MTH100A Math 1 (Linear Algebra) Monsoon Semester – 2023-24

Tutorial Exercise for Tuesday 20230912

1. Let V be a vector space. Prove the following (see Proposition 7):
   A. The additive inverse vector of any vector $\mathbf{u}$ is unique; we use the notation $-\mathbf{u}$ for the inverse vector.
   B. $0\mathbf{u} = \mathbf{0}$ for every vector $\mathbf{u}$
   C. $c\mathbf{0} = \mathbf{0}$ for every scalar c
   D. Cancellation Law, i.e. show that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} = \mathbf{w}$.

2. Give an example of a set X and an operation involving elements of X, which does not satisfy the cancellation law. Briefly justify your answer.

3. Verify the properties of a vector space for the space C[0,1] of continuous real-valued functions defined on the closed interval [0,1] using the field of real numbers as the underlying field of scalars. *You may assume that the sum of continuous functions and scalar multiples of continuous functions are also continuous functions.*

4. Show that the set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2}: a,b \in \mathbb{Q}\}$ is a field. **Remark**: Note that $\mathbb{Q}[\sqrt{2}]$ is a subset of $\mathbb{R}$; the wording for this situation is: $\mathbb{Q}[\sqrt{2}]$ is a ***subfield*** of $\mathbb{R}$. (***Hint: The key step is to show that nonzero elements of $\mathbb{Q}[\sqrt{2}]$ have multiplicative inverses in $\mathbb{Q}[\sqrt{2}]$.***)

5. a) Is $\mathbb{R}$ a vector space over $\mathbb{Q}$? Justify your answer in brief.
   b) Is $\mathbb{C}$ a vector space over $\mathbb{R}$? Justify your answer in brief.
   c) Can you generalize the answers to a) and b) above to a statement about fields and vector spaces ? Explain briefly.

6. *Modular arithmetic and fields*: Let n be a fixed but arbitrary positive integer, $n \geq 2$. Put $\mathbb{Z}_n = \{0,1,2,\ldots,n-1\}$. Define the operations of modular addition and modular multiplication on $\mathbb{Z}_n$ by: $x \oplus y = (x+y) \pmod n$ and $x \otimes y = (xy) \pmod n$. **N**B: Recall that $z \pmod n$ = **remainder** after division of z by n for all $z \in \mathbb{Z}$. Note that we have $0 \leq$ remainder $< n$, i.e. $z \pmod n \in \mathbb{Z}_n$ for all $z \in \mathbb{Z}$.
   a) Show that if $x \in \mathbb{Z}_n$, then x has an inverse in $\mathbb{Z}_n$ with regard to the operation $\oplus$ (i.e. additive inverse)..
   b) ***We have already shown in class that $\mathbb{Z}_2$ is a field***. Now show that $\mathbb{Z}_3$ and $\mathbb{Z}_5$ are fields. (***Hint: You may assume that $\oplus$ and $\otimes$ satisfy closure, associativity, commutativity and distributivity on $\mathbb{Z}_n$. This is straightforward but a ittle lengthy. Also see the hint for Q4***.)
   c) Are $\mathbb{Z}_4$ and $\mathbb{Z}_6$ fields ? Justify your answer briefly.
   d) Can you generalize the above to state a condition for $\mathbb{Z}_n$ **not to be a field**? Briefly justify your statement.