Vector Space Examples – 4

4. The space R^∞ of real sequences is a vector space over $\mathbb{R}$, i.e.

   $R^\infty = \{< a_n > : < a_n >$ is a sequence with real number terms$\}$.

- Of more interest than R^∞ itself, is c, the subset of convergent sequences. It is also a vector space.
- **Note**: the above are examples of “sequence” spaces. Sequence spaces also play a major role in the study of signals, specifically discrete or digital signals. Exercise: *Verify the vector space axioms for* $\mathbb{R}^\infty$.

Examples of Vector Spaces - 5

5. The space $\mathbb{R}_n[t]$ of polynomials of degree $\leq$ n with real coefficients. (NB: the zero polynomial, which technically does not have any degree, is regarded as an element of $\mathbb{R}_n[t]$ for all $n = 0,1,2,\ldots.$)
6. The space $\mathbb{R}[t]$ of all polynomials with real coefficients.

- **Note**: These two examples are closely related to each other. We can see that $\mathbb{R}_n[t]$ is actually a subset of $\mathbb{R}[t]$ (for all n). *Verification of the axioms is left as an exercise*.

Examples of Vector Spaces - continued

Remark: In all the examples above, we have taken the base field to be $\mathbb{R}$. By taking $\mathbb{C}$ as the base field, we can obtain analogous vector spaces of n-tuples, matrices, functions, sequences and polynomials. There are some major advantages with the complex field, so these examples are often used in applications. Howeve, we will not go into them much. Vector spaces over other fields are used in specialized branches. We have already mentioned vector spaces over $\mathbb{Z}_2$ being used in computer science.

Subspaces

- **Motivation:** We may have noticed that many of the example vector spaces were subsets of another example:

1. $\mathbb{R}_1[t]$ is a subset of $\mathbb{R}_2[t]$ which is a subset of $\mathbb{R}_3[t]$, etc. Moreover, all of these are subsets of the space R[t] of all polynomials on $\mathbb{R}$: $\mathbb{R}_0[t] \subseteq \mathbb{R}_1[t] \subseteq \mathbb{R}_2[t] \subseteq \ldots\ldots\ldots \subseteq \mathbb{R}[t]$.

2. $\mathbb{R}^\infty$ and c are both vector space over $\mathbb{R}$ and $c \subseteq \mathbb{R}^\infty$.

3. An example from Lay: C[0,1] is a subset of the vector space of all real-valued functions with domain [0,1]. The standard notation for the set of all functions with domain X and codomain Y is Y^X. Clearly: $C[0,1] \subseteq \mathbb{R}^{[0.1]}$.

$\mathbb{R}^{[0.1]}$ is not much used, but you can show it is a vector space over $\mathbb{R}$ in the same way as C[0,1].

Definition of Subspace

- **Definition**: Let V be a vector space over the field F. A (vector) subspace of V is a non-empty subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication in W being defined in the same way as for V.

Does a Vector Space Always Have Subspaces ?

1. For any vector space V, the subset consisting of the zero vector alone is a subspace of V, called the zero subspace.
2. V is of course a subspace of itself. Subspaces other than V and $\{\mathbf{0}\}$ are known as proper subspaces.