Homogeneous Systems - 1

Suppose that we have row-reduced the coefficient matrix A to an RREF matrix R:

- The leading entries in each non-zero row of R correspond to pivot columns. The corresponding variables are referred to as **basic variables**. Variables corresponding to non-pivot columns, if any, are referred to as **free variables**.
- If we write the matrix equation $R\mathbf{x} = \mathbf{0}$ as a linear system, we can obtain the general solution of the system (recall that the system $R\mathbf{x} = \mathbf{0}$ is equivalent to the original system $A\mathbf{x} = \mathbf{0}$). **The general solution is best expressed in column vector form.**
- **Remark:** This was the last slide in Monday's lecture; we then did an example.

Homogeneous Systems - 2

- **Observation 2**: if the number of non-zero rows r of R is less than the number of variables n, then the system has a non-trivial solution as follows:
  - Express basic variables in terms of free variables. When expressed in vector form, the number of distinct vectors on the RHS is equal to the number of free variables.
  - Free variables behave like parameters - i.e. we can choose any values for them, and each such choice gives a solution. So we get ***infinitely*** many solutions.
- **Observation 3** (Special case of above): if A is an $m \times n$ matrix with $m < n$, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ ***must have a*** non-trivial solution (in fact, infinitely many solutions). This is because in this case, there have to be free variables.

Homogeneous Systems - 3

- **Observation 4**: If the number of non-zero rows of R is equal to the number of variables (i.e. number of columns), then there are no free variables, and the system has a unique solution (only the trivial solution of all zeros).
- **Proposition 3**: If A is a square matrix, then A is row equivalent to the identity matrix if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- **Remark**: ***We will prove the result of Proposition 3 later, as part of a more comprehensive proposition. However, you can try to prove it yourself; we can also make use of this result in other justifications and proofs.***

Homogeneous Systems - Summary

1. System is always consistent
2. If the system has a unique solution, then it is the trivial solution of all zeroes – in this case the RREF is either the n×n identity matrix I_n itself or has I_n as its upper portion with only zero rows below
3. Else, the system contains free variables and has infinitely many solutions (one of which is the trivial solution); this happens when number of non-zero rows in the RREF is less than the number of variables.
4. If number of equations is less than the number of variables, then the system has infinitely many solutions. This is a special case of point 3.

Non-Homogeneous Systems

In this case, we work with the augmented matrix and reduce it to an RREF matrix, say R:

- **Proposition 4** (Existence and Nature of Solutions): The system is consistent if and only if the rightmost column of R **<u>is not a pivot column</u>**, i.e. if there is no row of the form

  [0 ……… 0 b] with b <u>non-zero</u>.

If the system is consistent, then it has either (i) a unique solution if there are no free variables or (ii) infinitely many solutions when there is at least one free variable.

Remark: The main idea behind the above proposition will be explained with the help of examples. You may try the proof as an exercise or refer to the textbook.

Observation 5: The non-homogeneous system $A\mathbf{x} = \mathbf{b}$ can be inconsistent in either of the two cases of the associated homogenous system $A\mathbf{x} = \mathbf{0}$ having unique solution or infinitely many solutions.