MTH100A Math 1 (Linear Algebra) Monsoon Semester – 2023-24

Tutorial Exercise for Tuesday 20230905

1. Determine the inverse of the given matrix A ***using row reduction***.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

2. TRUE or FALSE ? Justify your answer – proof if TRUE or counter-example if FALSE.

   a) The sum of two invertible matrices (square matrices of the same order) is always invertible.

   b) If matrices A and B commute, then invertibility of A implies invertibility of B.

3. Suppose AB = AC, where B and C are $n \times p$ matrices and A is an invertible $n \times n$ matrix. Show that B = C. Is this true, in general, when A is not invertible ? Justify your answer (proof if true, counter-example if false).

4. **Observation 1 in Invertible Matrices - Quick Review** (L07 on Monday 20230821) states that if the inverse of A exists, it is unique. Can you prove this ?

5. Consider a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

   a) Using Theorem 1 (VIT) and Corollary 1.1, show that A is invertible if and only if $ad - bc \neq 0$.

   b) Hence determine an expression (formula) for A^{-1}.

6. Construct a 2×2 matrix A with all non-zero entries such that the solution set of the system $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(5,-3)$ and the origin. Now find a non-zero vector $\mathbf{b}$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ **is not a line in** $\mathbb{R}^2$ **parallel to the solution** set of $A\mathbf{x} = \mathbf{0}$. Explain why this does not contradict Observation 6 (see lecture slides for L06 on Friday 20230818).

7. Given an $m \times n$ matrix A and an $n \times p$ matrix B, the product AB is given by the rule $AB = [A\mathbf{v}_1\ A\mathbf{v}_2\ \ldots..\ A\mathbf{v}_p]$ in column form where $B = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots..\ \mathbf{v}_p]$ in column form. Construct an example to illustrate this rule. The matrix A in your example should be at least 3×3 and B should be at least 3×2. Then prove the rule in the general case.

8. a) Show that an elementary matrix E obtained by replacement of a row R_i of I by $R_i + kR_j$, where $j < i$, is a unit lower triangular matrix.

   b) Show that the product of two unit lower triangular matrices is again a unit lower triangular matrix.

c) Show that if A is a unit lower triangular matrix, then A is invertible and A^{-1} is also a unit lower triangular matrix.

9. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system A**x** = **b**, where **b** is given below, using the LU decomposition obtained in part a).

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

10. For each of the following, clearly state TRUE or FALSE. Then, justify your answer (proof if TRUE, counter-example if FALSE).

a) For any square matrix A, if A^k is invertible for some positive integer $k > 1$, then A itself is invertible.

b) If a 3×3 square matrix A satisfies $A^3 = \mathbf{0}$, then $A = \mathbf{0}$. Here **0** indicates the zero matrix.

11. Consider the system $\mathbb{R}^{3\times 3}$ of 3×3 (square) matrices with real entries. A non-zero matrix A is said to be a **zero-divisor** if there exists some non-zero matrix B such that $AB = \mathbf{0}$, the zero matrix.

a) If A is invertible, then it cannot be a zero-divisor. TRUE or FALSE ? Justify your answer.
b) If A is not invertible, then it must be a zero-divisor. TRUE or FALSE ? Justify your answer.

12. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system A**x** = **b**, for $\mathbf{b}_1$ and $\mathbf{b}_2$ given below, using the LU decomposition obtained in part a). Take $\mathbf{b}_1$ and $\mathbf{b}_2$ as column vectors. Explain the difference in the answers for these two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \qquad \mathbf{b}_1 = (1, 4, 5) \quad \mathbf{b}_2 = (3, 7, 15)$$

13. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system A**x** = **b**, where **b** is given below, using the LU decomposition obtained in part a). Take **b** as a column vector.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = (4, 9, 14)$$