Example of an Inconsistent System

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 2 & 5 & 9 & 9 \\ 3 & 6 & 12 & 13 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad (*)$$

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Note that in this case, the associated homogeneous system actually has infinitely many solutions. The RREF matrix of the co-efficient matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 9 \\ 3 & 6 & 12 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It has $\bar{v} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ as a general solution.

Check: $A\bar{v} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 9 \\ 3 & 6 & 12 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

The RREF matrix R of the augmented matrix $[A : \bar{b}]$ is of the form $[S : \bar{b}_1]$, where S is the RREF matrix of ~~A~~ the coefficient matrix, and $\bar{b}_1$ is the last column of R, obtained from $\bar{b}$ by row-reduction. Let us write this as a linear system $S\bar{x} = \bar{b}$ or

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$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow$$

$$x_1 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1 \longrightarrow \text{!!!}$$

This equation is not TRUE for any x_1, x_2, x_3. In other words, the system CANNOT HAVE ANY SOLUTIONS.

INCONSISTENT!

An example with unique solution:

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 10 \\ 3 & 4 & 8 & 15 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_1 - 2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \text{(Gaussian Phase Complete)}$$

$$\xrightarrow[R_1 \to R_1 - 2R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_1 - R_2]{R_1 \to} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution is $\bar{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 4 \\ 10 \\ 15 \end{bmatrix}$ ✓

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2 useful properties of RREF matrices:

1. If R is an ~~RREF matrix~~ $m \times n$ RREF matrix, then the $m \times k$-matrix obtained by deleting the <u>last</u> $n-k$ columns is also an RREF matrix (not true for deleting leading $n-k$ columns), $0 < k < n$.

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 & 6 \end{bmatrix} \xrightarrow{\text{take } k=3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \to \text{RREF}$$

But $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} \to$ not RREF.

2. The RREF matrix R of a matrix A has a $\bar{0}$ column if and only if A has a $\bar{0}$ column in the same position.