The Axiomatic Approach - 1

- **INTRODUCTION:** Since approximately the beginning of the 20th century, mathematics has been dominated by the "axiomatic approach". In the axiomatic approach, the focus is on sets and axioms. Initially, a list of axioms is proposed, and it is stipulated that the set and its elements should obey or follow the axioms. The nature of the elements is not specified; it is only required that they should obey the axioms. All the sets governed by a specific list of axioms are designated as *systems* or *spaces* or *structures* with a specific name. Different lists of axioms lead to differently named systems or spaces. These are regarded as *abstract* or *general* systems.
- **How An Axiomatic System Works**: There are two aspects to this: a) results (theorems) are proved starting only with the axioms and using only logical reasoning; b) Previously known mathematical objects are now thought of as particular, concrete examples of a specific abstract system and the known theorems of the abstract system are applied to solve problems related to those objects.

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- **HISTORY:** We will not go deeply into how the axiomatic approach became so prominent. Broadly speaking, as the study of mathematics progressed, mathematicians relized that seemingly different mathematical objects, e.g. numbers, funtions, curves, and many others, behaved in the same way. They found that they were proving and re-proving very similar results for very different objects. This led to the further realization that it was not the nature of the objects but ***the laws which governed their behaviour*** that mattered. In the axiomatic approach, the focus is on behaviour, not on objects. Gradually the axiomatic approach gathered strength, though at different times and to a different extent in different branches of mathematics. It is generally recognized that the axiomatic approach has led to tremendous advances in mathematics since the beginning of the 20th century, and it is now accepted as the standard approach.

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- **Advantages:**

a) **Rigour:** Since mathematical proofs (arguments) are constructed from the fixed axioms by using logical reasoning only, they are more rigorous and do not suffer from the confusion which was earlier prevalent;

b) **Unification:** Since there are many different examples of a specific abstract space or structure, the same theorems apply to all of them and do not have to be re-proved time and again.

c) **Ease of Proof:** It has been found that it is frequently much easier to prove theorems in the abstract setting than for concrete or specific mathematical objects (which were studied earlier).

d) **Novelty:** By searching for new examples of a given abstract space or system, many new mathematical objects have been found which have later been found useful in the sciences as well.

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- **Interesting Systems:** Different branches of mathematics study different abstract systems – they have different names and are not necessarily called spaces.
  - **Topology** studies topological spaces and metric spaces;
  - **Measure Theory** studies measurable spaces and measure spaces;
  - **Functional Analysis** studies Banach spaces and Hilbert spaces;
  - **Discrete Mathematics** studies graphs, posets, lattices, etc;
  - **Algebra** studies groups, rings, fields, etc.
- **Commonality:** Several well-known sets are examples of more than one abstract structure and can take advantage of theorems from many branches. For instance, the real number system $\mathbb{R}$ is a metric space, a topological space, a measure space, a Banach space, a field, etc.

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- **Applying the Axiomatic Approach to Linear Algebra:** In Linear Algebra, we work with **scalars** and **vectors**; the concerned abstract systems we study are: **fields** and **vector spaces**. Vector spaces are of greater importance in LA, but since the axioms for vector spaces mention fields, we will commence with a very brief summary of fields.
- **Fields**: An informal definition of a field is that it is an algebraic system with addition and multiplication of elements, in which universal addition, subtraction, multiplication and division is possible, except that division by the zero element (usually written as 0) is not possible. An axiomatic definition is given on the next slide.

Formal Definition of Field

- A **field** is a *non-empty* set F with two binary operations called addition and multiplication (usual symbols + for addition and . or * or sometimes omitted altogether for multiplication). For any c,d,e ∈ F, the following properties hold:

A. **Closure under the Operations:** $c + d \in F$ and $c.d \in F$.

B. **The following properties hold for the operations:**

a) Associative property: $(c + d) + e = c + (d + e)$ and $(c.d).e = c.(d.e)$

b) Commutative property: $c + d = d + c$ and $c.d = d.c$

c) Zero property and identity (or unity) property: there exists a "zero" element 0 and an "identity" or "unit" element 1 (not the same as 0) which satisfy: $0 + c = c + 0 = c$ and $1.c = c.1 = c$

d) Every $c \in F$ has an additive inverse d and a multiplicative inverse e (provided $c \neq 0$) such that $c + d = 0$ and $c.e = 1$

e) Distributive property: $c.(d + e) = c.d + c.e$

Examples of Fields

- **Examples:** The most common examples of fields are the rational number system $\mathbb{Q}$, the real number system $\mathbb{R}$, and the complex number system $\mathbb{C}$. However, there are other less common examples, some of which will appear in tutorial exercises.
- **Some Sets (Systems) Which Are Not Fields:**
  - The integers $\mathbb{Z}$: reason – multiplicative inverses do not exist; for example, the integer 2 does not have a multiplicative inverse in $\mathbb{Z}$;
  - The system $\mathbb{R}^{2\times 2}$ of 2×2 matrices with real entries; firstly, not all are invertible; secondly, even if we consider only invertible matrices, two of the axioms are not satisfied: (i) the commutative property for multiplication does not hold; (ii) the sum of two invertible matrices need not be invertible.

Fields: What You Need to Know

- **Properties:** The following hold in every field F:
  - Since $1 \neq 0$, F must have at least two elements
  - The zero element is unique, and the unit element is unique
  - The additive inverse of every $c \in F$ is unique and is usually denoted by $-c$
  - The multiplicative inverse of every $c \in F$ ($c \neq 0$) is unique and is usually denoted by c^{-1}
  - $0.c = 0$ for every $c \in F$
  - F has no zero divisors (**Definition**: An element $c \in F$ ($c \neq 0$) is said to be a **zero divisor** if there exists an element $d \in F$, ($d \neq 0$), such that $c.d = 0$).

Exercise: *You can try to prove the second to sixth properties above. The first property is an axiom, and does not need to be proved. I have simply stated it here in the interest of clarity. The underlying reason is to ensure that the set {0} is not allowable as a field.*

Formal Definition of Vector Space

- A **vector space** is a non-empty set V of objects called **vectors** *together with an associated field* F of scalars, with two operations called addition and scalar multiplication. For $\mathbf{u}, \mathbf{v} \in V$ and $c,d \in F$, the following properties hold:

A. **Closure under the Operations:** $\mathbf{u} + \mathbf{v} \in V$ and $c\mathbf{u} \in V$.

B. **The following properties hold for addition:**

a) associative property: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

b) identity property: there exists a "zero" vector which satisfies

$$\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} \quad \text{for all vectors } \mathbf{u} \in V$$

c) Every vector $\mathbf{u} \in V$ has an additive inverse vector $\mathbf{v} \in V$ such that $\mathbf{u} + \mathbf{v} = \mathbf{0}$

d) Commutative property: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

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C. The following additional properties are satisfied:

a) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

b) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

c) $c(d\mathbf{u}) = (cd)\mathbf{u}$

d) $1\mathbf{u} = \mathbf{u}$ (where 1 indicates the unit element of F)