Elementary Matrices

- An $m \times m$ (square) matrix is said to be an **elementary matrix** if it is obtained from the identity matrix I_m by an elementary row operation.
- **Proposition 5:** If e is an elementary row operation and E is the $m \times m$ elementary matrix $e(I_m)$, then, for every $m \times n$ matrix A, $e(A) = EA$.
- In other words, applying an elementary row operation is the same as **left multiplication** by the corresponding elementary matrix.
- **Proof:** Left as an exercise. The three types of elementary row operations have to be treated separately.
- **Hint:** ***Before trying a symbolic proof, do some examples. Take a fixed (numerical) 3×3 matrix A, and perform an elementary row-operation of each type on A. Then construct the corresponding elementary matrices, and left multiply A by them. Then right multiply A by them, and observe the results. Finally, try to construct a general proof. This requires algebra with matrix entries – a general matrix is $A = [a_{ij}]$ and the algebra uses these a_{ij} symbols.***

Elementary Matrices - II

- **Proposition 6**: Every elementary matrix is invertible.

Proof: Let E be any elementary matrix, and let e be its corresponding elementary row operation. Then there is another row operation f of the same type that reverses the action of e. Let F be the elementary matrix corresponding to f. Then:

$$FE = (FE)I = F(EI) = f(e(I)) = I$$

Similarly, $EF = I$, so F is E^{-1}

Remark: *(i) We have already seen the main idea above in the proof that row-equivalence is an equivalence relation (TUT01). (ii) The above shows hat the inverse of an elementary matrix is also an elementary matrix (of the same type).*

Very Important Theorem – Ver 1.0

- **Theorem 1**: The following are equivalent for an $m \times m$ square matrix A:

  a. A is invertible

  b. A is row equivalent to the identity matrix

  c. The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

  d. The system of equations $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in R^m.