NOTES FOR EXTRA SESSION

20230906_WED

Corollaries to VIT - 1

- **Corollary 1.3**: Suppose a square matrix A is factored as a product of square matrices, i.e. $A = A_1A_2......A_n$ with $n \geq 2$. Then A is invertible if and only if each A_i is invertible.
  - **Note:** The above Corollary 1.3 applies only if the matrices A_i are ***all square matrices***. It is certainly possible to factor a square matrix as a product of rectangular matrices. The above result cannot be applied to such a factorization - non-square matrices are never invertible. ***The backward direction, i.e. that if each A_i is invertible, then so is their product, was noted in the Quick Review at the beginning. So we only need to show that if A is invertible, then so is each A_i.***

Proof of Corollary 1.3

- **Corollary 1.3:** Let a square matrix $A = A_1A_2......A_n$ (***all square matrices***). Then A is invertible if and only if each A_i is invertible.

Proof: The fact that the product of invertible matrices is invertible was covered previously (***Observation 4 on Monday 20230821***). So we have only to prove the ~~backward direction [⇐]~~. FORWARD DIRECTION [⇒]

Given: A is invertible.

RTP: Each A_i is invertible.

We will first show that the last matrix in the product, i.e. A_n is invertible. Let $\mathbf{v}$ be any solution of the homogeneous system $A_n\mathbf{x} = \mathbf{0}$. Then $A_n\mathbf{v} = \mathbf{0}$.

Multiplying on the left by $A_1A_2......A_{n-1}$, we get:

$A_1A_2......A_{n-1}A_n\mathbf{v} = \mathbf{0} \Rightarrow A\mathbf{v} = \mathbf{0}$. Since A is invertible, multiplying on the left by A^{-1}, we get $(A^{-1}A)\,\mathbf{v} = \mathbf{0} \Rightarrow I\mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = \mathbf{0}$

In short, the homogeneous system $A_n\mathbf{x} = \mathbf{0}$ has only the trivial solution. ***Hence, by VIT***, A_n is invertible.

Proof of Corollary 1.3 - continued

- **Corollary 1.3:** Let a square matrix $A = A_1A_2 \ldots\ldots A_n$ (***all square matrices***). Then A is invertible if and only if each A_i is invertible.

Proof (continued): Now putting $A_1A_2 \ldots\ldots A_{n-1} A_n = A$ and multiplying on the right by A_n^{-1}, we get $A_1A_2 \ldots\ldots A_{n-1} = A A_n^{-1} = B$ (say).

The matrix B being a product of invertible matrices is invertible.

Thus by what we have shown above, the last matrix in the factorization of B, namely A_{n-1} is invertible.

By repeating this step, we successively prove that $A_n, A_{n-1}, \ldots, A_2$ are invertible. Hence, $C = A_2 \ldots\ldots A_{n-1} A_n$ is invertible.

But $A = A_1C$, where A and C are invertible.

Hence, $A_1 = A C^{-1}$, being the product of invertible matrices, is also invertible.

This completes the proof.

Uniqueness of RREF Matrix

- **Corollary 1.5:** The RREF matrix of any given matrix is unique, i.e. a matrix cannot be row-equivalent to two distinct RREF matrices. Alternatively, two distinct RREF matrices cannot be row-equivalent to each other.
- **Remark 1**: We had earlier stated the above as a remark: *Remark 2 after Proposition 2*. It can be proved using VIT, and is therefore formally stated as a corollary.
- **Remark 2**: However, the proof is difficult and so it is optional: ***you are not required to be familaiar with the proof.*** A proof is presented in Appendix A of the textbook (David C. Lay), which you may refer to.
- ***A different proof is presented below (this was done in a special session on Wednesday 20230906).***

Corollary 1.5 : PROOF

We will prove two lemmas from which the Corollary will follow directly.

We start with ~~[illegible]~~ some terminology (for this proof only, as a convenience).

Defn: Two $m \times n$ matrices ~~[illegible]~~ A, B are system-equivalent if the solution sets of their corresponding homogeneous system $A\bar{x} = \bar{0}$ and $B\bar{x} = \bar{0}$ are equal.

Remark: Clearly, system-equivalence is an equivalence relation on $\mathbb{R}^{m \times n}$.

Lemma 1.5.1 :- If B is row-equivalent to A, then B is system-equivalent to A.

PROOF: We have essentially proved this already (See Q 7 a) in TUT 02). Details are left to the student.

Lemma 1.5.2: Let A and B be distinct $m \times n$ RREF matrices, i.e. $A \neq B$. Then, there exists a vector $\bar{v}$ s.t. $\bar{v}$ is a solution of exactly one of the two corresponding homogeneous systems $A\bar{x} = \bar{0}$ and $B\bar{x} = \bar{0}$.

Two Preliminary Remarks.

1. The vector $\bar{v}$ in the above is necessarily non-zero, since $\bar{0}$ is a solution of every homogeneous system.

2. Corollary 1.5 follows easily from the two Lemmas 1.5.1 & 2. If the two distinct RREF matrices are row-equivalent, then by Lemma 1.5.1, they must be system-equivalent. But the vector $\bar{v}$ in Lemma 1.5.2 is in one solution set and not the other, so the two solution sets cannot be equal, i.e. A and B are not system-equivalent.

We now proceed to prove Lemma 1.5.2.
We will keep m = number of rows fixed, but arbitrary. We will show that the result holds for all n = no. of columns, by induction. So, the result holds for all choices of m, and for all n.

Recall the Principle of Mathematical Induction (PMI).
Let $\Gamma(i)$ be a family of statements involving i, for each positive integer i.

IF: (i) $\Gamma(1)$ is TRUE (Base case)
(ii) If $\Gamma(n-1)$ is true where $(n-1)$ is any positive integer, then $\Gamma(n)$ is TRUE (Inductive Step)

THEN: $\Gamma(n)$ is TRUE for all $n \in \mathbb{Z}^+$.

Remark: Note that PMI contains an outer IF-THEN, and an inner if-then. This is like nested "if-then" statements in program code, and frequently causes ~~problems~~ difficulties to students.

For our proof of Lemma 1.5.2, the statement $\Gamma(i)$ is: If A and B are two distinct $m \times i$ RREF matrices, then there is a vector $\bar{v}$ which is a solution of one of the two associated homogeneous systems, but not of the other.

Proof of the Base Case:-

Let $n=1$. Then, the only possible RREF matrices are $A = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Then, the vector $[1]$ is a solution of $B\bar{x} = \bar{0}$ but not of $A\bar{x} = \bar{0}$.

~~So~~ $\Gamma(1)$ is TRUE

Proof of the Inductive Step:

We may assume that A and B are both non-zero $m \times n$ matrices, since if one is non-zero and other is the zero matrix, they cannot have the same solution set.

Here is the Inductive Hypothesis (IH).

$\Gamma(n-1)$: If C and D are any two distinct $m \times (n-1)$ RREF matrices then there exists a vector $\bar{u}$ which is a solution of say $C\bar{x} = \bar{0}$ but not of $D\bar{x} = \bar{0}$

The Proposition we have to prove is: If $\Gamma(n-1)$, then $\Gamma(n)$.

Given: $\Gamma(n-1)$ is TRUE (assumption)

RTP: $\Gamma(n)$ is TRUE

So let $A \neq B$ be ~~two~~ $m \times n$ RREF matrices. Let A' and B' be ~~two~~ the two $m \times (n-1)$ matrices obtained by deleting the n-th column of A and B respectively. Then A' and B' are $m \times (n-1)$ RREF matrices. (We have noted this fact earlier).

We consider two cases:–

Case 1. $A' \neq B'$

Case 2. $A' = B'$

Case 1. ~~Sin~~ Then the (IH) applies to A' and B'. So there exists ~~an~~ a vector $\bar{u} \in \mathbb{R}^{n-1}$ s.t. WOLOG $A'\bar{u} = \bar{0}$ but $B'\bar{u} \neq 0$.

Let $\bar{v}$ be the vector in $\mathbb{R}^n$ obtained by ~~ad~~ adjoining 0 in the n-th position to ~~$\bar{u}$~~ $\bar{u}$.

Then: $A\bar{v} = A'\bar{u} = \bar{0}$ but $B\bar{v} = B'\bar{u} \neq \bar{0}$.

So we are done in this case.

Case 2 : $A' = B'$.

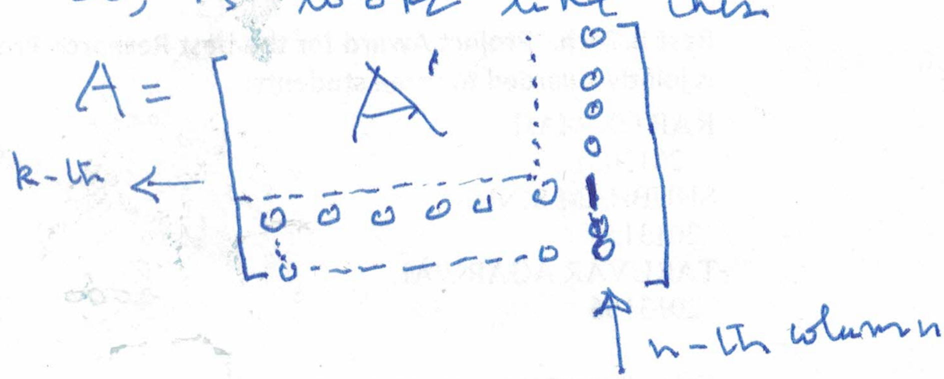
~~This splits into~~ In this case, A and B differ only in their last column (the n-th column). We split this further into 2 cases.

Case 2.1. The n-th column in both A and B is a pivot column corresponding to a basic variable.

Suppose the k-th row is the last <u>non-zero</u> row in A' and B'.

Then, the pivot element in ~~both~~ the ~~A and B~~ n-th column of both A and B must come in the $(k+1)$-st row.

So, A looks like this



B looks exactly the same, except with B' in place of A'.

But $A' = B'$

So $A = B$, contradiction!

~~So~~ This completes Case 2.1

Case 2.2. The n-th column in A is not a pivot column, i.e. it corresponds to a free variable in the homogeneous system $A\bar{x}=\bar{0}$.

Since there is a free variable, $A\bar{x}=\bar{0}$ has non-trivial solutions. We consider the non-trivial solution vector $\bar{v}$ which is attached to the free variable x_n. $\left(\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right)$

Since x_n is a free variable, the corresponding equation in the linear system $A\bar{x}=\bar{0}$ is the dummy equation $x_n = x_n$.

Hence, if $\bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, then $v_n = 1$. (1)

IMPORTANT

We now split Case 2.2 into two subcases.

~~[crossed out]~~ Subcase 2.2a:—

x_n is a basic variable in $B\bar{x}=\bar{0}$

Recall that x_n is the last variable in $B\bar{x}=\bar{0}$. So the equation corresponding to x_n in the linear system $B\bar{x}=\bar{0}$ is $x_n=0$.

In other words, every solution vector of $B\bar{x}=\bar{0}$ has 0 as its entry in the n-th position.

$\therefore$ $\bar{v}$ is not a solution of $B\bar{x}=\bar{0}$, since $v_n=1$ by (1)

So we have found the required $\bar{v}$ in this subcase.

Subcase 2.2b:

x_n is a free variable in $B\bar{x}=\bar{0}$ also.

We will show that $\bar{v}$ is not a solution of $B\bar{x}=\bar{0}$.

Suppose BWOC that $B\bar{v}=\bar{0}$.

Now, $B\neq A_3$, but the difference occurs in the n-th column only (since $B'=A_3'$).

Hence, for some k, $a_{kn} \neq b_{kn}$ (2)

(i.e. A and B must differ in the n-th entry in row k).

Since $A\bar{v} = \bar{0} = B\bar{v}$, equating the k-th entries in $A\bar{v}$ and $B\bar{v}$, we get:

$$a_{k1}v_1 + a_{k2}v_2 + \dots + a_{kn}v_n$$
$$= b_{k1}v_1 + b_{k2}v_2 + \dots + b_{kn}v_n$$

$$\Rightarrow a_{kn}v_n = b_{kn}v_n \quad (\text{since } A' = B')$$

$$\Rightarrow a_{kn} = b_{kn}, \text{ since } v_n = 1 \text{ by } (1)$$

(3)

But (3) contradicts (2)

Thus, our supposition that $B\bar{v} = \bar{0}$ must be FALSE.

Hence, $B\bar{v} \neq \bar{0}$.

So this ~~case~~ subcase also is done.

Proof: Completion

This completes the proof

If $\Gamma(n-1)$ is TRUE, then $\Gamma(n)$ is TRUE, i.e. the inductive step.

So, by PMI, $\Gamma(n)$ is TRUE for all $n \in \mathbb{Z}^+$.

COROLLARY 1.5.3:- If two $m \times n$ matrices A and B are system-equivalent, they must be row-equivalent.

Proof: Let R_1 and R_2 be the RREF matrices of A and B respectively, and suppose BWOC that $R_1 \neq R_2$. By Corollary 1.5, there is a vector $\bar{v}$ s.t. $R_1\bar{v} = \bar{0}$ but $R_2\bar{v} \neq \bar{0}$. But $R_1\bar{v} = \bar{0}$ $\Rightarrow A\bar{v} = \bar{0} \Rightarrow B\bar{v} = \bar{0} \Rightarrow R_2\bar{v} = \bar{0}$ $\Rightarrow\!\Leftarrow$.

$\therefore R_1 = R_2$ and so A and B are row-equivalent.