Q2

$x_0 \; \longrightarrow \; x_1 \quad \longleftrightarrow \quad -1 \; \longrightarrow \; 1 \qquad$ {coordinate transform}

$$\boxed{\begin{aligned} y &= ax + b \\ &= \frac{2}{x_1 - x_0}\,x - \frac{x_1 + x_0}{x_1 - x_0} \end{aligned}}$$

→ maps the domain to $[-1, 1]$; the de remains the same as the right hand side $\equiv 0$.

suppose ① $u_{NN}(\omega, b) = \sum_{j=1}^{N} a_j \sin(\omega_j y + b_j)$

(II) $v(k) = \sin(k\pi y)$

writing the residual for → $\boxed{u_{yy} = 0}$

↳
$$R_k^{(1)} = \int_{-1}^{1} \sin(k\pi y)\,\frac{d^2}{dy^2}\left(\sum_{j=1}^{N} a_j \sin(\omega_j y + b_j)\right) dy$$

$$= -\int_{-1}^{+1} \sin(k\pi y)\left\{\sum_{j=1}^{N} a_j \omega_j^2 \sin(\omega_j y + b_j)\right\} dy$$

→ exchange the summ', integral.

$$= -\sum_{j=1}^{N} \int_{-1}^{+1} \left[\underbrace{a_j \omega_j^2 \sin(\omega_j y + b_j)\sin(k\pi y)}_{\text{product of sines}}\, dy\right]$$

hence we

$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

$$= -\sum_j \int_{-1}^{1} \frac{a_j \omega_j^2}{2}\left\{\cos((\omega_j - k\pi)y + b_j) - \cos((\omega_j + k\pi)y + b_j)\right\} dy$$

$$= -\frac{1}{2}\sum_j a_j \omega_j^2 \left\{ \frac{\sin((\omega_j - k\pi)y + b_j)}{\omega_j - k\pi}\Bigg|_{-1}^{1} - \frac{\sin((\omega_j + k\pi)y + b_j)}{\omega_j + k\pi}\Bigg|_{-1}^{1} \right\}$$

$$= -\frac{1}{2}\sum_j a_j \omega_j^2 \left\{ \frac{\sin((\omega_j - k\pi) + b_j)}{\omega_j - k\pi} + \frac{\sin((\omega_j - k\pi) - b_j)}{\omega_j - k\pi} - \left(\frac{\sin(\omega_j + b_j + k\pi)}{\omega_j + k\pi} - \frac{\sin(-\omega_j + b_j - k\pi)}{\omega_j + k\pi}\right)\right\}$$

↳ we can use the expansion of $\sin(A+B) = \sin A\cos B + \cos A \sin B$

$$= -\frac{1}{2}\sum_j a_j \omega_j^2 \Bigg\{ \frac{1}{\omega_j - k\pi}\Big(\sin(\omega_j + b_j)\cos(k\pi) - \cos(\omega_j + b_j)\cancel{\sin(k\pi)} + \sin(\omega_j - b_j)\cos(k\pi) - \cancel{\sin(k\pi)}\cos(\omega_j - b_j)\Big)$$
$$- \frac{1}{\omega_j + k\pi}\Big(\sin(\omega_j + b_j)\cos(k\pi) + \cancel{\sin(k\pi)}\cos(\omega_j + b_j) - \sin(b_j - \omega_j)\cos(k\pi) + \cancel{\sin(k\pi)}\cos(b_j - \omega_j)\Big)\Bigg\}$$

$$= -\frac{1}{2}\sum a_j \omega_j^2 \left\{ \frac{1}{\omega_j - k\pi}\left\{\sin(\omega_j + b_j)(-1)^k + \sin(\omega_j - b_j)(-1)^k\right\} - \frac{1}{\omega_j + k\pi}\left\{\sin(\omega_j + b_j)(-1)^k + \sin(\omega_j - b_j)(-1)^k\right\}\right\}$$

{take common}

$$\frac{-1}{2}\sum_j a_j w_j^2 (-1)^k \times 2\sin(w_j)\cos(b_j)\left\{\frac{1}{w_j-k\pi}-\frac{1}{w_j+k\pi}\right\}$$

$$= -\sum_j \frac{a_j w_j^2 (-1)^k \sin(w_j)\cos(b_j)}{w_j^2 - k^2\pi^2}(2k\pi)$$

$$= (-1)^{k+1} 2k\pi \sum_{j=1}^{N} \frac{a_j w_j^2 \sin(w_j)\cos(b_j)}{(w_j^2 - k^2\pi^2)}$$

final form of the residual:

Physics informed loss $= 4k^2\pi^2\left(\sum_{j=1}^{N} \frac{a_j w_j^2 \sin(w_j)\cos(b_j)}{w_j^2 - k^2\pi^2}\right)^2$

boundary loss $= \left(\sum_j a_j \sin(-w_j + b_j) - 0\right)^2 + \left(\sum_j a_j \sin(w_j + b_j) - T_{int}\right)^2$

if known

for one bar case.

if we have two bar case then →

we have $u_{1\,NN}$, $u_{2\,NN}$ with $a_j\ w_j\ b_j$ / $a_j'\ w_j'\ b_j'$

boundary loss = (will have to flux term's difference at interface) + (will have predicted temp. difference at the interface)