MTH100 – Section A   Math 1 (Linear Algebra)   Monsoon Semester – 2023-24

Tutorial exercises for the week commencing Monday 21st August 2023

1. Find the solution set in vector form for the homogeneous system A**x** = **0** given A below. NB: A must be row-reduced to an RREF matrix in order to give the solution in standard form.

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

2. a) Row reduce the augmented matrix of the system given below to an RREF matrix:

$$\begin{aligned} 3x + 2y + 7z + 9w &= 7 \\ 6x + 14y + 22z + 15w &= 13 \\ x + 4y + 5z + 2w &= 2 \end{aligned}$$

b) Is the system consistent or inconsistent ? If consistent, express the solution in the form of a vector **u** which is a solution of the non-homogeneous system plus scalar multiples of vector(s) which are solutions of the associated homogeneous system.

3. Repeat Q2, both parts a) and b), for the non-homogeneous system A**x** = **b**, where A and **b** are given below.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

b = (3, –3, 1) taken as a column vector

4. Row reduce the augmented matrix of the system given below to an RREF matrix:

$$\begin{aligned} x + 5y - 3z &= -4 \\ -x - 4y + z &= 3 \\ -2x - 7y \phantom{{}+z} &= a \end{aligned}$$

b) For what values of a is the above system consistent and for what values of a is it inconsistent ? Justify your answer.

5. Is it possible for a non-homogeneous system A**x** = **b**, **b** ≠ **0**, to be inconsistent when the associated homogeneous system A**x** = **0** has a unique solution (i.e. only the trivial solution) ? Answer YES or NO, and justify your answer. If YES, construct an example and verify. If NO, explain with reference to suitable propositions. (***Note: See Observation 5 in Friday's lecture.***).

6. a) Find the values of x for which the following matrix is an augmented matrix corresponding to a consistent system.

$$A = \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix}$$

b) Find the RREF of the matrix formed by replacing x in A by π.

7. Recall the following from the lecture on Monday (slightly abbreviated): **Observation 1:** If we obtain a row equivalent matrix to the coefficient matrix, then the solution sets of the two linear systems are the same. ***In fact, that is why we defined the elementary row operations in the way we did !***

a) Make the above observation rigorous by proving the following: If the matrix B has been obtained from the matrix A by an elementary row operation, then the vector **v** is a solution of the homogeneous system A**x** = **0** if and only if **v** is a solution of the homogeneous system B**x** = **0**.

b) Formulate the nonhomogeneous version of the above statement, and then prove it.

SOLUTIONS FOLLOW – MAY NOT BE IN THE SAME ORDER.

Tutorial - 2

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7/3 & -10/3 \end{bmatrix} \quad R_3 \to R_3 - \frac{7}{3} R_2$$

$$= \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2/3 \\ R_3 \to \frac{3}{7} R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 2R_2 \\ R_2 \to R_2 + \frac{4}{3} R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 15/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{bmatrix} \quad R_1 \to R_1 - \frac{1}{3} R_3$$

The reduced system will be:-

$$x + \cancel{z}\,\frac{15}{7}\,w = 0 \Rightarrow x = -\frac{15}{7}\,w$$

$$y + \left(-\frac{4}{7}\right) w = 0 \Rightarrow y = \frac{4}{7}\,w$$

$$z - \frac{10}{7}\,w = 0 \Rightarrow z = \frac{10}{7}\,w$$

$$\bar{x} = \begin{bmatrix} -15/7 \\ 4/7 \\ 10/7 \\ 1 \end{bmatrix} w$$

Q(2) $[A|b] = \left[\begin{array}{cccc|c} 3 & 2 & 7 & 9 & 7 \\ 6 & 14 & 22 & 15 & 13 \\ 1 & 4 & 5 & 2 & 2 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 3 & 2 & 7 & 9 & 7 \\ 0 & 10 & 8 & -3 & -1 \\ 0 & 10/3 & 8/3 & -1 & -1/3 \end{array}\right]$ $\quad R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - \frac{1}{3}R_1$

$\sim \left[\begin{array}{cccc|c} 3 & 2 & 7 & 9 & 7 \\ 0 & 10 & 8 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\quad R_3 \to R_3 - \frac{R_2}{3}$

$\sim \left[\begin{array}{cccc|c} 1 & 2/3 & 7/3 & 3 & 7/3 \\ 0 & 1 & 4/5 & -3/10 & -1/10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\quad R_1 \to \frac{R_1}{3}$, $R_2 \to R_2/10$

(RREF) $\sim \left[\begin{array}{cccc|c} 1 & 0 & 9/5 & 16/5 & 12/5 \\ 0 & 1 & 4/5 & -3/10 & -1/10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\quad R_1 \to R_1 - \frac{2}{3}R_2$

b) Since the rightmost column of the RREF matrix of $[A|b]$ is not a pivot column, therefore $A\mathbf{x} = b$ is consistent.

The reduced system is

$$x + \frac{9}{5}z + \frac{16}{5}w = \frac{12}{5} \Rightarrow x = \frac{12}{5} - \frac{9}{5}z - \frac{16}{5}w$$

$$y + \frac{4}{5}z - \frac{3}{10}w = -1/10 \Rightarrow y = -1/10 - \frac{4}{5}z + \frac{3}{10}w$$

$$\overline{\mathbf{x}} = \begin{bmatrix} 12/5 \\ -1/10 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} -16/5 \\ 3/10 \\ 0 \\ 1 \end{bmatrix} w, \quad z, w \in \mathbb{R}$$

$= u + v$, where u is a solution of $A\overline{\mathbf{x}} = b$ and v is a solution of $A\overline{\mathbf{x}} = \overline{0}$.

$$Au = \begin{bmatrix} 3 & 2 & 7 & 9 \\ 6 & 14 & 22 & 15 \\ 1 & 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 12/5 \\ -1/10 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 2 \\ 0 \end{bmatrix}$$

The reduced system of $Ax = 0$ is

$$x + \frac{9}{5}z + \frac{16}{5}w = 0 \Rightarrow x = -\frac{9}{5}z - \frac{16}{5}w$$

$$y + \frac{4}{5}z - \frac{3}{10}w = 0 \Rightarrow y = -\frac{4}{5}z + \frac{3}{10}w$$

$$v = \bar{x} = \begin{bmatrix} -9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} -16/5 \\ 3/10 \\ 0 \\ 1 \end{bmatrix} w, \quad z, w \in \mathbb{R}$$ is a solution of $Ax = 0$

Q3) $[A \mid b] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & -2 & 1 \end{array}\right] \qquad R_2 \to R_2 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & 5 \end{array}\right] \qquad R_3 \to R_3 - \frac{2}{3} R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2 \to \frac{R_2}{3} \\ R_3 \to \frac{R_3}{5} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \to R_1 + R_2$$

<u>RREF</u> $$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 + 2R_3 \\ R_1 \to R_1 - R_3 \end{array}$$

Since the rightmost column of the RREF matrix of $[A \mid b]$ is a pivot column, therefore $A\mathbf{x} = b$ is inconsistent.

The reduced system is

$$\begin{aligned} x + z &= 0 \\ y - z &= 0 \\ 0 &= 1 \end{aligned} \quad \to \text{(not possible)}$$

Therefore, the system $A\mathbf{x} = b$ is inconsistent.

4) a) $\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & +3 \\ -2 & -7 & 0 & a \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & a-8 \end{array}\right]$ $\quad R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & a-5 \end{array}\right]$ $\quad R_3 \to R_3 - 3R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & a-5 \end{array}\right]$ $\quad R_1 \to R_1 - 5R_2$ (RREF)

(b) System is consistent if $a-5=0$, i.e. $\boxed{a=5}$

$$x + 7z = 1 \Rightarrow x = 1 - 7z$$
$$y - 2z = -1 \Rightarrow y = -1 + 2z$$
$$0 = 0$$

$$\bar{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$$

For $a \neq 5$, the last equation will become $0 = a-5 \ (\neq 0)$ which is not possible. Hence, system is inconsistent for $a \neq 5$

Q(5) Is it possible for a non-homogenous system $Ax=b$, $b\neq 0$, to be inconsistent when the associated homogenous system $Ax=0$ has a unique (trivial) solution? Answer Yes or No, and justify your answer. If yes, construct an example and verify. If No, explain with references to suitable propositions and theorems.

<u>Sol^n:</u> Yes, a nonhomogenous system $Ax=b$, $b\neq 0$ may ~~need~~ ~~not~~ be inconsistent when the associated homogenous system $Ax=0$ has a trivial solution. For example,

$$\begin{aligned} 5x + 3y &= 1 \\ 5x + 2y &= 1 \\ 5x + y &= 2 \end{aligned}, \quad A = \begin{bmatrix} 5 & 3 \\ 5 & 2 \\ 5 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$[A:b] \sim \left[\begin{array}{cc|c} 5 & 3 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{array}\right] \qquad \begin{aligned} &R_2 \to R_2 - R_1 \\ &R_3 \to R_3 - R_1 \end{aligned}$$

$$\sim \left[\begin{array}{cc|c} 5 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \begin{aligned} &R_3 \to R_3 - 2R_2, \ R_2 \to -R_2 \\ &\text{and } R_1 \to R_1 - 3R_2 \end{aligned}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \text{RREF}$$

Clearly the system $Ax=0$ has a unique trivial solution, however, the reduced system for $Ax=b$ is

$$\begin{aligned} x &= 1 \\ y &= 0 \\ 0 &= 1 \end{aligned} \rightarrow \text{not possible, i.e. } Ax=b \text{ is inconsistent.}$$

It can be observed that if $m \leq n$ and $Ax = 0$ has unique trivial solution, then all pivot columns of the RREF matrix of the augmented matrix $[A:b]$ will correspond to the columns of A, i.e the rightmost column of the RREF matrix of $[A:b]$ is not a pivot column. Hence, in this case system will be consistent.

[Remark: The example we have constructed has $m > n$.]

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[redacted]

This was given as an exam question some time back.

[redacted] Q6 →

(a) (5 marks) Find the values of x for which the following is an augmented matrix corresponding to a consistent system:

$$\begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix}$$

(b) (5 marks) Determine the RREF of the matrix formed by substituting x with π in the matrix in part (a).

Solution.

(a) We reduce the given augmented matrix to echelon form:

$$\begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & -15 & 6 & x^3 - 4x \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & 0 & 0 & x^3 - 4x + 3x^2 \end{bmatrix}$$

The corresponding linear system is consistent if and only if the augmented column is not a pivot column. This condition holds if is x is a root of the polynomial $x^3 + 3x^2 - 4x$. Now

$$\begin{aligned} x^3 + 3x^2 - 4x &= x(x^2 + 3x - 4) \\ &= x(x+4)(x-1) \end{aligned}$$

Therefore the given matrix is an augmented matrix corresponding to a consistent linear system when $x = 0$, $x = -4$ or $x = 1$.

(b) We find the RREF of the matrix

$$\begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 4 & -23 & 10 & \pi^3 \end{bmatrix}$$

Using part (a), we know that

$$\begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 4 & -23 & 10 & \pi^3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi \end{bmatrix}$$

We continue with the row reduction process to reduce the matrix to RREF:

$$\begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{5}R_2} \begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 1 & -\frac{2}{5} & \frac{\pi^2}{5} \\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1+2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{\pi^2}{5} \\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi \end{bmatrix}$$

$$\xrightarrow{R_3 \to \dfrac{1}{\pi^3+3\pi^2-4\pi}R_3} \begin{bmatrix} 1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{\pi^2}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2-\dfrac{\pi^2}{5}R_3} \begin{bmatrix} 1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5} \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1-\left(\pi+\dfrac{2\pi^2}{5}\right)R_3} \begin{bmatrix} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

PROVE:

7a

Q7 (a) If the matrix B has been obtained from the matrix A by an elementary row operation, then the vector $\bar{v}$ is a solution of $A\bar{v} = \bar{0}$ if and only if $\bar{v}$ is a solution of $B\bar{v} = \bar{0}$.

Answer: Notation: Let ~~A~~ $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, let e be an elementary row operation, and let $B = [b_{ij}] = e(A)$. Let $\bar{v} \in \mathbb{R}^n$,

$$\bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

Proof: Forward Direction [$\Rightarrow$]

Given: $A\bar{v} = \bar{0}$ (1)

RTP: $B\bar{v} = \bar{0}$ (2)

Since e affects only one or two rows of A, we only need to consider the affected entries of $B\bar{v}$. We treat the different cases for e separately.

Case 1: e is a scale operation, say $R_i \to cR_i$ (c a non-zero scalar).

Then, the i-th entry of $B\bar{v}$ =

$b_{i1}v_1 + b_{i2}v_2 + \cdots + b_{in}v_n$

$= ca_{i1}v_1 + ca_{i2} + \cdots + ca_{in}v_n$

(PTO)

$= c(a_{i1}v_1 + \cdots + c_{in}v_n) = c \cdot 0$ (because of (1))
$= 0$, as required.

Case 2: e is an interchange, say $R_i \leftrightarrow R_k$.
We only need to consider the i-th and k-th entries of $B\bar{v}$. The i-th entry is
$b_{i1}v_1 + \cdots + b_{in}v_n = a_{k1}v_1 + \cdots + a_{kn}v_n = 0$,
because of (1). k-th entry can be handled similarly.

Case 3: e is a replacement, say $R_i \to R_i + cR_k$ for some scalar c, ~~Only~~ We need to consider only the ith entry of $B\bar{v}$
$= b_{i1}v_1 + \cdots + b_{in}v_n = (a_{i1} + ca_{k1})v_1 + \cdots + (a_{in} + ca_{kn})v_n = (a_{i1}v_1 + \cdots + a_{in}v_n) + c(a_{k1}v_1 + \cdots + a_{kn}v_n) = 0 + c \cdot 0$
$= 0$, because of (1). So $B\bar{v} = \bar{0}$ in all cases.
This completes the forward direction.

[$\Leftarrow$] Backward direction:

Given: $B\bar{v} = \bar{0}$ (3)

RTP: $A\bar{v} = 0$ (4)

Using the notation of the solution to Q4 of TUT 01, we see that if $B = e(A)$, then $A = e^{-1}(B)$. So, applying the forward direction to the matrices B and $e^{-1}(B)$, we get the required result.

7 b): Re-formulate the statement of 7a) for the ~~...~~ nonhomogeneous case, and prove the statement. (7c)

Ans: Since b) is very similar to a), the statement and proof will be presented more briefly. (Backward direction same as for a), omitted).

Statement: For a nonhomogeneous system with augmented matrix $[A:\bar{b}]$, $\bar{b} \neq \bar{0}$, $\bar{v}$ is a solution of $[A:\bar{b}]$ if and only if $\bar{v}$ is a solution of $e[A:\bar{b}]$, for any elementary row operation e.

Proof: Note that $e[A:\bar{b}] = [e(A):e(\bar{b})]$ (regard $\bar{b}$ as an $m\times 1$-matrix).

Put $\bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ for convenience.

[$\Rightarrow$] Forward direction. Given: $A\bar{v} = \bar{b}$ ①

RTP: $e(A)\bar{v} = e(\bar{b})$ ②

Case 1: Scaling. $e = R_i \to cR_i$, $c \neq 0$.

i-th entry of $e(A)\bar{v} = ca_{i1}v_1 + \cdots + ca_{in}v_n$

$= c(a_{i1}v_1 + \cdots + a_{in}v_n) = cb_i$, as required.

Case 2: Interchange. $e = R_i \leftrightarrow R_k$,

i-th entry of $e(A)\bar{v} = a_{k1}v_1 + \cdots + a_{kn}v_n = b_k$

$=$ i-th entry of $e(\bar{b})$, Similarly for k-th entry of $e(A)\bar{v}$.

Case 3: Replacement: $e = R_i \to R_i + cR_k$.

i-th entry of $e(A)\bar{v} = (a_{i1} + ca_{k1})v_1 + \cdots (a_{in} + ca_{kn})v_n = (a_{i1}v_1 + \cdots a_{in}v_n) + c(a_{k1}v_1 + \cdots + a_{kn}v_n)$

$= b_i + cb_k =$ i-th entry of $e(\bar{b})$.