Q5)

A) I will try to compute the posterior using the multiplication of the prior of the mean with the likelihood and then complete the square in the exponent. $p(\theta|x) = p(\theta) \cdot p(x|\theta)$; putting in the necessary data from the problem statement we have something like the following :

$$N(\mu, \sigma^2) \cdot N(\mu_0, \sigma_0{}^2) = c \cdot exp(-(x_1-\mu)^2/2\sigma^2 - (\mu-\mu_0)^2/2\sigma_0^2)$$

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(a) $$P(\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(\frac{-(\mu-\mu_0)^2}{2\sigma_0^2}\right)$$

$$P(x|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

$$P(\mu|x) = \frac{P(\mu)\,P(x|\mu)}{\int_{-\infty}^{\infty} P(\mu)\,P(x|\mu)\,d\mu} = \frac{P(\mu)\,P(x|\mu)}{P(x)}$$

$$\propto P(\mu)\,P(x|\mu)$$

$$= \frac{1}{2\pi\sqrt{\sigma_0^2\sigma^2}} \exp\left\{\frac{-\mu^2+2\mu\mu_0-\mu_0^2}{2\sigma_0^2} - \frac{x^2-2\mu x+\mu^2}{2\sigma^2}\right\}$$

$$= (\text{Constant}) \exp\left\{\frac{-\mu^2\sigma^2+2\mu\mu_0\sigma^2-\mu_0^2\sigma^2-\sigma_0^2x^2+2\mu\sigma_0^2x-\mu^2\sigma_0^2}{2\sigma_0^2\sigma^2}\right\}$$

$$\propto \exp\left(\frac{-\mu^2(\sigma^2+\sigma_0^2)+2\mu(\mu_0\sigma^2+\sigma_0^2x)}{2\sigma_0^2\sigma^2} - (\mu_0^2\sigma^2+\sigma_0^2x^2)\right)$$

$$\propto \exp\left\{\frac{-\mu^2+2\mu\left(\frac{\mu_0\sigma^2+\sigma_0^2x}{\sigma^2+\sigma_0^2}\right)-\left(\frac{\mu_0\sigma^2+\sigma_0^2x}{\sigma^2+\sigma_0^2}\right)^2}{2\left(\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}\right)}\right\}$$

$$\times \exp\left\{-\frac{\mu_0^2\sigma^2+\sigma_0^2x^2}{2\sigma_0^2\sigma^2}\right\}$$

$$\propto \exp\left\{\frac{-\left(\mu-\overbrace{\left(\frac{\mu_0\sigma^2+x\sigma_0^2}{\sigma^2+\sigma_0^2}\right)}^{\mu_1}\right)^2}{\underbrace{2\left(1/\sigma_0^2+1/\sigma^2\right)}_{\sigma_1^2}}\right\}$$

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(b) $P(x|x_1) = \int_\theta P(x,\theta|x_1)\,d\theta = \int_\theta P(x'|x_1,\theta)\,P(\theta|x_1)\,d\theta$

$$= \int_\theta P\left(x|\theta\right) P(\theta|x_1)\,d\theta$$

$$\int_\theta P\left(x|\theta\right) P(\theta|x_1)\,d\theta = \int_\theta P(x = x_1-\theta+\theta|\theta)\,P(\theta|x_1)\,d\theta$$

$$= \int_\theta P(x = x_1+\theta\,|\,\theta)\,P(\theta|x_1)\,d\theta = \int_\theta P(x-\theta = x_1\,|\theta)\,P(\theta\,|\,x_1)\,d\theta$$

$$= \int_\theta \underbrace{P(x-\theta|\theta)\,P(\theta|x_1)}\,d\theta$$

↳ convolution of two func^n s.

$$\Rightarrow\quad x|x_1 \sim N\left(\mu_1, \sigma_x^2 + \tau_1^2\right)$$

given

calculated in part (a)

calculated in part (a)

(e) $\mu \sim N(\mu_0, \sigma_0^2)$ $\quad x_i|\mu \sim N(\mu, \sigma^2)$

$$P(\mu/\underline{x}) \propto P(\mu)\, P(\underline{x}|\mu) = P(\mu)\, P(x_1|\mu)\, P(x_2|\mu) \cdots P(x_n|\mu)$$

$$= \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{\frac{-(\mu-\mu_0)^2}{2\sigma_0^2}\right\} \times \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-(x_i-\mu)^2}{2\sigma^2}\right\}$$

$$= \frac{1}{(2\pi)^{n+1/2}\sqrt{\sigma_0^2\sigma^{2n}}} \exp\left\{\frac{-\mu^2+2\mu\mu_0-\mu^2}{2\sigma_0^2} - \sum_{i=1}^{n}\frac{x_i^2-2\mu x_i+\mu^2}{2\sigma^2}\right\}$$

$$\propto \exp\left\{\frac{-\mu^2(\sigma^2+n\sigma_0^2)+2\mu(\mu\sigma^2+\sigma_0^2x_1+\cdots\sigma_0^2x_n)-(\mu_0^2\sigma^2+\cdots\sigma_0^2x_n^2)}{2\sigma_0^2\sigma^2}\right\}$$

$$\propto \exp\left\{\frac{-\mu^2+2\mu\left(\frac{\mu_0\sigma^2+\sum\sigma_0^2x_i}{\sigma^2+n\sigma_0^2}\right)-\left(\frac{\mu_0\sigma^2+\sum\sigma_0^2x_i}{\sigma^2+n\sigma_0^2}\right)^2}{2\,\frac{\sigma_0^2\sigma^2}{\sigma^2+n\sigma_0^2}}\right\}$$

$$\times \exp\left\{-\frac{\mu_0^2\sigma^2+\sum\sigma_0^2x_i^2}{2\sigma_0^2\sigma^2}\right\}$$

$$\propto \exp\left\{-\frac{\left(\mu-\frac{\mu_0\sigma^2+\sum\sigma_0^2x_i}{\sigma^2+n\sigma_0^2}\right)^2}{2\,\frac{\sigma_0^2\sigma^2}{\sigma^2+n\sigma_0^2}}\right\}$$

$$\mu_1 = \sigma_1^2\left(\frac{\mu_0}{\sigma_0^2}+\frac{\bar{x}}{\sigma^2/n}\right) \quad \bar{x} \Rightarrow \text{sample mean}$$

$$\sigma_1^2 = \left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)^{-1}$$

$$P(\tilde{x}|x) = \frac{1}{(\sqrt{2\pi})^m} \det\left(\sigma^2 I + \tau_n^2 ii^T\right)^{-1/2} \cdot \exp\left(-\frac{1}{2}(\tilde{x}-\mu_n i)^T (\sigma^2 I + \tau_n^2 ii^T)^{-1} (\tilde{x} - \mu_n i)\right)$$

$$\tilde{x} = \begin{bmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

I is an $m \times m$ matrix.

$\mu_n i$ (posterior mean of μ)

$\sigma^2 I + \tau_n^2 ii^T$ (τ_n = posterior variance of μ)

and the whole term is the co variance matrix.

extending the same logic as the case where $m=1$

↳ we can be a little inventive too → we can consider the sufficient statistics of the data that is $T(x) = \bar{x}$ = sample mean of $\{x_1, \dots x_n\}$

can convert the whole problem into the univariate case. again.