Review of Friday's Lecture

- Definition: For m×n matrices A and B, B is **row equivalent** to A if B can be obtained from A by a **finite sequence of elementary row operations.**
- **Proposition 1:** Gauss-Jordan Elimination row-reduces any m×n matrix A to an RREF matrix OR there exists an RREF matrix row-equivalent to A.
- **Proposition 2:** Row equivalence is an **equivalence relation** on the set $\mathbb{R}^{m\times n}$ of real m×n matrices.
- **Remark 2**: The RREF matrix of any matrix is unique. Alternatively, two distinct RREF matrices

Application to Determinants

- **Determinants**: We will not formally discuss determinants till later. But we note the following: If A is an $n \times n$ matrix, and B is an echelon form $n \times n$ matrix obtained from A by Gaussian reduction, ***without applying any scaling operations***, then $\det(A) = (-1)^k \det(B) = (-1)^k b_{11} b_{22} \ldots b_{nn}$, where k = number of interchange operations applied. ***This is the preferred algorithm to calculate the determinant.*** That is why in software for matrix calculations, the two phases of the Row Reduction algorithm are carried out separately; we can obtain the determinant on the way.

Application to Solving Linear Systems - 1

- We will now see how the RREF matrix can be used with a few simple additional steps to solve linear systems.
- Consider a linear system in matrix form $A\mathbf{x} = \mathbf{b}$. If $\mathbf{b} = \mathbf{0}$, then the system is said to be **homogeneous**. A homogeneous system always has the trivial solution consisting of all zeroes, i.e. the zero vector $\mathbf{0}$. . If $\mathbf{b} \neq \mathbf{0}$, the system is said to be **non-homogeneous**. A non-homogeneous system may or may not have any solutions. A system which has at least one solution is said to be **consistent**. Otherwise, it is said to be **inconsistent**.

Solving Linear Systems - 2

- We will work directly with matrices: the **coefficient matrix** A for homogeneous systems, and the **augmented matrix** [A:**b**] for non-homogeneous systems. NB: The augmented matrix is obtained by putting a column corresponding to **b** as an additional column, i.e. the (n + 1)-st column.
- **Observation 1:** If we obtain a row equivalent matrix to either the coefficient matrix (in the case of a homogeneous system) or the augmented matrix (in the non-homogeneous case), then the solution sets of the two linear systems are the same. (This is expressed by saying that the systems are equivalent). ***In fact, that is why we defined the elementary row operations in the way we did !***

Homogeneous Systems - 1

Suppose that we have row-reduced the coefficient matrix A to an RREF matrix R:

- The leading entries in each non-zero row of R correspond to pivot columns. The corresponding variables are referred to as **basic variables**. Variables corresponding to non-pivot columns, if any, are referred to as **free variables**.
- If we write the matrix equation $R\mathbf{x} = \mathbf{0}$ as a linear system, we can obtain the general solution of the system (recall that the system $R\mathbf{x} = \mathbf{0}$ is equivalent to the original system $A\mathbf{x} = \mathbf{0}$). **The general solution is best expressed in column vector form.**