MTH100 - MONSOON 2023

NOTES FOR MATRIX INVERSION

An example for finding the inverse of a matrix by row-reduction.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}.$$

We work with

$$[A : I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2]{R_3 \to} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to (-1)R_3]{R_2 \to (-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\qquad\qquad\qquad I \qquad\qquad\qquad A^{-1}$$

Check:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

<u>Remark</u>: This method is preferable to the adjoint/determinant formula, which requires approx. $n!$ calculations, Gauss-Jordan elimination as above ~~requires~~ requires approx. $\frac{3}{2}n^3$ calculations.

MTH 100 - MONSOON ~~2018~~ ~~2019~~ 2023

Notes for V I T - final part

(a) is equivalent to (d)

(a) $\Rightarrow$ (d). Given: A is invertible.

RTP: $A\bar{x} = \bar{b}$ has at least one solution for any choice of $\bar{b}$.

Proof: Let $\bar{b}$ be any arbitrary but fixed vector. Since A is invertible, put $\bar{v} = A^{-1}\bar{b}$. Then, $A\bar{v} = A(A^{-1}\bar{b})$

$$= I\bar{b} = \bar{b}.$$

In other words, $\bar{v}$ is a solution of $A\bar{x} = \bar{b}$ as required.

(d) $\Rightarrow$ (a). Given: $A\bar{x} = \bar{b}$ has a solution for every choice of $\bar{b}$.

RTP: A is invertible (recall A is $m \times m$ square)

We consider the m vectors:

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \bar{e}_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(PTO)

Proof (cont'd):-

4

i.e. $\bar{e}_i$ is the vector which has its i-th coordinate $= 1$ and all other coordinates 0.

[These vectors are called the standard basis vectors, and play an important role – they will be considered in more detail later.]

So, now, by the given condition, the non-homog. system $A\bar{x} = \bar{e}_i$ has a solution for $i = 1, 2, \ldots, m$. Let $\bar{v}_1, \ldots, \bar{v}_m$ be the solution vectors, i.e. $A\bar{v}_1 = \bar{e}_1$, $A\bar{v}_2 = \bar{e}_2$, $\ldots$, $A\bar{v}_m = \bar{e}_m$.

Construct the matrix B which has $\bar{v}_1, \ldots, \bar{v}_m$ as its columns, i.e. $B = [\bar{v}_1\ \bar{v}_2\ \ldots\ \bar{v}_m]_{m\times m}$,

$$\therefore AB = A[\bar{v}_1\ \bar{v}_2\ \ldots\ \bar{v}_m]$$
$$= [A\bar{v}_1\ A\bar{v}_2\ \ldots\ A\bar{v}_m]^{*}$$

~~(recall tutorial problem)~~

$$= [\bar{e}_1\ \bar{e}_2\ \ldots\ \bar{e}_m] = I_m$$

Since $AB = I$, A has a right inverse.

$\therefore$ By Corollary 1.2, A has an inverse, i.e. A is invertible, as required.

* This is a standard result in matrix algebra.

* If the product AB is defined, and $B = [\bar{v}_1 \bar{v}_2 \ldots \bar{v}_k]$ in column form, then AB in column form is $[A\bar{v}_1 \ldots A\bar{v}_k]$.

See Lay Section 1.2