|• नमः सिद्धम् |

Aryan Jain - 2022111

Linear Algebra

[MTH-100]

1. Linear Algebra.

→ straightness → (SL or plane)

Homogeneous Diffrn. Eqn. [E1]
→ Sol S1, S2
→ S1 + S2 ∈ E1 soln.
→ k * S1 ∈ E1 soln.
(scaling)

(I) Linear Eqn.

$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$

$b, a_i \rightarrow$ coeff $\in \mathbb{R}$ or $\mathbb{Z}$

$x_i \rightarrow$ vars.

Eg: $F = \frac{9}{5} C + 32$ ✓

$x + y + z = 0$ ✓

$\cancel{x^2 + y^2 = 1}$

$\cancel{\sin x = y}$

2.

$x + y = 1$
$x - y = 1$

$AX = B$

Augmented Matrix → $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 1 \end{array}\right]$

$\begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

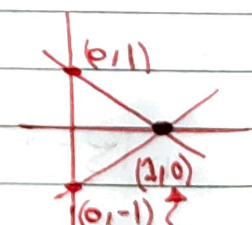
(II) System of Linear Eqn.

Colln. of 1 or more Lin Eqns in SAME variables $(x_1, x_2, \dots x_n)$

⇓

soln. is a list or a n tuple $(s_1, s_2, \dots, s_n)$ of numbers that make each statement TRUE when each $x_i \rightarrow s_i$ ✓

Eg: $x + y = 1$
$x - y = 1$



unique soln (1, 0)

3. System Of Equations

consistent or inconsistent / No Soln

consistent → 1 soln. or ∞ soln

inconsistent / No Soln ⇓ (∵ Contradicting Eqns.)

∵ 2 SL → intersect at max 1 point
2 Planes → intersect along a SL

(III) Equivalent L.E. → having the same soln. set

soln. set → set of all possible solns.

Aryan Jain - 2022111

4. ROW REDUCTION?

① eg:- (a) $x_1 - 2x_2 + x_3 = 0$

$2x_2 - 8x_3 = 8$

$5x_1 - 5x_3 = 10$

⇓

Matrix Form

$Ax = B$

$$\begin{bmatrix} +1 & -2 & +1 \\ 0 & +2 & -8 \\ +5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$$

$A \cdot X = B$ — column vector (0, 8, 10)

⇓

(b) augmented matrix $= \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$

Elementary Row Operations (on Matrices)

- $Row_1 \longrightarrow Row_1 \pm k \cdot Row_2$ (Replace)
- $R_1 \leftrightarrow R_2$ (Interchange) ; $M \to (-)M$
- $R_1 \to k \cdot R_1$ (Scaling) ; $(k \neq 0)$

⇓

(c) Row operations on augmented matrix

(i) $R_3 \to R_3 - 5R_1$ $\begin{bmatrix} 1 & -2 & 1 & 0 \\ \boxed{0} & 2 & -8 & 8 \\ \boxed{0} & 10 & -10 & 10 \end{bmatrix}$

: make all entries in 1st column, except 1st entry = 0

(ii) $R_3 \to R_3 - 5R_2$ $\begin{bmatrix} 1 & -2 & 1 & 0 \\ \boxed{0} & 2 & -8 & 8 \\ \boxed{0} & \boxed{0} & 30 & -30 \end{bmatrix}$

Echelon Form

Make all entries below the leading entry in each row = 0

ie first non zero entry.

Aryan Jain - 2022111

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ & & b_3 \\ 0 & c_2 & c_3 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_2]{\text{for Echelon form}} \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & c_2 & c_3 \\ 0 & 0 & b_3 \end{bmatrix}$$

Page No.:
Date:
YOUVA

Echelon form ? @ Row Echlon form

A rectangular matrix with 2 following properties :- make N upper Δ matrix

(I) Each leading entry of a row is in a coloumn to the right of the leading entry of row above it.

NOTE: ⇒ leading entries need NOT make an exact diagnol.

(II) All non zero rows are any rows of all zeros. → using interchange ($R_i \leftrightarrow R_j$) ; bring that row to last

* Leading Entry [of a row] : leftmost non zero entry

NOTE:

* Non zero row : contains atleast one non Zero entry

Augmented Coloumn [Last coloumn]
→ Pivot ∃ ⇒ Inconsistant system of Eqn.
→ does NOT has Pivot ⇒ Consistant system

* PIVOT ⇒ If matrix in row Echlon form ⇒ leading entry in each non zero row = Pivot

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}\right\} \Longrightarrow \begin{aligned} R_3 &\to R_3 - 5R_1 \\ R_3 &\to R_3 - 5R_2 \end{aligned} \Longrightarrow \left\{\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 30x_3 &= -30 \end{aligned}\right.$$

Eliminating x_1, x_2

Equivalent System of Eqn.

⇓ via BACK Substn.

$x_3 = -1$

$x_2 = 0$

$x_1 = 1$

Thus each elementary row operation on the augmented matrix corresponds to an operation of the system of eqns, which produces a sytem EQUIVALENT to orignal system

In the end we get a system more easy to solve.

NOTE: If every column other than the augmented column contains a pivot ⇒ unique soln.

Aryan Jain - 202211

Page No. 4

② Eg :- $x_1 - 2x_2 + x_3 = 0$

$2x_2 - 8x_3 = 8$

$5x_1 = 10$

$\Downarrow$

Augmented matrix :

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & 0 & 10 \end{bmatrix}$$

$\Downarrow$

$R_3 \to R_3 - 5R_1$ ✓

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -5 & 10 \end{bmatrix}$$

$\Downarrow$

$R_3 \to R_3 - 5R_2$ ✓

Echelon form. $\rightarrow$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ \boxed{0} & 2 & -8 & 8 \\ \boxed{0} & \boxed{0} & 35 & -30 \end{bmatrix}$$

$\therefore$ Consistant, unique soln.

Ans.

Equivalent system.

$x_1 - 2x_2 + x_3 = 0$

$2x_2 - 8x_3 = 8$

$35x_3 = -30$

$\Downarrow$

$x_3 = -6/7$

$x_2 = \square$

$x_1 = \square$

Ans.

via Back Substn.

③ Eg :- $2x_1 + 3x_2 + 5x_3 + x_4 = 6$

$7x_1 + 7x_3 - 7x_4 = 5$

$3x_2 + 3x_3 + 3x_4 = 0$

$\Rightarrow$ Augmented Matrix

$$\begin{bmatrix} 2 & 3 & 5 & 1 & 6 \\ 7 & 0 & 7 & -7 & 5 \\ 0 & 3 & 3 & 3 & 0 \end{bmatrix}$$

$R_2 \to \frac{1}{7}R_2$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 5/7 \\ 2 & 3 & 5 & 1 & 6 \\ 0 & 3 & 3 & 3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 5/7 \\ 0 & 3 & 3 & 3 & 32/7 \\ 0 & 3 & 3 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 5/7 \\ \boxed{0} & 3 & 3 & 3 & 32/7 \\ 0 & \boxed{0} & 0 & 0 & -32/7 \end{bmatrix}$$

Echlon form.

$\Rightarrow$ Inconsistant system of eqns.

Ans.

Aryan Jain - 2022111

Eg ④ :- $2x_1 + 3x_2 + 5x_3 + x_4 = 1$

Aryan Jain - 202211

M T W T F S S
Page No.: 5 YOUVA
Date: 12/11/22

$7x_1 + 7x_3 - 7x_4 = 7$

$3x_2 + 3x_3 + 3x_4 = 9$

$$\begin{bmatrix} 2 & 3 & 5 & 1 & 1 \\ 7 & 0 & 7 & -7 & 7 \\ 0 & 3 & 3 & 3 & 9 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{R_2 \to R_2/7,\ R_3 \to R_3/3} \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 2 & 3 & 5 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 3 & 3 & 3 & -1 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & \boxed{1} \\ 0 & 3 & 3 & 3 & \boxed{-1} \\ 0 & 0 & 0 & 0 & \boxed{0} \end{bmatrix}$$

NO pivot

3rd Augmented Coloumn does NOT have an Pivot
↳ infinite soln. (Consistant)

⇓

Consistant ; ∞ many soln

Ans. ⇓

$$\begin{cases} x_1 + x_3 - x_4 = 1 \\ 3x_2 + 3x_3 + 3x_4 = -1 \end{cases}$$

$x_1, x_2 \to$ Basic Variables

$x_3, x_4 \to$ free variables

(for arbit value of x_3, x_4 ; $\to x_1, x_2$ fixed)

→ 4 varb ; 2 eqns. ⇒ ∞ soln.

Basic Variables → corresponding to coloum with pivot

(N/L)

Free Variables → corresponding to coloums without pivot
(~ parameter)

Echlon form is ⟹ NOT unique

{Q : Coloumn having NO pivot corresponds to a free variable.}

Aryan Jain - 2022111

NOTE:- Matrix 1 $\longleftrightarrow$ Matrix 2

$\Downarrow$

Row Equivalent $\Rightarrow$ if via row operations $M_1 \rightleftarrows M_2$

Aryan Jain - 202211

Reduced Row Echlon Form →

[or reduced row Echelon form]

$\Rightarrow$ Any Echelon form satisfying →

(1) Each pivot should be = 1

(11) All other entries in that (pivot वाला) coloumn = 0 → both up & down

* RREF may have entries other than 0 and 1 in coloumns other than ones with pivots.

NOTE: Pivot posns. are the same in any ~~two~~ echlon form of a given matrix

contd.

Eg (3) $R_3 \to -7/32 R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & -1 & 5/7 \\ 0 & 3 & 3 & 3 & 32/7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow R_1 \to R_1 - 5/7 R_3$

$R_2 \to R_2 - 32/7 R_3$

: Go backwards (for making RREF)

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 5/7 \\ 0 & 3 & 3 & 3 & 32/7 \\ 0 & 0 & 0 & 0 & -32/7 \end{bmatrix}$$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$\Downarrow$

$R_2 \to \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & \boxed{0} & 1 & -1 & \boxed{0} \\ \boxed{0} & \boxed{1} & 1 & 1 & \boxed{0} \\ \boxed{0} & \boxed{0} & 0 & 0 & 1 \end{bmatrix} \leftarrow \text{RREF}$$

* Theorum :-

Each matrix is row equivalent to one & only one reduced echelon matrix.

$\Downarrow$

[*] any matrix A $\xleftrightarrow[\text{equivalents}]{\text{row}}$ U [reduced] Echlon form +1 matrix

$\Downarrow$

U = [reduced] Echlon form of A

Pivot posns: are the same in any echlon form of a given matrix ($\because$ RREF is unique)

Pivot Posns: in any matrix?

non echelon form matrix
- → pivots ⇒ NOT defined
- → pivot posns ⇒ defined

Eg: $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

matrix (at random) — NOT Echeleon.

consider any Echelon form — Echelon.

~~pivots~~

pivot posns (unique) ⇓ corresponding to pivots in any Echelon form.

pivots (unique ∀ Echelon forms)

A **pivot position** in a matrix U is a location in U that corresponds to a leading 1 in RREF form of U

A **pivot column** is a column that contains a pivot posn.

for Uniformity → Explicit procedure for computer programming
random भी कर सकते हैं

Page No. YOUVA
Date 14/11/22

The Row Reduction Algorithm?

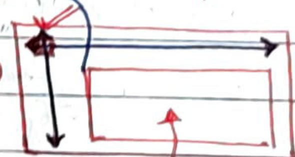
[step 1] begin with the left most non zero column
This is a pivot column ; pivot posn. is at the top

[step 2] Select a non zero entry in the pivot column as a PIVOT
If necessary interchange rows to move this entry into the pivot posn.

Zero नहीं होना चाहिए pivot posn पे ✗
⇓
Non Zero Entry से Pivot बनाओ
⇓
clear all entries below the pivot (make them 0)
⇓

[step 3] use row operations to create zeros in all posns. below the pivot.

[step 4] Cover / Ignore row 1 & column 1 [containg the pivot] ; & all rows above it (if any) ; Apply (→) step 1–3 on the submatrix remaing ; repeat until there are no more non zero rows ✓



Hide row 1 & column 1

Now apply this algorithm on this small matrix.

$S_1 \rightarrow S_4$:- forward face phase (to (echlon form)) ✓

S_5 :- backward face phase ; ie rightmost first (to(RREF))

↓

[step 5] Begining with the rightmost column ; & working upward, & to the left ; create zeros above each pivot.
If the pivot is NOT 1, make it 1 by a scaling operation.

Solutions of linear Systems?

(i) basic variables → linear system ke augmented matrix ke pivot coloums ke corresponding variables

v/s

free variables → baki saare variables

Consistant system ⇒ if containing free variables ⇒ ∞ many soln.

Free varb = parameter ; soln is in parametric form. ⇓ GENERAL SOLUTION

Imp.

PTR:

$0x_1 + 0x_2 + 0x_3 = b$

⇒ $b = 0$ ⇒ Soln = $\forall \in R$ → consistant

⇒ $b \neq 0$ ⇒ Soln = ϕ → inconsistant

Existance & Uniqueness Theorem?

A linear system is consistant if & only if:— the rightmost coloumn of the augmented matrix is the augmented coloumn is NOT a pivot coloumn. ie; iff the echlon form of augmented matrix has no row of the form $[0 \ 0 \ 0 \ldots b]$; $b \neq 0$

NOTE: Consistant Linear System

→ NO free variables ⇒ $\exists$ a unique soln.

or

→ ≥ 1 free variables ⇒ $\exists$ ∞ly many solns.

Aryan Jain - 202211

Page No 9

Eg (1):

Pivot ≠ 0

$$\begin{bmatrix} 2 & 1 & 11 & -1 \\ 0 & 3 & 3 & -3 \\ 2 & 0 & 10 & 2 \end{bmatrix}$$

(1) Pivot ≠ 0 ✓ (Else make an row interchange)

(2) clear all entries below it. $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 11 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

ab ispe yehi algorithm lagao

$R_3 \to R_3 + \frac{1}{3}R_2$

$$\begin{bmatrix} 2 & 1 & 11 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \infty\text{ly many solutions.}$$

same as 0 = 0

$R_2 \to \frac{1}{3}R_2$

$$\begin{bmatrix} 2 & 1 & 11 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 11R_2$

$$\begin{bmatrix} 2 & -10 & 0 & 10 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$

$$\begin{bmatrix} 1 & -5 & 0 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 5R_2$

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in RREF ✓

Ans.

NOTE: If augmented column ⟹ ± combination of any 2 or 3 other columns ⟹ Consistent

Using Row Reduction to Solve a linear System?

step (1) Write augmented matrix of the system

↓ [Row Reduction Algorithm]

step (2) Echlon form of augmented matrix

consistant ← → inconsistant / No soln.

⇓ consistant: step (3)

⇓ inconsistant: STOP

step (3) convert to RREF?

step (4) write corresponding system of equations (from RREF matrix of step 4)

step (5) Rewrite each non zero equation from step 4 so that its one basic variable is expressed in terms of any free variable appearing in the equation.

Elementary Matrices?

Applying an elementary row operation on a matrix = multiplying the matrix on the left by an ELEMENTARY MATRIX.

(1) Replacement?

$$\boxed{R_i \rightarrow R_i + c R_j}$$

$$\begin{bmatrix} 1 & & c & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{matrix} \\ \rightarrow i^{th} \text{ row} \\ \\ \\ \end{matrix}$$

j^{th} column

Left multiply by Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

NOTE:

$$\begin{bmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{bmatrix} (i < j) \qquad \begin{bmatrix} 1 & & \\ & 1 & \\ c & & 1 \end{bmatrix} (i > j)$$

Eg:

$$\begin{bmatrix} 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \begin{bmatrix} 0 & 0 & -1 & -2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{matrix} \rightarrow i=1 \\ \\ \end{matrix}$$

$j = 2$

Matrix Multiply Row to Column

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & -2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

justify

$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} R_1 + aR_2 \\ bR_1 + R_2 \end{bmatrix} \equiv \begin{bmatrix} R_1 + (-2)R_2 \\ R_2 \end{bmatrix}$ (LET)

$a = -2$
$b = 0$ Ans.

(II) Interchange :- $R_i \leftrightarrow R_j$

Make this interchange in the identity matrix & left multiply.

$E \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 \\ & & 1 & \ddots 0 & \\ & & & & 1 \end{bmatrix}$

(III) Scaling :- $\Rightarrow$ uss row wale "1" ko "c" bna do identity matrix me. $(R_i \to cR_i)$

$E = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & 1 \end{bmatrix}$

Equivalance of Systems :-

$\boxed{A \cdot x = b}$

$EA \cdot x = Eb$

$\Downarrow$

$\boxed{A' \cdot x = b'}$

$\Downarrow$

$(E^{-1}A')x = (E^{-1}b')$

Both have the same solution set bcz. E is invertible.

; $EA = A'$; $Eb = b'$

Inverse of Elementary Matrix :-

(I) Replacement Operation :-

$(c) \longrightarrow (-c)$

$R_1 \to R_1 + cR_2 = R_1'$

Inverse $\Rightarrow R_1' \to (R_1') - cR_2$

$= (R_1 + \cancel{cR_2}) - \cancel{cR_2}$

$= R_1$

(II) Interchange Operation :- Self inverse

$R_j' = R_i \leftrightarrow R_j = R_i' \Rightarrow R_i' \leftrightarrow R_j'$

$\equiv R_j \quad \equiv R_i$

(III) Scaling :- $(c) \to (1/c)$

n Tupple = $(\square_1, \square_2, \ldots, \square_n)$ real nos ← list of n coordinates

R^n ← n tupples

Vectors as Ordered Lists or n-tuples

NOTE → We will temporarily use the word "vector" to refer to an ordered list of numbers.

(I) [Vector ~ n tuples or $n \times 1$ column vectors/matrix or n ordered list parenthesized]

Zero Vector $\equiv \mathbf{0}$ → vector with all entries $= 0$

eg (I): $(1, 1, 2, 3) \in R^4$; $\begin{bmatrix}1\\1\\2\\3\end{bmatrix} \in R^4$ ≠ ≡ by identification [& NOT Equality]

eg (II): $\vec{0} \equiv \mathbf{0} \in R^4$

arrow → vector; [bold font]

$R^n \equiv n$ dimensional space

(II) R^n = list of n tuples ||| list of all $n \times 1$ column vectors ||| list of all $1 \times n$ row vectors

Equality of vectors, scalar multiplication, Vector addition → defined ENTRY (component) by ENTRY (component) in R^n

eg (III): $(1, 2) + (3, 4) = (4, 6)$

eg (IV): $3 \cdot (1, 2) = (3, 6)$

Algebric Properties of R^n?

$\forall u, v, w \in R^n$ (vectors) & c, d → scalars

(I) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$ → Commutativity of Addition

(II) $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$ → Associativity "

(III) $\bar{u} + \mathbf{0} = \mathbf{0} + \bar{u} = \bar{u}$ → Additive Identity ($\mathbf{0}$)

(IV) $\bar{u} + (-\bar{u}) = (-\bar{u}) + (u) = \mathbf{0}$ → Additive Inverse ; where $-u = (-1)(u)$

(V) $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$ → Distributivity over addition

(VI) $c(d\bar{u}) = (cd)\bar{u}$ → Order in which scalar multiplication is done does NOT matter

(VII) $1 \cdot \bar{u} = \bar{u}$ → Multiplication Identity (1)

Verification: TP: $\boxed{\overline{u} + \overline{v} = \overline{v} + \overline{u}}$

Let; $\vec{v}$, $\vec{u} \in \mathbb{R}^n$

& $\vec{u} = (u_1, u_2, u_3 --, u_n) \in \mathbb{R}^n$ → $u_i \in \mathbb{R}$

$\vec{v} = (v_1, v_2 -- v_n) \in \mathbb{R}^n$

Then, $\vec{u} + \vec{v} = (u_1, u_2, u_3, --, u_n) + (v_1, v_2 -- v_n)$

$= (\boxed{u_1 + v_1}, u_2 + v_2, --, u_n + v_n)$

$= (v_1 + u_1, v_2 + u_2, v_3 + u_3, --, v_n + u_n)$

$= (v_1, v_2, v_3 --, v_n) + (u_1, u_2, u_3, --, u_n)$

$= \vec{v} + \vec{u}$ ∴ QED

How to Proof: ⇒ 2 steps ✓

(step 1): Prove that RHS makes sense ie $\vec{y}$ is defined.

(step 2): Show that the defination / sum is independent of order.

Generalisation of associativity of vector addition

Linear Combination:

commutativity

Given;

$\vec{v_1}, \vec{v_2}, -\vec{v_p} \in \mathbb{R}^n$ (Vectors)

Scalars → $c_1, c_2, --, c_p$

then;

vector $\vec{y} = c_1\vec{v_1} + c_2\vec{v_2} + --- + c_p\vec{v_p}$

is called LINEAR COMBINATION of $v_1 -- v_p$ with weights of coeff(s) $c_1, ---, c_p$

Induction Theorum - PMI:

step 1: Check wether the given statement is true for $n=1$

step 2: Assume the given statement $P(n)$ is also true $\forall\ k \leq n;\ k \in \mathbb{N}$

step 3: Prove the result is true for $P(n+1)$ $P(k+1)$ ✓

Part [II]: Proof? For $p=1$, $c_1\bar{r}_1$ is obviously defined

If

For all $1 \le k \le p-1$, we assume $(c_1\bar{r}_1 + c_2\bar{r}_2 + \dots + c_k\bar{r}_k)$ is defined (PMI theorem) $\Downarrow$

then the term $c_1\bar{r}_1 + c_2\bar{r}_2 + \dots + c_p\bar{r}_p$ can be defined as

"$\underbrace{(c_1\bar{r}_1 + c_2\bar{r}_2 + \dots + c_{p-1}\bar{r}_{p-1})}_{\text{defined by PMI theorem; = vector}} + c_p\bar{r}_p$"

which is defined as addition of two vectors is defined

$\therefore$ QED

Defination of Matrix - Vector Product?

(Eg.(1) of linear combination)

Let; $A \equiv M \times M$ matrix

$\therefore$ columns of matrix A; $\vec{a}_1, \vec{a}_2, \vec{a}_3 \dots, \vec{a}_M \in R^M$

[$\vec{a}_i$(s) are vectors in R^M]

$\Downarrow$

ie if $a_1, a_2 \dots, a_M$ are columns of A matrix then

Eg:-

$A \to$ matrix $\begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array}$ $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

$\vec{x} \to$ vector $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$A\vec{x} \boxed{:=} [\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_M] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_M\vec{a}_M$$

$$A\vec{x} = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= 1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \text{ Ans.}$$

or $= \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ Ans.

Let; B matrix $= [\vec{x} \ \vec{y}]$; $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix}$; $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix}$

then; $AB = A \cdot [\vec{x} \ \vec{y}]$

$$= [A\vec{x} \quad A\vec{y}]$$

$$= \left[\sum_{i=1}^{M} x_i\vec{a}_i \quad \sum_{i=1}^{M} y_i\vec{a}_i\right] \text{ Ans.}$$

Eg: $A = [\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3]$

$$B = [\bar{b}_1 \ \bar{b}_2 \ \bar{b}_3] \equiv \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$AB = [A\bar{b}_1 \ A\bar{b}_2 \ A\bar{b}_3]$$

$$\boxed{\bar{b}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}}$$

$$A\bar{b}_1 = A\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \vec{a}_1 \cdot b_{11} + \vec{a}_2 \cdot b_{21} + \vec{a}_3 \cdot b_{31}$$

$[a_1 \ a_2 \ a_3]$

$$\text{Column}_k(AB) = \sum_{i=1}^{M} \vec{a}_i b_{ik}$$

Permutation? ⇒ reording of a bijection defined from a set to itself (any fixed order)

M T W T F S S
Page No.:
Date:
YOUVA

- σ ⇒ function ; bijection from $\{1,2,3,--,p\}$ to $\{1,2,3,--,p\}$
- $\vec{V_j}$ ⇒ element in domain $\in [p]$
- $\vec{V}_{\sigma(j)}$ ⟶ Image of V_1 in funtion σ

$$\boxed{[p] = \{1,2,3,---,p\}}$$

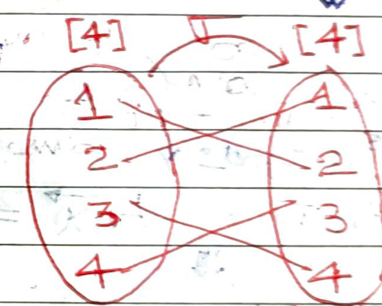
⟸ Genratization of Associative Law

∴ We prove this by induction on p;

↳ PMI theorum, assume that this claim is true ∀ k subsets of $[p]$

↳ if $S \subset \{1,2,3,--,p\}$ & $|S| \leq p-1$; any random permutⁿ $\sigma : S \to S$ is a bijection then; $\sum_{k\in S} \vec{V}_k = \sum_{k \in S} \vec{V}_{\sigma(k)}$

Let $\sigma : [p] \to [p]$ be any permutation



$\sigma(1) = 2$, $\sigma(3) = 4$,
$\sigma(2) = 1$, $\sigma(4) = 3$

Then $\sum_{j=1}^{p} \vec{V}_{\sigma(j)} = \left(\sum_{j=1}^{p-1} \vec{v}_{\sigma(j)}\right) + \vec{v}_{\sigma(p)}$

Eg:- $\sum_{j=1}^{p-1 \to 3} \vec{v}_{\sigma(j)}$

$= \vec{v}_{\sigma(1)} + \vec{v}_{\sigma(2)} + \vec{v}_{\sigma(3)}$

$= \vec{v}_2 + \vec{v}_1 + \vec{v}_4$

$= \sum_{j=1}^{4} \vec{v}_j \; ; j \neq \sigma(p)$

$$= \left(\sum_{j=1;\, j \neq \sigma(p)}^{p} \vec{v}_j\right) + \vec{v}_{\sigma(p)}$$

Case (i) $\boxed{\sigma(p) \neq p}$

$$= \left(\sum_{j=1;\, j\neq\sigma(p)}^{p-1} \vec{v}_j + \vec{v}_p\right) + \vec{v}_{\sigma(p)}$$

$$= \left(\left(\sum_{j=1;\, \cancel{j\neq\sigma(p)}}^{p-1} \vec{v}_j\right) + \cancel{\vec{v}_{\sigma(p)}}\right) + \vec{v}_p$$

by associative law

$$= \left(\sum_{j=1}^{p-1} \vec{v}_j\right) + \vec{v}_p$$

$$= \sum_{j=1}^{p} \vec{v}_j \quad ; \therefore QED$$

case (ii) $\boxed{\sigma(p) = p}$

$$= \left(\sum_{j=1;\, j\neq \cancel{\sigma(p)}\, p}^{p} \vec{v}_j\right) + \vec{v}_{\cancel{\sigma(p)}\, p}$$

$$= \left(\sum_{j=1}^{p-1} \vec{v}_j\right) + \vec{v}_p$$

$$= \sum_{j=1}^{p} \vec{v}_j \quad ; \therefore QED$$

Ans.

Theorem :

$A \Rightarrow m \times n$ matrix ; $\vec{a}_1, \vec{a}_2, \vec{a}_3 - - , \vec{a}_n$ columns of A

$\vec{b} \in R^m$

then:
matrix eqn $\Rightarrow A\vec{x} = \vec{b}$
has same soln set as $x_1\vec{a}_1 + x_2\vec{a}_2 + - - + x_n\vec{a}_n = b$

which in turn has the same solution set as system of linear eqn(s.) having augmented matrix $[\vec{a}_1 \ \vec{a}_2 - - - \vec{a}_n \ \vec{b}]$ or simply $[A \ \vec{b}]$

Homogeneous Linear Systems :

∞ soln (trivial + non trivial solns.) or 1 soln. [trivial soln ($\vec{0}$)]

Linear Sys. of form $A\vec{x} = 0$; $A_{m \times n}$ matrix

NOTE: A homogeneous system can NEVER be inconsistant ; bcz. $\vec{0}$ is always a soln.

FACT: homogeneous eqn ($A\vec{x} = 0$) can have non trivial solns {Case of ∞ soln.} iff the eqn has atleast 1 free variable

∵ free varb. नहीं है $\Rightarrow$ unique soln $\Rightarrow \vec{0}$ (trivial soln)

Parametric Vector Form :

$\Rightarrow$ soln set of linear systems expr. Explicitly in terms of vectors & variables & fixed weights.

Homogeneous system $\rightarrow$ weights $\Rightarrow$ all free variable

$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x} = \text{free varb}_1 \begin{bmatrix} \ \end{bmatrix} + \text{free varb}_2 \begin{bmatrix} \ \end{bmatrix} + -$

eg(1): [homogeneous system]

$2x_1 + 3x_2 - 4x_3 + x_4 = 0$
$x_2 - 3x_3 + 2x_4 = 0$

$\Rightarrow \begin{bmatrix} 2 & 3 & -4 & 1 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3/2 & -2 & 1/2 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 5/2 & -5/2 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{bmatrix}$

General Soln is :
$x_1 = -\frac{5}{2}x_3 + \frac{5}{2}x_4$
$x_2 = 3x_3 - 2x_4$
& $x_3, x_4 \rightarrow$ free variables

∴ In Paramateric Vector form ;

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{x} = x_3 \begin{bmatrix} -5/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5/2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Ans.

$+ \dots + x_m \vec{a}_m = b$

matrix

A homogeneous ... can NEVER be ... nt; ... is always

...ms iff the

...c system expr. ... of vectors ... ights

$[\ ] + \text{free var}_2 [\ ] + \dots$

$\Rightarrow \begin{bmatrix} 1 & 3/2 & -2 & 1/2 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 5/2 & -5/2 & 0 \\ 1 & -3 & 2 & 0 \end{bmatrix}$

|नमः सिद्धम|

Matrices

(I) Addition?

$A, B, C \rightarrow m \times m$ matrix ; $r, s \rightarrow$ scalars

- $A + B = B + A$; (commutative)
- $A + (B + C) = (A + B) + C$; (associative)
- $A + 0 = A$; (additive identity)

$$\boxed{\begin{aligned} b_1 &= b_2 \\ \Downarrow\ ab_1 &= ab_2 \end{aligned}}$$

- $r(A+B) = rA + rB$
- $(r+s)A = rA + sA$ } (Distributive law)
- $r(sA) = (rs)A$

(II) Matrix Multiplication?

$A_{m \times m}$ & $B_{m \times p}$; $C := (AB)_{m \times p}$, $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

AB is defined

- $A(BC) = (AB)C$; (associative law)
- $A(B+C) = AB + AC$; (left distributive law)
- $(B+C)A = BA + CA$; (right ")
- $r(AB) = (rA)B = A(rB)$; $\forall$ scalars "r"
- $I_m \cdot A_{m \times m} = A_{m \times m} \cdot I_m = A$; [multiply identity]

Matrix Multiplication is also associative.

(III) Powers of a Matrix?

$A^k = \underbrace{A \dots A}_{k \text{ times}}$

NOTE: $m \times m$ Matrix $\sim$ m Vectors (only for addition of matrices)

$AB \neq BA$; in matrices

Matrix Transpose ⇒ rows ⟳ coloumns

(IV) Transpose ?

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$ ⟹ Transpose operation respects Addition
- $(\gamma A)^T = \gamma(A^T)$; $\gamma \to$ any scalar
- ** $(AB)^T = B^T A^T$

A new perspective :-

Let $\underset{\text{matrix}}{B} \xrightarrow{\text{coloumns}} \vec{b}_1, \vec{b}_2, ---, \vec{b}_p$; then

$$AB = [A\vec{b}_1 \quad A\vec{b}_2 \quad --- \quad A\vec{b}_p]$$

∴ Each Coloumn of AB is a linear combination of coloumns of A using weights from the corresponding coloumn of B

⇓

PROOF :

Eg : $A = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$B = [\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3]$

$\therefore AB = [A\vec{b}_1 \quad A\vec{b}_2 \quad A\vec{b}_3]$

$$A\vec{b}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3] = \vec{a}_1 b_{11} + \vec{a}_2 b_{21} + \vec{a}_3 b_{31}$$

$$= b_{11}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_{21}\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + b_{31}\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

⇓

$$\therefore (A\vec{b}_1)_3 = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}$$

$$\equiv (AB)_{31} \qquad \llcorner \sum_{k=1}^{3} a_{3k} b_{k1}$$

$K \to$ coloumn index } at
$l \to$ row index } random

M	T	W	T	F	S	S
Page No.						YOUVA
Date						

Let; $A = (a_{ij})$ be an $m \times n$ matrix
$B = (b_{ij})$ be an $n \times p$ matrix
$\hookrightarrow$ coloumns $\vec{b}_1, \vec{b}_2 \cdots \vec{b}_p$

$b_k \to k^{th}$ coloumn of B matrix
$a_j \to j^{th}$ coloumn of A matrix

thus; $(AB)_{lk} = \sum_{j=1}^{n} a_{lj} b_{jk}$ ✓

row l, col^{mn} k entry
(or)
l^{th} entry of k^{th} coloumn

$\Downarrow$

$A = [\bar{a}_1 \quad \bar{a}_2 \quad -- \quad \bar{a}_m]$

$AB = [A\vec{b}_1 \quad A\vec{b}_2 - A\vec{b}_k - A\vec{b}_p]$

$$A\vec{b}_k = [\bar{a}_1 \quad \bar{a}_2 \quad --- \bar{a}_m]\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{bmatrix} = b_{1k}\vec{a}_1 + b_{2k}\vec{a}_2 + \cdots + b_{mk}\vec{a}_m$$

$$\vec{b}_k = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

$$= b_{1k}\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2k}\begin{bmatrix} a_{12} \\ \vdots \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots$$

$\Downarrow$

$\therefore [A\vec{b}_k]_l$ = l^{th} entry of $A\vec{b}_k$ $= (AB)_{lk} = b_{1k} a_{l1} + b_{2k} a_{l2} + \cdots$

$$= \sum_{j=1}^{n} a_{lj} \cdot b_{jk}$$

$\therefore$ QED

$(AB)_{lk}$ $\Downarrow$
l^{th} entry of $A\vec{b}_k$ (coloumn vector) $= \sum_{j=1}^{n} b_{jk} a_{lj}$

$\therefore A\vec{b}_k$ is k^{th} coloumn of AB

$\Downarrow$

Similarly, each row of AB is a linear combination of rows of B using weights from corresponding row of A

ie $row_i(AB) = [row_i(A)][B]$

Proof:

Aryan Jain - 202211

$B = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_M \end{bmatrix}$ $\quad A = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_M \end{bmatrix}$

$\Rightarrow A^T = [\bar{a}_1^T \ \bar{a}_2^T \ - - \ \bar{a}_M^T]$

$B^T A^T = B^T [\bar{a}_1 \ \bar{a}_2 \ - - \ \bar{a}_M]$

$= [B^T \vec{a}_1^T \ B^T \vec{a}_2^T \ - - - \ B^T \vec{a}_M^T]$

$\Rightarrow (B^T A^T)^T = AB = \begin{bmatrix} \bar{a}_1 B \\ \bar{a}_2 B \\ \vdots \\ \bar{a}_M B \end{bmatrix}$

$B^T \vec{a}_K^T$

$[a_1 \ a_2 \ a_3] \xrightarrow{T} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\Downarrow$

K^{th} row of AB $= row_K(AB) = [col_K (AB)^T]^T$

$= [col_K (B^T A^T)]^T$

$= [B^T \cancel{a_K}]^T$

$col_K(A^T)$

$= row_K(A) \cdot B$

(Q1) 1^{st} & 2^{nd} col^{mn}(s) of B, $\bar{b}_1$ & $\bar{b}_2$ equal; what can you say about 1^{st} & 2^{nd} col^{m}(s) of AB?

Ans.) EQUAL Ans.

$(AB)_1 = (AB)_2$ $\therefore$ QED

as $\vec{b}_1 = \vec{b}_2 \Rightarrow A\vec{b}_1 = A\vec{b}_2$

(Q2) $\vec{b}_2$ = second column of B = all $\emptyset$; what can you say about 2^{nd} col^{mn} of AB?

Ans.) $A\cancel{\vec{b}_2}^{0} = \emptyset$

$\therefore$ All zero Ans.

Invertible Matrices?

$A_{M \times M} \longrightarrow$ invertible iff $\exists\, B_{M \times M}$ such that $AB = BA = I$

[DEFINATION]

Aryan Jain - 202211

- Inverse of $A_{M\times M}$ is unique, if exists & is denoted by A^{-1}

Uniqueness is a consequence of Associative law

Proof: ⇓

$$\begin{cases} AB = BA = I \\ AC = CA = I \end{cases}$$

$$B = IB = (CA)B = C\underbrace{(AB)}_{I} = C$$

⇓

$B = C \Rightarrow A^{-1}$ is unique

- Singular Matrix

≡ Non Invertible Matrix $(|A| = 0)$

Non Singular Matrix

≡ Invertible Matrix $(|A| \neq 0)$

$\boxed{\text{Singularity} \sim \emptyset}$

- Matrix is Invertible iff $|A| \neq 0$

Proof: (I) $\underbrace{A \equiv \text{Invertible}}_{\Rightarrow A^{-1} \text{ exists}} \Rightarrow \exists B$ such that $AB = BA = I$ (or $AA^{-1} = I$) $\Rightarrow$ det(A) $|A||B| = 1$ (or $|A|\cdot|A^{-1}| = 1$) ⇒ $|A| \neq 0$

A invertible ⇒ $|A| \neq 0$

(II) $\boxed{A \cdot \text{adj}(A) = |A| \cdot I} \xrightarrow[\text{If } |A| \neq 0]{} A \cdot \left\{\frac{1}{|A|}\text{adj}(A)\right\} = I$

$|A| \neq 0 \Rightarrow A^{-1}$ exists

$\exists B = \frac{1}{|A|}\text{adj}(A)$ such that $AB = BA = I$

⇓

A is invertible

⇓

$A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$$\boxed{A^{-1} = \frac{\text{adj}(A)}{|A|}}$$

$\boxed{\text{adj}(A) = (A^c)^T \Leftarrow \text{(cofactor matrix of A) Transpose}}$

say about 2^{nd} col^{mn} of AB?

Theorem →

① If A is a invertible matrix, then A^{-1} is invertible & $(A^{-1})^{-1} = A$

② A, B → M×M invertible matrices ; $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

Observe that

Proof :- $(AB)(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{I}A^{-1} = AA^{-1} = I$

& $(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I$

Aryan Jain - 2022111

Matrices :-

Proposition :- If A is an invertible $n \times n$ matrix, then for each $b \in \mathbb{R}^n$; the equation $A\bar{x} = \bar{b}$ has the unique soln $\vec{x} = A^{-1}\vec{b}$.

Pf :- $A\vec{x} = \vec{b} \iff \vec{x} = A^{-1}\vec{b}$

Proposition :-

Let A be a $m \times n$ matrix & let A' be an echlon form of A.
Then $\exists$ an invertible $m \times m$ matrix E such that
$A' = EA$

&

Further, E can be expressed as a product of elementary matrices

Proof: Let $A\vec{x} = \vec{b}$ and $A'\vec{x} = \vec{b}'$ be two equivalent systems of Eqn(s).
In other words, $\exists$ a sequence of row operations which reduces $[A\vec{b}]$ to $[A'\vec{b}]$
Let the corresponding elementary be $E_1, E_2, ---, E_p$
Then $[A' \ \vec{b}'] = E_p E_{p-1} --- E_1 \ [A\vec{b}]$

$= E[A \ \vec{b}]$

$= [EA \ \ E\vec{b}]$

$[A\vec{b}] = [\bar{a}_1 \ \bar{a}_2 \ --- \ \bar{a}_n \ \ \bar{b}]$ — Matrix multiplication is column wise

$\Rightarrow E[A\vec{b}] = E[\bar{a}_1 \ \bar{a}_2 --- \bar{a}_n \ \bar{b}]$

$= [E\bar{a}_1 \ \ E\bar{a}_2 \ --- \ E\bar{a}_n \ \ \bar{b}] = [EA \ \ E\vec{b}]$

$\therefore [A' \ \vec{b}'] = [EA \ \ E\vec{b}] \Rightarrow A' = EA$ and $\vec{b}' = E\vec{b}$

$\therefore A\vec{x} = \vec{b}$
$\Rightarrow EA\vec{x} = E\vec{b}$
$\Rightarrow A'\vec{x} = \vec{b}$

$\therefore A'\vec{x} = \vec{b}' \Rightarrow EA\vec{x} = E\vec{b}$
$\Rightarrow E^{-1}EA\vec{x} = E^{-1}E\vec{b} \Rightarrow A\vec{x} = \vec{b}$

$\Rightarrow A'\vec{x} = \vec{b} \iff A\vec{x} = \vec{b}$

$\square$ QED

Aryan Jain - 2022111

Theorem :-

Each matrix is row equivalent to one & only one reduced Echlon matrix.

Existance & Uniqueness Theorums?

Linear system ⇒ consistant iff the rightmost/augmented coloumn of the augmented matrix is NOT a pivot coloumn.

ie iff an Echlon form of aug.^m matrix has no row of the form $[0\ 0 - - - - 0\ b]$; $b \neq 0$

⇓

Consistant Linear System

Unique Soln ⟵ or ⟶ ∞ many soln.

⇑ No free variables | ⇑ ≥1 free variables

Theorum : An $m \times m$ matrix A is invertible iff A is row eq^v. to I_m

; & in this case, any sequence of elementary row operations that reduces A to I_m ; also transforms I_m to A^{-1}

$$E_p \cdot E_{p-1} \cdot - - - \cdot E_1 A = I_m \Rightarrow A^{-1}$$

$$E_p \cdot E_{p-1} \cdot - - - \cdot E_1 \cdot I = A^{-1}$$

Algorithm for finding A^{-1}?

(I) row reduce the augmented matrix $[A\ I]$

(II) If ; $A_{m \times m}$ is row eq^v. to I_m then $[A\ I] \sim [I\ A^{-1}]$

Or A does NOT have an inverse

Eg: $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$; find A^{-1} if exists?

Ans :-

$$[A\ I] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1/2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & -1/2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 & -1 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1/2 & 1 \end{bmatrix} = [I\ A^{-1}]$$

$\underbrace{\quad}_{A \xrightarrow[E]{} I}$ $\underbrace{\quad}_{I \xrightarrow[E]{} A^{-1}}$

$$\Downarrow \begin{bmatrix} 0 & 1/2 & -1 \\ -2 & 1 & -1 \\ 1 & -1/2 & 1 \end{bmatrix} = A^{-1} \text{ Ans}$$

{ bi-implication → prove 2 statements
$a \xleftrightarrow{\text{iff}} b \Rightarrow a \xrightarrow{\text{if}} b$ and $b \xrightarrow{\text{if}} a$ }

$\boxed{\text{iff} \Rightarrow \text{if \& only if}}$

M T W T F S S
Page No.:
Date:
YOUVA

PROOF:-

[Part 1]

Assume A is invertible & prove that it is row equivalent to Identity matrix

A ⟶ unique RREF(A') (Let)

A' = rref of a sq. matrix

[sq. matrix का RREF ∴ It has pivots along its diagonal (if exist)]

⇓

It can be either I or have a row of zeros at last

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3x3 → 3 pivots; 3x3 → <3 pivots

A' = rref of A ⇒ ∃ an invertible matrix E such that $EA = A'$
& E is the product of elementary matrices which row reduce A to A'

⇓

If $A' \neq I$, then A' must have a non pivot coloumn ⇒ The eqn $A'\vec{x} = \vec{0}$ has a free varb. ie ∴ has a Non trivial soln.

⇒ $\exists$ a $\vec{y} \in \mathbb{R}^n$; $\vec{y} \neq \vec{0}$ such that $A'\vec{y} = 0$

⇓

$(EA)(\vec{y}) = 0$

$\cancel{(EA)^{-1}(EA)}\vec{y} = \vec{0}$ ⇐ E & A invertible ↳ EA is invertible

∴ $A' = I$ □ ⇐ $\vec{y} = \vec{0}$ ✳ contradiction

QED

[Part 2]

We assume that that RREF of A is I and show that A is invertible.

Let A' be the RREF of A ; by proposition stated Earlier ; ∃ an invertible matrix E such that $A' = EA$; where E is product of elementary matrices

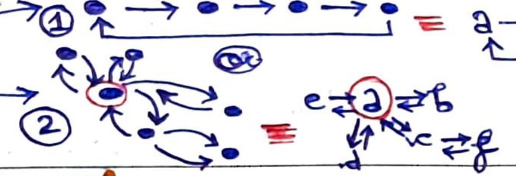
∴ $A' = I$;

$EA = I \Rightarrow A = E^{-1}$ — E or E^{-1} ↳ invertible matrices

↳ A is invertible ✓

~~ABBABAB~~ □

NOTE: for such proves;



$\therefore$ There must be a bi-directional (any) path from each to each

Invertible Matrix Theorem:

Let A be a $m \times n$ matrix; Then the following statements are equivalent

i.e.; Then the following statements are either all True or all False

(a) A is an invertible matrix

(b) A is row equivalent to $m \times m$ identity matrix

(c) A has m pivot posn(s).

(d) The eqn $A\vec{x} = \vec{0}$ has only the trivial soln.

(e) The eqn $A\vec{x} = \vec{b}$ has at least one soln. for each $\vec{b} \in \mathbb{R}^n$

(f) There is an $m \times m$ matrix C such that $CA = I$

(g) There is an $m \times m$ matrix D such that $AD = I$

(h) A^T is an invertible matrix

Proof :-

(I) (a) $\Leftrightarrow$ (b) :- done

(II) (b) $\Rightarrow$ (c) :- A is row eqv. to I_m ($\because I_m \equiv$ RREF) $\Rightarrow$ m pivots in RREF(I_m)

$\Downarrow$

A has pivot posns. in all row $\Rightarrow$ m pivot posn(s).

$\square$ QED

(III) (c) $\Rightarrow$ (b) :- A has m pivot posns $\Rightarrow$ RREF of A has m pivots $\xrightarrow{\because A = m \times m \text{ matrix}}$ RREF of $A_{m \times m} \equiv I_m$

$\Downarrow$

[m colms & m pivots]

$\square$

(IV) (b) $\Rightarrow$ (f) :- If A is row eqv. to I_m (Echlon form)

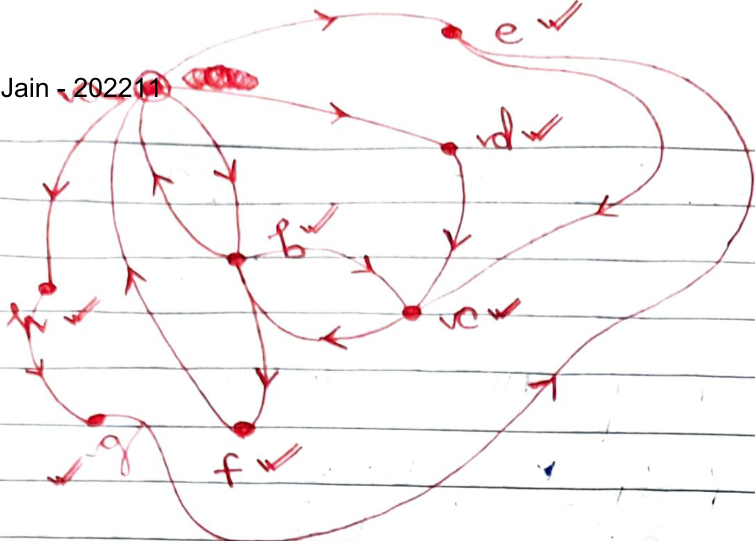
$\Downarrow$

by proposition stated earlier; $\exists$ an invertible matrix E (sq matrix $\Rightarrow m \times m$) such that $\underbrace{EA = I}$; & E is a product of elementary matrices

$\square$

(V) (a) $\Rightarrow$ (d) :- $A\vec{x} = \vec{0} \Rightarrow A^{-1} A\vec{x} = A^{-1}\vec{0} \Rightarrow \vec{x} = \vec{0}$

$\square$



$A\vec{x} = \vec{b}$

(VI) (a) $\Rightarrow$ (e) :- Let $\vec{b} \in \mathbb{R}^m \Rightarrow A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = \vec{b}$

$\Downarrow$

$\therefore \vec{x} = A^{-1}\vec{b}$ is an soln. of $A\vec{x} = \vec{b}$ $\square$

$\boxed{\text{RREF of any } M \times M \text{ sq. matrix} = I_M}$

(VII) (h) $\Rightarrow$ (g) :- A^T = invertible

$\therefore$ by proposition stated earlier $\exists$ an invertible matrix E such that $EA^T = \text{RREF}_{m\times m} \to I_m$; where E is a product of elementary matrices.

$(A^T)^T \cdot E^T = I_m^T \quad (T)$

$\Downarrow$

$A \cdot E^T = I_m$

$\square$

(VIII) (a) $\Rightarrow$ (h) :- A = invertible $\Rightarrow \exists$ an invertible matrix E such that $EA = I \Rightarrow A^T \cdot E^T = I$

$\Downarrow$

$EAE = E \Rightarrow E^{-1}(EAE) = E^{-1}(E) \Rightarrow AE = I \Rightarrow E^T \cdot A^T = I$

A^T = Invertible $\Leftarrow A^T E^T = E^T A^T = I_m$

& $(A^T)^{-1} = E^T$

$\begin{bmatrix} A \to B \\ \equiv \text{(contrapositive)} \\ A' \leftarrow B' \end{bmatrix}$

$\therefore$ c is NOT true $\Downarrow$

(IX) (d) $\Rightarrow$ (c) $\equiv$ $\boxed{\text{NOT (c)} \Rightarrow \text{NOT (d)}}$ $\Rightarrow \therefore A_{m\times m}$ does NOT have m pivot pos^n(s) $\Downarrow$

Atleast 1 col^m of $A_{m\times m}$ is NOT a pivot column.

(d) is Not True $\square$ $\Leftarrow A\vec{x} = \vec{0}$ has ∞ many soln $\Leftarrow A\vec{x} = \vec{0}$ has a free variable $\Leftarrow$

$[A \ \vec{0}]$

(X) (g) $\Rightarrow$ (e) :- $AD = I \Rightarrow A(D\vec{b}) = \vec{b}$

$\Downarrow$

$\vec{x} = D\vec{b}$ is a soln of eqn. $A\vec{x} = \vec{b}$

$\square$ QED

(XI) (e) $\Rightarrow$ (c) :-

$\equiv$ $\boxed{\text{not (c)} \Rightarrow \text{not (e)}}$

$\because$ A does NOT have m pivot posns.

$\Downarrow$

{m×m RREF होगा} $\Rightarrow$ RREF of A (say A') has atleast a row of zeros $\Downarrow$

$A' \equiv$ something like $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A'\vec{x} = \vec{e}_m$ has NO soln

$\equiv$ $[A' \ \vec{e}_m]$ $\because$ will have a pivot in augmented coloumn.

$\vec{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

We know that; $\exists$ an invertible matrix E such that $EA = A' \Rightarrow A = E^{-1}A'$

$A'\vec{x} = \vec{e}_m \Leftrightarrow E^{-1}A'\vec{x} = E^{-1}e_m \Leftrightarrow A\vec{x} = E^{-1}e_m$ has NO soln.

$\vec{b} = E^{-1}e_m$; yeh ek particular case है! $\Rightarrow$ $\exists$ a $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ has NO soln.

$\therefore$ (e) $\equiv$ false $\Leftarrow$

$\Downarrow$

NOT (e)

$\square$ QED

EXTRA.

~~(e) $\Rightarrow$ (c) :- $A\vec{x} = \vec{b}$ has ≥ 1 soln for all $\vec{b} \in \mathbb{R}^m$~~

~~$\Downarrow$~~

~~augmented colm of $[A \ \vec{b}]$ is never a pivot colm.~~ $\Leftarrow$ ~~Existance & uniquness thrm.~~

~~$\Downarrow$~~

~~A must have m pivot pos(s)~~ $\Rightarrow$ In case of ∞ many soln; RREF has ≥ 1 rows of all zeros $\Rightarrow$ < m pivots!

(XII) (f) $\Rightarrow$ (a) : $\exists$ a matrix $C_{m\times m}$ such that $CA = I$

$\Downarrow$

$A^T C^T = I$

$X D = I$

$\Updownarrow$

X is invertible

$\Updownarrow$

X^T is invertible

$\therefore$ A^T satisfies condition (g) $\leftarrow$

$\Downarrow$

As (g) $\Rightarrow$ (h) proved / holds :- $(A^T)^T$ is invertible

$\Downarrow$

$A \equiv$ invertible

$\Downarrow$

(a) holds $\square$

NOTE: $(\text{Not } C \Rightarrow \text{Not } B)$ is same as $(B \Rightarrow C)$

$C' \Rightarrow B' \Leftrightarrow B \Rightarrow C$

taking negation

NOTE: $\boxed{B \Rightarrow C}$

∀ PROOFS

- we can NOT assume C ✗ ← ie [C ki statement True man kar proof nahi kar sakte!]
- we must assume B & get to C ka claim statement ✓

[XII]

Alternatively; $\because A^T C^T = I \Rightarrow A^T \vec{x} = \vec{b}$ has a soln $\forall\, \vec{b} \in \mathbb{R}^m$

$$A^T(C^T \vec{b}) = \vec{b}$$

$\therefore \vec{x} = C^T \vec{b}$ is a soln.

$\Downarrow$

(A^T) satisfies condition (e) ✓

$\Downarrow$

and as (e) $\Rightarrow$ (h)

$\Downarrow$

$(A^T)^T$ is invertible $\Rightarrow$ A is invertible $\Rightarrow$ (a) holds ✓

□ QED

LU Factorisation →

If A is a $m \times n$ matrix which can be expressed as $A = LU$; where $L \equiv$ lower Δ sq. matrix & $U \equiv$ matrix in Echlon form; then $A = LU$ is called LU factorisation of A.

factorisation of matrices → Matrix = $\text{Matrix}_1 \times \text{Matrix}_2$
↳ something like; $(6 = 2 \times 3)$

Assumption on A?

It is assumed that A can be reduced to echlon form by using only row replacements of the type $R_i \to R_i + cR_j$; where i is strictly greater than j (in other words row i is below row j.)

$j > i$

U → Echlon form of A

If an $m \times n$ matrix A can be reduced to an echlon form using only such row operations; then $\exists$ a sequence of lower Δ matrices $E_1, E_2 \text{ --- } E_p$ such that $E_p E_{p-1} \text{ --- } E_1 \cdot A = U$

↑ Echlon form of A

Not all Matrices can be factorized

∞ many possible LU factors

:- [Elementary Matrix $\equiv$ Lower Δ Matrix; if $i > j$]
We can only use rows above the row being replaced

Hence; $A = I \cdot E_1^{-1} \cdot E_2^{-1} \cdot \text{ --- } E_p^{-1} U$

As the product of lower Δ matrices is lower Δ matrix; the product $L = E_1^{-1} E_2^{-1} \text{ --- } E_p^{-1}$ is a lower Δ matrix

This gives us an LU-factorisation for the matrix A.

NOTE: { Interchange & Scaling NOT allowed
Only row replacement by higher rows allowed } → A to its Echlon form }

Prove that $L_1 \cdot L_2 = LTM$

Inverse of lower Δ Elem. Matrix $\equiv$ Lower Δ matrix

Eg:-

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \longrightarrow E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$$

Algorithm for finding LU factorisation :-

[Left Multiply → EA → row operations
Right Multiply → AE → coloumn operations]

① Reduce A to an echlon form U by a sequence of row replacement operations of the form above, if possible.

② Place entries in L such that the same sequence of row operations reduces L to I (or Alternatively, perform flipped col^{mn} operations on I in the same sequence ✓)

ie To get to L do "reverse operations" on I ✓

"Flipped" Coloumn Operations ⟺
→ flip indeces ; EA (row oper^n) ⟶ AE (coloumn operation)
→ $(c) \to (-c)$; as inverse

$$R_i \to R_i + cR_j \implies C_j \to C_j - cC_i$$

① Konse Row/Col को replace Karna है ?
② Kis Row/Col से करना है ?

To Understand more on flipped Col^{mn} operations ?

for $R_2 \to R_2 - \frac{3}{2}R_1$;

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 7 & 1 \\ 3 & -2 & 0 \\ 1 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \\ 0 & -25/2 & -3/2 \\ 1 & 5 & 3 \end{bmatrix}$$

a_{21}

[row operation] ∴ The pre-multiplying matrix which correspond to $R_2 \to R_2 - \frac{3}{2} R_1$ a_{ij} [i] [i] [j]
⇓
corresponds to post multiplication [coloumn operations]
$C_1 \to C_1 - 3/2 C_2$ a_{ij} [j] [j] [i] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ -\frac{3}{2}R_1 + R_2 \\ R_3 \end{bmatrix}$$

consider ; post multiplying by same matrix ?

$$\begin{bmatrix} 2 & 7 & 1 \\ 3 & -2 & 0 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -17/2 & 7 & 1 \\ 6 & -2 & 0 \\ -13/2 & 5 & 3 \end{bmatrix}$$

$\parallel$ $[C_1 \quad C_2 \quad C_3]$ $\qquad$ $\parallel$ $[C_1 - 3/2\, C_2 \quad C_2 \quad C_3]$

Aryan Jain - 2022111

Ex ①:- $A = \begin{bmatrix} 2 & 7 & 1 \\ 3 & -2 & 0 \\ 1 & 5 & 3 \end{bmatrix}$

Ans:)

for Echlon form U,

$\sim \begin{bmatrix} 2 & 7 & 1 \\ 0 & -25/2 & -3/2 \\ 0 & 3/2 & 5/2 \end{bmatrix}$

$R_2 \to R_2 - \frac{3}{2} R_1 \Rightarrow C_1 \to C_1 + \frac{3}{2} C_2$; $l_{21} = \frac{3}{2}$

$R_3 \to R_3 - \frac{1}{2} R_1 \Rightarrow C_1 \to C_1 + \frac{1}{2} C_3$; $l_{31} = \frac{1}{2}$

for L ; perform corresponding flipped Colⁿ Operations ;

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$

$-(-3/25)$

$\sim \begin{bmatrix} 2 & 7 & 1 \\ 0 & -25/2 & -3/2 \\ 0 & 0 & 58/25 \end{bmatrix}$ = U

$R_3 \to R_3 + \frac{3}{25} R_2 \Rightarrow C_2 \to C_2 - \frac{3}{25} C_3$; $l_{32} = -3/25$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & -3/25 & 1 \end{bmatrix} = L$

corresponding to each ; $\forall$; $R_x \to R_x - c R_y$

$\hookrightarrow [a_{xy}]_L = c$

remember to keep the "−" sign in L

Or

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & -3/25 & 1 \end{bmatrix}$

$\because$ L, U $\sim \Delta_{\text{matrix}}$ ⇒ we can solve given system of equations by $\boxed{\text{BACK SUBSTN.}}$

⇓

from Δ-gmi. Matrix ; write corresponding system of equations

⇓

& solve directly the equations

Solving a system of Eqⁿˢ using LU factorisation ?

① We write the eqⁿ: $A\vec{x} = \vec{b}$ as $LU\vec{x} = \vec{b}$

② Put $\vec{y} = U\vec{x}$

③ Solve, the pair of eqⁿ (1)

$L\vec{y} = \vec{b}$

$U\vec{x} = \vec{y}$

in this order

contd

Ex ①:- $A = \begin{bmatrix} 2 & 7 & 1 \\ 3 & -2 & 0 \\ 1 & 5 & 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & -3/25 & 1 \end{bmatrix}$; $U = \begin{bmatrix} 2 & 7 & 1 \\ 0 & -25/2 & -3/2 \\ 0 & 0 & 58/25 \end{bmatrix}$

(I) solving ; $L\vec{y} = \vec{b} \Rightarrow [L \; \vec{b}]$

⇓

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 3/2 & 1 & 0 & 0 \\ 1/2 & -3/25 & 1 & 3 \end{bmatrix}$

RREF Or back substn

$y_1 = 1$

$\frac{3}{2} y_1 + y_2 = 0$

$\frac{1}{2} y_1 + \left(\frac{-3}{25}\right) y_2 + y_3 = 3$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 58/25 \end{bmatrix}$

⇓

$\vec{y} = \begin{bmatrix} 1 \\ -3/2 \\ 58/25 \end{bmatrix}$

(II) Solve ; $U\vec{x} = \vec{y}$

$\vec{x} \Longrightarrow [U \; \vec{y}]$

$x_1 = 0 \Leftarrow 2x_1 + 7x_2 + x_3 = 1$

$x_2 = 0 \Leftarrow -\frac{25}{2} x_2 - \frac{3}{2} x_3 = -3/2$

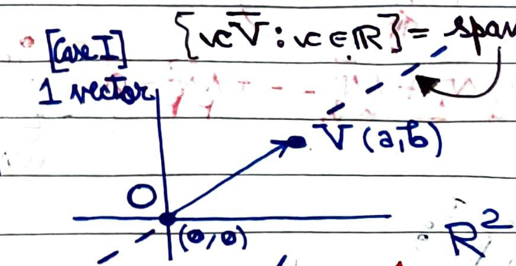
$\frac{58}{25} x_3 = \frac{58}{25} \Rightarrow x_3 = 1$ ← back substn.

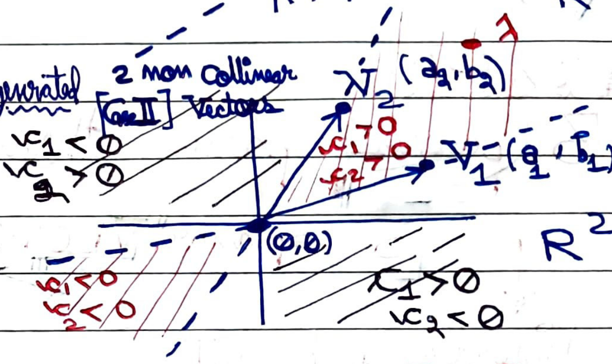
Ans. soln. $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

→ collection of all linear combinations

Linear Span?

If $\bar{v}_1, ---, \bar{v}_p \in \mathbb{R}^m$, then the set of all linear combinations of $\bar{v}_1, ---, \bar{v}_p$ is denoted by span$\{\bar{v}_1, -, -, ---, \bar{v}_p\}$ & is called subset of $\mathbb{R}^m$ spanned / generated by $\bar{v}_1, \bar{v}_2 --- \bar{v}_p$ ✓



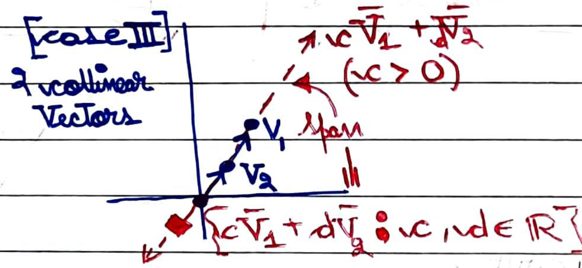


① NOTE:-

$\mathbb{R}^2 \rightarrow$ 1 vector $\rightarrow$ SL

$\hookrightarrow$ 2 vectors

- collinear $\rightarrow$ SL
- non collinear $\rightarrow \mathbb{R}^2$ complete vector



② NOTE:

$\mathbb{R}^3 \rightarrow$ 3 coplanar vectors $\rightarrow$ plane ✓

3 points in a plane passing through (0,0,0)

$\hookrightarrow$ 3 non coplanar vectors $\rightarrow \mathbb{R}^3$ complete ✓

3 points in a plane NOT passing through (0,0,0)

* Span$\{\bar{v}_1, \bar{v}_2 ---, \bar{v}_p\}$ is the collection of all vectors that can be written as $c_1\bar{v}_1 + c_2\bar{v}_2 + --- + c_p\bar{v}_p$; with $c_1 -- c_p \rightarrow$ arbit. scalars ✓

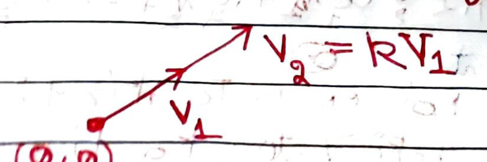
[line in 2D & plane in 3D]

Q: min^m 2 linearly independent vector to span $\mathbb{R}^2$
minimum 3 "linearly independent vectors" req^d to span $\mathbb{R}^3$
$\hookrightarrow$ M dimensional space $\rightarrow \geq M$ linearly INDEPENDENT vectors ✓

In 2D; collinear lines in $\mathbb{R}^2 \rightarrow L_1 = \lambda L_2 \rightarrow$ scaled vectors $\Rightarrow$ scalar multiples of each other

Linear Dependency?:

$\hookrightarrow$ Linearly Dependent

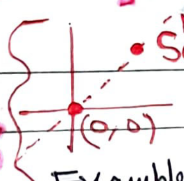


* if non trivial soln $\Rightarrow \exists$ Linear dependency

$V\vec{x} = \vec{0}$

$\Updownarrow$

$x_1\bar{v}_1 + x_2\bar{v}_2 - - + x_p\bar{v}_p = \vec{0}$



$\Rightarrow$ 2 points
- 1 non trivial soln
- 1 origin

Example in $\mathbb{R}^2$:-

$\Downarrow$

only 1 unique line

In $\mathbb{R}^3$;

In $\mathbb{R}^3$,

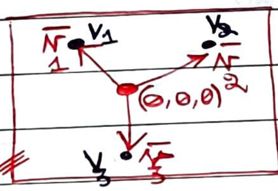
3 points passing through a plane through (0, 0, 0)

$\Updownarrow$

[3 "coplanar" "VECTORS" ($\equiv$ ↗)]

$\Downarrow$

linearly Dependent



coplanar VECTORS

NOTE :-

M linearly independent vectors $\longrightarrow$ span $\mathbb{R}^M$

eg: $\hat{i}, \hat{j}, \hat{k} \Rightarrow$ span $\mathbb{R}^3$
$\hookrightarrow$ can write any point in $\mathbb{R}^3$ using $\hat{i}, \hat{j}, \hat{k}$

⊕ Linear Dependence / Independence?

An Ordered set of vectors $\{\bar{v}_1, \bar{v}_2 - - - \bar{v}_p\} \in \mathbb{R}^n$ is said to be linearly independent if the vector equation $x_1\bar{v}_1 + x_2\bar{v}_2 + - - - x_p\bar{v}_p = \vec{0}$ has ONLY the trivial soln.

$\Updownarrow$

The Ordered set $\{\bar{v}_1, \bar{v}_2 - - \bar{v}_p\}$ is said to be linearly dependent if $\exists$ weights $c_1, - - -, c_p$ such that $c_1\bar{v}_1 + c_2\bar{v}_2 + - - - + c_p\bar{v}_p = \vec{0}$

atleast one $c_k \neq 0$ NOT all Zero

$\Leftrightarrow$ Atleast कोई एक ऐसा c_r exist करे ; $c_r \neq 0$ • soln हो / satisfy करे ; $c_1\bar{v}_1 + c_2\bar{v}_2 + - - + c_p\bar{v}_p = \vec{0}$

OR

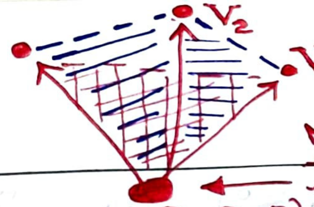
Koi ek vector; kin hi bhi baki Vectors ka Linear Comb^n हो

Aryan Jain - 2022111

3 Coplanar Vectors (N/L)

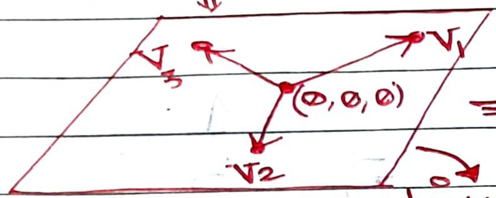
$\Updownarrow$

3 points passing through a plane



$\Rightarrow$ Linearly INDEPENDENT vectors $\Rightarrow$ $\therefore$ span $\mathbb{R}^3$

Non trivial soln. does NOT exists



$\Rightarrow \vec{v}_1 = \lambda \vec{v}_2 + \mu \vec{v}_3$ ✓ $\Rightarrow (-1)\bar{v}_1 + \lambda \bar{v}_2 + \mu \bar{v}_3 = \vec{0}$

(a family of lines)

$\Downarrow$

Linearly DEPENDENT vectors $\Leftarrow$ $\exists$ a Non trivial soln ; $\vec{x} = (-1, \lambda, \mu)$ $\Leftarrow$ $x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \vec{0}$ (or) $V\vec{x} = \vec{0}$

NOTE: if you take ANY 3 independent directions $\Rightarrow$ 3 linearly independent vectors

$\Updownarrow$

you can span the entire $\mathbb{R}^3$ ✓

Given $\vec{b} \in \mathbb{R}^m$ ✓

& $\bar{v}_1, \bar{v}_2 \text{ --- } \bar{v}_p \in \mathbb{R}^m$ ✓

[Q.] Does $\vec{b} \in \text{span}[\bar{v}_1, \bar{v}_2 \text{ --- }, \bar{v}_p]$

Ans.) Does $\vec{b} \in$:- set of all linear combination of $\bar{v}_1, \bar{v}_2 \text{ -- } \bar{v}_p$

does $\vec{b} \in x_1\bar{v}_1 + x_2\bar{v}_2 + \text{---} + x_p\bar{v}_p$; $x_j \to$ arbit scalars

$\Updownarrow$

Does $x_1\bar{v}_1 + x_2\bar{v}_2 + \text{---} + x_p\bar{v}_p = \vec{b}$ (or) $V \cdot \vec{x} = \vec{b}$

is "$\bar{b}$" a linear combination of $\{\bar{v}_1 \text{---} \bar{v}_p\}$

Look at the augmented matrix of $[\bar{v}_1 \ \bar{v}_2 \text{ -- } \bar{v}_p \ \bar{b}]$ ✓

- If soln. Exists $\Rightarrow \vec{b} \in \text{span} \{\bar{v}_1 \text{---} \bar{v}_p\}$ ✓
- If soln. does NOT exist $\Rightarrow \vec{b} \notin \text{span} \{\bar{v}_1 \text{---} \bar{v}_p\}$ ✗

Aryan Jain - 2022111

(Q.) How do we know if an given collection of vectors is linearly independent?

Eg: $\vec{V}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$; $\vec{V}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$; $\vec{V}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{bmatrix}$

Ans) Solving: $V \cdot \vec{x} = 0$

$V = [\bar{V}_1 \ \bar{V}_2 \ \bar{V}_3]$

$\Downarrow$

$x_1 \bar{V}_1 + x_2 \bar{V}_2 + x_3 \bar{V}_3 = 0$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ or (x_1, x_2, x_3)

$\Downarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 3 & 4 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_4 \to R_4 - R_1,\ R_2 \to R_2 - 2R_1} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

$R_4 \to -\frac{1}{2} R_4$, $R_2 \leftrightarrow R_4$, $R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & -3 \to 0 & 1 \to 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ All 3 columns are pivot columns ⇒ $V\vec{x} = 0$ has only the trivial soln.

$\Downarrow$

∴ $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are linearly independent

Ans.

eg: soln of differ^n eq^n like $y'' = y \rightarrow y = e^x$ & $y = e^{-x}$

In Vector space (2 Dimensional) $\Downarrow$ R^2

Page No.:
Date: 06/12/22

YOUVA

Abstract Vector Space?

A vector space (real vector space) is a non empty set V of objects, called vectors on which are defined two operations, called ADDITION and MULTIPLICATION by scalars (real numbers), subject to 10 axioms/rules listed below.

These axioms must hold $\forall$ vectors $\bar{u}$, $\bar{v}$ & $\bar{w}$ in V & $\forall$ scalars c, d.

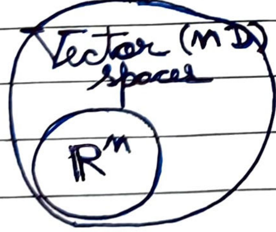
[closure (addition) $\Rightarrow$ Sum of vectors = vectors]

1. The sum of $\bar{u}$ & $\bar{v}$, denoted by $\bar{u}+\bar{v}$ is in V
2. $\bar{u}+\bar{v} = \bar{v}+\bar{u}$ ← commutative
3. $u+(\bar{v}+\bar{w}) = (\bar{u}+\bar{v})+\bar{w}$ ← associativity
4. There is a zero vector $\vec{0}$ in V such that $\bar{u}+\bar{0} = \bar{u}$ ← additive identity

(we need to prove that $\vec{0}$ is unique; axiom states there exists $\vec{0}$ & NOT that its unique)

5. for Each u in V, there is a vector "$-u$" in V such that $(\bar{u}) + (-\bar{u}) = \bar{0}$ ← additive inverse
6. The scalar multiple u by c; cu is in V (scalar multiplication) closusre
7. $c(\bar{u}+\bar{v}) = c\bar{u}+c\bar{v}$
8. $(c+d)\bar{u} = c\bar{u}+d\bar{u}$
9. $c(d\bar{u}) = (cd)\bar{u}$
10. $1 \cdot \bar{u} = \bar{u}$ ← multiplicative identity

Generalisation of $R^n \Rightarrow$ Vector Space ✓



Q: jis pe yeh axioms lag gye $\Rightarrow$ hai a Vector space structure ✓

Examples of Vector Spaces ?

(I) Euclidean spaces $\mathbb{R}^n$

(II) $M_{m\times n}(\mathbb{R}) \rightarrow$ set of all $m \times n$ matrices having

(III) $\mathbb{C} \sim \mathbb{R}^2$ as a real vector space

complex nos. — ISOMORPHISM (Bijective correspondance)

Defination of addition in that vector space ✓

(IV) Let $\mathbb{R}^N$ be the space of all infinite sequences of real numbers.

defination of scalar multiplication in $\mathbb{R}^N$

$\{y_k\} \in \mathbb{R}^N$; $\{y_k\} = (y_1, y_2, \ldots, y_\infty)$ $\quad \{y_1+z_1, y_2+z_2, \ldots\}$

∴ $\{y_k\} \oplus \{z_k\} = \{y_k+z_k\}$ is formed by adding corresponding terms of $\{y_k\}$ & $\{z_k\}$

& ∴ $c\{y_k\} = \{c \cdot y_k\} = \{cy_1, cy_2, \ldots\}$

Sequence → Any random sequence: [AP | GP | AGP | HP -- etc.]

$\mathbb{R}^N$ ← [capital N = Natural Nos.] $\sim \mathbb{R}^\infty$; $\infty = \{1, 2, 3, \ldots, \infty\}$

If we allow the terms to be indexed by $\mathbb{Z}$, the space obtained is sometimes called the space of Discrete time signals ✓

Checking Axioms

① $\{y_k\} \oplus \{z_k\} = \{y_k+z_k\} \in \mathbb{R}^N$ ← sequence ✓

② $\{y_k\}\oplus\{z_k\} = \{y_k+z_k\} = \{z_k+y_k\} = \{z_k\}\oplus\{y_k\}$

③ $\{x_k\} + (\{y_k\}+\{z_k\}) = \{x_k\}+\{y_k+z_k\} = \{x_k+(y_k+z_k)\} = \{x_k+y_k\}+\{z_k\}$
$= (\{x_k\}+\{y_k\})+\{z_k\}$

④ $\{0_k\} = (0, 0, 0, \ldots\}$; $\{y_k\}+\{0\} = \{y_k\}$

⑤ $\{-y_k\} = -1\cdot\{y_k\}$ ✓

⑥ $c\{y_k\} = \{cy_k\}$, which is a sequence $\Rightarrow \in \mathbb{R}^N$ (∞)

⑦ $c\{y_k+z_k\} = \{c(y_k+z_k)\} = \{cy_k+cz_k\} = \{cy_k\}+\{cz_k\}$ ← $c\{y_k\}+c\{z_k\}$ ✓

⑧ $(c+d)\{y_k\} = \{(c+d)y_k\} = \{cy_k\}+\{dy_k\} = c\{y_k\}+d\{y_k\}$

⑨ $c\{dy_k\} = \{cdy_k\} = cd\{y_k\}$ ✓

⑩ $\{1_k\} = (1, 1, 1, \ldots)$ ✓

$poly_1 = poly_2$
$\hookrightarrow$ corresponding coeff equal

M T W T F S S
Page No.: YOUVA
Date:

[Polynomial Vector Space]

(V) for $n \geq 0$, the set $\mathbb{P}_n$ of polynomials of degree at most n consi of all polynomials of the form

$(a_0, a_1 - - a_n) \rightarrow$ Real nos.

$p(t) = a_0 + a_1 t + a_2 t^2 + - - - + a_n t^n$; $t \rightarrow$ var. taking real values

$$\Downarrow$$

$(a_0, a_1, a_2 - - a_n) \Rightarrow$ given tupple $\Rightarrow \sim R^{n+1}$

[Isomorphism]

& zero polynomial $\longrightarrow$ all coeff $\equiv 0$
$(0, 0, 0 - -, 0)$

polynomials are not thought as functions

$$\Downarrow$$

we are not Evaluating "t"

degree of non zero $p(t) \rightarrow$ highest power of t whose coeff $\neq 0$

\# check axioms ; ① Closed under addition?

✓ $p(t) + f(t) = (p_0 + f_0) + (p_1 + f_1)t + - - - (p_n + f_n)t^n$
$= (p+f)(t) \in \mathbb{P}_n$ ✓

② Closed under multiplication?

✓ $c\, p(t) = (cp_0) + (cp_1)t + (cp_2)t^2 + - - - + (cp_n)t^n$
$\hookrightarrow \in \mathbb{P}^n$

[Function Vector Space]

(VI) Let D be a non empty set ; Let $\mathbb{V}$ be the set of all real valued functions defined on set D

$D [n] \xrightarrow{f} R^n$

$$\Downarrow$$

V:- collection of all possible real valued functions "f"

$\mathbb{V} = \{f : f : D \rightarrow R$ is a function$\}$

check axioms ; ① for $f_1(x)$ and $f_2(x) \in \mathbb{V}$; $h(x) = f_1(x) + f_2(x) \in \mathbb{V}$
② for $f(x) \in \mathbb{V}$; $g(x) = c \cdot f(x) \in \mathbb{V}$

$\therefore$ Vector Space ✓

$$\Downarrow$$

Are there any restrictions on D?

M T W T F S S
Page No.: | YOUVA
Date:

NOTE:-

If all 10 axioms hold on (✱)

$\Downarrow$

(✱) has vector space structure ✓

:- need not check all 10 axioms; just check the 2 closure axioms ✓

Why?

If;

(I) Closed under Addition ; $\bar{u}+\bar{v} \in V$; for $\bar{u}$ & $\bar{v} \in V$

and

(II) Closed under multiplication ; $c\bar{u} \in V$; for $\bar{u} \in V$

→ Just Ensure इतना

→ ∴ V has vector space structure ✓

NOTE:

Ek bade already known subspace ka subset → vector space is also an vector space

Eg: $W \subset V$ ⇒ W pe same multiplication & addition ⇒ $W \equiv$ subspace of V → W is also an vector space ✓

(all continuous function) (all real valued function)

Subspace $\subseteq$ **original वाला Vector space**

[Subset]

All subspaces of vector space are an vector space

✱ H is subspace of V; iff H closed under addition & multiplication

⇕ proposition:-

Let H be a non empty subset of V that has 2 properties:

(a) H is closed under vector addition, ie $\forall$ $\bar{u}$ & $\bar{v}$ in H, the sum $(\bar{u}+\bar{v})$ is in H

(b) H is closed under multiplication by scalars, ie $\forall$ $\bar{u} \in H$ & $\forall$ scalars c; the vector $c\bar{u} \in H$

Then, H is a subspace of V

→ proof / verification?

(I) Addition Closure → given

(II) Commutativity : $\bar{v}, \bar{u} \in H$; $\bar{v}+\bar{w} = \bar{w}+\bar{v}$ by commutativity holds in V

IIIrd; other Axioms ✓

redefining addition & scalar multiplication?

NOTE that; $\varnothing$ नहीं है; $(v+w)$ में additive inverse nahi milega! ✗

⇒ Redefine ⊕ & ⊛ to preserve/make vector space structure

Page No.: Date: YOUVA

Example :-

$V = (0, \infty)$ ⇐ Vector space structure on the set of positive real nos.

Let; $v, w \in V$

define;

$v \oplus w = vw$; $w > 0$ { defined such so that; $(v)\left(\frac{1}{v}\right) = 1$

$c \circledast v = v^c$; $c \to$ any scalar $\in \mathbb{R}$

redefined addition in vector space

redefined scalar multiplication in vector space

Checking 10 axioms :-

(I) $v \oplus w = vw > 0 \Rightarrow \in V$

(II) $v \oplus w = v \cdot w = w \cdot v = w \oplus v$

(III) $u \oplus (v \oplus w) = u \oplus vw = uvw = uv \oplus w = (u \oplus v) \oplus w$

(IV) $\boxed{\vec{0} \equiv 1 > 0}$; $\bar{u} \oplus \bar{0} = \bar{u}(\cancel{\bar{0}})^{1} = \bar{u}$

(V) $\bar{u} + (-\bar{u}) = \vec{0} \to 1$ $\therefore (v)\left(\frac{1}{v}\right) = 1$

↳ $(-\bar{u}) \equiv \frac{1}{u}$; $\left(-\bar{u}\right)$ ⇓ $\vec{0}$

(VI) $c \circledast \bar{w} = w^c \in V$, $w > 0$

(VII) $c \circledast (\bar{u} \oplus \bar{v}) = c \circledast (uv) = (uv)^c = u^c \cdot v^c$

$= (c \circledast \bar{u}) \cdot (c \circledast \bar{v}) = (c \circledast \bar{u}) \oplus (c * \bar{v})$

(VIII) $(c+d) \circledast \bar{u} = u^{c+d} = u^c \cdot u^d = u^c \oplus u^d = (c \circledast \bar{u}) \oplus (d \circledast \bar{u})$

(IX) $c \circledast (d \circledast \bar{u}) = c \circledast (u^d) = u^{cd} = cd \circledast u$

(X) $1 \circledast w = w^1 = w$

$\therefore$ All axioms hold ⇒ Vector space ✓

Defination :- Let $(V, +, \cdot)$ be a vector space.
A subset $W \subset V$ is said to be a subspace of V if W is a vector space under the same operations of addition & scalar multiplication

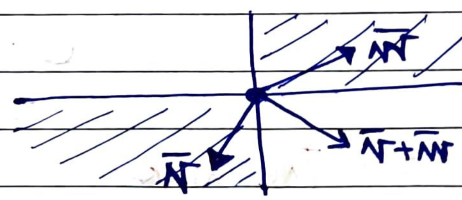
Practise Question :-

Q1) Why are the following sets NOT vector spaces?

(i) Let V be the first quadrant in the x-y plane ie
$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 , y \geq 0 \right\}$$

Ans.) $v = (1,1) \in V$
as for $c = -1$; $cv = (-1,-1) \notin V$; is out of vector space
$\therefore$ Not closed under multiplication $\Rightarrow$ Not a vector space

(ii) Let W be the union of first & third quadrant in xy plane
ie $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right.$



Ans.) $\bar{v} = (-3,-3) \in W$
$\bar{w} = (4,2) \in W$
$\bar{v}+\bar{w} = (1,-1) \notin W \Rightarrow$ not closed under addition

(iii) Let D be the unit disk in x-y plane ie
$$D = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2+y^2 \leq 1 \right\}$$

Ans.) a vector multiplied by a scalar $c > 1$; will go outside the disk
ie $c\bar{v} = [cx, cy] \notin D \Rightarrow$ Not closed under scalar multiplication
$\forall c > 1$

$(cx)^2 + (cy)^2$
$c^2 \cdot (x^2+y^2) \Rightarrow c^2 > 1$
$\in [1,\infty)$ $\quad \in [0,1]$
May be $> 1 \Rightarrow \in D$

(iv) A line in $\mathbb{R}^2$ or $\mathbb{R}^3$ which does NOT pass through $(0,0)$?

M	T	W	T	F	S	S

Page No.: YOUVA
Date:

Ans.) $\vec{0} \notin$ vector space $\Rightarrow \therefore$ NOT a vector space!

Proposition:- Let V be a vector space; The zero vector in V is unique

proof:- Let $\bar{0}$ & $\bar{z}$ both be zero vectors $\{\bar{0} \neq \bar{z}\}$
Let $\bar{a}$ be any element in the vector space V

$$\left.\begin{aligned} \bar{a} + \bar{0} &= \bar{a} \\ \bar{a} + \bar{z} &= \bar{a} \end{aligned}\right\}$$

RHS same; evaluating LHS

$$\bar{a} + \bar{0} = \bar{a} + \bar{z}$$

$\Downarrow$ $+(-\bar{a})$ both sides

$$(-\bar{a}) + \bar{a} + \bar{0} = (-\bar{a}) + (\bar{a}) + (\bar{z})$$

$$\Downarrow$$

by associative law;
by additive identity;

$$(\bar{a} - \bar{a}) + \bar{0} = (\bar{a} - \bar{a}) + \bar{z}$$

$$\Downarrow$$

$$\bar{0} + \bar{0} = \bar{0} + \bar{z}$$

$$\Downarrow$$

$$\boxed{\bar{0} = \bar{z}} \Rightarrow$$ $\exists$ only 1 zero vector & it is unique

Note :- The zero subspace $\{0\}$ consisting of only $\vec{0}$ is subspace of any vector space

Closed under addition: $\vec{0} + \vec{0} = \vec{0}$

closed under scalar multiplication; $\vec{0} \times \underset{\text{scalar}}{c} = \vec{0}$

Proposition :- Let V be a vector space; for every $\bar{u}$ in V; $\exists$ a unique $(-\bar{u})$ called negative or additive inverse of $\bar{u}$ such that $(\bar{u}) + (-\bar{u}) = \bar{0}$

Page No.: YOUVA
Date:

proof:- for $\bar{u}, \bar{a}, \bar{b} \in V$

let $\bar{a}, \bar{b}$ be additive inverses $(\bar{a} \neq \bar{b})$

$$\bar{u} + \bar{a} = \vec{0}$$
$$\bar{u} + \vec{b} = \vec{0}$$

RHS same ⇓ ; Equating LHS

$$\bar{u} + \bar{a} = \bar{u} + \bar{b} \Rightarrow \bar{a} + (\bar{u} + \bar{a}) = \bar{a} + (\bar{u} + \bar{b})$$

$$\hookrightarrow (\bar{a} + \bar{u}) + \bar{a} = (\bar{a} + \bar{u}) + \bar{b} \Rightarrow \vec{0} + \vec{a} = \vec{0} + \vec{b}$$

$$\hookrightarrow \vec{a} = \vec{b}$$

$\therefore$ additive inverse is unique.

Proposition :- Let V be a vector space, then

① $0\bar{u} = \bar{0}$
② $(-\bar{u}) = -1 \times \bar{u}$ $\Big\}\ \forall\ \bar{u} \in V$
③ $c\bar{0} = \bar{0}$, $\forall$ scalars c

proof: ① $\bar{u} + \bar{0} = \bar{u}$

$\bar{u} + 0(\bar{u}) = (1+0)\bar{u} = \bar{u}$

Equating LHS

$$\bar{u} + \bar{0} = \bar{u} + 0(\bar{u}) \xrightarrow[+(-\bar{u})]{} \cancel{(-\bar{u}) + (\bar{u})} + \bar{0} = \cancel{(-\bar{u}) + \bar{u}} + 0\bar{u}$$

$$\Downarrow$$

$$\bar{0} = 0\bar{u} \quad \square$$

QED

② $\bar{u} + \underbrace{(-1)\bar{u}}_{} = \bar{0}$ $\quad (1-1)\bar{u} = 0\bar{u} = \bar{0}$

$(\bar{u}) + (-\bar{u}) = \bar{0}$

$\square$
QED

③ Let $\bar{u} \in V \Rightarrow \bar{u} + (-\bar{u}) = \bar{0}$

$$c\,\bar{0} = c(\bar{u} - \bar{u}) = (c\bar{u}) - (c\bar{u}) = \bar{0}$$

$$\Downarrow$$

$$c\,\bar{0} = \bar{0}$$

$\square$
QED

- Given; vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \in$ vector space V ; we define

$$\boxed{\sum_{i=1}^{n} \bar{v}_i = \bar{v}_n + \sum_{i=1}^{n-1} \bar{v}_i}$$ ✓

Proposition : $\sigma : [n] \to [n]$ be a permutation (bijection)

Aryan Jain - 202211

Let $\vec{v_1}, \vec{v_2}, ---, \vec{v_M} \in$ vector space $\mathbb{V}$

proved before already

$$\sum_{j=1}^{n} \bar{v}_j = \sum_{j=1}^{n} v_{\sigma(j)}$$ $\implies$ order of addition of multiple vectors does NOT matter

Defination :

Let $\bar{v}_1, \bar{v}_2 --- \bar{v}_p \in$ vector space $\mathbb{V}$ (distinct elements of) & let $c_1, c_2 -- c_p$ be scalars ; the vector $c_1\bar{v}_1 + c_2\bar{v}_2 + --- + c_p\bar{v}_p$ is called Linear Comb^n of vectors $\bar{v}_1, --, \bar{v}_p$

नमः सिद्धम ॥

Page No.:
Date: 12/12/22

Defination :- Let S be a finite subset of vector space $\mathbb{V}$; (1) If S is the empty set, we define span S to be singleton set {0}. (2) If S is non empty, we define span S to be the set of all elements of $\mathbb{V}$ that can expressed as linear combination of elements of S

Proposition :- Let S be a finite subset of $\mathbb{V}$; then (span S) is a **SUBSPACE** of $\mathbb{V}$

ie S → any set / collection of vectors / points
Span S → collection of all linear combination of elements from S

Note : S → need not be a vector space (Imp). ; (S any ⊆ of $\mathbb{V}$)
(but) span S = vector subspace ; (for sure)

if [S ⇒ empty set] ⇒ span S = {0}

Proof :- Let $S = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_M\} \subset V$

Let $\bar{v}, \bar{w} \in$ span S

& let

∃ scalars $c_1, c_2, \dots, c_M$ & $d_1, d_2, \dots d_M$;

such that

$$\bar{v} = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_M\bar{v}_M$$
$$\bar{w} = d_1\bar{v}_1 + d_2\bar{v}_2 + \dots + d_M\bar{v}_M$$

(i) Closure under addition ?

$$\bar{v} + \bar{w} = \Sigma c_i\bar{v}_i + \Sigma d_i\bar{v}_i = \Sigma(c_i + d_i)\bar{v}_i \in \text{span } S$$

∴ QED

(ii) Closure under multiplication ?

$$\lambda\bar{v} = \lambda c_1\bar{v}_1 + \lambda c_2\bar{v}_2 + \dots + \lambda c_M\bar{v}_M$$
$$= \gamma_1\bar{v}_1 + \gamma_2\bar{v}_2 + \dots + \gamma_M\bar{v}_M \in \text{span } S$$

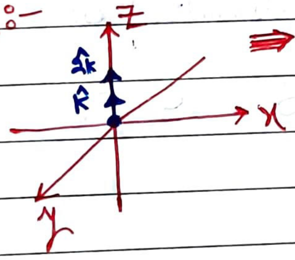
□

Proposition

① Let S be a finite subset V. Then span S is the intersection of all subspaces of V which contain S

② Let S be a infinite subset of V
We define Span S to be intersection of all subspaces of V which contain S

✳ Eg :- ①



$\Rightarrow$ Intersection of all planes containing $2\hat{k}$ & $\hat{k}$ $= z$ axis

ie required subspace

$V \to$ complete 3D space / स्वयं Vector space

$S \equiv \{\hat{k}, 2\hat{k}\}$; $\boxed{S \subset V}$

$W \to$ any plane containing $\hat{k}$ & $2\hat{k}$ ie any plane passing through z axis

span $S = c_1(\hat{k}) + c_2(2\hat{k}) = z$ axis

$\Downarrow$

intersection of all planes $\in W$ $=$ z axis $=$ span S

①
proof :- Let W be any subspace of V which contains S ; Then any linear combination of elements of S is contained in W $\therefore$ span $S \subset W$

set S Contained in subspace W
$\equiv$
$(\bar{v}, \bar{u})$ Let VLOG

(i) $\bar{v} + \bar{u} \in W \ \forall \ \bar{v}, \bar{u} \in W$
(ii) $c\bar{v} \in W$

$c\bar{v} \in W$ if $\bar{v} \in W$
$d\bar{u} \in W$ if $\bar{u} \in W$
$\Downarrow$
[span S] $c\bar{v} + d\bar{u} \in W$

such possible all subspaces W_k contain span S
$\downarrow$
$\cap W_k \equiv$ intersection of such W will contain span S
$\downarrow$

* How is intersection exactly Equal to span S ?

As span S is itself a subspace containing S ;
$\therefore$ span S equals to the intersection of all subspaces which contain S

M T W T F S S
Page No.: YOUVA
Date:

Defination: An ordered set of vectors $\{\bar{v}_1, \bar{v}_2 - - \bar{v}_p\} \in \mathbb{R}^n$ is said to be linearly independent if

→ the vector eq^n $V\bar{x} = \bar{0} \Leftrightarrow \bar{v}_1 x_1 + \bar{v}_2 x_2 + - - \bar{v}_p x_p = \bar{0}$ has only the trivial soln.

or

→ if $\exists$ weights $c_1, c_2, - - -, c_p$ not all zero ; such that $c_1\bar{v}_1 + - - - + c_p\bar{v}_p = \bar{0}$

Examples: $\mathbb{V}$:→ all real valued functions defined on $\mathbb{R}$

(1.) $S = \{1, \sin^2 x, \cos^2 x\}$

Ans] Linearly Dependent Set

as $\sin^2 x + \cos^2 x - 1 = 0 \Rightarrow \bar{x} = (1, 1, -1)$ is soln of $S\bar{x} = \bar{0}$

or

$\exists$ $c_1 = 1, c_2 = 1, c_3 = -1$ ✓

Imp.

(2.) $S_2 = \{1, \sin x, \cos x\}$

Ans.) Linearly Independent Set

we solve the equation: $\underbrace{c_1(1) + c_2(\sin x) + c_3(\cos x)}$ $= 0$

Evaluating LHS at $x = 0 \Rightarrow c_1 + c_3 = 0$ —(1)

at $x = \frac{\pi}{2} \Rightarrow c_1 + c_2 = 0$ —(2)

at $x = \pi \Rightarrow c_1 - c_3 = 0$ —(3)

(1) & (3) : $c_1 = 0$

↳ (2) : $c_2 = 0$

↳ (1) or (3) : $c_3 = 0$

→ only trivial soln exist for $S\vec{x} = 0$

↳ no non zero c_k exist!

Set S = Linearly Independent

iff $\Rightarrow$ bi implication ($\Leftrightarrow$) $\Rightarrow$ S linear dependent $\Leftrightarrow$ v_j linear combn.

Page No.: Date: 15/12/22 YOUVA

Characterization of Linearly Dependent Sets ?

An indexed set $S = \{v_1, v_2, ---, v_p\}$ of two or more is linearly dependent (if & only if) iff at least one of the vectors in S is linear combination of the others.

In fact, if S is linearly dependent & $v_1 \neq 0$, then some v_j $(j>1)$ is a linear combination of preceding vectors $v_1, v_2, ---, v_{j-1}$

While Writing proof; Define every symbol you use. ***

proof: (i) S ≡ linearly dependent [given] ; v_j ≡ linear combn. of others [To prove]

Assume that one of the vectors in S is a linear combination of others; ie $\exists\, j \in [p]$ such that $v_j = c_1 v_1 + --- + c_{j-1} v_{j-1} + c_{j+1} v_{j+1} + --- + c_p v_p$; $c_k \in \mathbb{R}$ || $S = \{v_1, v_2, ---, v_p\}$

Consider Eqn. $V\bar{x} = \bar{0}$ or $c_1 v_1 + --- + c_p v_p = 0$ $\Rightarrow$ satisfy

put $c_j = -1$

This is a non trivial soln of $V\vec{x} = 0$ or $\exists$ a weight $c_j \neq 0$ such that ▭ has a soln.

$\Rightarrow$ S are Linearly dependent vectors

(ii) v_j ≡ linear combn. of others & $\bar{v}_1 \neq 0$ [given] ; S ≡ linearly dependent [To prove]

$\exists$ scalars $c_1, c_2 -- c_p$; such that not all zero $c_1 v_1 + --- + c_p v_p = 0$

[Q] preceding ? Ans:] if $c_p v_p$, 0, $c_{p-1} v_p$, $c_j v_j$; $c_j \neq 0$

we only need one $c_k \neq 0$; for linear dependence

Let $j \in \{1, --, p\}$ be the longest index for which $c_j \neq 0$

Then; $\exists\, c_k(s)$ not all zero, such that only preceding vectors contribute

$c_1 v_1 + c_2 v_2 + -- + c_j v_j = 0$

$\frac{c_1}{c_j} v_1 + \frac{c_2}{c_j} v_2 - - + \dots$ we can divide by c_j $[c_p/c_j] v_p = 0$

Aryan Jain - 2022111

Page No.:
Date:

dividing all coeff(s) by c_j & rearranging terms, we get,

$$\left(-\frac{c_1}{c_j}\right)v_1 - \left(\frac{c_2}{c_j}\right)v_2 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right)v_{j-1} = v_j$$

$$\Downarrow$$

$v_j \equiv$ linear combination of preciding vector $\square$

Basis

M T W T F S S
Page No.: YOUVA
Date:

Defination :-

Let V be a vector space ; A set of vectors $B \subset V$ is said to be a basis of V if
(I) B is linearly independent
(II) B spans V

NOTE :- [Whenever B is a finite set ; V is finite dimensional]

Spanning set given ⇒ uss se basis kese banaye
Ek-Ek vector utha
⇒ If linearly independent vector wrt other vector selected for basis ⇒ rakhlo Basis me
↳ else ; hata do
⇓ ie pre existing linearly independent set
$\{v_1, --, v_m\}$ me
w vector add kar rhe
& add iff ie linearly independent of $\{v_1 - -, v_m\}$
ie add if $w \notin span(\{v_1, --, v_m\})$

Proposition :-

Let $\{v_1 --- v_m\}$ be a linearly independent set in a vector space V. If $w \in span(\{v_1 -- v_m\})$ then the set $\{v_1, ---, v_m, w\}$ is linearly independent

↳ proof :- Consider the Eqn ;

$$x_1 \cdot v_1 + x_2 \cdot v_2 + --- + x_m \cdot v_m + x_{m+1} \cdot w = 0$$

suppose, if possible that this equation has a non trivial solution say $c_1 v_1 + --- + c_m v_m + c_{m+1} w = 0$
assume

ie $\exists$ $c_1, c_2, ---, c_{m+1}$; not all zero
such

case (I) :- $c_{m+1} \neq 0$; ÷ by c_{m+1} & rearrange terms

$$\Rightarrow w = -\frac{c_1}{c_{m+1}} v_1 + --- + -\frac{c_m v_m}{c_{m+1}}$$

$\in span\{v_1 --- v_m\}$ *

Aryan Jain - 2022111

M T W T F S S
Page No.:
Date:
YOUVA

case (II) : $c_{m+1} = 0$

$\Downarrow$

$c_1 v_1 + \dots + c_m v_m = 0$ (F) $\ast$

$\Downarrow$

~~$\exists$ a non trivial soln of $V\vec{x} = \vec{0}$~~

$\Longrightarrow$

Aryan Jain - 202211

$\setminus \rightarrow$ set difference

set $A - \{0,0\}$ $\Updownarrow$ set $A \setminus \{0,0\}$

M T W T F S S
Page No.:
Date:
YOUVA

Spanning Set Theorum

[Basis + Extra vector]

S $\rightarrow$ spanning set
↳ ie spans H $(H \subseteq V)$

⇓ if we remove Extra vectors ⇒ set $T \subseteq S$; $T \rightarrow$ Basis of H
ie still span H

ie if:
$v_k \in span(v_1, v_2, --, v_{k-1})$
we can drop v_k from S
& new set S' will yet span H

Theorum:- Let V be a vector space. Let $S = \{v_1, v_2 --- v_p\}$ be a set in V and let $H = span\{v_1, v_2 --- v_p\}$

$v_k \rightarrow$ given vector $\in S$ ie a linear combination of other vectors $\in S$

$v \rightarrow$ any random vector
↳ $\notin S$ and
↳ $\in span(S) \equiv H$

claim ① If one of the vectors in S, say v_k is a linear combination of the remaining vectors in S; then the set formed from S by removing v_k will still span H

we will prove that "set $S - \{v_k\}$" will yet span H
(as $v_k \rightarrow$ linear combn of it; we can remove it)

claim ② If $H \neq \{0\}$, some subset of S is a basis for H.

① ⇒ proof: $\exists$ scalars not all 0, such that $v_k = c_1v_1 + c_2v_2 + -- + c_{k-1}v_{k-1} + c_{k+1}v_{k+1} + --- + c_pv_p$

Let $v \in H$ ($= span\{v_1 -- v_p\}$). Then $\exists$ scalars $d_1, d_2, ---, d_p$ not all zero such that

$$v = d_1v_1 + d_2v_2 + --- d_{k-1}v_{k-1} + d_k v_k$$

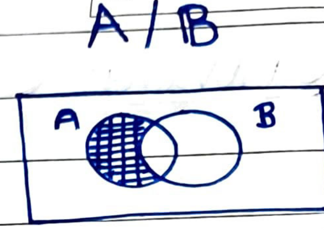
⇓

$$v_k \rightarrow (c_1v_1 + --- + c_pv_p)$$

$$= \gamma_1 v_1 + ---- + \gamma_{k-1} v_{k-1}$$

↳ $v \in span\{v_1, --, v_{k-1}\}$
$\in H$

Aryan Jain - 2022111



② → proof: we prove this by recursively removing elements from S:

Let $S_0 = S$

~~[illegible]~~ If $v_n \in \text{span}\{v_1 - - - v_{n-1}\}$

then; $S_1 = S_0 \setminus \{v_n\}$ otherwise $S_1 = S_0$ ← if $v_n \notin \text{span}\{v_1 - - v_n\}$

Similarly: Assuming that S_k has been constructed

To construct S_{k+1}, if $v_{k+1} \in \text{span}\, S_k$; then $S_{k+1} = S_k \setminus \{v_{k+1}\}$

otherwise, $S_{k+1} = S_k$

if span S_k me v_{k+1} नहीं है।

CLAIM:- In the end, S_n is a basis ← because of claim ①, S_n will span H

If S_n is linearly dependent, then by characterisation of linear dependent sets (we proved earlier); one of the vectors in S_n is a linear combn. of other/preceding vectors

But; since we removed all such vectors in this construction

⇓

① $H = \text{span}\, S_n$ & ② $S_n \equiv$ linearly independent set

⇓

S_n is basis of H

M T W T F S S
Page No.: Date: YOUVA

Fundamental Subspace?

• **Nul A** = The null space of an $m \times n$ matrix A, is the set of all solutions of the homogeneous linear system $A\vec{x} = \vec{0}$

$$\text{Nul } A = \{\bar{x} \in \mathbb{R}^n : A\bar{x} = \bar{0}\} \subset \mathbb{R}^n$$

• **Col A** = The coloumn space of a $m \times n$ matrix A is the set of all linear combinations of the coloumns of A.

span of any set of vectors
↳ subspace

ie if $A = [\bar{a}_1 \ \bar{a}_2 - - - - - - \bar{a}_m]$

then $\text{Col } A = \text{span}\{\bar{a}_1, \bar{a}_2 - - -, \bar{a}_n\} \subset \mathbb{R}^m$

$A\vec{x} \rightarrow$ linear combination

$\vec{b} \rightarrow$ if also linear cbn $\Rightarrow A\vec{x} = \vec{b}$ has soln

or $\Rightarrow \vec{b} \in$ span ie $\vec{b} \in$ col space (A)

III^{ly}:

• **Row A** = The row space of a $m \times n$ matrix A [written as Row A] is the set of all linear combinations of rows A

ie $\text{Row } A = \text{col}(A^T) \subset \mathbb{R}^n$

∴ $\underbrace{\text{Nul } A, \ \text{Nul}(A^T), \ \text{Col } A, \ \text{Row } A}$

↳ fundamental subspaces associated with matrix A

Eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix} \Rightarrow$ find fundamental subspaces of A?

Ans:) (I) to find Null space of A, solve $A\bar{x} = \bar{0}$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -12 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 0 \longrightarrow x_1 = 5x_3$$
$$-3x_2 - 12x_3 = 0 \Rightarrow x_2 = -4x_3$$

$\bar{x} = x_3 \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$ Ans. $\Rightarrow \text{Nul}A = \underset{\text{set}}{\infty} = \left\{ \lambda \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} ; \lambda \in \mathbb{R} \right\}$

(II) Col A = span{(1,4), (2,5), (3,0)} $= \mathbb{R}^2$

$$\boxed{\vec{b} \in \text{Col}A \Leftrightarrow A\vec{x} = \vec{b} \text{ has a soln.}}$$

If $\vec{b} \in \mathbb{R}^2$, the eqn $A\bar{x} = \bar{b}$ always has a soln. [as x_3 is free variable]

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 0 & b_2 \end{bmatrix}$$

Defination :-

Let S be a finite subset of a vector space V. If S is empty set, we define span S to be the singleton set {0}. If S is non empty, we define span S to be the set of all elements of V that can be expressed as linear combination of elements of S.

- Let S be a finite subset of V;
  Then span S is subspace of V ✓

- Let S be a finite subset of V;
  Then span S is the intersection of all subspaces of V which contain S ✓

- ~~Let S be an infinite subset of V, then define S~~

proof : { $A \subseteq B$ & $B \subseteq A$ ⟹ $A = B$ } ∵ span S is subspace of V.

- ~~Proposition:~~

~~Let B be a mxn matrix in RREF, having k first columns where [illegible]~~
~~Then [illegible] is a basis for Col B~~
~~first columns of RREF~~

- ~~Theorem:~~

~~Let A be a mxn matrix.~~
~~The first columns of A form a basis for Col A.~~

Aryan Jain - 202211

$Col B \subset Span\{\bar e_1, \bar e_2, ..., \bar e_k\}$ & $Span\{\bar e_1, \bar e_2, ..., \bar e_k\} \subset Col B$

$Col B = Span\{\bar e_1, \bar e_2, ..., \bar e_k\}$

~~Theorem 2~~ ~~proof~~

Part (I) Since $\vec b = b_1 \bar e_1 + b_2 \bar e_2 + ... + b_k \bar e_k \in span(\bar e_1, \bar e_2, ... \bar e_k)$

$\therefore$ any column of $B \in span(\bar e_1 ... \bar e_k)$

$\Downarrow$ How?

$\therefore Col B \subset Span\{\bar e_1, \bar e_2, ... \bar e_k\}$

Part (II) Since $\{\bar e_1 ... \bar e_k\}$ are columns of B

$\Downarrow$

$span(\bar e_1, ..., \bar e_k) \in col B$

$\Rightarrow span(\bar e_1, \bar e_2, ..., \bar e_k) = col B$

$\square$

Theorem

Proof:- Let B be the RREF of A, then $\exists$ an invertible matrix E such that $\boxed{EA = B}$

$\Rightarrow \underset{rref}{B} = [E\bar a_1 \; ... \; E\bar a_m]$

Let $A = [\bar a_1 \; ... \; \bar a_m]$

$I = [\bar e_1 \; \bar e_2 \; ... \; \bar e_m]$ Let $\bar a_{i_1}, ..., \bar a_{i_k}$ be the pivot columns of A; where $1 \le i_1 < ... < i_k \le m$

$E\bar a_{i_1} = \bar e_1, \; E\bar a_{i_2} = e_2, ..., E\bar a_{i_k} = \bar e_k$

Claim: $\bar a_{i_1}, ..., \bar a_{i_k}$ are linearly independent

$\Rightarrow$ proof: we consider the eq^n: $c_1 \bar a_{i_1} + c_2 \bar a_{i_2} + ... + c_k \bar a_{i_k} = 0$

$\Downarrow$

Then $E(c_1 \bar a_{i_1} + c_2 \bar a_{i_2} + ... + c_k a_{i_k}) = 0$ where $c_1, ..., c_k \in \mathbb{R}$

$\Downarrow$

$\sum_{j=1}^{k} c_j E a_{i_j} \equiv c_1 \bar e_1 + c_2 \bar e_2 + ... + c_k \bar e_k = 0$

$\Downarrow$ $\Downarrow$

$(c_1, c_2, ..., c_k, \underbrace{0 ... 0}_{m-k \text{ entries}}) = 0$ or $I_m \vec x = 0$ which has only the trivial soln.

$\Downarrow$

$c_1 = c_2 = ... = c_k = 0$

Aryan Jain - 2022111

M T W T F S S
Page No.:
Date:
YOUVA

Next; we show that $\text{Col}\,A = \text{span}\{a_{i_1}, ---, a_{i_k}\}$

part ① Clearly; $\text{span}\{\bar{a}_{i_1} --- \bar{a}_{i_k}\} \subset \text{Col}\,A$ ✓

part ② Let $\vec{a} \in \text{Col}\,A$ linear combn. of columns of A

$\Downarrow \because EA = B$

Then, $E\vec{a} \in \text{Col}\,B$

$\therefore E\vec{a} \in \text{span}\{\bar{e}_1, ---, \bar{e}_k\}$

$\Downarrow$

$E\vec{a} = c_1\bar{e}_1 + c_2\bar{e}_2 + --- + c_k\bar{e}_k$ for some $c_1, c_2 --- c_k \in \mathbb{R}$

$\Downarrow$

$$\vec{a} = c_1 E^{-1}\bar{e}_1 + c_2 E^{-1}\bar{e}_2 + --- + c_k E^{-1} e_k$$
$$= c_1 \bar{a}_{i_1} + c_2 \bar{a}_{i_2} + --- + c_k \bar{a}_{i_k}$$
$$\in \text{span}\{\bar{a}_{i_1}, --, \bar{a}_{i_k}\}$$

$\therefore \text{Col}\,A \subset \text{span}\{\bar{a}_{i_1}, ---, \bar{a}_{i_k}\}$

from ① & ② $\Rightarrow$ $\text{Col}\,A = \text{span}\underbrace{\{\bar{a}_{i_1}, ---, \bar{a}_{i_k}\}}_{\text{linearly independent vectors}}$ ∎

$\Downarrow$

BASIS