MTH100 MATH 1 Section A Monsoon Semester – 2023-24

Tutorial exercise for the Week commencing 20230814

1. Reduce the following matrix to an RREF matrix using elementary row operations:

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 1 \end{bmatrix}$$

2. Reduce the following matrix to an RREF matrix using elementary row operations:

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

3. Explicitly describe all non-zero 2×2 RREF matrices. You may also try to do this for 2×3 and 3×3 RREF matrices.

4. Verify that row-equivalence is an equivalence relation on the set $\mathbb{R}^{m\times n}$ of all m by n matrices with real entries.

5. Show that if E is an equivalence relation on a set X, then any two distinct equivalence classes must be disjoint. Also, show that every element of X has to belong to an equivalence class. NB: the equivalence class of any element $a \in X$ is the set of all elements of X which are related to a; the formal definition is:

   $[a] = \{ x \in X: x\,E\,a$, i.e. x is related to a under the relation E$\}$

6. Show that if $\mathbb{P}$ is a partition of a set X, then there exists an equivalence relation E on X such that the equivalence classes correspond to the parts of the given partition $\mathbb{P}$. (*Q6 is the converse of Q5.*)

7. Define a relation T on the real number system $\mathbb{R}$ by xTy if $y - x \in \mathbb{Z}$, the set of integers. Is T an equivalence relation? Justify your answer. If yes, can you find a special representative in each equivalence class, just as we could do for row-equivalence of matrices?

SOLUTIONS FOLLOW

(NOT IN SAME ORDER)

Q4. Verify that row-equivalence is an equivalence relation on the set $\mathbb{R}^{m\times n}$ of all $m\times n$-matrices with real entries.

ANS: Recall that a matrix B is said to be row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations. Clearly, a matrix B cannot be row-equivalent to a matrix A if they have different sizes.

So, let E be the row-equivalence relation on $\mathbb{R}^{m\times n}$ (m and n fixed but arbitrary positive integers).

Precisely, AEB if there exist finitely many elementary row operations, $e_1, e_2, \ldots, e_k$ s.t. $B = e_k(e_{k-1}(\cdots(e_1(A))$

$= (e_k e_{k-1} \cdots e_1)A$.

We will now verify the three ~~equiv~~ relation properties for E.

Let A, B, C be arbitrary matrices in $\mathbb{R}^{m\times n}$.

(i) Reflexive Property: For any A, let $e: R_1 \longrightarrow 1.R_1$ (scaling by 1.)

Then, $e(A) = A$.

$\therefore AEA$. Hence, E is reflexive, as required.

(PTO)

(ii) Symmetric Property.

Recall that when elementary row operations were defined, there was a remark that if e is any elementary row operation, there exists an elementary row operation e_1 which reverses the action of e, i.e. for any matrix A, $(e_1 e) A = A$. You were asked to verify this remark (has to be done separately for each of the 3 types of e-operation). This reversing e-operation is called the inverse of e, and usually denoted by e^{-1}.

[In fact, e^{-1} is the same type of operation as e; however, we don't use this fact here.]

So now suppose AEB holds.

Then, there exist e-operations $e_1, \dots, e_k$ such that $(e_k e_{k-1} \cdots e_1) A = B$.

Apply the e-operations $e_1^{-1}, e_2^{-1}, \dots, e_k^{-1}$ in reverse order to B.

$$(e_1^{-1} e_2^{-1} \cdots e_k^{-1}) B = (e_1^{-1} e_2^{-1} \cdots e_k^{-1})(e_k \cdots e_1) A.$$

Since $e_j^{-1} e_j$ for any j and any matrix D leaves D unchanged, we get: $(e_1^{-1} \cdots e_k^{-1}) B = A$.

$\therefore$ BEA, proving symmetry.

(iii) Transitive Property.

Suppose AEB and BEC. Then, there exist e-operations $e_1, \ldots, e_j$ s.t. $B = (e_j \cdots e_1)A$ and e-operations $f_1, \ldots, f_k$ s.t. $C = (f_k \cdots f_1)B$. So now,

$$(f_k \cdots f_1 e_j \cdots e_1)A = (f_k \cdots f_1)((e_j \cdots e_1)A)$$
$$= (f_k \cdots f_1)B = C.$$

$\therefore$ AEC holds, as required to prove transitivity.

Explanatory Remarks: 1. If e is an e-operation, and A is a matrix in $\mathbb{R}^{m \times n}$, then $e(A)$ is again a matrix in $\mathbb{R}^{m \times n}$. In short, e can be regarded as a function,

$$e : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{m \times n}.$$

Applying a finite sequence of e-operations is therefore just the same as composition of functions. As you would have learned, composition of functions obeys the associative property. We have used this in the proof of (ii) and (iii).

2. Consider the identity function $I : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{m \times n}$ such that $I(A) = A$ for all $A \in \mathbb{R}^{m \times n}$. As we saw in the proof of (i), I is an e-operation, and for any e, $e^{-1}e = I = ee^{-1}$.

Q5. Show that if E is an equivalence relation on a set X, then any two distinct equivalence classes are disjoint. Also, show that every element of X belongs to an equivalence class.

Answer: The equivalence class of any element $a \in X$ is defined as $[a] = \{x \in X : a E x\}$

We prove the second statement first.

Since aEa holds (Reflexive property), $a \in [a]$, i.e. every $a \in X$ lies in an equivalence class.

For the main ~~part~~ result to be proved, recall the following: A ~~result of~~ statement of the form: If p, then q or p implies q, can be briefly written as: $p \Rightarrow q$. (1)

The contra-positive of (1) is the statement: $(\text{not } q) \Rightarrow (\text{not } p)$. (2)

A statement and its contra-positive are equivalent: we may either prove $p \Rightarrow q$ OR $(\text{not } q) \Rightarrow (\text{not } p)$.

For this ~~proposition~~ result, we will prove the contra-positive

Q5. Continued

(5b)

The given statement is:

If $[a] \neq [b]$, then $[a] \cap [b] = \phi$ (3)

We will prove its contra-positive:

If $[a] \cap [b] \neq \phi$, then $[a] = [b]$ (4)

So, suppose $[a] \cap [b] \neq \phi$,
and let $c \in [a] \cap [b]$.

(We have to show that $[a] = [b]$.
The standard way to show that two sets are equal is to show that each one is a subset of the other. We will also do it this way.)

So, suppose $x \in [a]$. (5)
Then $x E a$. Also, $c \in [a]$, so $a E c$. Hence $x E c$ (transitive property).
But $c \in [b]$, so $c E b$.
Hence, $x E b$ (transitivity again).
$\therefore x \in [b]$ (6)
Since x was arbitrary, (5) and (6) show
$[a] \subseteq [b]$ (7)
Similarly, $[b] \subseteq [a]$ (8).
$\therefore [a] = [b]$, (9), as required.

Q6. Show that if P is a partition of a set X, then there exists an equivalence relation E on X such that the equivalence classes under E correspond to the parts of the partition P. [Parts are assumed to be non-empty.] (6a)

Answer: Let $P = \{P_i : i \in \mathcal{I}\}$ be a partition of X. Then: $X = \bigcup_{i \in \mathcal{I}} P_i$

and $P_i \cap P_j = \emptyset$ if $i \neq j$. (1)

Define a relation E on X by:

$x E y$ if ~~x and~~ $y \in P_i$ ~~for some i~~ where $x \in P_i$,

i.e. x and y belong to the same part of the partition. This is well-defined since any $z \in X$ belongs to <u>exactly</u> <u>one</u> <u>part</u> of the partition.

Now, to show E is an equivalence relation.

(i) Reflexive Property: For $x \in X$, $x \in P_i$ for some i, and so $x E x$.

(ii) Symmetric Property: If $x E y$, then for some part of the partition P_i, $x \in P_i$ and $y \in P_i$. But then, $y \in P_i$ and $x \in P_i$, i.e. $y E x$.

Q 6 cont'd

(iii) ~~Sat~~ Transitive Property:

Suppose $x E y$ and $y E z$.

Then, $x \in P_i$ and $y \in P_i$ for some i,

and $y \in P_j$ and $z \in P_j$ for some j.

But since y can belong to exactly one part of the partition, $i = j$, and so

$x \in P_i$ and $z \in P_i$,

i.e. $x E z$.

[The proof that E ~~is~~ an equivalence relation is ~~actual~~ very, very easy. The problem ~~exactly~~ really only requires you to define E correctly.]

Finally, we need to show that the equivalence classes of E are the same as the parts of the partition. (*)

Given an equivalence class $[x]$, let P_i be the part of $\mathcal{P}$ s.t. $x \in P_i$.

Then, $y \in [x] \Rightarrow x E y \Rightarrow y \in P_i$

$\Rightarrow [x] \subseteq P_i$. (1)

Conversely, $y \in P_i \Rightarrow x E y \Rightarrow y \in [x]$

$\Rightarrow P_i \subseteq$ (2)

From (i) and (ii), $[x] = P_i$.

Conversely, given $P_i \in \mathcal{P}$, let $x \in P_i$.

Then, $[x] = P_i$.

This proves the statement (*).

7a

Q7. Define a ~~re~~ relation T on $\mathbb{R}$ by xTy if $y-x \in \mathbb{Z}$. Is T an equivalence relation? If YES, can you find a special representative in each equivalence class?

<u>Ans:</u> YES, T is an equivalence relation. JUSTIFY / VERIFY:-

(i) Reflexive Property: if $x \in \mathbb{R}$, then $x-x=0 \in \mathbb{Z}$, so xTx.

(ii) Symmetric Property: Suppose xTy. Then, $y-x=k \in \mathbb{Z}$. But then $x-y=-k \in \mathbb{Z}$. So yTx.

(iii) Transitive Property: Suppose xTy and yTz. Then, $y-x=k \in \mathbb{Z}$, and $z-y=m \in \mathbb{Z}$,

Hence, $z-x=(z-y)+(y-x)=m+k \in \mathbb{Z}$, i.e. xTz,

Finding a special representative in each equivalence class:-

the basic idea is very simple. Regard a real number x as a point on the real line, By moving

Q7 – cont'd.

up or down ~~.....~~ in steps of length 1, we find ourselves in the half-open interval $[0, 1)$ at some point y, ~~which~~ for which $x T y$. y is the special representative.
Let's make this precise.
For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ = the <u>floor</u> of x = largest integer $\leq x$.
Put $r_x = x - \lfloor x \rfloor$ so that $r_x \in [0, 1)$. Also, $x - r_x = \lfloor x \rfloor \in \mathbb{Z}$,

so $r_x T x$,
For $x \in \mathbb{R}$, $r_x \in [x]$. Also,
if $r_1, r_2 \in [0, 1)$, $r_1 < r_2$, then
$r_2 - r_1 < 1 \Rightarrow r_1 T r_2$ <u>does not hold</u>. In short, each equivalence class $[x]$ contains exactly one ~~....~~ element of $[0, 1)$, which is the special representative.

Remark: It is very helpful when working with a particular equivalence, if each equivalence class contains a <u>special representative</u>. Special only means <u>easily identifiable</u>. Unfortunately, this doesn't always happen.

3a

Q3. Explicitly describe all 2×2 RREF matrices. You may also try to do this for 2×3 and 3×3 ~~...~~ RREF matrices.

Ans: The description will be provided by rectangular arrays of the desired sizes. The symbols used in the arrays would be 0, 1 and $*$ (star or asterisk). The star symbol can indicate any real number (including 0); if an array has multiple stars, each ~~one~~ one could be filled by a different real number. Thus, there are infinitely many RREF matrices for any $m \geq 1$ and $n > 1$. The ~~different~~ number of differently-patterned arrays rises rapidly as m and n increase. Hence, the challenge of this problem is to find a systematic method which can generate all possible patterns, rather than trying to do the problem

Cont'd

in an ad hoc way which only works for small m and n.

One possible approach is presented here. Since every non-zero row of an RREF has a leading entry (=1), a way to proceed is column-wise:- each ~~of~~ column j is either a pivot column or not. The case that j is a pivot column is considered first, then the case that j is not a pivot column is considered second.

A. 2×2 matrices

j=1 – Pivot Column – j=2 Pivot column

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (1)$$

– j = 2 not pivot column

$$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix} \quad (2)$$

j = 1 – not pivot column j=2 pivot column

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (3)$$

Only 3 patterns.

3e

B. 2×3 matrices

$j=1$ - pivot

$\begin{bmatrix} 1 & & \\ 0 & & \end{bmatrix}$

$j=2$ pivot

$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ 1

$j=2$ not pivot

$\begin{bmatrix} 1 & * & \\ 0 & 0 & \end{bmatrix}$

$j=3$ pivot

$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 2

$j=3$ not pivot

$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ 3

$j=1$ not pivot

$\begin{bmatrix} 0 & & \\ 0 & & \end{bmatrix}$

$j=2$ pivot

$\begin{bmatrix} 0 & 1 & \\ 0 & 0 & \end{bmatrix}$

$j=3$ pivot

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 4

$j=3$ not pivot

$\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ 5

$j=2$ not pivot

$\begin{bmatrix} 0 & 0 & \\ 0 & 0 & \end{bmatrix}$

$j=3$ pivot

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 6

Here, we get 6 patterns.

C. 3×2 matrices (not in the question, but let us do)

$j=1$ pivot

$\begin{bmatrix} 1 & \\ 0 & \\ 0 & \end{bmatrix}$

$j=2$ pivot

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ 1

$j=2$ not pivot

$\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ 2

$j=1$ not pivot

$$\begin{bmatrix} 0 & \\ 0 & \\ 0 & \end{bmatrix} \rightarrow j=2 \text{ pivot } \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (3)$$

So, 3 patterns.

D: 3×3 matrices

$j=1$ pivot

$$\begin{bmatrix} 1 & & \\ 0 & & \\ 0 & & \end{bmatrix}$$

$j=2$ pivot

$$\begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ 0 & 0 & \end{bmatrix}$$

$j=3$ pivot

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

~~j~~

$j=3$ not pivot

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

$j=2$ not pivot

$$\begin{bmatrix} 1 & * & \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}$$

$j=3$ pivot

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (3)$$

$j=3$ not pivot

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (4)$$

$j=1$ not pivot.

$$\begin{bmatrix} 0 & & \\ 0 & & \\ 0 & & \end{bmatrix}$$

Now, we can use a short cut! The remaining part has to be a 3×2 RREF matrix, and we know from C. that there are only 3 ~~basi~~ patterns. We can finish as follows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (5) \qquad \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (6) \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (7)$$

Totally, 7 patterns.

Tutorial 1 Solution.

Solution 1

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

$R_2 \rightarrow \frac{R_2}{4}$

$$= \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 3R_2$; $R_3 \rightarrow R_3 - 5R_2$

$$= \begin{bmatrix} 1 & 2 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & -3/2 \end{bmatrix}$$

$R_3 \rightarrow \frac{2R_3}{-3}$

$$= \begin{bmatrix} 1 & 2 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 3/2 R_3$; $R_2 \rightarrow R_2 - R_3/2$

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution 2:

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$; $R_3 \rightarrow R_3 - 3R_1$

$$= \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix}$$

$R_2 \rightarrow R_2/3$

$$= \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 7 & -7 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 7R_2$; $R_1 \rightarrow R_1 + 2R_2$

$$= \begin{bmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 7/3 & -10/3 \end{bmatrix}$$

$R_3 \rightarrow \frac{3}{7} R_3$

$$= \begin{bmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{bmatrix}$$

$R_2 \rightarrow R_2 + \frac{4}{3} R_3$; $R_1 \rightarrow R_1 - \frac{1}{3} R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 15/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{bmatrix}$$