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Q3 (a)

C ~> not a full rank matrix, we use the inversion lemma on this to get $C^{-1} = \sigma^{-2}(I - WM^{-1}W^T)$

$$M \equiv W^TW + \sigma^2 I$$

plug in the MLE estimates for W, σ^2

$$C^{-1} = \frac{1}{\sigma^2}\left[I - U_M(L_M - \sigma^2 I)^{0.5}(L_M - \cancel{\sigma^2 I} + \cancel{\sigma^2 I})^{-1}(L_M - \sigma^2 I)^{0.5} U_M^T\right]$$

$$= \frac{1}{\sigma^2}\left[I - U_M(L_M - \sigma^2 I)^{0.5}(L_M)^{-1}(L_M - \sigma^2 I)^{0.5} U_M^T\right]$$

now the problem reduces to prove that:

$$(L_M - \sigma^2 I)^{0.5} L_M^{-1} (L_M - \sigma^2 I)^{0.5} = \mathrm{diag}\left(1 - \frac{\sigma^2}{\lambda_1}, 1 - \frac{\sigma^2}{\lambda_2} \ldots\right)$$

note $L_M =$ is a diagonal matrix $d(\lambda_1, \lambda_2 \ldots \lambda_m)$

$$(L_M - \sigma^2 I) = \mathrm{diag}(\lambda_i) - \mathrm{diag}(\sigma^2)$$

$$= \mathrm{diag}(\lambda_i - \sigma^2) \qquad (i = 1, \ldots, M)$$

note $L_M^{-1} = \mathrm{diag}(\lambda_1^{-1}, \lambda_2^{-1} \ldots \lambda_M^{-1}) = \mathrm{diag}(\lambda_i^{-1})$

$$(i = 1, \ldots, M)$$

note $\mathrm{diag}(a_i) \cdot \mathrm{diag}(b_i) \cdot \mathrm{diag}(c_i) = \mathrm{diag}(a_i b_i c_i)$

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using the following identities the product of $\longrightarrow$

$$(L_M - \sigma^2 I)^{0.5} (L_M^{-1}) (L_M - \sigma^2 I)^{0.5}$$

$$\equiv \mathrm{diag}\left((\lambda_i - \sigma^2)^{0.5} \lambda_i^{-1} (\lambda_i - \sigma^2)^{0.5} \right)$$

$$(i = 1, \dots, M)$$

$$= \mathrm{diag}\left((\lambda_i - \sigma^2) \times \frac{1}{\lambda_i} \right)$$

$$= \mathrm{diag}\left(1 - \sigma^2 / \lambda_i \right) \quad (i = 1, \dots, M)$$

hence $(L_M - \sigma^2 I)^{0.5} L_M^{-1} (L_M - \sigma^2 I)^{0.5} = \mathrm{diag}\left(1 - \frac{\sigma^2}{\lambda_1}, 1 - \frac{\sigma^2}{\lambda_2}, \dots \right)$

(b) in this problem we have to prove that $\longrightarrow$

$$\ln |WW^T + \sigma^2 I| = (D - M) \ln(\sigma^2) + \sum_{i=1}^{M} \ln(\lambda_i)$$

let's take the matrix $(WW^T + \sigma^2 I)$ to be C

$$|C| = |U_M (L_M - \sigma^2 I)^{0.5} (L_M - \sigma^2 I)^{0.5} U_M^T + \sigma^2 I|$$

$$= |U_M (L_M - \sigma^2 I) U_M^T + \sigma^2 I|$$

use matrix determinant lemma $\longrightarrow$

$$|A + UWV^T| = |W^{-1} + V^T A^{-1} U| \cdot |W| \cdot |A|$$

$$\Rightarrow \det\left((L_M - \sigma^2 I)^{-1} + \frac{1}{\sigma^2} \mathrm{diag}(1, 1, \dots) \right) \det\left(\mathrm{diag}(\lambda_i - \sigma^2) \right) \det(\sigma^2 I)$$

$$\Rightarrow \det\left(\frac{\cancel{\sigma^2} + \lambda_i - \cancel{\sigma^2}}{(\lambda_i - \sigma^2)(\sigma^2)} \right) \det\left(\mathrm{diag}(\lambda_i - \sigma^2) \right) \det(\sigma^2 I)$$

$$\ln |C| = \ln \left(\frac{\prod_{i=1}^{M} \lambda_i}{\sigma^{2M} \cancel{\prod_{i=1}^{M} (\lambda_i - \sigma^2)}} \cdot \cancel{\prod_{i=1}^{M} (\lambda_i - \sigma^2)} \cdot \sigma^{2D} \right)$$

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$$\ln|C| = \ln\left(\prod_{i=1}^{M} \lambda_i\right) + \ln\left(\sigma^{2(D-M)}\right)$$

$$\ln\left|WW^T + \sigma^2 I\right| = \sum_{i=1}^{M} \ln(\lambda_i) + (D-M)\ln\left(\sigma^2\right)$$