces with 3 or 5 vectors
(d) The possible orders of subspaces of V are
1, 2, 4, 8, 16

(e) Let F^n be the space of n-tuples, with entries from $\mathbb{Z}_2$.
To prove that any subspace W of F^n has 2^k elements where k can be $0, 1, 2, \ldots n$.

If $W = \{0\} \Rightarrow W$ has $2^0 = 1$ element
If $W = F^n \Rightarrow W$ has 2^n elements

Again proceeding as above. If we add a non-zero element to the zero subspace, we get a subspace W_1 with $2^1 = 2$ elements.

Adding another non-zero element, which is not in W_1, we get a subspace W_2 with $2^2 = 4$ elements

We can have subspaces like these, until we get a subspace with 2^n elements.

Thus F^n can have subspaces of order 2^k, $k = 0, 1, 2 \ldots n$.