MTH100A Math 1 (Linear Algebra) Monsoon Semester – 2023-24

Tutorial Exercise for Tuesday 20230919

1. Given the following vectors in $\mathbb{R}^3$: $\mathbf{u} = (1,3,5)$, $\mathbf{v} = (1,4,6)$, $\mathbf{w} = (2, -1, 3)$ and $\mathbf{b} = (6,5,17)$.
   a) Does $\mathbf{b} \in W = \text{span}\,\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$
   b) If the answer to b) is yes, express $\mathbf{b}$ as a linear combination of $\mathbf{u},\mathbf{v},\mathbf{w}$.

2. Let U and W be two subspaces of the vector space V. Show that $U \cap W$ is also a subspace of V.

3. Obtain an LU decomposition of the symmetric matrix A given below. Find four conditions on a, b, c, d to ensure that an LU decomposition exists, i.e. A can be reduced to an echelon form matrix **<u>without row interchanges</u>**.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

4. In the following is W a subspace of V? Base field is $\mathbb{R}$ in all. Justify your answer.
   a. $V = \mathbb{R}_n[t]$ = vector space of polynomials of degree $\leq n$, $W = \{p(t) \in V: \deg p(t) = n\} \cup \{\mathbf{0}(t)\}$. Here $\mathbf{0}(t)$ indicates the zero polynomial.
   b. $V = \mathbb{R}^3$, $W = \{(x,y,z): x, y, z \in \mathbb{Q}\}$.
   c. $V = \mathbb{R}^3$, $W = \{(x,y,z): xy = 0\}$.
   d. $V = \mathbb{R}^3$, $W = \{(x,y,z): x^2 + y^4 + z^6 = 0\}$

5. Consider the space V of all 2×2 matrices over $\mathbb{R}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.
   - All symmetric matrices (**Definition**: For any m×n matrix $A = [a_{ij}]$, its **transpose** is the n×m matrix $B = [b_{ij}]$, given by $b_{ij} = a_{ji}$. The standard notation for the transpose of A is A^T. A matrix is symmetric if $A = A^T$.)
   - All A such that AB = BA where B is some fixed matrix in V
   - All A such that BA = 0 where B is some fixed matrix in V
   - Would the above results hold for all n×n matrices where n is a general positive integer

6. Consider the space V of all n×n matrices over $\mathbb{R}$. and let W be the subset consisting of all upper triangular matrices.
   a) Show that W is a subspace of V.
   b) Show further that W satisfies closure with regard to products and multiplicative inverses, i.e. if $A, B \in W$, then $AB \in W$, and if $A \in W$ happens to be invertible, then $A^{-1} \in W$.

7. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$ (YES/NO) ? Justify your answer briefly.

8. Let $V = \{x \in \mathbb{R} : x > 0\}$. Define addition for V by $x \oplus y = xy$, and scalar multiplication by any $\alpha \in \mathbb{R}$ by $\alpha * x = x^{\alpha}$.

(a) (7 marks) Verify the closure axioms, the commutative, zero and inverse properties for addition, and the property $1 * x = x$ for all $x \in V$.
(***Remark***: *V is in fact a vector space over the field* $\mathbb{R}$. *However, you need not verify the other properties of a vector space.)*
(b) (3 marks) Is V a subspace of $\mathbb{R}$ regarded as a vector space over itself (YES/NO) ? Justify your answer clearly.

(*This question was given as an exam problem for a previous batch*.)

9. Prove Remark 6 related to linear dependence/independence : Any list which contains a linearly dependent list is linearly dependent.

10. Prove Remark 7 related to linear dependence/independence : Any subset of a linearly independent set is linearly independent .

11. Determine whether the given matrices in the vector space $\mathbb{R}^{2\times 2}$ are linearly dependent or linearly independent.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

12. In the vector space $V = C[0, 2\pi]$, determine whether the given vectors (i.e. functions) are linearly dependent or linearly independent :

$f_1(x) = 1$, $f_2(x) = \sin(x)$, $f_3(x) = \sin(2x)$.

(You must justify your answer.)

13. Let $F = \mathbb{Z}_2$, and consider the vector space $V = F^4$, the space of all ordered 4-tuples with entries from F.

a) Suppose $\mathbf{v} \in V$, $\mathbf{v} \neq \mathbf{0}$. What can you say about the additive inverse of $\mathbf{v}$?
b) Consider the vectors $\mathbf{v_1} = (1, 0, 1, 0)$, $\mathbf{v_2} = (1, 1, 0, 0)$ and $\mathbf{v_3} = (0. 0. 1. 1)$. Determine Span $\{\mathbf{v_1}, \mathbf{v_2}\}$ and Span $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.
c) Construct subspaces U and W of V which have 3 and 5 vectors, respectively.
d) Apply what you have learned from a), b), and c) to state and prove a result about the possible orders of subspaces of V. ***Note***: For any finite set X, the **orde**r of X is the number of elements in X, notation $|X|$.
e) Generalize your result in d) to subspaces of F^n for any arbitrary positive integer n. Can you prove your result ? ***Hint:*** *Attempt a bounded stepwise ascent type of proof (you may recall that the proof of* ***Corollary 1.3*** *was a bounded stepwise descent).*