MTH100A **Math 1 (Linear Algebra)** **Monsoon Semester – 2023-24**

Tutorial Exercise for Tuesday 20230905

1. Determine the inverse of the given matrix A ***using row reduction***.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

2. TRUE or FALSE ? Justify your answer – proof if TRUE or counter-example if FALSE.

   a) The sum of two invertible matrices (square matrices of the same order) is always invertible.

   b) If matrices A and B commute, then invertibility of A implies invertibility of B.

3. Suppose AB = AC, where B and C are $n \times p$ matrices and A is an invertible $n \times n$ matrix. Show that B = C. Is this true, in general, when A is not invertible ? Justify your answer (proof if true, counter-example if false).

4. **Observation 1 in Invertible Matrices - Quick Review** (L07 on Monday 20230821) states that if the inverse of A exists, it is unique. Can you prove this ?

5. Consider a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

   a) Using Theorem 1 (VIT) and Corollary 1.1, show that A is invertible if and only if $ad - bc \neq 0$.

   b) Hence determine an expression (formula) for A^{-1}.

6. Construct a 2×2 matrix A with all non-zero entries such that the solution set of the system $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through (5,–3) and the origin. Now find a non-zero vector **b** such that the solution set of $A\mathbf{x} = \mathbf{b}$ **is not a line in** $\mathbb{R}^2$ **parallel to the solution** set of $A\mathbf{x} = \mathbf{0}$. Explain why this does not contradict Observation 6 (see lecture slides for L06 on Friday 20230818).

7. Given an $m \times n$ matrix A and an $n \times p$ matrix B, the product AB is given by the rule $AB = [A\mathbf{v}_1 \; A\mathbf{v}_2 \; \ldots\ldots \; A\mathbf{v}_p]$ in column form where $B = [\mathbf{v}_1 \; \mathbf{v}_2 \; \ldots\ldots \; \mathbf{v}_p]$ in column form. Construct an example to illustrate this rule. The matrix A in your example should be at least 3×3 and B should be at least 3×2. Then prove the rule in the general case.

8. a) Show that an elementary matrix E obtained by replacement of a row R_i of I by $R_i + kR_j$, where $j < i$, is a unit lower triangular matrix.

   b) Show that the product of two unit lower triangular matrices is again a unit lower triangular matrix.

c) Show that if A is a unit lower triangular matrix, then A is invertible and A^{-1} is also a unit lower triangular matrix.

9. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is given below, using the LU decomposition obtained in part a).

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

10. For each of the following, clearly state TRUE or FALSE. Then, justify your answer (proof if TRUE, counter-example if FALSE).

a) For any square matrix A, if A^k is invertible for some positive integer $k > 1$, then A itself is invertible.

b) If a 3×3 square matrix A satisfies $A^3 = \mathbf{0}$, then $A = \mathbf{0}$. Here $\mathbf{0}$ indicates the zero matrix.

11. Consider the system $\mathbb{R}^{3\times 3}$ of 3×3 (square) matrices with real entries. A non-zero matrix A is said to be a **zero-divisor** if there exists some non-zero matrix B such that $AB = \mathbf{0}$, the zero matrix.

a) If A is invertible, then it cannot be a zero-divisor. TRUE or FALSE ? Justify your answer.
b) If A is not invertible, then it must be a zero-divisor. TRUE or FALSE ? Justify your answer.

12. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, for $\mathbf{b}_1$ and $\mathbf{b}_2$ given below, using the LU decomposition obtained in part a). Take $\mathbf{b}_1$ and $\mathbf{b}_2$ as column vectors. Explain the difference in the answers for these two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \qquad \mathbf{b}_1 = (1, 4, 5) \quad \mathbf{b}_2 = (3, 7, 15)$$

13. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is given below, using the LU decomposition obtained in part a). Take $\mathbf{b}$ as a column vector.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = (4, 9, 14)$$

Tutorial- 3

Q(1) $[A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1/2 & -1/2 & -5/2 & 0 & 1 \end{array}\right]$ $R_3 \to R_3 - \frac{5}{2}R_1$

$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & -3/2 & 1 & -1/2 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1/4 & -5/2 & 1/4 & 1 \end{array}\right]$ $R_1 \to R_1 - \frac{1}{2}R_2$, $R_3 \to R_3 + \frac{1}{4}R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$ $R_1 \to R_1/2$, $R_2 \to R_2/2$, $R_3 \to R_3 \times -4$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$ $R_1 \to R_1 + \frac{3}{4}R_3$, $R_2 \to R_2 - \frac{1}{2}R_3$

$[I \quad | A^{-1}]$

$$A^{-1}A = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$AA^{-1} = I$

Q(2) a) Let $A = I_{n\times n}$ (identity matrix) and $B = (-I_{n\times n})$
Then, A and B are invertible. But $A+B = O_{n\times n}$ is not invertible. No, sum of two interovible matrices need not be invertible.

b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} = I_{2\times2}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times2}$. Clearly, A is invertible.
$AB = B = BA$, ie A and B commute. But B is not invertible as $|B| = 0$.

Also, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, ie B is not row equivalent to ~~the~~ 2×2 identity matrix, hence B is not invertible.
Hence, if $AB = BA$ and A^{-1} exists $\not\Rightarrow B^{-1}$ exists.

Q(3) Let $AB = AC$. As A is invertible therefore A^{-1} exists s.t. $AA^{-1} = I_{n\times n} = A^{-1}A$.

Now, $AB = AC \Rightarrow A^{-1}AB = A^{-1}AC$

$\Rightarrow IB = IC \quad (AA^{-1} = I = A^{-1}A)$

$\Rightarrow \boxed{B = C} \quad$ (as $IB = B$ & $IC = C$)

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Clearly, A is not invertible as seen in Q(2). Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

i.e $AB = AC$, but $B \neq C$.

Hence, $AB = AC \not\Rightarrow B = C$, if A is not invertible.

Q(4) Let A^{-1} exist. Then,

$$AA^{-1} = I_{n\times n} = A^{-1}A.$$

Let B & C be inverse of A. Then, we have

$$AB = BA = I_{n\times n} \text{ and } AC = CA = I_{n\times n}$$

$\Rightarrow AB = AC$

$\Rightarrow B = C$ (as A is invertible see Q(3) solution).

Hence, inverse of an invertible matrix is unique.

Q(5) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Leftrightarrow$ $[A \mid I] \sim [I \mid A^{-1}]$

If $a = 0$ then $[A \mid I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim [I \mid A^{-1}]$

$\Leftrightarrow A \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$ (as $a = 0$)

$\Leftrightarrow \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$ has 2 pivots

$\Leftrightarrow b$ & c are nonzero $\Leftrightarrow bc \neq 0$

$\Leftrightarrow ad - bc \neq 0$ as $a = 0$

If $a \neq 0$ then $[A \mid I] \sim [I \mid A^{-1}]$

$\Leftrightarrow A \sim \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$ has 2 pivots

$\Leftrightarrow \frac{ad - bc}{a} \neq 0$

$\Leftrightarrow ad - bc \neq 0$ as $a \neq 0$.

8. [redacted] Construct a 2×2 matrix A with all non-zero entries such that the solution set of t system **A**x = **0** is the line in $\mathbb{R}^2$ through (5,–3) and the origin. Now find a non-ze vector **b** such that the solution set of **A**x = **b is not a line in** $\mathbb{R}^2$ **parallel to the soluti** set of **A**x = **0**. Explain why this does not contradict Observation 6 ([redacted]e slides f
[redacted]

Q6

Answer: Since the system $A\bar{x} = \bar{0}$ has a non-trivial solution, it must have a free variable when reduced to RREF matrix. Also, if the first column is a zero column, then every entry of A cannot be non-zero. Hence, the RREF matrix of A, say R, looks like:

$$\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}.$$ Converting this to a linear system, we get

$$x_1 = -c\,x_2$$
$$x_2 = x_2.$$

So, the solution set is of the form $\bar{x} = x_2 \begin{bmatrix} -c \\ 1 \end{bmatrix}$

Since we are given that (5, −3) is on the solution set, which is a line through the origin, we get $x_2 \begin{bmatrix} -c \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ for some

(

[redacted] (cont'd)

choice of x_2.

Converting to a ~~linear~~ pair of equations, we get:

$$\left.\begin{aligned} -cx_2 &= 5 \\ x_2 &= -3 \end{aligned}\right| \Rightarrow (-c)(-3) = 5$$

$$\Rightarrow c = \frac{5}{3}$$

$$\therefore R = \begin{bmatrix} 1 & \frac{5}{3} \\ 0 & 0 \end{bmatrix}$$

A could be any positive matrix which is row-equivalent to R.

An obvious choice is:

$$A = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}.$$

$$R \xrightarrow{R_1 \to 3R_1} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_2 + R_1]{R_2 \to} \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}$$

[There are infinitely many such A's obviously.]

[redacted] (conclusion)

We need to find a non-zero vector $\bar{b}$ such that the solution set of $A\bar{x}=\bar{b}$ is not a line parallel to the solution set of $A\bar{x}=\bar{0}$.

Note that the solution set of $A\bar{x}=\bar{0}$ is the line $\bar{x} = t\begin{bmatrix}-c\\1\end{bmatrix} = t\begin{bmatrix}-\frac{5}{3}\\1\end{bmatrix}$ in

paramater form,

or $3x+5y=0$ ⓘ in usual form.

So, we simply select any vector $\bar{b}$ not on this line.

For example, consider the system $A\bar{x}=\bar{b}$ when $\bar{b}=\begin{bmatrix}1\\2\end{bmatrix}$.

The augmented matrix is $[A:\bar{b}]$

$$= \left[\begin{array}{cc:c} 3 & 5 & 1 \\ 3 & 5 & 2 \end{array}\right] \xrightarrow[R_2-R_1]{R_2\to} \left[\begin{array}{cc:c} 3 & 5 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

Clearly, this system is not consistent.

So, its solution set $S=\emptyset$, the empty set. S is definitely not a line parallel to the line ⓘ which is the solution set of $A\bar{x}=\bar{0}$

Why does this not contradict Observation 6?

Recall that Observation 6 holds provided the system $A\bar{x}=\bar{b}$ has at least one solution.

[redacted]

[redacted] Given an m×n matrix A and an n×p matrix B, the product AB is given by the rule AB = $[A\mathbf{v}_1\ A\mathbf{v}_2\ \ldots\ A\mathbf{v}_p]$ in column form where B = $[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_p]$ in column form. Construct an example to illustrate this rule. The matrix A in your example should be at least 3×3 and B should be at least 3×2. Then prove the rule in the general case.

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \end{bmatrix}$.

So $B = [\bar{v}_1\ \bar{v}_2]$ where $\bar{v}_1 = (1,3,1)$ and $\bar{v}_2 = (2,4,1)$ in column form.

Then: $AB = \begin{bmatrix} 5 & 7 \\ 9 & 13 \\ 16 & 23 \end{bmatrix}$ ① and

$[A\bar{v}_1\ A\bar{v}_2] = \begin{bmatrix} 5 & 7 \\ 9 & 13 \\ 16 & 23 \end{bmatrix}$ ②

From ① and ②, $AB = [A\bar{v}_1\ A\bar{v}_2]$ in column form, as required

General Proof: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$, $AB = C = [c_{ij}]_{m\times p}$.

Then, for any fixed k, $1 \le k \le p$,

(PTO)

8a

[redacted] Q7.

The k-th column of C is:

$$\begin{bmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{bmatrix}_m = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} b_{jk} \\ \sum_{j=1}^{n} a_{2j} b_{jk} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} b_{jk} \end{bmatrix}_m \qquad (3)$$

[The hint makes the calculation easier.]

OTOH, if $\bar{v}_k$ is the k-th column of B, then $\bar{v}_k = \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix}_n$ and so

$$A\bar{v}_k = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} b_{jk} \\ \sum_{j=1}^{n} a_{2j} b_{jk} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} b_{jk} \end{bmatrix} \qquad (4)$$

Comparing (3) and (4), the k-th column of C is $A\bar{v}_k$. Since k was arbitrary, $C = [A\bar{v}_1, \dots, A\bar{v}_p]$ in column form.

[redacted]

Q8 a) $I_{n\times n} = [a_{il}]_{n\times n}$, $a_{il} = \begin{cases} 1 & i=l \\ 0 & i\neq l \end{cases}$, $\forall\, 1\le i, l\le n$

$I \sim E$: $R_i \to R_i + kR_j$, $j<i$

$E = [a'_{il}]_{n\times n}$

$$a'_{il} = a_{il} + k\, a_{jl} = \begin{cases} a_{ii} + k a_{ji} & i=l \\ a_{ij} + k a_{jj} & l=j \\ 0 & i\neq l,\ j\neq l \end{cases}$$

$$= \begin{cases} 1 & i=l \\ k & l=j \\ 0 & i\neq l,\ j\neq l \end{cases}$$

$$\Rightarrow E = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ j \\ \vdots \\ i \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & j & i & & n \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & - & 0 \\ 0 & 1 & 0 & 0 & - & 0 \\ 0 & 0 & 1 & 0 & - & 0 \\ 0 & 0 & k & 1 & - & 0 \\ & & & & 1 & 0 \\ 0 & 0 & 0 & 0 & -- & 1 \end{bmatrix} \end{array}$$

all diagonal elements are 1 and E is a lower triangular matrix and hence, E is a unit lower triangular matrix.

b) Let $A = [a_{ij}]$ $B = [b_{ij}]$ be $n\times n$ unit lower triangular matrices. By definition

$$a_{ij} = b_{ij} = 0, \quad i<j$$

Let $C = AB = [C_{ij}]$, where

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

<u>Case I</u>: $k \le i < j$.

In this case, $b_{kj} = 0$, for $k, j \in \{1, 2 - - n\}$ with $k<j$.

<u>Case II</u>: $i<k$.

In this case, $a_{ik} = 0$, for $i, k \in \{1, 2 - n\}$ with $i<k$.

Hence, for $i < j$ we have

$$C_{ij} = \sum_{k=1}^{i} a_{ik} b_{kj} + \sum_{k=i+1}^{n} a_{ik} b_{kj}$$

$$= 0 + 0$$

(from case I as $i<j$ so $b_{kj} = 0 \ \forall\ 1 \le k \le i$) (from Case II as $k > i$)

$$= 0$$

Now, $C_{ii} = \sum_{k=1}^{i-1} a_{ik} b_{ki} + a_{ii} b_{ii} + \sum_{k=i+1}^{n} a_{ik} b_{ki}$

$$= \underset{\text{(from case I)}}{0} + 1 \times 1 + \underset{\text{(from case II)}}{0} \quad \left(\text{as } a_{ii} = b_{ii} = 1 \ \forall\ i \le n \right\}$$

$$= 1$$

Hence, $C = AB$ is a unit lower triangular matrix.

c) Let $A = \begin{bmatrix} 1 & & & 0 \\ a_{21} & 1 & & \\ a_{31} & a_{32} & 1 & \\ \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} \cdots & 1 \end{bmatrix}$ be a unit lower triangular matrix.

$Ax = 0 \Rightarrow x_1 = 0$

$a_{21} x_1 + x_2 = 0 \Rightarrow x_2 = 0$

$a_{31} x_1 + a_{32} x_2 + x_3 = 0 \Rightarrow x_3 = 0$

$\vdots$

$a_{n1} x_1 + a_{n2} x_2 + \cdots + x_n = 0 \Rightarrow x_n = 0$

Hence, the system $Ax = 0$ has only trivial sol^n which implies that A is invertible.

As $AA^{-1} = I = [c_{ij}]$, $c_{ii} = 1 \ \forall 1 \le i \le n$, $c_{ij} = 0 \ \forall \ i \ne j$
$1 \le i, j \le n$

Let $A = [a_{ij}]$ be unit lower triangular matrix.

Then, $a_{ij} = 0 \quad \forall \ i < j$ and $a_{ii} = 1 \quad \forall \ i$

Let $A^{-1} = [b_{ij}]$. Then,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{i} a_{ik} b_{kj} \qquad (\text{as } a_{ik} = 0 \ \forall \ k > i$$

Now, $c_{1j} = a_{11} b_{1j} = b_{1j} \qquad (\text{as } a_{11} = 1)$

$= 0 \ \forall \ j > 1 \quad -① \quad (\text{as } c_{1j} = 0 \ \forall \ j > 1)$

$c_{2j} = a_{21} b_{1j} + a_{22} b_{2j}$

$= a_{21} b_{1j} + b_{2j} \qquad (\because a_{22} = 1)$

$= b_{2j} \qquad \forall \ j > 1 \quad (\text{as } b_{1j} = 0)$ see ①

$\Rightarrow b_{2j} = 0 \quad \forall \ j > 2 \quad (\text{as } c_{2j} = 0 \ \forall \ j > 2)$

Now, $c_{3j} = a_{31} b_{1j} + a_{32} b_{2j} + a_{33} b_{3j}$

$= b_{3j} \ \forall \ j > 2 \left(\text{as } b_{1j} = b_{2j} = 0 \ \text{and } a_{33} = 1 \quad \forall \ j > 2 \right)$

$\Rightarrow c_{3j} = b_{3j} = 0 \ \forall \ j > 3 \quad [\text{as } c_{3j} = 0 \ \forall \ j > 3]$

Proceeding in this manner we can get

$b_{ij} = c_{ij} = 0 \quad \forall \quad i < j.$

Also, $c_{ii} = \sum_{k=0}^{i-1} a_{ik} b_{ki} + a_{ii} b_{ii}$

$1 = 0 + a_{ii} b_{ii} \qquad \left(\text{as } b_{ki} = 0 \ \forall \ k < i \right)$

$\Rightarrow 1 = b_{ii} \qquad (\text{as } a_{ii} = 1 \quad \forall \ i)$

Hence, $A^{-1} = [b_{ij}]$ is a unit lower triangular matrix.

[redacted]

8 a) Show that an elementary matrix E obtained by replacement of a row R_i of I by $R_i + kR_j$, where $j < i$, is a unit lower triangular matrix.

b) Show that the product of two unit lower triangular matrices is again a unit lower triangular matrix.

c) Show that if A is a unit lower triangular matrix, then A is invertible and A^{-1} is also a unit lower triangular matrix.

ALTERNATIVE PROOF OF 2nd part of [redacted] Q8 c.

Let A be a unit lower triangular matrix. As shown above, A is invertible. Hence, A is row-equivalent to I, i.e.

$(E_p \cdots E_1)A = I$, where the E_i are elementary matrices. (1)

However, from a), each E_i is also unit lower triangular. (*)

Now, by Corollary 1.1 to VIT, the same sequence of elementary operations which reduces A to I, also reduces I to A^{-1}, i.e. $(E_p \cdots E_1)I = A^{-1}$ (2),

or $A^{-1} = E_p E_{p-1} \cdots E_1 = E$, say (3)

Using Q (b), *, and the fact that elementary matrices are invertible, E is unit lower triangular, proving the result.

Q([redacted] a) 9.

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 1/2 & 3/2 & 1/2 \\ 0 & 1/2 & 1/2 & 3/2 \end{bmatrix} \qquad \begin{array}{ll} R_2 \to R_2 - \frac{R_1}{2} & (1) \\ R_3 \to R_3 - R_1/2 & (2) \\ R_4 \to R_4 - \frac{R_1}{2} & (3) \end{array}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{3}{2} & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 4/3 \end{bmatrix} \qquad \begin{array}{ll} R_3 \to R_3 - \frac{R_2}{3} & (4) \\ R_4 \to R_4 - \frac{R_2}{3} & (5) \end{array}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = U \qquad R_4 \to R_4 - \frac{1}{4} R_3 \quad (6)$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/4 & 1 \end{bmatrix} \qquad R_4 \to R_4 + \frac{1}{4} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 \\ 0 & 1/3 & 1/4 & 1 \end{bmatrix} \qquad \begin{array}{l} R_4 \to R_4 + \frac{R_2}{3} \\ R_3 \to R_3 + R_2/3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1 & 0 \\ \frac{1}{2} & 1/3 & 1/4 & 1 \end{bmatrix} \qquad \begin{array}{l} R_4 \to R_4 + R_1/2 \\ R_3 \to R_3 + R_1/2 \\ R_2 \to R_2 + R_1/2 \end{array}$$

$$= L$$

It can be verified that $LU = A$.

$$Ax = b \Rightarrow LUx = b$$
$$\Rightarrow Ux = L^{-1}b = y$$

$$Ly = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1 & 0 \\ 1/2 & 1/3 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow y_1 = 1$$

$$\frac{y_1}{2} + y_2 = -1 \Rightarrow y_2 = -1 - \frac{1}{2} = -\frac{3}{2}$$

$$\frac{y_1}{2} + \frac{y_2}{3} + y_3 = -1 \Rightarrow y_3 = -1 - \frac{1}{2} - \frac{1}{3} \times \left(\frac{-3}{2}\right)$$
$$= -1 - \frac{1}{2} + \frac{1}{2} = -1$$

$$\frac{y_1}{2} + \frac{y_2}{3} + \frac{y_3}{4} + y_4 = 1$$

$$\Rightarrow y_4 = 1 - \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4}$$

$$y = \begin{bmatrix} 1 & \frac{-3}{2} & -1 & \frac{5}{4} \end{bmatrix}^T$$

$$Ux = y$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3/2 \\ -1 \\ 5/4 \end{bmatrix}$$

$\Rightarrow$ ① $2x_1 + x_2 + x_3 + x_4 = 1$, ② $\frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = -\frac{3}{2}$

③ $\frac{4}{3}x_3 + \frac{1}{3}x_4 = -1$ ④ $\frac{5}{4}x_4 = \frac{5}{4} \Rightarrow \boxed{x_4 = 1}$

$$\Rightarrow \frac{4}{3}x_3 = -1 - \frac{1}{3}x_4$$
$$= -1 - \frac{1}{3} = -\frac{4}{3}$$
$$\Rightarrow \boxed{x_3 = -1}$$

Now, from ② $\frac{3}{2}x_2 = \frac{-3}{2} - \frac{1}{2} + \frac{1}{2} = -\frac{3}{2}$

$$\Rightarrow x_2 = -1$$

From ①, $2x_1 + x_2 + x_3 + x_4 = 1$

$$\Rightarrow 2x_1 = 1 - 1 + 1 + 1 = 2$$
$$\Rightarrow \boxed{x_1 = 1}$$

Hence $x = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T$

TUT 03

~~10 (a)~~ 10(a)

Q10 (a) For any square matrix A, if A^k is invertible for some positive integer k, then A itself is invertible.

Ans: TRUE.

This immediately follows from Corollary 1.3 (see ~~notes~~ slides for L09 on Monday 20230828).

Remark: The following ~~discussion~~ justification is WRONG.

Put $B = A^k = \underbrace{A.A.\cdots.A}_{k \text{ times}}$.

Since B is invertible (given),

$B^{-1} = (A.A.\cdots.A)^{-1} = \underbrace{A^{-1}.A^{-1}\cdots.A^{-1}}_{k \text{ times}}$.

Hence, A is invertible.

This wrong proof is using Observation 4 in the Quick Review of Invertible Matrices (L07, Monday 20230821). But this already

Q10(a) – cont'd

assumes that the factors are invertible. Hence, this proof is a circular proof (already assumes what is to be proved).

(b) If a 3×3-matrix satisfies $A^3 = [0]$, then $A = [0]$.

Ans. FALSE. Here is one counter-example to show it is not TRUE.

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Then: $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [0]$.

$\therefore A^3 = A[0] = [0]$. But $A \neq [0]$.

Remark: Finding counter-examples is a matter of trial and error. Since we know from 10(a), that A cannot be invertible, a good guess would be to take A an RREF matrix not equal to I_3. This is a rather easy counter-example — there are many.

S [redacted]

Q [redacted] a) If A is invertible, it cannot be

11. a zero-divisor (TRUE or FALSE)?

Justify.

Ans: TRUE.

Suppose BWOC that A is invertible, but is a zero-divisor. Then there exists a non-zero matrix B such that $AB = [0]$

Multiplying on the left by A^{-1}, we get: $A^{-1}(AB) = A^{-1}[0]$

$$\Rightarrow (A^{-1}A)B = [0]$$

$$\Rightarrow IB = [0]$$

$$\Rightarrow B = [0] \quad \Rightarrow\!\Leftarrow$$

Since B was assumed non-zero.

b) If A is not invertible, it must be a zero-divisor (TRUE or FALSE)

Justify.

Ans: TRUE.

Since A is not invertible, the homogeneous system $A\bar{x} = \bar{0}$ has a non-trivial solution, say $\bar{v}$.

This follows from IVT.

Put $B = [\bar{v} \;\; \bar{v} \;\; \bar{v}]$, i.e. the matrix with $\bar{v}$ as its columns. Then $AB = A[\bar{v} \;\; \bar{v} \;\; \bar{v}]$ $= [\bar{0} \;\; \bar{0} \;\; \bar{0}] = [0]$.

Q5(a) Find an LU decomposition of

12.

(5)

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_3 \to} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$ (Using short cut method)

Check: $LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} =$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} = A,$$ as required.

(b) Solve $A\bar{x} = \bar{b}_1 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

Step 1. Solve $L\bar{y} = \bar{b}$

$y_1 = 1$

$2y_1 + y_2 = 4 \Rightarrow y_2 = 2$

$3y_1 + y_2 + y_3 = 5 \Rightarrow y_3 = 0$

[redacted](b) cont'd:-

Now solve $U\bar{x} = \bar{y}$ or $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} x_1 + 2x_2 + 5x_3 &= 1 \\ 2x_2 + 6x_3 &= 2 \\ 0 &= 0 \end{aligned}$$

Solve by backward substitution:-

$$2x_2 = 2 - 6x_3$$

$$x_1 = 1 - 2x_2 - 5x_3$$

x_3 is like a parameter.

(i) Put $x_3 = 0 \Rightarrow 2x_2 = 2 \Rightarrow x_2 = 1$

and $x_1 = 1 - 2(1) = -1$

Check $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, as reqd.

Putting $x_3 = 1$, we get $\bar{x} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ as another solution.

Infinitely many solutions.

Now solve $A\bar{x} = \bar{b}_2 = \begin{bmatrix} 3 \\ 7 \\ 15 \end{bmatrix}$

[redacted] b) cont'd [redacted]

$L\bar{y} = \bar{b}_2$:- $y_1 = 3$

$2y_1 + y_2 = 7 \Rightarrow y_2 = 1$

$3y_1 + y_2 + y_3 = 15 \Rightarrow y_3 = 5$

NOW $U\bar{x} = \bar{y}$ becomes

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

or
$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= 3 \\ 2x_2 + 6x_3 &= 1 \\ 0 &= 5 \end{aligned}$$
$\rightarrow$ NOT CONSISTENT

Explanation:

From U we see that A is not invertible. $\therefore$ the homogeneous system $A\bar{x} = \bar{0}$ has infinitely many solutions. The non-homogeneous system $A\bar{x} = \bar{b}$, therefore, could either have infinitely many solutions or be inconsistent. First case happened for $\bar{b}_1$; second case happened for $\bar{b}_2$.

[redacted]: a) Get an LU decomposition [redacted]

13. of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 6 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[3R_2]{R_3 \to R_3 -} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ (short-cut)

Check: $LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} = A,$

as reqd.

(b) Now solve $A\bar{x} = \bar{b} = \begin{bmatrix} 4 \\ 9 \\ 14 \end{bmatrix}$

$L\bar{y} = \bar{b}$ becomes $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

or $y_1 = 4$

$2y_1 + y_2 = 9 \Rightarrow y_2 = 1$

$y_1 + 3y_2 + y_3 = 14 \Rightarrow y_3 = 7$

$U\bar{x} = \bar{y}$, i.e. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = 4$

$2x_2 - x_3 - x_4 = 1$

$4x_3 + 3x_4 = 7$

We treat x_4 as a parameter.
For example, $x_4 = 1$ gives $x_3 = 1$

$\Rightarrow 2x_2 = 1 + x_3 + x_4 \Rightarrow x_2 =$

$\therefore x_1 = 4 - x_2 - x_3 - x_4 = \frac{1}{2}$

Check: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 14 \end{bmatrix}$

Since x_4 is a parameter, there a
infinitely many solutions. Since A
fewer rows than variables, $A\bar{x} = \bar{0}$
necessarily has infinitely many
Hence, if $A\bar{x} = \bar{b}$ is consistent, it
has infinitely many solutions.

Tutorial- 3

Q(1) $[A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1/2 & -1/2 & -5/2 & 0 & 1 \end{array}\right]$ $R_3 \to R_3 - \frac{5}{2}R_1$

$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & -3/2 & 1 & -1/2 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1/4 & -5/2 & 1/4 & 1 \end{array}\right]$ $R_1 \to R_1 - \frac{1}{2}R_2$, $R_3 \to R_3 + \frac{1}{4}R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$ $R_1 \to R_1/2$, $R_2 \to R_2/2$, $R_3 \to R_3 \times -4$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$ $R_1 \to R_1 + \frac{3}{4}R_3$, $R_2 \to R_2 - \frac{1}{2}R_3$

$[I \mid A^{-1}]$

$$A^{-1}A = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$AA^{-1} = I$

Q(2) a) Let $A = I_{n\times n}$ (identity matrix) and $B = (-I_{n\times n})$

Then, A and B are invertible. But $A+B = O_{n\times n}$ is not invertible. No, sum of two intervible matrices need not be invertible.

b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} = I_{2\times 2}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$. Clearly, A is invertible.

$AB = B = BA$, ie A and B commute. But B is not invertible as $|B| = 0$.

Also, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, ie B is not row equivalent to ~~the~~ 2×2 identity matrix, hence B is not invertible.
Hence, if $AB = BA$ and A^{-1} exists $\not\Rightarrow$ B^{-1} exists.

Q(3) Let $AB = AC$. As A is invertible therefore A^{-1} exists s.t. $AA^{-1} = I_{n\times n} = A^{-1}A$.

Now, $AB = AC \Rightarrow A^{-1}AB = A^{-1}AC$

$$\Rightarrow IB = IC \quad (AA^{-1} = I = A^{-1}A)$$

$$\Rightarrow \boxed{B = C} \quad (\text{as } IB = B \ \& \ IC = C)$$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Clearly, A is not invertible as seen in Q(2). Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \cancel{= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

i.e $AB = AC$, but $B \neq C$.

Hence, $AB = AC \not\Rightarrow B = C$, if A is not invertible.

Q(4) Let A^{-1} exist. Then,

$$AA^{-1} = I_{n\times n} = A^{-1}A.$$

Let B & C be inverse of A. Then, we have

$$AB = BA = I_{n\times n} \text{ and } AC = CA = I_{n\times n}$$

$\Rightarrow AB = AC$

$\Rightarrow B = C$ (as A is invertible see Q(3) solution).

Hence, inverse of an invertible matrix is unique.

Q(5) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Leftrightarrow$ $[A \mid I] \sim [I \mid A^{-1}]$

If $a = 0$ then $[A \mid I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim [I \mid A^{-1}]$

$\Leftrightarrow A \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$ (as $a=0$)

$\Leftrightarrow \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$ has 2 pivots

$\Leftrightarrow$ b & c are nonzero $\Leftrightarrow bc \neq 0$

$\Leftrightarrow ad - bc \neq 0$ as $a = 0$

If $a \neq 0$ then $[A \mid I] \sim [I \mid A^{-1}]$

$\Leftrightarrow A \sim \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$ has 2 pivots

$\Leftrightarrow \frac{ad-bc}{a} \neq 0$

$\Leftrightarrow ad - bc \neq 0$ as $a \neq 0$.

8. [redacted] Construct a 2×2 matrix A with all non-zero entries such that the solution set of t
system $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through (5,–3) and the origin. Now find a non-ze
vector **b** such that the solution set of $A\mathbf{x} = \mathbf{b}$ **is not a line in** $\mathbb{R}^2$ **parallel to the soluti**
set of $A\mathbf{x} = \mathbf{0}$. Explain why this does not contradict Observation 6 ([redacted]re slides f
[redacted]

Q6

Answer: Since the system $A\bar{x} = \bar{0}$ has a non-trivial solution, it must have a free variable when reduced to RREF matrix. Also, if the first column is a zero column, then every entry of A cannot be non-zero. Hence, the RREF matrix of A, say R, looks like:

$$\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}.$$ Converting this to a linear system, we get

$$x_1 = -c\,x_2$$
$$x_2 = x_2.$$

~~or~~

So, the solution set is of the form $\bar{x} = x_2 \begin{bmatrix} -c \\ 1 \end{bmatrix}$

Since we are given that (5, -3) is on the solution set, which is a line through the origin,

we get $x_2 \begin{bmatrix} -c \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ for some

(

[redacted] (cont'd)

choice of x_2.

Converting to a ~~linear~~ pair of equations, we get:

$$\left.\begin{aligned} -cx_2 &= 5 \\ x_2 &= -3 \end{aligned}\right| \Rightarrow (-c)(-3) = 5$$

$$\Rightarrow c = \frac{5}{3}$$

$$\therefore R = \begin{bmatrix} 1 & \frac{5}{3} \\ 0 & 0 \end{bmatrix}$$

A could be any positive matrix which is row-equivalent to R.

An obvious choice is:

$$A = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}.$$

$$R \xrightarrow{R_1 \to 3R_1} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_2 + R_1]{R_2 \to} \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}$$

[There are infinitely many such A's obviously.]

[redacted] (conclusion)

We need to find a non-zero vector $\bar{b}$ such that the solution set of $A\bar{x} = \bar{b}$ is not a line parallel to the solution set of $A\bar{x} = \bar{0}$.

Note that the solution set of $A\bar{x} = \bar{0}$ is the line $\bar{x} = t\begin{bmatrix} -c \\ 1 \end{bmatrix} = t\begin{bmatrix} -\frac{5}{3} \\ 1 \end{bmatrix}$ in paramater form,

or $3x + 5y = 0$ (i) in usual form.

So, we simply select any vector $\bar{b}$ not on this line.

For example, consider the system $A\bar{x} = \bar{b}$ when $\bar{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The augmented matrix is $[A : \bar{b}]$

$$= \left[\begin{array}{cc:c} 3 & 5 & 1 \\ 3 & 5 & 2 \end{array}\right] \xrightarrow[R_2 - R_1]{R_2 \to} \left[\begin{array}{cc:c} 3 & 5 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

Clearly, this system is not consistent. So, its solution set $S = \emptyset$, the empty set. S is definitely not a line parallel to the line (i) which is the solution set of $A\bar{x} = \bar{0}$

Why does this not contradict Observation 6?

Recall that Observation 6 holds provided the system $A\bar{x} = \bar{b}$ has at least one solution.

[redacted]

[redacted] Given an m×n matrix A and an n×p matrix B, the product AB is given by the rule AB = [Av_1 Av_2 ….. Av_p] in column form where B = [v_1 v_2 ….. v_p] in column form. Construct an example to illustrate this rule. The matrix A in your example should be at least 3×3 and B should be at least 3×2. Then prove the rule in the general case.

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \end{bmatrix}$.

So $B = [\bar{v}_1 \ \bar{v}_2]$ where $\bar{v}_1 = (1,3,1)$ and $\bar{v}_2 = (2,4,1)$ in column form.

Then: $AB = \begin{bmatrix} 5 & 7 \\ 9 & 13 \\ 16 & 23 \end{bmatrix}$ ① and

$[A\bar{v}_1 \ A\bar{v}_2] = \begin{bmatrix} 5 & 7 \\ 9 & 13 \\ 16 & 23 \end{bmatrix}$ ②

From ① and ②, $AB = [A\bar{v}_1 \ A\bar{v}_2]$ in column form, as required.

General Proof: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, $AB = C = [c_{ij}]_{m \times p}$.

Then, for any fixed k, $1 \le k \le p$,

(PTO)

8a

[redacted] Q7.

The k-th column of C is:

$$\begin{bmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{bmatrix}_m = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} b_{jk} \\ \sum_{j=1}^{n} a_{2j} b_{jk} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} b_{jk} \end{bmatrix}_m \quad (3)$$

[The hint makes the calculation easier.]

OTOH, if $\bar{v}_k$ is the k-th column of B, then $\bar{v}_k = \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix}_n$ and so

$$A\bar{v}_k = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} b_{jk} \\ \sum_{j=1}^{n} a_{2j} b_{jk} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} b_{jk} \end{bmatrix} \quad (4)$$

Comparing (3) and (4), the k-th column of C is $A\bar{v}_k$. Since k was arbitrary, $C = [A\bar{v}_1, \ldots, A\bar{v}_p]$ in column form.

[redacted]

Q8 a) $I_{n\times n} = [a_{il}]_{n\times n}$, $a_{il} = \begin{cases} 1 & i=l \\ 0 & i\neq l \end{cases}$, $\forall\, 1\le i, l\le n$

$I \sim E$ $\quad R_i \to R_i + kR_j$, $j<i$

$E = [a'_{il}]_{n\times n}$

$$a'_{il} = a_{il} + k\,a_{jl} = \begin{cases} a_{ii} + k a_{ji} & i=l \\ a_{ij} + k a_{jj} & l=j \\ 0 & i\neq l,\ j\neq l \end{cases}$$

$$= \begin{cases} 1 & i=l \\ k & l=j \\ 0 & i\neq l,\ j\neq l \end{cases}$$

$$\Rightarrow E = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ j \\ \vdots \\ i \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & j & i & \cdots & n \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & k & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \end{array}$$

all diagonal elements are 1 and E is a lower triangular matrix and hence, E is a unit lower triangular matrix.

b) Let $A = [a_{ij}]$ $B = [b_{ij}]$ be $n\times n$ unit lower triangular matrices. By definition

$$a_{ij} = b_{ij} = 0, \quad i<j$$

Let $C = AB = [c_{ij}]$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Case I: $k \le i < j$.

In this case, $b_{kj} = 0$, for $k, j \in \{1, 2 - - n\}$ with $k<j$.

Case II: $i<k$.

In this case, $a_{ik} = 0$, for $i, k \in \{1, 2 - n\}$ with $i<k$.

Hence, for $i < j$ we have

$$c_{ij} = \sum_{k=1}^{i} a_{ik} b_{kj} + \sum_{k=i+1}^{n} a_{ik} b_{kj}$$

$$= 0 \quad + \quad 0$$

(from case I as $i<j$ so $b_{kj} = 0 \ \forall \ 1 \le k \le i$) (from Case II as $k > i$)

$$= 0$$

Now, $c_{ii} = \sum_{k=1}^{i-1} a_{ik} b_{ki} + a_{ii} b_{ii} + \sum_{k=i+1}^{n} a_{ik} b_{ki}$

$$= \underset{\text{(from case I)}}{0} + 1 \times 1 + \underset{\text{(from case II)}}{0} \quad (\text{as } a_{ii} = b_{ii} = 1 \ \forall \ 1 \le i \le n)$$

$$= 1$$

Hence, $C = AB$ is a unit lower triangular matrix.

c) Let $A = \begin{bmatrix} 1 & & & 0 \\ a_{21} & 1 & & \\ a_{31} & a_{32} & 1 & \\ \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} \cdots & 1 \end{bmatrix}$ be a unit lower triangular matrix.

$Ax = 0 \Rightarrow x_1 = 0$

$a_{21} x_1 + x_2 = 0 \Rightarrow x_2 = 0$

$a_{31} x_1 + a_{32} x_2 + x_3 = 0 \Rightarrow x_3 = 0$

$\vdots$

$a_{n1} x_1 + a_{n2} x_2 + \cdots + x_n = 0 \Rightarrow x_n = 0$

Hence, the system $Ax=0$ has only trivial sol^n which implies that A is invertible.

As $AA^{-1} = I = [c_{ij}]$, $c_{ii} = 1 \ \forall 1 \le i \le n$, $c_{ij} = 0 \ \forall \ i \ne j$
$1 \le i, j \le n$

Let $A = [a_{ij}]$ be unit lower triangular matrix.

Then, $a_{ij} = 0 \quad \forall \ i < j$ and $a_{ii} = 1 \quad \forall \ i$

Let $A^{-1} = [b_{ij}]$. Then,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{i} a_{ik} b_{kj} \quad (\text{as } a_{ik} = 0 \ \forall k > i$$

Now, $c_{1j} = a_{11} b_{1j} = b_{1j} \qquad (\text{as } a_{11} = 1)$

$= 0 \quad \forall \ j > 1 \quad —① \quad (\text{as } c_{1j} = 0 \ \forall \ j > 1)$

$$c_{2j} = a_{21} b_{1j} + a_{22} b_{2j}$$

$$= a_{21} b_{1j} + b_{2j} \quad (\because a_{22} = 1)$$

$= b_{2j} \quad \forall \ j > 1 \quad (\text{as } b_{1j} = 0)$ see ①

$\Rightarrow b_{2j} = 0 \quad \forall \ j > 2 \quad (\text{as } c_{2j} = 0 \ \forall \ j > 2)$

Now, $c_{3j} = a_{31} b_{1j} + a_{32} b_{2j} + a_{33} b_{3j}$

$= b_{3j} \ \forall \ j > 2 \ (\text{as } b_{1j} = b_{2j} = 0 \ \forall \ j > 2$ and $a_{33} = 1)$

$\Rightarrow c_{3j} = b_{3j} = 0 \quad \forall \ j > 3 \quad [\text{as } c_{3j} = 0 \ \forall \ j > 3]$

Proceeding in this manner we can get

$$b_{ij} = c_{ij} = 0 \quad \forall \quad i < j.$$

Also, $c_{ii} = \sum_{k=0}^{i-1} a_{ik} b_{ki} + a_{ii} b_{ii}$

$1 = 0 + a_{ii} b_{ii} \quad (\text{as } b_{ki} = 0 \ \forall \ k < i)$

$\Rightarrow 1 = b_{ii} \qquad (\text{as } a_{ii} = 1 \quad \forall \ i)$

Hence, $A^{-1} = [b_{ij}]$ is a unit lower triangular matrix.

[redacted]

8

a) Show that an elementary matrix E obtained by replacement of a row R_i of I by $R_i + kR_j$, where $j < i$, is a unit lower triangular matrix.

b) Show that the product of two unit lower triangular matrices is again a unit lower triangular matrix.

c) Show that if A is a unit lower triangular matrix, then A is invertible and A^{-1} is also a unit lower triangular matrix.

ALTERNATIVE PROOF OF 2nd part of [redacted] Q8 c.

Let A be a unit lower triangular matrix. As shown above, A is invertible. Hence, A is row-equivalent to I, i.e.

$(E_p \cdots E_1) A = I$, where the E_i are elementary matrices. ①

However, from a), each E_i is also unit lower triangular. (*)

Now, by Corollary 1.1 to VIT, the same sequence of elementary operations which reduces A to I, also reduces I to A^{-1}, i.e. $(E_p \cdots E_1) I = A^{-1}$ ②,

or $A^{-1} = E_p E_{p-1} \cdots E_1 = E$, say ③

Using Q (b), *, and the fact that elementary matrices are invertible, E is unit lower triangular, proving the result.

Q([redacted] a) $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

9.

$$A \sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 1/2 & 3/2 & 1/2 \\ 0 & 1/2 & 1/2 & 3/2 \end{bmatrix} \qquad \begin{array}{ll} R_2 \to R_2 - \frac{R_1}{2} & (1) \\ R_3 \to R_3 - R_1/2 & (2) \\ R_4 \to R_4 - \frac{R_1}{2} & (3) \end{array}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & \frac{3}{2} & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 4/3 \end{bmatrix} \qquad \begin{array}{ll} R_3 \to R_3 - \frac{R_2}{3} & (4) \\ R_4 \to R_4 - \frac{R_2}{3} & (5) \end{array}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = U \qquad R_4 \to R_4 - \frac{1}{4}R_3 \quad (6)$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/4 & 1 \end{bmatrix} \qquad R_4 \to R_4 + \frac{1}{4}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 \\ 0 & 1/3 & 1/4 & 1 \end{bmatrix} \qquad \begin{array}{l} R_4 \to R_4 + \frac{R_2}{3} \\ R_3 \to R_3 + R_2/3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1 & 0 \\ \frac{1}{2} & 1/3 & 1/4 & 1 \end{bmatrix} = L \qquad \begin{array}{l} R_4 \to R_4 + R_1/2 \\ R_3 \to R_3 + R_1/2 \\ R_2 \to R_2 + R_1/2 \end{array}$$

It can be verified that $LU = A$.

$$Ax = b \Rightarrow LUx = b$$
$$\Rightarrow Ux = L^{-1}b = y$$

$$Ly = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1 & 0 \\ 1/2 & 1/3 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$\Rightarrow y_1 = 1$

$\frac{y_1}{2} + y_2 = -1 \Rightarrow y_2 = -1 - \frac{1}{2} = -\frac{3}{2}$

$\frac{y_1}{2} + \frac{y_2}{3} + y_3 = -1 \Rightarrow y_3 = -1 - \frac{1}{2} - \frac{1}{3} \times \left(-\frac{3}{2}\right)$

$= -1 - \frac{1}{2} + \frac{1}{2} = -1$

$\frac{y_1}{2} + \frac{y_2}{3} + \frac{y_3}{4} + y_4 = 1$

$\Rightarrow y_4 = 1 - \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4}$

$y = \begin{bmatrix} 1 & -\frac{3}{2} & -1 & \frac{5}{4} \end{bmatrix}^T$

$Ux = y$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3/2 \\ -1 \\ 5/4 \end{bmatrix}$$

$\Rightarrow$ ① $2x_1 + x_2 + x_3 + x_4 = 1$, ② $\frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = -\frac{3}{2}$

③ $\frac{4}{3}x_3 + \frac{1}{3}x_4 = -1$ ④ $\frac{5}{4}x_4 = \frac{5}{4} \Rightarrow \boxed{x_4 = 1}$

$\Rightarrow \frac{4}{3}x_3 = -1 - \frac{1}{3}x_4$

$= -1 - \frac{1}{3} = -\frac{4}{3}$

$\Rightarrow \boxed{x_3 = -1}$

Now, from ② $\frac{3}{2}x_2 = -\frac{3}{2} - \frac{1}{2} + \frac{1}{2} = -\frac{3}{2}$

$\Rightarrow x_2 = -1$

From ①, $2x_1 + x_2 + x_3 + x_4 = 1$

$\Rightarrow 2x_1 = 1 - 1 + 1 + 1 = 2$

$\Rightarrow \boxed{x_1 = 1}$

Hence $x = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T$

TUT 03

~~10(a)~~ 10a

Q10 (a) For any square matrix A, if A^k is invertible for some positive integer k, then A itself is invertible.

Ans: TRUE.

This immediately follows from Corollary 1.3 (See ~~notes~~ slides for L09 on Monday 20230828).

Remark: The following justification is WRONG.

Put $B = A^k = \underbrace{A.A.\cdots.A}_{k \text{ times}}$.

Since B is invertible (given),

$B^{-1} = (A.A.\cdots.A)^{-1} = \underbrace{A^{-1}.A^{-1}\cdots.A^{-1}}_{k \text{ times}}$.

Hence, A is invertible.

This wrong proof is using Observation 4 in the Quick Review of Invertible Matrices (L07, Monday 20230821). But this already

Q10(a) – cont'd

assumes that the factors are invertible. Hence, this proof is a circular proof (already assumes what is to be proved).

(b) If a 3x3-matrix satisfies $A^3 = [0]$, then $A = [0]$.

Ans. FALSE. Here is one counter-example ~~to~~ show it is not TRUE.

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Then: $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [0]$.

$\therefore A^3 = A[0] = [0]$. But $A \neq [0]$.

Remark: Finding counter-examples is a matter of trial and error. Since we know from 10(a), that A cannot be invertible, a good guess would be to take A an RREF matrix not equal to I_3. This is ~~an~~ a rather easy counter-example – there are many.

S[redacted]

Q[redacted] a) If A is invertible, it cannot be
11. a zero-divisor (TRUE or FALSE)?
Justify.

Ans: TRUE.
Suppose BWOC that A is invertible, but is a zero-divisor. Then there exists a non-zero matrix B such that $AB = [0]$
Multiplying on the left by A^{-1}, we get: $A^{-1}(AB) = A^{-1}[0]$

$$\Rightarrow (A^{-1}A)B = [0]$$

$$\Rightarrow IB = [0]$$

$$\Rightarrow B = [0] \quad \Rightarrow\!\Leftarrow$$

Since B was assumed non-zero.

b) If A is not invertible, it must be a zero-divisor (TRUE or FALSE)
Justify.

Ans: TRUE.
Since A is not invertible, the homogeneous system $A\bar{x} = \bar{0}$ has a non-trivial solution, say $\bar{v}=$
This follows from VIT.
Put $B = [\bar{v} \ \bar{v} \ \bar{v}]$, i.e. the matrix with $\bar{v}$ as its columns. Then $AB = A[\bar{v} \ \bar{v} \ \bar{v}]$
$= [\bar{0} \ \bar{0} \ \bar{0}] = [0]$.

Q(5)(a) Find an LU decomposition of (5)

12.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_3 \to} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$ (Using short cut method)

Check: $LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} =$

$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} = A$, as required.

(b) Solve $A\bar{x} = \bar{b}_1 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

Step 1. Solve $L\bar{y} = \bar{b}$

$y_1 = 1$

$2y_1 + y_2 = 4 \Rightarrow y_2 = 2$

$3y_1 + y_2 + y_3 = 5 \Rightarrow y_3 = 0$

[redacted]
[redacted](h) cont'd:-

Now solve $U\bar{x} = \bar{y}$ or $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} x_1 + 2x_2 + 5x_3 &= 1 \\ 2x_2 + 6x_3 &= 2 \\ 0 &= 0 \end{aligned}$$

Solve by backward substitution:-

$$2x_2 = 2 - 6x_3$$

$$x_1 = 1 - 2x_2 - 5x_3$$

x_3 is like a parameter.

(i) Put $x_3 = 0 \Rightarrow 2x_2 = 2 \Rightarrow x_2 = 1$

and $x_1 = 1 - 2(1) = -1$

Check $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, as reqd.

Putting $x_3 = 1$, we get $\bar{x} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ as another solution.

Infinitely many solutions.

Now solve $A\bar{x} = \bar{b}_2 = \begin{bmatrix} 3 \\ 7 \\ 15 \end{bmatrix}$

[redacted] ii) cont'd [redacted]

$L\bar{y} = \bar{b}_2$:- $\quad y_1 = 3$

$$2y_1 + y_2 = 7 \Rightarrow y_2 = 1$$

$$3y_1 + y_2 + y_3 = 15 \Rightarrow y_3 = 5$$

Now $U\bar{x} = \bar{y}$ becomes

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

or
$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= 3 \\ 2x_2 + 6x_3 &= 1 \\ 0 &= 5 \end{aligned} \quad \Big| \rightarrow \text{NOT CONSISTENT}$$

Explanation:

From U we see that A is not invertible. $\therefore$ the homogeneous system $A\bar{x} = \bar{0}$ has infinitely many solutions. The non-homogeneous system $A\bar{x} = \bar{b}$, therefore, could either have infinitely many solutions or be inconsistent. First case happened for $\bar{b}_1$; second case happened for $\bar{b}_2$.

[redacted]: a) Get an LU decomposition [redacted]

13. of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 6 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[3R_2]{R_3 \to R_3 -} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ (Short-cut)

Check: $LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} = A,$

as reqd.

(b) Now solve $A\bar{x} = \bar{b} = \begin{bmatrix}4\\9\\14\end{bmatrix}$

$L\bar{y} = \bar{b}$ becomes $\begin{bmatrix}1&0&0\\2&1&0\\1&3&1\end{bmatrix}\begin{bmatrix}y_1\\y\\y_3\end{bmatrix} = \begin{bmatrix}4\\9\\1\end{bmatrix}$

or $y_1 = 4$

$2y_1 + y_2 = 9 \Rightarrow y_2 = 1$

$y_1 + 3y_2 + y_3 = 14 \Rightarrow y_3 = 7$

$U\bar{x} = \bar{y}$, i.e. $\begin{bmatrix}1&1&1&1\\0&2&-1&-1\\0&0&4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$

$$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 4$$
$$2x_2 - x_3 - x_4 = 1$$
$$4x_3 + 3x_4 = 7$$

We treat x_4 as a parameter.
For example, $x_4 = 1$ gives $x_3 = 1$

$\Rightarrow 2x_2 = 1 + x_3 + x_4 \Rightarrow x_2 =$

$\therefore x_1 = 4 - x_2 - x_3 - x_4 = \frac{1}{2}$

Check: $\begin{bmatrix}1&1&1&1\\2&4&1&1\\1&7&2&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\\frac{3}{2}\\1\\1\end{bmatrix} = \begin{bmatrix}4\\9\\14\end{bmatrix}$

Since x_4 is a parameter, there a
infinitely many solutions. Since
fewer rows than variables, $A\bar{x} = \bar{0}$
necessarily has infinitely many
Hence, if $A\bar{x} = \bar{b}$ is consistent, it
has infinitely many solutions.